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## A CLASS OF FUZZY MEASURES BASED ON TRIANGULAR NORMS†

### A general framework for the combination of uncertain information

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An axiomatic approach to a broad class of fuzzy measures in the sense of Sugeno is presented via the concept of triangular norm (*t*-norm for short). Fuzzy measures are actually set functions which are monotonic with respect to set inclusion. Triangular norms and conorms are semi-groups of the unit interval which have been thoroughly studied in the literature of functional equations. The proposed class encompasses probability measures, Zadeh's possibility measures and the dual notion of necessity measures. Any set function of the class can be expressed in terms of a density, and constructively defined out of this density. This feature makes the proposed framework attractive from a practical point of view for the representation and manipulation of subjective evidence. The link between *t*-norm and *t*-conorm based set functions and Shafer's belief functions is investigated.

INDEX TERMS: Triangular norm; probability; possibility; necessity; belief function; fuzzy measure; fuzzy set; uncertainty.

## I. INTRODUCTION

The term "fuzzy measure" was coined by Sugeno<sup>30</sup> in 1972 to denote a mapping from a  $\sigma$ -algebra to the unit interval whose key property is monotonicity with respect to set inclusion. Obviously, this property is far weaker than the usual additivity property of probability measures. In other words, a fuzzy measure is not a measure in the sense of classical mathematics, and a probability measure is a particular case of a fuzzy measure. In 1977, L. A. Zadeh<sup>36</sup> introduced a noticeable family of fuzzy measures, named "possibility measures", which proves to be

very relevant for representing some aspects of natural languages.‡ Independently, Shafer<sup>29</sup> developed a class of fuzzy measures called belief functions, which are, roughly speaking, superadditive probabilities. All of the aforementioned authors share the purpose of representing subjective evidence or uncertainty in the form of numerical values and finding rational rules for combining these values while dropping the too restrictive probabilistic setting.

Triangular norms come from the study of statistical metric spaces. They can be encountered in the Menger triangle inequality<sup>17,25</sup>

$$F_{pr}(x+y) \geq T(F_{pq}(x), F_{qr}(y))$$

†A joint and extended version of two unpublished papers (Prade,<sup>21</sup> Dubois<sup>4</sup>) respectively presented at a CNRS Round Table at the University of Lyon, France (June 1980) and at the Second International Seminar on Fuzzy Sets, Linz, Austria (September 1980).

‡Strictly speaking, possibility measures on non-finite domains are fuzzy measures (see definition in III.1) only under suitable conditions on the associated possibility "distribution" (density).

where  $F_{pr}$ ,  $F_{pq}$ ,  $F_{qr}$  are probability distribution functions of random variables which represent distances between locations  $p$  and  $r$ ,  $p$  and  $q$ ,  $q$  and  $r$  respectively, and  $T$  is a triangular norm, i.e. a semi-group operator of the unit interval satisfying some additional requirements, which are recalled in a further section. It is worth noticing that in his works on probabilistic geometry Menger,<sup>18</sup> in a paper written in French, was the first to coin the term "ensemble flou" (the French counter-part of the English term "fuzzy set"), which he later translated into "hazy set" (Menger<sup>19</sup>). According to Menger, a "probability of membership" is assigned to each element. In Zadeh's framework, the right concept is rather a "possibility of membership."

That both Zadeh and Sugeno on the one hand, and Menger on the other hand, make use of such a term as "fuzzy" (or some synonym) should not be regarded as a fortuitous event. The purpose of this paper is to demonstrate that quite general class of fuzzy measures can be defined in a very natural way via the concepts of triangular norms and conorms. The latter appear as soon as we admit the following assumption; the grade of uncertainty of a union of disjoint (mutually exclusive) events can be obtained by combination of the grades of uncertainty of each of the events. Besides it is worth noticing that a triangular norm (resp: conorm) also provides a good model for fuzzy-set-theoretic intersections (resp: union), as will be briefly shown later. One of the problems in dealing with monotonic set functions is that they are very difficult to handle at a practical level, because they do not necessarily derive from some sort of a density. Shafer meets the same problem with his approach. Sugeno<sup>30</sup> had to resort to a special parametered family of set functions and arbitrary combination rules. The proposed approach directly faces this difficulty by laying bare the most general combination rules for grades of uncertainty of events. In other words,  $t$ -norm or  $t$ -conorm based set functions underlie a density from which they can be constructively generated.

After some background on triangular norms has been recalled, triangular norm and conorm based set functions are successively investigated. It is proved that some of these set functions simultaneously stem from a  $t$ -norm and a  $t$ -conorm. In each case, the "densities" are exhibited. Lastly, an attempt to build a bridge between Shafer's approach and ours is provided.

## II. TRIANGULAR NORMS AND CONORMS

In this section, we briefly survey some material in functional equations which are helpful for the derivation of our results.

### 1. Triangular Norms

DEFINITION 1 (Menger<sup>17</sup>) A triangular norm  $T$  (a  $t$ -norm for short) is a two place real-valued function whose domain is the unit square  $[0, 1] \times [0, 1]$ , and which satisfies the following conditions

$$T1: T(0, 0) = 0; T(a, 1) = T(1, a) = a \\ \text{(boundary conditions)}$$

$$T2: T(a, b) \leq T(c, d) \text{ whenever } a \leq c; b \leq d \\ \text{(monotonicity)}$$

$$T3: T(a, b) = T(b, a) \text{ (symmetry)}$$

$$T4: T(a, T(b, c)) = T(T(a, b), c) \text{ (associativity)}$$

It is patent that  $T$  is a semi-group of  $[0, 1]$  with identity 1.

N.B. Let  $A$  and  $B$  be two fuzzy sets (Zadeh<sup>34</sup>) over a universe  $X$ , i.e. membership functions  $\mu_A$  and  $\mu_B: X \rightarrow [0, 1]$  are given. The membership function of the intersection  $A \cap B$  can be pointwisely defined as  $\mu_{A \cap B} = I(\mu_A, \mu_B)$ . It is easy to see that  $T1$  through  $T4$  are reasonable requirements for the mapping  $I$ . Such axioms are a subset of those given by Bellman and Giertz<sup>3</sup>. Thus a  $t$ -norm is a very general model for fuzzy set intersection (see Refs. 1, 5, 14, 22).

Simple, but important triangular norms are the minimum operator ( $\min(a, b)$ ) the product operator ( $a \cdot b$ ), the so-called  $T_m$  operator  $\max(0, a + b - 1)$ , the so-called  $T_w$  operator (defined by the boundary condition and  $T_w(a, b) = 0, \forall (a, b) \in [0, 1]^2$ ).

We have the following inequality

$$T_w(a, b) \leq \max(0, a + b - 1) \leq a \cdot b \leq \min(a, b) \quad (1)$$

Moreover, for any  $t$ -norm  $T$ , it holds:

$$T_w(a, b) \leq T(a, b) \leq \min(a, b) \quad (2)$$

Strong results have been obtained for characterizing a large class of  $t$ -norms by means of one place functions. Namely, Ling<sup>16</sup> has

proved that for any continuous  $t$ -norm satisfying the Archimedean property  $T5$

$$T5: T(a, a) < a, \quad \forall a \in (0, 1)$$

there exists a continuous and decreasing function  $f: [0, 1] \rightarrow [0, +\infty]$  such that

$$T(a, b) = f^*(f(a) + f(b)) \quad (3)$$

where  $f^*$  is the pseudo-inverse of  $f$ , defined by

$$f^*(a) = \begin{cases} 1 & \text{for } a \in [0, f(1)] \\ f^{-1}(a) & \text{for } a \in [f(1), f(0)] \\ 0 & \text{for } a \in [f(0), +\infty] \end{cases} \quad (4)$$

Note that "min" does not satisfy  $T5$ . Actually, min cannot be represented according to (3).  $f$  is called an additive generator of the  $t$ -norm  $T$  and is defined up to a positive multiplicative constant.

In the following we shall focus on two subclasses of continuous Archimedean  $t$ -norms:

— the *strict  $t$ -norms* i.e. continuous  $t$ -norms which are *strictly* increasing in each of their places. Schweizer and Sklar<sup>26</sup> have proved that any strict  $t$ -norm generator is such that  $f(0) = +\infty$ , and  $f(1) = 0$ . The corresponding  $t$ -norm is

$$T(a, b) = f^{-1}(f(a) + f(b)) \quad (5)$$

The typical strict  $t$ -norm is the product operator.

— The  $t$ -norms generated by functions  $f$  such that  $f(0)$  is finite and  $f(1) = 0$ . Note that given a sequence  $\{a_i\}_{i \in \mathbb{N}}$  of numbers in  $(0, 1)$ , there is some  $n_0 \in \mathbb{N}$  such that  $\sum_{i=1}^{n_0} f(a_i) > f(0)$ , so that  $T(a_0, a_1, \dots, a_{n_0}) = 0$  for a finite number  $n_0$ , so that  $T$  can be called a *nilpotent  $t$ -norm*.

The typical nilpotent  $t$ -norm is  $T_m$ .

*N.B.* Schweizer and Sklar<sup>26</sup> also provide a multiplicative representation for strict  $t$ -norms; stating  $h = e^{-f}$  in (5) yields

$$T(a, b) = h^{-1}(h(a) \cdot h(b)) \quad (6)$$

where  $h$ , a one-place strictly increasing  $[0, 1] \rightarrow [0, 1]$  mapping with  $h(0) = 0$ ,  $h(1) = 1$ , is called a multiplicative generator.

## 2. Triangular Co-norms

The dual class of mappings can be obtained by changing the boundary condition  $T1$  into

$$S1 \quad S(1, 1) = 1; S(0, a) = S(a, 0) = a$$

DEFINITION 2 A mapping  $S$  satisfying  $S1$  and  $T2$ – $T4$  is called a triangular conorm ( $t$ -conorm).

Any  $t$ -conorm  $S$  can be generated from a  $t$ -norm  $T$  through the transformation

$$S(a, b) = 1 - T(1 - a, 1 - b) \quad (7)$$

and conversely. For instance, transforming the four basic  $t$ -norms  $\min(a, b)$ ,  $ab$ ,  $\max(0, a + b - 1)$ ,  $T_w$  respectively yields  $\max(a, b)$ ,  $a + b - ab$ , ("probabilistic sum")  $\min(1, a + b)$  ("bounded sum"), and  $T_w^*$  (defined by  $S1$  and  $S(a, b) = 1 \forall (a, b) \in (0, 1]^2$ ).  $S$  and  $T$  in (7) are said to be *dual operators*.

A *strict  $t$ -conorm* is a continuous  $t$ -conorm which is increasing in both places. If  $f$  is a strict  $t$ -norm additive generator then  $\varphi(\cdot) = f(1 - \cdot)$  is a strict conorm additive generator. The corresponding conorm is  $\varphi^{-1}(\varphi(\cdot) + \varphi(\cdot))$ . Similarly  $k = e^{-\varphi}$  is a strict conorm multiplicative generator. The dual of a nilpotent  $t$ -norm will be called a nilpotent  $t$ -conorm;  $a + b - ab$  and  $\min(1, a + b)$  are typical strict and nilpotent  $t$ -conorms, respectively.

*N.B.* Conorms can be used for modelling fuzzy set unions, of course.

## 3. Negations

Searching for an extension of the usual fuzzy set complementation ( $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ ), Trillas<sup>31</sup> proposed the following axioms.

DEFINITION 3 A negation is a mapping  $C: [0, 1] \rightarrow [0, 1]$  such that

- C1  $C(0) = 1$
- C2  $C$  is involutive i.e.  $C(C(a)) = a$
- C3  $C$  is strictly decreasing
- C4  $\bar{C}$  is continuous.

If  $T(\cdot, \cdot)$  is a  $t$ -norm, then  $S(\cdot, \cdot) = C[T(C(\cdot), C(\cdot))]$  is a  $t$ -conorm and conversely  $T(\cdot, \cdot) = C[S(C(\cdot), C(\cdot))]$ .  $S$  and  $T$  are said to be *C-dual operators* (Alsina *et al.*<sup>1</sup>).

Trillas<sup>31</sup> has proved that for any negation  $C$ , there is a mapping  $t$  from  $[0, 1]$  to  $[0, +\infty]$  such

that  $t(0)=0$ ,  $t(1)<+\infty$ ,  $t$  is continuous and increasing and

$$C(a)=t^{-1}(t(1)-t(a)) \tag{8}$$

Note that there is always a unique number  $s \in (0, 1)$  such that  $C(s)=s$ , and we can check  $s = t^{-1}(t(1)/2)$ . Moreover, if  $C$  is a negation, then  $1 - C(1 - \cdot)$  is also a negation, distinct from  $C$  generally.

A negation generator  $t$  can be related to  $t$ -norms and  $t$ -conorms, since it is patent that the mapping  $f(\cdot)=t(1)-t(\cdot)$  is a nilpotent- $t$ -norm generator and that  $\varphi(\cdot)=t(\cdot)$  is a nilpotent  $t$ -conorm generator. Conversely, if  $f$  is a nilpotent  $t$ -norm generator, then  $t(\cdot)=f(0)-f(\cdot)$  is a negation generator.

Then, a given negation generator  $t$  generates a  $t$ -norm and its  $C$ -dual  $t$ -conorm by stating

$$\begin{aligned} C &= t^{-1}(t(1)-t) \\ T &= f^*(f(\cdot)+f(\cdot)) \text{ where } f = t(1)-t \\ S &= C\text{-dual of } T = t^*(t(a)+t(b)) \end{aligned} \tag{9}$$

If  $(C, T, S)$  all come from a generator  $t$ , it is easy to check that

$$\forall a \in [0, 1], T(a, C(a))=0; S(a, C(a))=1 \tag{10}$$

In a multivalued logic language, the contradiction and excluded middle laws hold. Conversely, for any nilpotent  $t$ -norm  $T$  (resp.  $t$ -conorm  $S$ ), there exists a negation  $C$  and a  $t$ -conorm  $S$  (resp.  $t$ -norm  $T$ ) such that (10) holds, (applying (8) and (9)). For instance,  $\max(0, a+b-1)$ ,  $\min(1, a+b)$ , and  $1-a$  stem from  $t(a)=a$ .

#### 4. Examples of Parametered Families

Starting from a parametered mapping which can act as an additive generator for a  $t$ -norm, we can build a parametered family  $\mathcal{F}$  of  $t$ -norms and the dual family  $\mathcal{S}$  of  $t$ -conorms (in the sense of (7)). If the  $t$ -norms of  $\mathcal{F}$  are nilpotent, there exist an associated family  $\mathcal{C}$  of negations and an associated family  $\mathcal{S}(\mathcal{C})$  of  $C$ -dual  $t$ -conorms (in the sense of (8) and (9)); then, another family  $\mathcal{C}'$  of negations and the associated family  $\mathcal{F}(\mathcal{C}')$  of  $C$ -dual  $t$ -norms (in the

sense of (8) and (9)) can be derived directly from the  $t$ -conorms of  $\mathcal{S}$ . We have  $\mathcal{C}' = \{1 - C(1 - \cdot) \mid C \in \mathcal{C}\}$ .

In the following some noticeable families are reviewed; their choice is motivated by historical reasons or by remarkable properties.

— *The  $T_p$  family* (Schweizer and Sklar<sup>27</sup>)

$$T_p(a, b) = [\max(0, a^{-p} + b^{-p} - 1)]^{-1/p}, \quad p \in \mathbb{R}$$

with the convention  $T_0 = \text{product}$ . For  $p$  finite and  $p \geq 0$ ,  $T_p$  is a strict  $t$ -norm; for  $p < 0$  it is a nilpotent  $t$ -norm. Moreover,  $T_{-1} = T_m$ ,  $T_{+\infty} = \min$ ,  $T_{-\infty} = T_w$ .

— *The Sugeno family* (Sugeno<sup>30</sup>)

$$S_\lambda(a, b) = \min(1, a + b + \lambda ab), \quad \lambda \geq -1$$

for  $\lambda > -1$  it is a nilpotent  $t$ -conorm.  $S_{-1}$  is the probabilistic sum,  $S_0$  is the bounded sum,  $S_\infty$  is  $T_w^*$ .

— *The Hamacher product* (Hamacher<sup>10</sup>)

$$H_\gamma(a, b) = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)}, \quad \gamma \geq 0$$

For  $\gamma < \infty$ , it is a strict  $t$ -norm.  $H_0(a, b) = (ab/(a+b-ab))$ ,  $H_1$  is the product, and  $H_{+\infty} = T_w$ . For  $\gamma = 2$ , the associated  $t$ -conorm is the Lorentz operator  $(a+b/1+ab)$ . The Hamacher product is the only strict Archimedean  $t$ -norm which can be written under the form of a rational function.

— *The Frank family* (Frank<sup>9</sup>)

$$F_s(a, b) = \text{Log}_s \left[ 1 + \frac{(s^a - 1)(s^b - 1)}{s - 1} \right], \quad s > 0$$

$F_s$  is a strict  $t$ -norm for  $\infty > s > 0$ .  $F_0 = \min$ ;  $F_1 = \text{product}$ ,  $F_{+\infty} = T_m$ .

Moreover the only  $t$ -norms  $T$  which satisfy

$$T(a, b) + S(a, b) = a + b$$

where  $S$  is the dual of  $T$ , are the  $t$ -norms of Frank's family (see the cited paper) or their ordinal sums (see Annex 1).

— The Yager family (Yager<sup>33</sup>)

$$Y_q(a, b) = 1 - \min\left(1, \left[(1-a)^q + (1-b)^q\right]^{1/q}\right) \quad q \geq 0$$

$Y_q$  is a nilpotent  $t$ -norm, for  $q > 0$ ;  $Y_1 = T_m$ ;  $Y_{+\infty} = \min$ ;  $Y_0 = T_w$ .

— A non-Archimedean family (Dubois and Prade<sup>7</sup>)

$$\sigma_\alpha(a, b) = \frac{ab}{\max(a, b, \alpha)}$$

$\sigma_\alpha$  is a  $t$ -norm without additive generator.  $\sigma_0 = \min(a, b)$ ;  $\sigma_1 = a \cdot b$ . It is easy to see that more generally

$$\begin{aligned} \sigma_\alpha(a, b) &= \min(a, b) \quad \forall a, b: \max(a, b) \geq \alpha \\ &= \frac{ab}{\alpha}, \quad \forall (a, b) \in [0, \alpha]^2 \end{aligned}$$

$\sigma_\alpha$  is an ordinal sum based on product (see Annex 1 for the definition of the notion).

The Annex 2 is devoted to the exhibition of generators, dual and examples of  $C$ -dual operators for each of the above mentioned families. One worth-noticing point is that the  $C$ -dual of Schweizer and Sklar's  $t$ -norm is the dual of Yager's. The same holds, exchanging dual and  $C$ -dual. Here, the two associated negations are exchanged in the transformation  $C(\cdot) \rightarrow 1 - C[1 - (\cdot)]$ .

### III SET FUNCTIONS BASED ON A TRIANGULAR CONORM

#### 1. Fuzzy Measures

Let  $X$  be a set, assumed to be finite, for the sake of simplicity.

The cardinal of  $X$  is  $n$ .

A fuzzy measure, in the sense of Sugeno<sup>30</sup> is a mapping  $g$  from a  $\sigma$ -algebra  $\beta$  of  $X$  to  $[0, 1]$ , such that

- i)  $g(\emptyset) = 0$ ;
- ii)  $g(X) = 1$
- iii)  $\forall A \in \beta, \forall B \in \beta, \text{ if } A \subseteq B, \text{ then } g(A) \leq g(B)$ .

*N.B.* When  $X$  is no longer finite, a continuity

axiom is added:

iv) if  $\{A_i\}_i$  is a sequence of nested subsets ( $\forall i, A_i \subseteq A_{i+1}$  or  $\forall i, A_i \supseteq A_{i+1}$ ), then

$$\lim_{i \rightarrow +\infty} g(A_i) = g\left(\lim_{i \rightarrow +\infty} A_i\right).$$

In the following  $\beta$  is taken as the set of subsets of  $X$ , denoted  $\mathcal{P}(X)$ . A fuzzy measure can be interpreted as follows:  $g$  is supposed to be associated with some parameter  $u$  whose range is  $X$ , and whose accurate value is unknown.  $g_u(A)$  is viewed as the grade of (possibly subjective) (un)certainty of the event "the value of  $u$  belongs to  $A$ ". Namely, if  $u \in A$  (resp.  $u \notin A$ ) is a known fact then  $g_u(A) = 1$  (resp.  $0$ ). Note that the converse is not assumed to be true. Axioms (i) and (ii) should be regarded as a consequence of  $u \in X$  and  $u$  does bear a value, respectively. Axiom (iii) means that  $u \in A$  is at least as uncertain as  $u \in B$  as soon as  $A$  is part of  $B$ . Clearly, the class of fuzzy measures subsumes that of probability measures. A fuzzy measure is not a measure in the usual sense.

It is important to be aware that such an interpretive framework is quite different from the fuzzy set interpretive framework. In the latter, the set is not well-defined, has no definite boundaries, and the grade of membership of well located elements  $x$  of  $X$  is assessed by  $\mu_A(x)$ . Here, the set  $A$  is crisp, but the value of  $u$  is unknown.

#### 2. Arrival of Conorms

Triangular conorms are very naturally met as long as we take for granted the following basic assumption regarding the fuzzy measure  $g$ :

$$\text{if } A \cap B = \emptyset, \text{ then } g(A \cup B) = g(A) * g(B) \quad (11)$$

where  $*$  is some operator under which  $[0, 1]$  is closed.

Equation (11) expresses that the grade of uncertainty of the union of disjoint events  $A$  and

$B$  only depends upon the grade of uncertainty of  $A$ , and the grade of uncertainty of  $B$ . For instance, a probability measure satisfies (11) when  $*$  is the sum. The basic assumption can be viewed as a natural requirement which is used in other contexts such that information theory (Kampé de Fériet<sup>11</sup>). Equation (11) is crucial for easy manipulation of fuzzy measures, since it permits getting the grades of uncertainty of composite events out of that of elementary ones. For instance, Sugeno<sup>30</sup> had to use  $* = S_2$  in order to be able to use fuzzy measures in applications. On the contrary, we try to keep the combination of information as general as possible.

However, the algebraic structure of  $\mathcal{P}(X)$  induces compatibility constraints on  $*$ , as is now demonstrated, by the following remarks:

+The associativity and commutativity of set union lead to require the associativity and commutativity of  $*$

$$+g(A) = g(A \cup \emptyset) = g(A) * g(\emptyset) = g(A) * 0$$

hence,  $\forall a \in [0, 1], a * 0 = a$ .

+Let  $(A, B)$  and  $(C, D)$  be two pairs of disjoint subsets of  $X$  such that  $A \subseteq C, B \subseteq D$  (hence  $g(A) \leq g(C)$  and  $g(B) \leq g(D)$ ), then

$$g(A \cup B) = g(A) * g(B) \leq g(C \cup D) = g(C) * g(D).$$

As a result  $*$  should be non-decreasing in each place; particularly  $1 * 1 = 1$ .

It is then obvious that in (11) the combination operator cannot be but a triangular conorm. If  $X$  were not finite, the continuity of  $g$  claims for the continuity of  $*$ .

### 3. Properties of Conorm Based Set Functions

As a consequence, the following inequalities hold:

$$\forall A, B, A \cap B = \emptyset \Rightarrow$$

$$\max(g(A), g(B)) \leq g(A \cup B) \leq T_w^*(g(A), g(B))$$

However, the first inequality is valid as soon as  $g$  is a fuzzy measure, for any pair of subsets  $A$  and  $B$ . This is due to axiom (iii). Besides (11) can be extended to non-disjoint events:

PROPOSITION 1 *If  $*$  is a conorm and  $g(\emptyset) = 0$ , then (11) is equivalent to*

$$\forall A, B \subseteq X, g(A \cup B) * g(A \cap B) = g(A) * g(B) \quad (12)$$

*Proof* (12)  $\Rightarrow$  (11) is obvious. To prove the converse, we first express  $A \cup B$  and  $B$  as a union of disjoint events:  $A \cup B = A \cup (\bar{A} \cap B)$ ;  $B = (\bar{A} \cap B) \cup (A \cap B)$  where  $\bar{A}$  denotes the complement of  $A$  in  $X$ . Then

$$\begin{aligned} g(A \cup B) * g(A \cap B) &= (g(A) * g(\bar{A} \cap B)) * g(A \cap B) \\ &= g(A) * g(B) \end{aligned}$$

using the associativity of  $*$

Q.E.D.

Another interesting consequence of (11) is, choosing  $B = \bar{A}$

$$\forall A \subseteq X, g(A) * g(\bar{A}) = 1 \quad (13)$$

Lastly, a conorm based set function  $g$  is uniquely defined by the knowledge of the conorm  $*$  and the values of  $g$  over the set of singletons of  $X$ . Indeed, let  $X = \{x_1 \dots x_n\}$ ,  $g_i = g(\{x_i\})$  and  $A = \{x_{i_1} \dots x_{i_p}\}$ .

$$g_*(A) = g_* \left( \bigcup_{j=1, p} \{x_{i_j}\} \right) = g_{i_1} * g_{i_2} \dots * g_{i_p} \quad (14)$$

where the notation  $g_*$  means " $g_*$  derives from  $*$ ", and the associativity of  $*$  is used. In order to meet axiom (ii), the  $g_i$ 's must satisfy

$$g_1 * g_2 * \dots * g_n = 1. \quad (15)$$

This is always trivially satisfied if  $g_i = 1$  for some  $x_i \in X$ . Equation (15) must be considered as a normalization similar to the one for probability densities. It is the ability of conorm based set functions to be defined completely by a reduced set of data (the "density"  $\{g_i\}_i$  and the conorm) which makes this concept attractive for practical use.

### 4. Examples

Choosing the bounded sum  $a * b = \min(1, a + b)$  (11) yields

$$A \cap B = \emptyset \Rightarrow g(A \cup B) = \min(1, g(A) + g(B))$$

Decomposing  $A$  as a disjoint union of singletons, we get

$$g_+(A) = \min\left(1, \sum_{x \in A} g(\{x\})\right)$$

(15) reads  $\sum_{i=1}^n g_i \geq 1$ ; letting this expression be an equality, we clearly recover probability measures. Similarly, letting  $a * b = \max(a, b)$  (12) yields with  $g = \Pi$

$$\forall A, \forall B \quad \Pi(A \cup B) = \max(\Pi(A), \Pi(B))$$

In other words, Zadeh's possibility measures<sup>36</sup> are recovered. On the contrary, the probabilistic sum  $a + b - ab$  does not lead to a well-known family of set functions and what is paradoxical, certainly not to probability measures. Such set functions are of the  $S_\lambda$  family,  $\lambda = 1$ , which Sugeno does not deal with. For possibility measures (13) reads

$$\max(g(A), g(\bar{A})) = 1 \quad (16)$$

and (15) reads

$$\max_{i=1, n} g_i = 1 \quad (17)$$

Using the probabilistic sum ( $a \hat{+} b = a + b - ab$ ) for  $*$  (13) reads:

$$(1 - g(A)) \cdot (1 - g(\bar{A})) = 0 \quad (18)$$

which is obviously equivalent to  $\max(g(A), g(\bar{A})) = 1$ ; note that (16) implies (17) (this can be shown by splitting  $A$  if  $g(A) = 1$  until some singleton is isolated).

### 5. More properties

The following results indicate some classes of conorms which generate set functions satisfying (16).

**PROPOSITION 2** *If  $*$  is a strict conorm, then  $g_*$  is such that  $\max(g_*(A), g_*(\bar{A})) = 1$ .*

*Proof* Let  $k$  be a multiplicative generator of  $*$ , i.e.  $k$  is a decreasing  $[0, 1] \rightarrow [0, 1]$  mapping with  $k(0) = 1, k(1) = 0$ .

$$g(A) * g(\bar{A}) = 1 = k^{-1}(k(g(A)) \cdot k(g(\bar{A}))) = 1$$

whence

$$k(g(A)) \cdot k(g(\bar{A})) = 0, \text{ i.e. one of } g(A), g(\bar{A}) \text{ is } 1$$

Q.E.D.

Set functions generated by the dual of Hamacher product, of Frank  $t$ -norm or of  $T_p$  for  $p > 0$  exemplify Proposition 2.

**PROPOSITION 3** *If  $*$  is a conorm smaller than or equal to the probabilistic sum ( $a * b \leq a + b - ab$ ), then  $\max(g_*(A), g_*(\bar{A})) = 1$ .*

*Proof*

$$1 = g_*(A) * g_*(\bar{A}) \leq g_*(A) + g_*(\bar{A}) - g_*(A) \cdot g_*(\bar{A}) \leq 1$$

the last inequality stems from  $g_*(A) \leq 1, g_*(\bar{A}) \leq 1$ . Thus (18) holds for  $g_*$ . Q.E.D.

The dual of the  $t$ -norms  $\sigma_x$  (see Annex 2) satisfy Proposition 3. Note that  $\sigma_x$  is not a strict  $t$ -norm.

All set functions compatible with Proposition 2 or 3 have densities such that  $g_i = 1$  for some  $x_i \in X$ . The normalization condition (15) is, in this case, strictly equivalent to  $\max_{i=1, n} g_i = 1$ . Since they behave similarly to possibility measures, set functions verifying (16) can be called "pseudo possibilities".

As a consequence these set functions are such that  $g(A) + g(\bar{A}) \geq 1$ . However, the next proposition indicates that this inequality holds for a broader class.

**PROPOSITION 4** *If  $*$  is a conorm smaller than or equal to the bounded sum ( $a * b \leq \min(1, a + b)$ ) then  $g_*(A) + g_*(\bar{A}) \geq 1$ .*

*Proof*

$$1 = g_*(A) * g_*(\bar{A}) \leq \min(1, g_*(A) + g_*(\bar{A}))$$

Q.E.D.

**PROPOSITION 5** *If  $*$  is a nilpotent conorm whose additive generator is  $t$  then  $t(g(A)) + t(g(\bar{A})) \geq t(1)$ .*

*Proof* Let  $a * b = t^*(t(a) + t(b))$  where  $t$  is a continuous increasing mapping from  $[0, 1]$  to



$[0, t(1)]$  and  $t^*$  is the pseudo inverse of  $t$ , such that

$$\begin{aligned} t^*(a) &= t^{-1}(a) & \text{if } a \leq t(1) \\ &= 1 & \text{if } a \geq t(1) \end{aligned}$$

Hence,  $g(A) * g(\bar{A}) = 1$  reads  $t(g(A)) + t(g(\bar{A})) \geq t(1)$  Q.E.D.

Let us give some examples:

Sugeno<sup>30</sup> considered the family of fuzzy measures defined by:

$$\text{if } A \cap B = \emptyset,$$

then

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda \cdot g_\lambda(A) \cdot g_\lambda(B) \quad \lambda > -1 \quad (19)$$

It is clear that, noticing  $g_\lambda(X) = 1$ , the  $g_\lambda$  are based on the conorm  $S_\lambda$ , i.e. (19) is a particular case of

$$\text{if } A \cap B = \emptyset$$

$$g(A \cup B) = \min(1, g(A) + g(B) + \lambda g(A)g(B)).$$

For  $\lambda \leq 0$ , these set functions satisfy  $g(A) + g(\bar{A}) \geq 1$ .

Similarly, the family of conorms defined by  $a * b = \min(1, (a^q + b^q)^{1/q})$ ,  $q > 0$  underlies set functions such that

$$[g(A)]^q + [g(\bar{A})]^q \geq 1$$

Proposition 4 applies only if  $q \geq 1$ .

Note that the normalization condition of a nilpotent conorm based set function reads

$$\sum_{i=1}^n t(g_i) \geq t(1) \quad (20)$$

i.e.

$$\text{for } * = S_\lambda, \quad t(a) = \frac{1}{\lambda} \text{Ln}(1 + \lambda a) \quad \lambda > 0$$

$$\text{and } \prod_{i=1}^n (1 + \lambda g_i) \geq 1 + \lambda \quad \text{holds}$$

the equality corresponds to the  $g_\lambda$ 's normalization condition (Sugeno<sup>30</sup>)

for  $* = Y_q$  (Yager's family)  $t(a) = a^q$

and  $\sum_{i=1}^n g_i^q \geq 1$  holds.

Lastly, the weakest constraint relating  $g(A)$  and  $g(\bar{A})$ , for any conorm  $*$  is  $\forall A \subseteq X, g(A) = 0 \Rightarrow g(\bar{A}) = 1$ , which is a consequence of  $g(A) * g(\bar{A}) = 1$ . Particularly,  $T_w^*$ -based set functions only satisfy this weak condition. Their normalization condition is  $\exists k, g_{i_k} = 1$  or  $\exists k, \exists l \neq k, g_{i_k} > 0$  and  $g_{i_l} > 0$ .

#### IV SET FUNCTIONS BASED ON T-NORMS

Let  $g$  be a fuzzy measure on  $X$ , and let  $C$  be a negation. We can build the set-function  $g_C$  defined by

$$\forall A \subseteq X, g_C(A) = C(g(\bar{A})) \quad (21)$$

It is clear that  $g_C$  is also a fuzzy measure, i.e. it satisfies axioms (i)–(iii). Now assume  $g$  is based on a conorm  $*$ . As a consequence  $g_C$  is ruled by an axiom which is dual to (11):

PROPOSITION 6  $\forall A, B \subseteq X$ , if  $A \cup B = X$ ,

then

$$g_C(A \cap B) = g_C(A) \perp g_C(B) \quad (22)$$

where  $\perp$  is a  $t$ -norm, the  $C$ -dual of the conorm  $*$ .

*Proof*  $A \cup B = X$  can be written  $\bar{A} \cap \bar{B} = \emptyset$ .

Hence,  $g(\bar{A} \cap \bar{B}) = g(\bar{A}) * g(\bar{B})$ . But using (21) and  $C$  involutive, it comes

$$C[g_C(A \cap B)] = C[g_C(A)] * C[g_C(B)]$$

i.e.

$$\begin{aligned} g_C(A \cap B) &= C[C[g_C(A)] * C[g_C(B)]] \\ &= g_C(A) \perp g_C(B) \end{aligned}$$

Q.E.D.

A set-function  $g$  is said to be based on a  $t$ -norm if it is a fuzzy measure which satisfies (22) for some  $t$ -norm  $\perp$ .

$t$ -norm based set functions have properties similar to those of conorm based set functions. More specifically, the following statements are true:

If  $g$  is based on  $\perp$ , then, corresponding to (12)

$$\forall A, \forall B \subseteq X, g(A \cup B) \perp g(A \cap B) = g(A) \perp g(B) \quad (23)$$

corresponding to (13), with  $B = \bar{A}$

$$g(A) \perp g(\bar{A}) = 0 \quad (24)$$

$g(A)$  can be uniquely defined from the knowledge of  $g(X - \{x_i\})$  for all  $x_i \in X$ . Denoting  $g(X - \{x_i\}) = \bar{g}_i$ , and  $A = \{x_{i_1}, \dots, x_{i_p}\}$ .

$$g(A) = g\left(\bigcap_{x \in A} (X - \{x\})\right) = \bar{g}_{i_{p-1}} \perp \bar{g}_{i_{p-2}} \perp \dots \perp \bar{g}_{i_1} \quad (25)$$

where  $\bar{A} = \{x_{i_{p-1}}, \dots, x_{i_1}\}$ .  $\bar{g}_i$   $i=1, n$  is called the co-density of  $g$ .

The normalization condition of the co-density is obtained by stating  $g(\emptyset) = 0$  which reads

$$\bar{g}_1 \perp \bar{g}_2 \perp \dots \perp \bar{g}_n = 0. \quad (26)$$

this condition is trivially satisfied as soon as  $\bar{g}_i = 0$  for some  $i$ .

if  $\perp$  is a strict  $t$ -norm, then  $\min(g(A), g(\bar{A})) = 0$ , for all  $A$ .

if  $\perp$  is between min and product ( $\min(a, b) \geq a \perp b \geq a \cdot b$ ) then

$$\min(g(A), g(\bar{A})) = 0, \text{ for all } A.$$

if  $g$  is such that  $\min(g(A), g(\bar{A})) = 0$  for all  $A$ , their normalization condition (26) is equivalent to  $\min_{i=1, n} \bar{g}_i = 0$ , i.e.  $\bar{g}_i = 0$  for some  $i$ .

if  $\perp$  is between min and  $T_m$  ( $\min(a, b) \geq ab \geq \max(0, a + b - 1)$ ) then  $g(A) + g(\bar{A}) \leq 1$

if  $\perp$  is a nilpotent  $t$ -norm whose additive generator is  $f$ , then

$$f(g(A)) + f(g(\bar{A})) \geq f(0); \sum_{i=1}^n f(\bar{g}_i) \geq f(0).$$

for any  $t$ -norm based set function, if  $g(A) = 1$ , then  $g(\bar{A}) = 0$ .

If  $g$  is based on a conorm  $*$ , then  $g_C$  has a co-density  $\bar{g}_i = C(g_i)$ ,  $\forall i = 1, n$ . This is just because  $\bar{g}_i = g_C(X - \{x_i\}) = C(g(\{x_i\})) = C(g_i)$ . Moreover, if  $g_1 * g_2 \dots * g_n = 1$  (normalization), then

$$C(g_1) \perp \dots \perp C(g_n) = C[g_1 * \dots * g_n] = 0$$

i.e.  $g$  is normalized if and only if  $g_C$  is normalized. In other words, from a given density, a conorm and its  $C$ -dual, we can build two set-functions.

Choosing  $\perp = \min$ , (23) yields  $g(A \cap B) = \min(g(A), g(B))$ ,  $\forall A, B \subseteq X$ . We find a special class of belief functions called "consonant" belief functions by Shafer<sup>29</sup>.

They are dual of possibility measures, and as suggested by Dubois and Prade<sup>6</sup> can be named "necessity measures" since the necessity  $N(A)$  of an event is the grade of impossibility of the opposite event ( $= C(\Pi(\bar{A}))$ ). See Prade<sup>24</sup> for a more detailed justification. Zadeh<sup>37</sup> also calls such measures "certainty measures". They were actually discovered by Shackle<sup>28</sup>, an English economist who interpreted  $N(\bar{A})$  as the grade of potential surprise of the event  $A$ . All  $t$ -norm based fuzzy measures  $g$  such that  $\min(g(A), g(\bar{A})) = 0$  can be called pseudo necessity measures.

Choosing  $\perp = T_m$  yields

$$g(A \cap B) = \max(0, g(A) + g(B) - 1), \text{ if } A \cup B = X \quad (27)$$

the normalization condition reads  $\sum_{i=1}^n \bar{g}_i \leq n - 1$ . Probability measures obviously satisfy (27), and are part of this class.

Choosing  $\perp = \text{product}$  does not yield something very well-known in spite of its simplicity. Lastly, as an example of parametered family, we can consider

$$g(A \cap B) = T_p(g(A), g(B)), \text{ for } A \cup B = X;$$

for  $p < 0$ ,  $T_p$  is a nilpotent  $t$ -norm. The normalization condition is

$$\sum_{i=1}^n (\bar{g}_i)^{-p} \leq n - 1.$$

## V FUZZY MEASURES DERIVING FROM A $T$ -NORM AND A CONORM

Given a conorm based fuzzy measure and a negation  $C$ , there may happen that

$$\forall A \subseteq X, g_C(A) \stackrel{\Delta}{=} C(g(\bar{A})) = g(A) \quad (28)$$

For instance, if  $g$  is a probability measure, then  $C(a) = 1 - a$  and (28) translates into  $g(A) + g(\bar{A}) = 1$ . Equation (28) expresses that  $g(A)$  and  $g(\bar{A})$  are dependent quantities, in that the knowledge of  $g(A)$  is equivalent to the knowledge of  $g(\bar{A})$ . Such fuzzy measures can be called "pseudo probabilities", and are the topic of this section.

### 1. Characterization

First, we notice that if there is a  $[0,1] \rightarrow [0,1]$  continuous mapping  $C$  such that  $\forall A, g(A) = C(g(\bar{A}))$ , then  $C$  is a negation in the sense of Definition 3.

Indeed

$$\begin{aligned} C(0) &= C(g(\bar{X})) = g(X) = 1 \\ g(A) &= C(g(\bar{A})) = C(C(g(A))), \end{aligned}$$

hence  $C$  is involutive

If  $A \subseteq B$  then  $g(\bar{A}) \geq g(\bar{B})$  and  $g(A) \leq g(B)$  hence  $C$  is non-increasing; non-increasingness of an involutive mapping implies its decreasingness.

In other words, if  $g$  is based on a conorm  $*$  and  $\forall A, g(A) = C(g(\bar{A}))$ , then  $g$  is also based on a  $t$ -norm  $\perp$  such that

$$a \perp b = C(C(a) * C(b)) \quad (29)$$

The same holds replacing  $t$ -norm by  $t$ -conorm and conversely.

More generally, a fuzzy measure is said to be based on a  $t$ -norm  $\perp$  and a  $t$ -conorm  $*$  if and only if

- i)  $g(\emptyset) = 0$
  - ii)  $g(X) = 1$
  - iii)  $\forall A, \forall B, A \cap B = \emptyset \Rightarrow g(A \cup B) = g(A) * g(B)$
  - iv)  $\forall A, \forall B, A \cup B = X \Rightarrow g(A \cap B) = g(A) \perp g(B)$
- (30)

As will be seen in the following there are pairs  $(*, \perp)$  which cannot be candidate for the generation of a single fuzzy measure.

Note that a consequence of (30) is, taking  $B = \bar{A}$

$$\forall A \subseteq X, g(A) * g(\bar{A}) = 1, g(A) \perp g(\bar{A}) = 0 \quad (31)$$

this implies  $g(A) = 0$  is equivalent to  $g(\bar{A}) = 1$ .

**PROPOSITION 7** Assume  $*$  is between max and probabilistic sum, or a strict conorm then for any  $t$ -norm  $\perp$ , if  $g$  is based on  $*$  and  $\perp$ , it is a Dirac measure, i.e.

$$\exists x_0 \in X, g(A) = 0 \text{ if } x_0 \notin A, g(A) = 1 \text{ if } x_0 \in A$$

*Proof* If  $*$  is between max and probabilistic sum, then from Proposition 3,  $\forall A, \max(g(A), g(\bar{A})) = 1$  the same holds from Proposition 2 if  $*$  is a strict conorm. In both cases (31) implies  $\forall A, g(A) \in \{0, 1\}$ . The only possible candidate is a Dirac measure.

Q.E.D.

In other words, fuzzy measures based on a  $t$ -conorm compatible with Proposition 7 cannot be based also on a  $t$ -norm, except being something trivial. Consequently, conorms between max and probabilistic sum, and strict conorms cannot generate pseudo probabilities, i.e. there is no negation  $C$  such that (28) holds; this is quite consistent with the fact that such conorms generate pseudo possibilities. Similarly,  $t$ -norms between product and min and strict  $t$ -norms generate pseudo necessities and can never generate pseudo probabilities, nor fuzzy measures such that (30) holds, except Dirac measures.

**PROPOSITION 8** If a fuzzy measure  $g$  is based on a nilpotent conorm  $*$  with generator  $t$ , and is such that  $\sum_{i=1}^n t(g_i) = t(1)$ , then  $g$  is also based on the  $C$ -dual of the conorm, say the  $t$ -norm  $\perp$ , where  $C$  is generated by  $t$ .

*Proof* If the normalization condition of  $g$  (20) holds with equality then, denoting  $A = \{x_{i_1} \dots x_{i_p}\}$  a non empty subset of  $X$ ,

$$\begin{aligned} g(A) &= t^* \left( \sum_{j=1}^p t(g_{i_j}) \right) = t^{-1} \left( t(1) - \sum_{j=p+1}^n t(g_{i_j}) \right) \\ &= t^{-1} \left( t(1) - t \left[ t^{-1} \left( \sum_{j=p+1}^n t(g_{i_j}) \right) \right] \right) \\ &= C(g(\bar{A})) \end{aligned}$$

Hence  $g_C$ , based on the  $C$ -dual of  $*$  is nothing but  $g$ .  
Q.E.D.

From a pair of  $C$ -dual nilpotent operators  $*$  and  $\perp$  and a density  $\{g_i, i=1, n\}$  such that  $\sum_{i=1}^n t(g_i) = t(1)$ , where  $t$  generates  $C$ ,  $*$  and  $\perp$ , we only give birth to a single set function, which is a pseudo probability.

On the contrary, starting from a pair of operators  $*$  and  $\perp$  such that

— either  $*$  is a strict conorm and  $\perp$  the dual  $t$ -norm.

— or  $*$  is between max and probabilistic sum and  $\perp$  the dual  $t$ -norm and a density  $\{g_i, i=1, n\}$  satisfying  $\max_{i=1, n} g_i = 1$ , we give birth to a pseudo possibility measure and a pseudo necessity measure.

## 2. Examples

### . Pseudo possibilities and necessities

The  $t$ -norms  $H_p$  (Hamacher),  $F_p$  (Frank),  $T_p$  ( $p > 0$ ) are strict. They generate families of pseudo necessity measures, and their dual families of pseudo possibility measures. For the family  $\sigma_\alpha$  of  $t$ -norms located between min and product, and the dual family, the same holds.

Particularly, from  $\{g_i, i=1, n\}$  such that  $\exists i: g_i = 1$ , "max" and "min" yields the set functions  $\Pi$  and  $N$  defined by

$$\forall A, \Pi(A) = \sup_{x_i \in A} g_i$$

(possibility measures, Zadeh<sup>36</sup>)

$$N(A) = 1 - \Pi(\bar{A}) = \inf_{x_i \notin A} 1 - g_i$$

(Necessity measures, Shafer,<sup>29</sup> Dubois and Prade;<sup>6</sup>)

also called certainty measures by Zadeh<sup>37</sup>)

### . A nilpotent conorm family: Sugeno's $S_\lambda$

Stating  $\bar{A} = B$  in (19) yields, for the  $S_\lambda$ -based set function  $g_\lambda$

$$g_\lambda(A) + g_\lambda(\bar{A}) + \lambda g_\lambda(A) \cdot g_\lambda(\bar{A}) = 1 \quad (32)$$

whence

$$g_\lambda(\bar{A}) = \frac{1 - g_\lambda(A)}{1 + \lambda \cdot g_\lambda(A)} = C_\lambda(g_\lambda(A)) \quad (\lambda > -1)$$

i.e. we find Sugeno's negation  $C_\lambda(a) = (1-a)/(1+\lambda a)$ .  $g_\lambda$  is also based on the  $C_\lambda$ -dual of  $S_\lambda$ , i.e. the  $t$ -norm

$$\max\left(0, \frac{a+b-1+\lambda ab}{\lambda+1}\right)$$

which can be found directly by using (19) and (12). More specifically,

$$\frac{1}{\lambda+1}(g_\lambda(A) + g_\lambda(\bar{A}) - 1 + \lambda g_\lambda(A)g_\lambda(\bar{A})) = 0$$

if and only if (32) holds. The case  $\lambda=0$  corresponds to probability measures

N.B.—The dual of  $S_\lambda$  is  $\max(0, (\lambda+1)(a+b-1) - \lambda ab)$ . This  $t$ -norm generates the set-function  $g'_\lambda$  such that

$$g'_\lambda(A) = 1 - g_\lambda(\bar{A})$$

and we have, if (19) holds, from (23)

$$\begin{aligned} \forall A, \forall B, (\lambda+1)(g'_\lambda(A \cap B) + g'_\lambda(A \cup B) - 1) \\ - \lambda g'_\lambda(A \cap B)g'_\lambda(A \cup B) \\ = (\lambda+1)(g'_\lambda(A) + g'_\lambda(B) - 1) - \lambda g'_\lambda(A)g'_\lambda(B) \end{aligned}$$

if  $A \cap B = \emptyset$  we get

$$g'_\lambda(A \cup B) = g'_\lambda(A) + g'_\lambda(B) - \frac{\lambda}{\lambda+1} g'_\lambda(A)g'_\lambda(B)$$

i.e.  $g'_\lambda = g_\mu$  with  $\mu = \frac{-\lambda}{1+\lambda}$

In other words, the dual of  $S_\lambda$  generates a set function of the same family as  $S_\lambda$ -based set functions. Namely, it is a  $S_\mu$ -based set function with  $\mu = (-\lambda/1+\lambda)$ . Of course  $g'_\lambda \neq g_\lambda$ , except if  $\lambda = 0$ .

### . Yager's family of nilpotent $t$ -norms

The family  $Y_q$  generates set functions which are also based on the  $C$ -dual family of  $Y_q$ , i.e. the conorms

$$a * b = 1 - \max(0, (1-a)^q + (1-b)^q - 1)^{1/q}$$

If  $g_q$  is such a set function we have

$$g_q(\bar{A}) = 1 - \sqrt[q]{1 - (1 - g_q(A))^q}$$

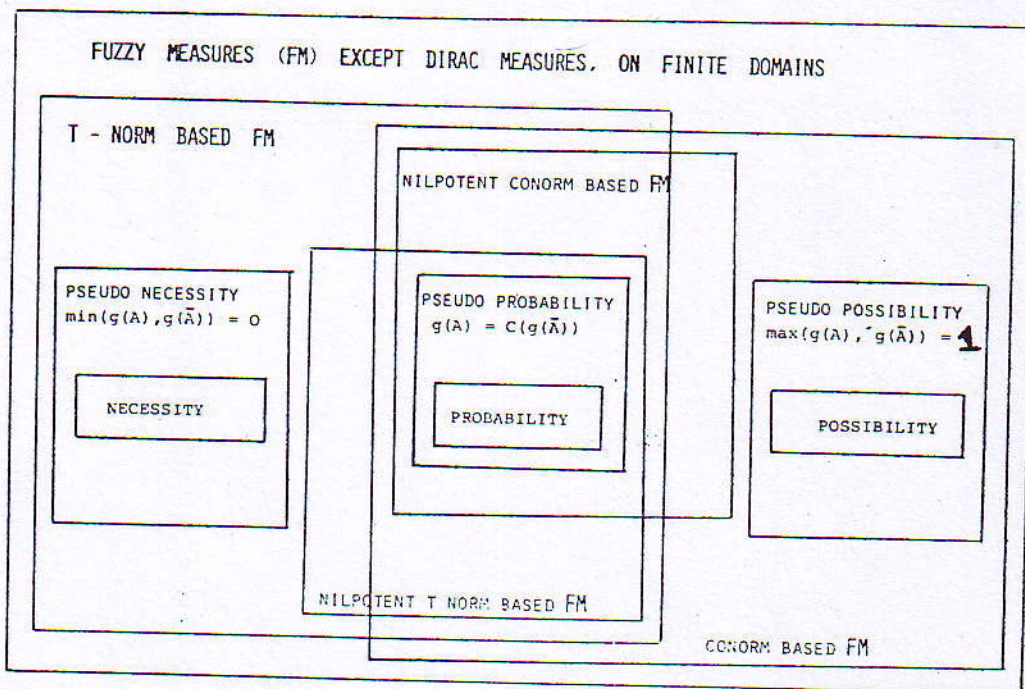
Note that the dual of  $Y_q$  is  $\min(1, a^q + b^q)^{1/q}$ ; denoting  $p = -q$ , this family is the C-dual of Schweizer and Sklar's  $T_p$  family for  $p < 0$ . Hence, the dual of  $Y_q$  generates a set function  $g_{T_p}$  based both on  $T_p$  and its C-dual for  $p < 0$ ;

we have

$$1 - g_q(\bar{A}) = g_{T_{-q}}(A) = \sqrt[q]{1 - g_{T_{-q}}(\bar{A})^q}.$$

### 3 Summary and Unsolved Questions

The following figure sums up the various classes of fuzzy measures encountered in this study, and points out the inclusion of sub classes.



Unsolved questions regarding such a classification include:

— If  $g$  is based on a  $t$ -norm  $\perp$  and a conorm  $*$ , and is not a Dirac measure, does there exist a negation  $C$  such that  $g(\bar{A}) = C(g(A))$ ?

We already know the answer for set functions concerned by Proposition 7, i.e. pseudo possibilities and pseudo necessities make disjoint classes (except for Dirac Measures). Besides, if  $g$  is based on a conorm smaller or equal to the bounded sum, denoting  $\bar{g}$  the set function based on the dual of the conorm, i.e.  $\bar{g}(A) = 1 - g(\bar{A})$  it is clear from Proposition 4 that

$$g(A) \geq \bar{g}(A), \quad \forall A \subseteq X.$$

Hence,  $g$  and  $\bar{g}$  are still consistent with possibility-

like and necessity-like meanings, respectively (possibility greater than necessity). Another question is more generally to compare  $g$  and  $g_C$  in the same spirit: when  $g \neq g_C$ , is there some inequality between them?

There is also a need for extending the presented framework to infinite universes. Let us hint some research lines for  $X = \mathbb{R}$ . Let  $p$  be a mapping  $\mathbb{R} \rightarrow \mathbb{R}^+$ , which will denote a density. We can evaluate

The probability of an interval  $[a, b] = A: P(A) = \int_a^b p(x) dx$  where  $p$  satisfies the identity  $\int_{-\infty}^{+\infty} p(x) dx = 1$ .

The possibility of an interval  $A: p$  is now such that

$$\sup p = 1, \quad \text{and} \quad \Pi(A) = \sup_{x \in A} p(x)$$

The necessity of  $A: p$  is still such that

$$\sup p = 1 \quad \text{and} \quad N(A) = \inf_{x \notin A} 1 - p(x)$$

We can extend these definitions when  $g$  is:

— a strict conorm based set function:  $p$  is such that  $\sup p = 1$ ; define

$$\prod_{x \in A} f(x) = \exp \int_a^b \text{Ln}(f(x)) dx$$

for  $f$  positive, whenever the integral exists.

Let  $k$  be a multiplicative generator of the strict conorm; then:

$$g(A) = k^{-1} \left( \prod_{x \in A} k(p(x)) \right)$$

— a strict  $t$ -norm based set function:  $p$  is such that  $\sup p = 1$ . Let  $h$  be a multiplicative generator of the  $t$ -norm; then

$$g(A) = h^{-1} \left( \prod_{x \notin A} h(p(x)) \right)$$

— a set function derived from nilpotent  $t$ -norm and conorm; if  $t$  is the additive conorm generator, then the density  $p$  must satisfy

$$\int_{-\infty}^{+\infty} t(p(x)) dx = t(1)$$

and we have

$$g(A) = t^{-1} \left[ \int_a^b t(p(x)) dx \right]$$

### V RELATIONSHIP WITH OTHER APPROACHES TO FUZZY MEASURES

In this section an attempt is made to exhibit links between the proposed framework and other approaches which can be found in the literature to tackle the same problem.

#### 1. Shafer's theory of evidence

A belief function according to Shafer<sup>29</sup> is a mapping  $\text{Bel}$  (as belief) from  $\mathcal{P}(X)$  to  $[0, 1]$  such that

- i)  $\text{Bel}(\emptyset) = 0$ ,
- ii)  $\text{Bel}(X) = 1$
- iii)  $\forall A_i, i = 1, n,$

$$\begin{aligned} \text{Bel} \left( \bigcup_{i=1}^n A_i \right) &\geq \sum_{i=1}^n \text{Bel}(A_i) \\ &- \sum_{i < j} \text{Bel}(A_i \cap A_j) \dots \\ &+ (-1)^{n+1} \text{Bel} \left( \bigcap_{i=1}^n A_i \right) \end{aligned} \quad (35)$$

for all  $n$ .

Axiom (iii) is called monotonicity of order  $\infty$ . For  $n=2, A_1=A, A_2=B$ , we get

$$\text{Bel}(A \cup B) \geq \text{Bel}(A) + \text{Bel}(B) - \text{Bel}(A \cap B) \quad (36)$$

One of the main results of Shafer<sup>29</sup> is the following characterization of belief functions: A belief function is uniquely defined through the specification of a mapping  $m: \mathcal{P}(X) \rightarrow [0, 1]$  called basic probability assignment, satisfying:

$$m(\emptyset) = 0, \quad \sum_{A \in \mathcal{P}(X)} m(A) = 1 \quad (37)$$

and we have

$$\text{Bel}(A) = \sum_{B \subseteq A} m(B), \quad \forall A \subseteq X \quad (38)$$

Equivalently  $m$  is obtained from  $\text{Bel}$  by

$$m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} \text{Bel}(B) \dagger \quad (39)$$

The quantity  $P1(A) = 1 - \text{Bel}(\bar{A})$  satisfies (36) with the reversed inequality, and is also uniquely characterized by means of the basic assignment  $m$ :

$$P1(A) = \sum_{B: B \cap A \neq \emptyset} m(B) \quad (40)$$

Clearly we have  $P1(A) \geq \text{Bel}(A)$ ,  $P1(A)$  is called a plausibility function.

Banon<sup>2</sup> proved that belief and plausibility functions are fuzzy measures. Moreover, using (36), the following identity holds:

$$\begin{aligned} \max(0, \text{Bel}(A) + \text{Bel}(B) - 1) &\leq \text{Bel}(A \cap B) \\ &\leq \min(\text{Bel}(A), \text{Bel}(B)) \end{aligned} \quad (41)$$

Similarly:

$$\begin{aligned} \max(P1(A), P1(B)) &\leq P1(A \cup B) \\ &\leq \min(1, P1(A) + P1(B)) \end{aligned} \quad (42)$$

Let us investigate which  $t$ -conorm or  $t$ -norm based set functions are belief or plausibility functions.

The upper limit in (41) yields a belief function which is a necessity measure.

$\dagger |A|$  denotes the cardinality of the set  $A$ .

The lower limit in (42) yields a plausibility function which is a possibility measure.

If (35) holds with equality, probability measures are recovered which are the only ones to be both a belief function and a plausibility function.

Further results motivate the following definition:

A  $t$ -norm  $\perp$  is said to be distributive if and only if

$$(a+b-a\perp b)\perp c = a\perp c + b\perp c - a\perp b\perp c \quad (43)$$

The same definition applies to  $t$ -conorms. Frank<sup>9</sup> proved that the only possible distributive  $t$ -norms are min, product and their ordinal sums (for instance the  $\sigma_x$  family). Dual results hold for conorms.

**PROPOSITION 9** *Set functions based on distributive  $t$ -norms (resp: conorms) are belief (resp. plausibility) functions.*

*Proof* It consists mainly in proving that if  $g$  is based on a distributive  $t$ -norm then

$$\forall A, m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} g(B)$$

defines a basic probability assignment. See Annex 3 for a complete proof; if  $g$  is a belief function then  $1-g(\bar{A})$ , based on the dual  $t$ -conorm is a plausibility function. Q.E.D.

For instance fuzzy measures based on product (or probabilistic sum) are belief functions (plausibility functions).

Another noticeable class of fuzzy measures is the one based on the Sugeno family of conorms  $S_\lambda$ . Banon<sup>2</sup> proved that for  $\lambda \geq 0$ ,  $g_\lambda$  based on  $S_\lambda$ , is a belief function. The basic assignment is

$$\forall A, m(A) = \lambda^{|A|-1} \prod_{x_i \in A} g_i$$

For  $\lambda \in (-1, 0]$ ,  $g_\lambda$  is a plausibility measure (Prade,<sup>20</sup> Dubois-Prade<sup>6</sup>), because  $1-g_\lambda(\bar{A})$  is based on  $S_\mu$ ,  $\mu = (-\lambda/1+\lambda)$ . From Proposition 9, we know that this result still holds when  $\lambda = -1$ .

Other links between Shafer's theory and the approach of this paper would be worth discovering: due to (41) and (42) we know that there are some  $t$ -norm (resp:  $t$ -conorm) based set

functions which are not belief functions. For instance  $g$  such that

$$A \cup B = X \Rightarrow g(A \cap B) = T_w(g(A), g(B))$$

$$(\text{resp: } A \cap B = \emptyset \Rightarrow g(A \cup B) = T_w^*(g(A), g(B))).$$

However, it is not known yet what are the belief functions which fit our framework, nor what conorm or  $t$ -norm based set functions are belief or plausibility functions.

## 2. Participation measures

Tsichritzis<sup>32</sup> studied so-called participation measures, which are fuzzy measures  $h$  such that  $h(\bar{A}) + h(A) = 1$ . This concept was motivated by the general definition of some evaluation index for the extent to which a subset  $A$  "participates" in  $X$ . An example of participation measure is a probability measure. Such a set function  $h$  is easily built from a fuzzy measure  $g$  by stating

$$h(A) = \frac{g(A) + 1 - g(\bar{A})}{2} \quad (44)$$

It can be checked that  $h(A)$  is a participation measure. Moreover if  $g$  is a possibility measure then  $h$  coincides with the concept of truth-value in some many valued logics (cf. Prade<sup>23</sup>). Note that we can choose  $g=h$  in (44) so that (44) is a characteristic form of participation measures.

The following fuzzy measures are clearly not participation measures

- pseudo probabilities, i.e. conorm-based set functions such that  $g(\bar{A}) = C(g(A))$  for some  $C$ , except probability measures
- pseudo possibilities, i.e. conorm-based set functions such that  $\forall A \max(g(A), g(\bar{A})) = 1$ , except Dirac measures.
- pseudo necessities, i.e.  $t$ -norm-based set functions such that  $\forall A \min(g(A), g(\bar{A})) = 0$ , except Dirac measures.

## CONCLUSION

A general class of fuzzy measures has been proposed and described for the combination of uncertain pieces of information. The most salient features of this framework are its intuitive appeal, its generality, and yet computational attractiveness. It is intuitive because it stems from the simple idea of combining disjoint pieces of

information. It is general because the only requirements are the compatibility of this combination with natural properties of set-theoretic operators. It is computationally attractive because any fuzzy measure of the concerned class, can be point-wisely defined, and constructively built up.

Two main types of fuzzy measures belong to this class. On the one hand, those for which the knowledge concerning an event is only weakly linked with the knowledge concerning the opposite event: for instance, the pseudo-possibility measures and their dual, the pseudo-necessity measures. On the other hand, those for which the knowledge concerning an event determines in a unique way the knowledge concerning the opposite event: here we get pseudo-probability measures. The links between the approach based on  $t$ -norms and conorms and Shafer's theory of evidence have been pointed out, although not completely clarified yet. One noticeable detail is that some belief functions different from probability measures are pseudo-probability measures, i.e.  $\text{Bel}(A)$  uniquely determines  $\text{Bel}(A)$ . An example has been provided by means of Sugeno's conorm based fuzzy measure.

It seems that  $t$ -norms have an important role to play in the forthcoming theories of uncertainty and vagueness, aimed at enlarged computerized information processing. In the sole fuzzy set theory, apart from supplying a general model of fuzzy set theoretic union and intersection (cf. section I), and a unifying framework for possibility and probability measures,  $t$ -norms and  $t$ -conorms are also proved as a useful tool for the setting of fuzzy  $\sigma$ -algebras (cf. Klement<sup>12,13,14,15</sup>), and the study of fuzzy arithmetic when the involved variables interact (cf. Dubois and Prade<sup>8</sup>). Lastly, the by now well-known extension principle stated by Zadeh,<sup>35</sup> extending in a "possibilistic" fashion, ordinary mappings to mappings having fuzzy arguments, can be generalized using  $t$ -norms and conorms. It is then possible to encompass into a single theory, for instance both the addition of fuzzy numbers (Dubois and Prade<sup>6</sup>) and that of random variables through a convolution.

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## ANNEX 1 ORDINAL SUMS

Let  $\{\mathcal{F}_n\}^n$  be a countable family of non-overlapping, closed, proper subintervals of  $[0, 1]$ . With each  $\mathcal{F}_n = [a_n, b_n]$ , associate a  $t$ -norm  $T_n$  such that  $\forall x \in (0, 1)$ ,  $T_n(x, x) < x$ . Let  $T$  be a function defined on  $[0, 1]^2$  via

$$T(x, y) = \begin{cases} a_n + (b_n - a_n) T_n\left(\frac{x - a_n}{b_n - a_n}, \frac{y - a_n}{b_n - a_n}\right), & (x, y) \in [a_n, b_n]^2 \\ \min(x, y) & \text{otherwise} \end{cases} \quad n = 1, 2, \dots$$

$T$  is called the ordinal sum of the  $T_n$ 's.  $T$  is a  $t$ -norm. More details can be found, for instance in Frank<sup>9</sup> from whom we borrow the above definition. Ordinal sums of conorms can also be defined in the same fashion, but changing min into max in the above statement and  $T_n$  in  $S_n$  with  $\forall x \in (0, 1)$ ,  $S_n(x, x) > x$ .

$\sigma_\alpha$  is an ordinal sum of the product. We have

$$\sigma_\alpha(a, b) = \frac{ab}{\max(a, \alpha, b)}$$

Let  $\mathcal{F}_1 = [0, \alpha]$ ,  $n = 1$ ,  $T_1 = \text{product}$

$$\text{— in } [0, \alpha]^2 \quad \sigma_\alpha(x, y) = \frac{xy}{\alpha} = 0 + \alpha \cdot \left(\frac{x-0}{\alpha-0}\right) \cdot \left(\frac{y-0}{\alpha-0}\right)$$

$$\text{— in } [\alpha, 1]^2 \quad \sigma_\alpha(x, y) = \min(x, y)$$

$$\text{— in } [0, \alpha] \times [\alpha, 1] \quad \text{or} \quad [\alpha, 1] \times [0, \alpha] \quad \sigma_\alpha(x, y) = \min(x, y).$$

Q.E.D.

The dual of  $\sigma_\alpha$  is an ordinal sum of the probabilistic sum.

ANNEX 2 SOME PARAMETERED FAMILIES OF T-NORMS, T-CONORMS AND NEGATIONS

Dual in the sense of 1-(.)

References and notations	t-norm	t-conorm	Additive generator of the t-norms	Negation	C-dual t-norm of the t-norm	C-dual t-conorm of the t-norm
Hamacher, <sup>10</sup> H <sub>γ</sub>	$\frac{ab}{\gamma + (1-\gamma)(a+b-ab)}$ $\gamma \geq 0$	$\frac{a+b-(2-\gamma)ab}{1-(1-\gamma)ab}$	$f(a) = \text{Ln} \left[ \frac{\gamma - (\gamma-1)a}{a} \right]$	(strict)	does not exist	does not exist
Sugeno, <sup>10</sup> S <sub>λ</sub>	$\max(0, \lambda + (a+b-1) - \lambda ab)$ $\lambda \geq -1$	$\min(1, a+b+\lambda ab)$	$\varphi(a) = \frac{1}{\lambda} \text{Ln}(1+\lambda a)$ (issued from the t-conorm)	$\frac{1-a}{1+\lambda a}$ (issued from the t-conorm)	$\max \left( 0, \frac{a+b-1+\lambda ab}{\lambda+1} \right)$	does not exist
Schweizer and Sklar, <sup>23</sup> T <sub>p</sub>	$\max(0, a^p + b^p - 1)^{1/p}$ $p \in \mathbb{R}$	$1 - \max(0, (1-a)^p + (1-b)^p - 1)^{1/p}$ $\lambda$	$f(a) =   -a^{-p}, p < 0$ $= a^{-p} - 1, p > 0$ of the t-norms	$\sqrt[p]{1-a^p}$ (issued from the t-norm)	does not exist	$\min(1, a^{-p} + b^{-p})^{1/p}$
Yager, <sup>23</sup> Y <sub>q</sub>	$1 - \min(1, (1-a)^q + (1-b)^q)^{1/q}$ $q > 0$	$\min(1, a^q + b^q)^{1/q}$	$f(a) = (1-a)^q$ of the t-norms	$\sqrt[q]{1-(1-a)^q}$ (issued from the t-norm)	does not exist	$1 - \max(0, (1-a)^q + (1-b)^q - 1)^{1/q}$
Frank, <sup>9</sup> F <sub>s</sub>	$\text{Log}_s \left[ 1 + \frac{(s^a-1)(s^b-1)}{s-1} \right]$ $s > 0$	$1 - \text{Log}_s \left[ 1 + \frac{(s^{1-a}-1)(s^{1-b}-1)}{s-1} \right]$	$f(a) = \text{Log}_s \left[ \frac{s-1}{s^a-1} \right]$ of the t-norms	(strict)	does not exist	does not exist
Dubois and Prade, <sup>7</sup> G <sub>s</sub>	$\frac{ab}{\max(a, b, z)}$ $x \in [0, 1]$	$\frac{a+b-ab - \min(a, b, 1-\alpha)}{\max(1-a, 1-b, \alpha)}$	does not exist	does not exist	does not exist	does not exist

### ANNEX 3 SET FUNCTIONS BASED ON DISTRIBUTIVE T-NORMS ARE BELIEF FUNCTIONS

A distributive  $t$ -norm is, from Frank<sup>9</sup> an ordinal sum of products, otherwise min, or product themselves. Hence, a distributive  $t$ -norm lies between product and min, and generates pseudo-necessity measures. Let  $g$  be such a fuzzy measure. We have proved that its codensity  $\{\bar{g}_i | i = 1, n\}$  is such that

$$\exists i: \bar{g}_i = 0$$

Let  $K = \{x_i | \bar{g}_i = 0\}$ . The elements of  $X$  can be ordered such that

$$K = \{x_{m+1}, \dots, x_n\}.$$

Obviously,

$$g(A) > 0 \Leftrightarrow A \supset K,$$

$$\text{since } g(A) = \bigwedge_{x_i \notin A} \bar{g}_i.$$

Let

$$m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} g(B). \quad (I)$$

Whenever  $g(A) = 0$ , it implies  $m(A) = 0$ . So  $m$  need to be evaluated for all subsets  $A$  of the form  $K \cup A'$ , with  $K \cap A' = \emptyset$ .

Note that

$$m(K) = \bigwedge_{i=1, m} \bar{g}_i \geq 0$$

where  $\bigwedge$  is the  $t$ -norm generating  $g$ .

Similarly,  $\forall j \in [1, m]$  (where  $[1, m]$  denotes the set  $\{1, 2, \dots, m\}$ )

$$m(K \cup \{x_j\}) = - \bigwedge_{i=1, m} \bar{g}_i + \bigwedge_{\substack{i=1, m \\ i \neq j}} \bar{g}_i \geq 0$$

$$\left( \bigwedge_{i=1, m} \bar{g}_i \text{ stands for } \bar{g}_1 \bigwedge \dots \bigwedge \bar{g}_m \right)$$

To establish the positivity of  $m(K \cup A')$  when  $A'$  is more than a singleton we need the following lemmas:

LEMMA 1 *If  $\bigwedge$  is a distributive  $t$ -norm, then the*

*operator  $*$  defined by  $a * b = a + b - a \bigwedge b$  is associative, and it is positive.*

*Proof*

$$\begin{aligned} (a * b) * c &= (a + b - a \bigwedge b) + c - (a + b - a \bigwedge b) \bigwedge c \\ &= a + b + c - a \bigwedge b - a \bigwedge c - b \bigwedge c + a \bigwedge b \bigwedge c \\ &= a * (b * c) \end{aligned}$$

using the associativity and distributivity properties of  $\bigwedge$ . The positivity of  $a * b$  when  $(a, b) \in [0, 1]^2$  is obvious. Q.E.D.

LEMMA 2

$$\forall n \in \mathbb{N}^+, \forall (a_1 \dots a_n) \in [0, 1]^n, \forall b \in [0, 1]$$

$$\begin{aligned} S_n &= b \bigwedge (a_1 * a_2 * \dots * a_n) \\ &= \sum_{i=1}^n (-1)^{i-1} \sum_{\substack{I \subseteq [1, n] \\ |I|=i}} b \bigwedge \left( \bigwedge_{j \in I} a_j \right) \end{aligned}$$

$$\text{Proof } S_n = b \bigwedge (a_1 * a_2 \dots * a_{n-2} * (a_{n-1} * a_n))$$

using associativity of  $*$

$$= b \bigwedge (A_1 * A_2 \dots * A_{n-2} * A_{n-1})$$

with  $A_i = a_i$ ,  $i \leq n-2$ ,  $A_{n-1} = a_{n-1} * a_n$

$$= \sum_{i=1}^n (-1)^{i-1} \sum_{\substack{I \subseteq [1, n] \\ |I|=i}} b \bigwedge \left( \bigwedge_{j \in I} A_j \right)$$

assuming Lemma 2 holds at order  $n-1$ .

All terms  $\bigwedge_{j \in I} A_j$  involving  $A_{n-1}$  can be changed as follows

$$\begin{aligned} b \bigwedge \left( \bigwedge_{j \in I} A_j \right) &= b \bigwedge \left( \bigwedge_{j \in I - \{n-1\}} a_j \right) \bigwedge (a_{n-1} \\ &\quad + a_n - a_{n-1} \bigwedge a_n) \end{aligned}$$

$$\begin{aligned} &= b \bigwedge \left( \bigwedge_{j \in I} a_j \right) + b \bigwedge \left( \bigwedge_{j \in (I - \{n-1\}) \cup \{n\}} a_j \right) \\ &\quad - b \bigwedge_{j \in I \cup \{n\}} a_j \end{aligned}$$

using the distributivity property. Inserting the above term into  $S_n$  yields the result. The recurrence is valid since Lemma 2 is obvious for  $n=1$ .  
Q.E.D.

It is now easy to figure out that from (I), when  $A=K \cup \{x_1, \dots, x_k\}$ , then

$$m(A) = \sum_{i=0}^k (-1)^i \sum_{\substack{J \subseteq \{1, \dots, k\} \\ |J|=k-i}} \perp \bar{g}_j \quad (k \leq m)$$

Noting  $G_k \stackrel{\Delta}{=} \bar{g}_{k+1} \perp \bar{g}_{k+2} \perp \dots \perp \bar{g}_m$

$$m(A) = G_k - \sum_{i=1}^k (-1)^{i-1} \sum_{\substack{J \subseteq \{1, \dots, k\} \\ |J|=i}} \left( \perp \bar{g}_j \right) \perp G_k$$

$= G_k - G_k \perp (\bar{g}_1 * \dots * \bar{g}_k) \geq 0, \quad \forall k$  using Lemma 2;  
for  $k=m, A'=\bar{K}$  and  $m(X) = \bar{g}_1 * \dots * \bar{g}_m \geq 0$ .

Lastly,

$$m(\emptyset) = 0, \quad \sum_{A \in X} m(A) = 1$$

because  $g(\emptyset) = 0$  and  $g(X) = 1$  respectively.

Thus,  $m(A)$  is a genuine basic assignment, and  $g$  is a belief function.  
Q.E.D.

N.B. When  $\perp =$  product

$$m(A) = (1 - \bar{g}_1) \cdot (1 - \bar{g}_2) \cdot \dots \cdot (1 - \bar{g}_k) \cdot \bar{g}_{k+1} \cdot \bar{g}_{k+2} \cdot \dots \cdot \bar{g}_m$$



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