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# Exact solutions for the coupled warping beam problem

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## Abstract

Exact solutions for coupled warping beam problem are developed in the present paper. The torsion and shear warping problems are considered together, and three beam formulations are considered, regarding the influence of the bi-moment and bi-shear terms. In the first case, where the bi-moment and bi-shear terms are neglected, the exact solutions have a simple polynomial form. In the second and third cases, the exact solution have a more elaborate form, consisting of hyperbolic terms. The coupling between the modes results from the lack of symmetry of the cross-section and is captured in the exact solution for all modes.

*Keywords:* Exact solution, Shear, Warping, Coupled, Thin-walled,

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## 1. Introduction

The effects of shear and warping deformations on the structural analysis of beam structures is often relevant, specially when dealing with short thin-walled beams. When the beam's section is not doubly symmetric, the differential equations that describe the variation of the displacements, rotations and warping fields along the beam axis are coupled. This implies that exact solutions of the system of differential equations presents a coupling between the bending, shear, torsion and warping modes. In practice, this coupling is not considered, as it usually amounts only for a small contribution in the interpolation of the displacement and strain fields. However, in order to obtain a solution with increased accuracy and avoid possible numerical locking problems, the coupling between these modes is considered in the present work and exact solutions are derived.

The uniform and non-uniform warping problems have been studied extensively in the literature. Ferradi et al. [11] developed a higher order beam finite element with warping eigen-modes. Later, Ferradi and Cespedes [10] added to their element shear warping eigen-modes. Pi and Trahair [13] studied the distortion and warping at beam supports. El Fatmi [7] and El Fatmi [8] developed a non-uniform warping beam theory including the effects of torsion and shear forces. El Fatmi and Ghazouani [9] studied a higher order composite beam theory built on the Saint-Venant's solution. Krenk and Damkilde [12] studied the warping of joints in I-beam assemblages. Pi et al. [14] developed a spatially curved-beam element with warping and Wagner effects. Banerjee et al. [3] developed an exact dynamic stiffness matrix of a bending-torsion coupled beam including warping. Addressi et al. [1] proposed an enriched beam finite element models with torsion and shear warping for the analysis of thin-walled structures. Battini and Pacoste [4] proposed a co-rotational beam elements accounting for warping effects in instability problems. Alsafadie et al. [2] developed a three-dimensional formulation of a mixed co-rotational thin-walled beam element incorporating warping deformation.

The present contribution focuses on the development of exact interpolation functions for the warping beam problem taking into account the coupling between the shear/bending, torsion and warping fields. This is achieved by writing the equilibrium equations as a function of the kinematic and warping fields

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via the elastic constitutive relations and then performing a suitable simplification in order to identify the general form of the coupled fields. This general form is then transformed to the usual interpolation functions relating the fields with the nodal degrees of freedom. Finally, the interpolation of the section strains, used for example in the computation of the stiffness matrix in a finite element analysis, is obtained.

The development of exact solutions poses advantages in the structural analysis of beams since it allows for an exact representation of the displacement and rotation fields and, consequently, of the section strains and stress-resultants. Perhaps most importantly, the use of exact interpolation functions for the strain fields completely avoids problems of numerical locking of the stiffness matrix in the context of finite element analysis. Also, the exact interpolation of the displacement and rotation fields completely avoids problems of numerical locking of the inertia (mass) matrix when performing dynamic and modal analysis.

The paper is organized as follows. The theory regarding the beam kinematics and the torsion and shear warping modes are briefly recalled in Sec. 2 for consistency and completeness. The exact solutions to the coupled warping beam problem are developed in Sec. 3. A number of numerical examples, showing the behavior of the solution for sections with and without axes of symmetry are presented in Sec. 4. Finally, some conclusions and suggestions for future work are drawn in Sec. 5.

## 2. Beam theory

In this section, the non-uniform coupled beam problem is briefly reviewed for consistency and completeness. The beam studied in the present work is considered to be a prismatic body with a generic section's domain  $\mathbb{D}$  and boundary  $\mathbb{C}$ . The beam's volume can be described by the parameter  $x_1$  running along its length and the parameters  $x_2$  and  $x_3$  mapping the section's domain (Fig. 1). The coordinate axes ( $x_2$ ,  $x_3$ ) are assumed to be placed at the section's center of gravity and aligned along the section's principal axes, i.e.:

$$\langle x_2 \rangle = \langle x_3 \rangle = \langle x_2 x_3 \rangle = 0 \quad (1)$$

where here and throughout the manuscript the following notations are adopted:

$$(\cdot)' = \frac{d(\cdot)}{dx_1} \quad (2)$$

$$\langle (\cdot) \rangle = \int_A (\cdot) dA \quad (3)$$

$$\nabla(\cdot) = \left( \frac{\partial(\cdot)}{\partial x_2}, \frac{\partial(\cdot)}{\partial x_3} \right) \quad (4)$$

$$\Delta(\cdot) = \frac{\partial^2(\cdot)}{\partial x_2^2} + \frac{\partial^2(\cdot)}{\partial x_3^2} \quad (5)$$

$$\frac{\partial(\cdot)}{\partial n} = n_2 \frac{\partial(\cdot)}{\partial x_2} + n_3 \frac{\partial(\cdot)}{\partial x_3} \quad (6)$$

$$(7)$$

The section's area and second moment of area along the principal axes are given by  $A = \langle 1 \rangle$ ,  $I_{22} = \langle x_2^2 \rangle$  and  $I_{33} = \langle x_3^2 \rangle$ , respectively. Considering the displacements fields  $u_1^p$ ,  $u_2^p$  and  $u_3^p$  of an arbitrary point  $p$  of the beam along the axes  $x_1$ ,  $x_2$  and  $x_3$ , respectively, the in-plane strains are defined as:

$$\varepsilon_{11}(x_1, x_2, x_3) = \frac{\partial u_1^p}{\partial x_1}(x_1, x_2, x_3) \quad (8)$$

$$\gamma_{12}(x_1, x_2, x_3) = \frac{\partial u_1^p}{\partial x_2}(x_1, x_2, x_3) + \frac{\partial u_2^p}{\partial x_1}(x_1, x_2, x_3) \quad (9)$$

$$\gamma_{13}(x_1, x_2, x_3) = \frac{\partial u_1^p}{\partial x_3}(x_1, x_2, x_3) + \frac{\partial u_3^p}{\partial x_1}(x_1, x_2, x_3) \quad (10)$$

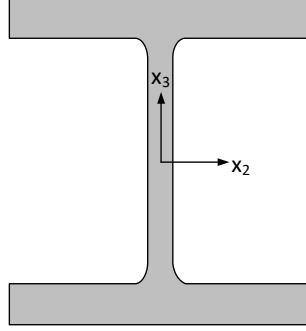


Figure 1: Section parameters  $x_2$  and  $x_3$  running alongside the section's principal axes.

As usual in beam theories, it's assumed that the out-of-plane stresses are identically null, that is:

$$\sigma_{22}(x_1, x_2, x_3) = \sigma_{33}(x_1, x_2, x_3) = \tau_{23}(x_1, x_2, x_3) = 0 \quad (11)$$

A linear elastic material response, with elastic modulus  $E$  and shear modulus  $G$ , is assumed and so, the constitutive relations relating the in-plane stresses ( $\sigma_{11}$ ,  $\tau_{12}$  and  $\tau_{13}$ ) with the in-plane strains ( $\varepsilon_{11}$ ,  $\gamma_{12}$  and  $\gamma_{13}$ ) can be obtained as:

$$\sigma_{11}(x_1, x_2, x_3) = E\varepsilon_{11}(x_1, x_2, x_3) \quad (12)$$

$$\tau_{12}(x_1, x_2, x_3) = G\gamma_{12}(x_1, x_2, x_3) \quad (13)$$

$$\tau_{13}(x_1, x_2, x_3) = G\gamma_{13}(x_1, x_2, x_3) \quad (14)$$

In the absence of volume forces and taking into account the null stresses in Eq. (11), the point wise equilibrium equations over the volume of the beam are given by:

$$\frac{\partial \tau_{12}}{\partial x_1} = 0 \quad (15)$$

$$\frac{\partial \tau_{13}}{\partial x_1} = 0 \quad (16)$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} = 0 \quad (17)$$

Finally, let  $n_2$  and  $n_3$  be the components along the axes  $x_2$  and  $x_3$  of the normal vector to the lateral surface of the beam, respectively. As the beam is considered to be prismatic, this vector has no component along the  $x_1$  axis and the stresses boundary condition can be written as:

$$\tau_{12}n_2 + \tau_{13}n_3 = 0 \quad (18)$$

### 2.1. Uniform torsion

When the beam is subjected to a uniform torsion moment  $M_1$ , a section located at  $x_1$  rotates about the axis  $x_1$  centered at the shear center ( $c_2, c_3$ ) by an angle  $\theta_1(x_1)$ , experiencing a constant torsional deformation  $\kappa_1 = \theta_1'(x_1)$  and warp in the axial direction via a warping function  $\omega : \mathbb{D} \rightarrow \mathbb{R}$ . The resulting displacement's field can be expressed as:

$$u_1^p(x_1, x_2, x_3) = \omega(x_2, x_3)\kappa_1(x_1) \quad (19)$$

$$u_2^p(x_1, x_2, x_3) = -(x_3 - c_3)\theta_1(x_1) \quad (20)$$

$$u_3^p(x_1, x_2, x_3) = +(x_2 - c_2)\theta_1(x_1) \quad (21)$$

The uniform torsion problem has been extensively studied in the literature. It can be shown that the warping function in torsion  $\omega$  is the unique function that satisfies the following conditions:

$$\Delta\omega(x_2, x_3) = 0, \forall(x_2, x_3) \in \mathbb{D} \quad (22)$$

$$\frac{\partial\omega}{\partial n}(x_2, x_3) = (x_3 - c_3)n_2 - (x_2 - c_2)n_3, \forall(x_2, x_3) \in \mathbb{C} \quad (23)$$

$$\langle\omega(x_2, x_3)\rangle = \langle x_2\omega(x_2, x_3)\rangle = \langle x_3\omega(x_2, x_3)\rangle = 0 \quad (24)$$

The relation in Eqs. (22) and (23) come from the equilibrium equations in the domain (Eq. 17) and the boundary (Eq. 18), respectively. The integral relations in Eq. (24) ensure the elastic uncoupling between the torsion moment, the axial force and the bending moments. More specifically, these relations guarantee that, even when the torsional deformation  $\kappa_1$  varies along the length of the beam, no axial force or bending moments are created while the material remains in the elastic domain.

Since the position of the shear center  $(c_2, c_3)$  is not known a priori, it's convenient to introduce the auxiliary warping function in torsion  $\bar{\omega} : \mathbb{D} \rightarrow \mathbb{R}$  as:

$$\bar{\omega}(x_2, x_3) = \omega(x_2, x_3) + c_3x_2 - c_2x_3 \quad (25)$$

From Eqs. (22 - 24), the corresponding boundary value problem can be obtained as:

$$\langle\bar{\omega}(x_2, x_3)\rangle = 0 \quad (26)$$

$$\Delta\bar{\omega}(x_2, x_3) = 0, \forall(x_2, x_3) \in \mathbb{D} \quad (27)$$

$$\frac{\partial\bar{\omega}}{\partial n}(x_2, x_3) = x_3n_2 - x_2n_3, \forall(x_2, x_3) \in \mathbb{C} \quad (28)$$

The boundary value problem in Eqs. (26 - 28) can be solved via the finite element. The shear center can then be computed as:

$$c_2 = \frac{\langle -x_3\bar{\omega}(x_2, x_3)\rangle}{I_{33}} \quad (29)$$

$$c_3 = \frac{\langle +x_2\bar{\omega}(x_2, x_3)\rangle}{I_{22}} \quad (30)$$

The original warping function can be recovered as:

$$\omega(x_2, x_3) = \bar{\omega}(x_2, x_3) - c_3x_2 + c_2x_3 \quad (31)$$

Finally, the torsion moment  $M_1$  can be related to the torsional deformation  $\kappa_1$  via:

$$M_1 = GJ\kappa_1 \quad (32)$$

where the section's stiffness in torsion  $J$  is given by:

$$J = I_{22} + I_{33} + (c_2^2 + c_3^2)A - \langle\nabla\omega \cdot \nabla\omega\rangle \quad (33)$$

## 2.2. Uniform shear

When the beam is subjected to a uniform shear force  $F_2$ , a section located at  $x_1$  rotates about the axis  $x_3$  by an angle  $\theta_3(x_1)$  and have a displacement  $u_2(x_1)$  in the direction of the axis  $x_2$ . Analogously, when the beam is subjected to a uniform shear force  $F_3$ , a section located at  $x_1$  rotates about the axis  $x_2$  by an angle  $\theta_2(x_1)$  and have a displacement  $u_3(x_1)$  in the direction of the axis  $x_3$ . In both cases, the beam experiences constant shear deformations,  $\gamma_2 = u_2'(x_1) - \theta_3(x_1)$  and  $\gamma_3 = u_3'(x_1) + \theta_2(x_1)$ , and warp in the axial direction via the warping functions  $\eta_2 : \mathbb{D} \rightarrow \mathbb{R}$  and  $\eta_3 : \mathbb{D} \rightarrow \mathbb{R}$ . The resulting displacement's field can be expressed as:

$$u_2^p(x_1, x_2, x_3) = u_2(x_1) \quad (34)$$

$$u_3^p(x_1, x_2, x_3) = u_3(x_1) \quad (35)$$

$$u_1^p(x_1, x_2, x_3) = x_3\theta_2(x_1) - x_2\theta_3(x_1) + (\eta_2(x_2, x_3) - x_2)\gamma_2 + (\eta_3(x_2, x_3) - x_3)\gamma_3 \quad (36)$$

The uniform shear problem also has been extensively studied in the literature. It can be shown that the warping functions in shear,  $\eta_2$  and  $\eta_3$ , are the unique functions that satisfy the following conditions:

$$\langle x_2\eta_2(x_2, x_3) \rangle = I_{22} \quad (37)$$

$$\langle x_3\eta_3(x_2, x_3) \rangle = I_{33} \quad (38)$$

$$\langle \eta_2(x_2, x_3) \rangle = \langle \eta_3(x_2, x_3) \rangle = 0 \quad (39)$$

$$\langle x_3\eta_2(x_2, x_3) \rangle = \langle x_2\eta_3(x_2, x_3) \rangle = 0 \quad (40)$$

$$\frac{\partial \eta_2}{\partial n}(x_2, x_3) = \frac{\partial \eta_3}{\partial n}(x_2, x_3) = 0, \forall (x_2, x_3) \in \mathbb{C} \quad (41)$$

$$\Delta \eta_2(x_2, x_3) + \frac{x_2 A_{22}}{I_{22}} + \frac{x_3 A_{23}}{I_{33}} = 0, \forall (x_2, x_3) \in \mathbb{D} \quad (42)$$

$$\Delta \eta_3(x_2, x_3) + \frac{x_2 A_{32}}{I_{22}} + \frac{x_3 A_{33}}{I_{33}} = 0, \forall (x_2, x_3) \in \mathbb{D} \quad (43)$$

where the shear areas  $A_{ij}$  are defined as:

$$A_{ij} = \left\langle \frac{\partial \eta_i}{\partial x_j}(x_2, x_3) \right\rangle \quad (44)$$

The relations in Eqs. (42 - 43) and (41) come from the equilibrium equations in the domain (Eq. 17) and the boundary (Eq. 18), respectively. The integral relations in Eq. (37 - 40) ensure the uncoupling between the shear forces, the axial force and the bending moments. More specifically, these relations guarantee that, even when the shear deformations  $\gamma_2$  and  $\gamma_3$  vary along the beam's length, no axial force or bending moments are created while the material remains in the elastic domain.

Since the shear areas  $A_{ij}$  are not known a priori, it's convenient to introduce the auxiliary warping functions in shear  $\bar{\eta}_2 : \mathbb{D} \rightarrow \mathbb{R}$  and  $\bar{\eta}_3 : \mathbb{D} \rightarrow \mathbb{R}$  as:

$$\bar{\eta}_2(x_2, x_3) = I_{22} \left( \frac{A_{33}\eta_2(x_2, x_3) - A_{23}\eta_3(x_2, x_3)}{A_{22}A_{33} - A_{23}A_{32}} \right) \quad (45)$$

$$\bar{\eta}_3(x_2, x_3) = I_{33} \left( \frac{A_{22}\eta_2(x_2, x_3) - A_{32}\eta_3(x_2, x_3)}{A_{22}A_{33} - A_{23}A_{32}} \right) \quad (46)$$

From Eqs. (37 - 43), the corresponding boundary value problem can be obtained as:

$$\langle \bar{\eta}_2(x_2, x_3) \rangle = \langle \bar{\eta}_3(x_2, x_3) \rangle = 0 \quad (47)$$

$$\Delta \bar{\eta}_2(x_2, x_3) + x_2 = 0, \forall (x_2, x_3) \in \mathbb{D} \quad (48)$$

$$\Delta \bar{\eta}_3(x_2, x_3) + x_3 = 0, \forall (x_2, x_3) \in \mathbb{D} \quad (49)$$

$$\frac{\partial \bar{\eta}_2}{\partial n}(x_2, x_3) = \frac{\partial \bar{\eta}_3}{\partial n}(x_2, x_3) = 0, \forall (x_2, x_3) \in \mathbb{C} \quad (50)$$

The boundary value problem in Eqs. (47 - 50) can be solved via the finite element method. The shear areas can then be computed as:

$$A_{22} = \frac{I_{22}^2 H_{33}}{H} \quad (51)$$

$$A_{33} = \frac{I_{33}^2 H_{22}}{H} \quad (52)$$

$$A_{23} = A_{32} = \frac{-I_{22} I_{33} H_{23}}{H} \quad (53)$$

where:

$$H_{ij} = \langle x_i \bar{\eta}_j(x_2, x_3) \rangle \quad (54)$$

$$H = H_{22}H_{33} - H_{23}H_{32} \quad (55)$$

The original warping functions can be recovered using:

$$\eta_2(x_2, x_3) = \frac{A_{22}\bar{\eta}_2(x_2, x_3)}{I_{22}} + \frac{A_{23}\bar{\eta}_3(x_2, x_3)}{I_{33}} \quad (56)$$

$$\eta_3(x_2, x_3) = \frac{A_{32}\bar{\eta}_2(x_2, x_3)}{I_{22}} + \frac{A_{33}\bar{\eta}_3(x_2, x_3)}{I_{33}} \quad (57)$$

Finally, the shear forces,  $F_2$  and  $F_3$ , can be related to the shear deformations,  $\gamma_2$  and  $\gamma_3$ , via the shear areas as:

$$F_2 = G(A_{22}\gamma_2 + A_{32}\gamma_3) \quad (58)$$

$$F_3 = G(A_{32}\gamma_2 + A_{33}\gamma_3) \quad (59)$$

It's noteworthy that, the symmetry of the section with respect to either the  $x_2$  or  $x_3$  axis implies  $A_{23} = 0$ , that is, the shear axes coincide with the bending (principal inertia) axes. Also, the symmetry of the section with respect to the  $x_2$  axis implies  $c_3 = 0$  and the symmetry of the section with respect to  $x_3$  axis implies  $c_2 = 0$ .

### 2.3. Combined beam theory

When the deformations in torsion and shear are allowed to vary along the beam axis, it's assumed that the warping kinematics aren't necessarily proportional to the deformations. In replacement, three new displacement functions  $\mu_k : [0, L] \rightarrow \mathbb{R}, k \in \{1, 2, 3\}$  are introduced. The combined non-uniform displacements field can be written as:

$$u_2^p(x_1, x_2, x_3) = u_2(x_1) - (x_3 - c_3)\theta_1(x_1) \quad (60)$$

$$u_3^p(x_1, x_2, x_3) = u_3(x_1) + (x_2 - c_2)\theta_1(x_1) \quad (61)$$

$$u_1^p(x_1, x_2, x_3) = u_1(x_1) + x_3\theta_2(x_1) - x_2\theta_3(x_1) + \Omega_k(x_2, x_3)\mu_k(x_1) \quad (62)$$

where the notation  $\Omega_1(x_2, x_3) = \omega(x_2, x_3)$ ,  $\Omega_2(x_2, x_3) = \eta_2(x_2, x_3) - x_2$  and  $\Omega_3(x_2, x_3) = \eta_3(x_2, x_3) - x_3$  has been introduced. Also, in Eq. (62) and throughout the manuscript, index notation is used to indicate summation on the repeated variable ( $k = 1, 2, 3$ ).

The resulting strains field can be obtained from Eqs. (8 - 10) as:

$$\varepsilon_{11}(x_1, x_2, x_3) = \varepsilon_1(x_1) + x_3\kappa_2(x_1) - x_2\kappa_3(x_1) + \Omega_k(x_2, x_3)\beta_k(x_1) \quad (63)$$

$$\gamma_{12}(x_1, x_2, x_3) = \frac{\partial \eta_2}{\partial x_2}\gamma_2(x_1) + \frac{\partial \eta_3}{\partial x_2}\gamma_3(x_1) + \left( \frac{\partial \omega}{\partial x_2} + c_3 - x_3 \right) \kappa_1(x_1) + \frac{\partial \Omega_k}{\partial x_2} \lambda_k(x_1) \quad (64)$$

$$\gamma_{13}(x_1, x_2, x_3) = \frac{\partial \eta_2}{\partial x_3}\gamma_2(x_1) + \frac{\partial \eta_3}{\partial x_3}\gamma_3(x_1) + \left( \frac{\partial \omega}{\partial x_3} + x_2 - c_2 \right) \kappa_1(x_1) + \frac{\partial \Omega_k}{\partial x_3} \lambda_k(x_1) \quad (65)$$

where  $\beta_k(x_1) = \mu_k'(x_1)$  and:

$$\varepsilon_1(x_1) = u_1'(x_1) \quad (66)$$

$$\kappa_2(x_1) = \theta_2'(x_1) \quad (67)$$

$$\kappa_3(x_1) = \theta_3'(x_1) \quad (68)$$

$$\lambda_1(x_1) = \mu_1(x_1) - \kappa_1(x_1) \quad (69)$$

$$\lambda_2(x_1) = \mu_2(x_1) - \gamma_2(x_1) \quad (70)$$

$$\lambda_3(x_1) = \mu_3(x_1) - \gamma_3(x_1) \quad (71)$$

With the adopted kinematic assumptions, the internal virtual work of the beam  $\delta U$  can be written as:

$$\begin{aligned} \delta U &= \int_0^L \langle \sigma_{11} \delta \varepsilon_{11} + \tau_{12} \delta \gamma_{12} + \tau_{13} \delta \gamma_{13} \rangle dx_1 \\ &= \int_0^L (F_k \delta \gamma_k + M_k \delta \kappa_k + B_k \delta \beta_k + V_k \delta \lambda_k) dx_1 \end{aligned} \quad (72)$$

The axial force  $F_1(x_1)$  and shear forces,  $F_2(x_1)$  and  $F_3(x_1)$ , are linked to the section's deformations  $\varepsilon_1(x_1)$ ,  $\gamma_2(x_1)$  and  $\gamma_3(x_1)$ , respectively, via the internal virtual work  $\delta U$  and can be computed as:

$$F_1(x_1) = \langle \sigma_{11} \rangle = EA\varepsilon_1(x_1) \quad (73)$$

$$F_2(x_1) = \left\langle \frac{\partial \eta_2}{\partial x_2} \tau_{12} + \frac{\partial \eta_2}{\partial x_3} \tau_{13} \right\rangle = GA_{22} \gamma_2(x_1) + GA_{23} \gamma_3(x_1) \quad (74)$$

$$F_3(x_1) = \left\langle \frac{\partial \eta_3}{\partial x_2} \tau_{12} + \frac{\partial \eta_3}{\partial x_3} \tau_{13} \right\rangle = GA_{23} \gamma_2(x_1) + GA_{33} \gamma_3(x_1) \quad (75)$$

Analogously, the torsional moment  $M_1(x_1)$  and bending moments,  $M_2(x_1)$  and  $M_3(x_1)$ , are linked to the section's deformations  $\kappa_1(x_1)$ ,  $\kappa_2(x_1)$  and  $\kappa_3(x_1)$ , respectively, via the internal virtual work  $\delta U$  and can be computed as:

$$M_2(x_1) = \langle +x_3 \sigma_{11} \rangle = EI_{33} \kappa_2 \quad (76)$$

$$M_3(x_1) = \langle -x_2 \sigma_{11} \rangle = EI_{22} \kappa_3 \quad (77)$$

$$M_1(x_1) = \left\langle \left( \frac{\partial \omega}{\partial x_3} + x_2 - c_2 \right) \tau_{13} + \left( \frac{\partial \omega}{\partial x_2} + c_3 - x_3 \right) \tau_{12} \right\rangle = GJ \kappa_1(x_1) \quad (78)$$

The bi-moment terms  $B_k(x_1)$  are the section's resultants related to the warping field deformations  $\beta_k(x_1)$  via the internal virtual work  $\delta U$  and can be computed as:

$$B_k(x_1) = \langle \Omega_k \sigma_{11} \rangle = EW_{kp} \beta_p(x_1) \quad (79)$$

where:

$$W_{kp} = \langle \Omega_k(x_2, x_3) \Omega_p(x_2, x_3) \rangle \quad (80)$$

Finally, the bi-shear terms  $V_k(x_1)$  are the section's resultants related to the warping strain functions  $\lambda_k(x_1)$  via the internal virtual work  $\delta U$  and can be computed as:

$$V_1(x_1) = \left\langle \frac{\partial \Omega_1}{\partial x_2} \tau_{12} + \frac{\partial \Omega_1}{\partial x_3} \tau_{13} \right\rangle = G(I_p - J) \lambda_1(x_1) + Gc_3 A \lambda_2(x_1) - Gc_2 A \lambda_3(x_1) \quad (81)$$

$$V_2(x_1) = \left\langle \frac{\partial \Omega_2}{\partial x_2} \tau_{12} + \frac{\partial \Omega_2}{\partial x_3} \tau_{13} \right\rangle = +Gc_3 A \lambda_1(x_1) + G(A - A_{22}) \lambda_2(x_1) - GA_{23} \lambda_3(x_1) \quad (82)$$

$$V_3(x_1) = \left\langle \frac{\partial \Omega_3}{\partial x_2} \tau_{12} + \frac{\partial \Omega_3}{\partial x_3} \tau_{13} \right\rangle = -Gc_2 A \lambda_1(x_1) - GA_{23} \lambda_2(x_1) + G(A - A_{33}) \lambda_3(x_1) \quad (83)$$

where:

$$I_p = I_{22} + I_{33} + (c_2^2 + c_3^2)A \quad (84)$$

When the warping strain fields  $\lambda_k$  are identically null, i.e. when the warping fields  $\mu_k$  are identically equal to the section deformations related to the warping functions  $\Omega_k$ , and in the absence of distributed



loads the virtual work equation in Eq. (72) can be integrated by parts and, considering the fundamental lemma of calculus of variations, the following equilibrium equations are obtained:

$$F_1'(x_1) = 0 \quad (85)$$

$$F_2'(x_1) - B_2''(x_1) = 0 \quad (86)$$

$$F_3'(x_1) - B_3''(x_1) = 0 \quad (87)$$

$$M_1'(x_1) - B_1''(x_1) = 0 \quad (88)$$

$$M_2'(x_1) - F_3(x_1) + B_3'(x_1) = 0 \quad (89)$$

$$M_3'(x_1) + F_2(x_1) - B_2'(x_1) = 0 \quad (90)$$

When the warping fields  $\mu_k$  are independent of the section deformations related to the warping functions  $\Omega_k$  and in the absence of distributed loads, the following additional equilibrium equations are obtained:

$$B_1'(x_1) - V_1(x_1) = 0 \quad (91)$$

$$B_2'(x_1) - V_2(x_1) = 0 \quad (92)$$

$$B_3'(x_1) - V_3(x_1) = 0 \quad (93)$$

### 3. Exact solution

When dealing with beams subjected to torsion, it's common to study three cases. In the first case, known as Saint-Venant's beam theory, the contributions of the bi-shear  $V_1$  and bi-moment  $B_1$  terms in the internal virtual work are neglected, leading to the assumption of a constant torsional deformation field  $\kappa_1$  along of the beam axis. In the second case, known as Vlasov's beam theory, the contribution of the bi-shear  $V_1$  to the internal virtual work is neglected, while the contribution of the bi-moment  $B_1$  is considered, implying a variation of torsional deformation  $\kappa_1$  along the beam axis. In third case, known as Bencoter's beam theory, the contributions of the bi-shear  $V_1$  and bi-moment  $B_1$  are considered in the internal virtual work and, hence, the deformation fields  $\kappa_1$  and  $\mu_1$  vary along the beam length. In the present work, exact solutions are obtained for the three cases while considering all three fundamental warping modes ( $\omega$ ,  $\eta_2$  and  $\eta_3$ ). Throughout the manuscript,  $x_k^p$  denotes the  $k$ -th component of field  $x$  evaluated at node  $p$ .

#### 3.1. Case 1

In the first case, the contributions of the bi-moment  $B_k$  and the bi-shear  $V_k$  components to the internal virtual work are neglected. From the constitutive relations in Eqs. (79 - 83), the warping related deformations are constant along the beam axis and equal to the warping displacement fields  $\mu_k$ , i.e.  $\beta_k(x_1) = 0$  and  $\lambda_k(x_1) = 0$ . Therefore, each node of the beam has as degrees of freedom  $\{u_1, u_2, u_3, \theta_1, \theta_2, \theta_3\}$ . Combining the elastic constitutive relations and the equilibrium equations we have:

$$u_1''(x_1) = 0 \quad (94)$$

$$\theta_1''(x_1) = 0 \quad (95)$$

$$\mathbf{K}_s (\mathbf{d}_s''(x_1) + \mathbf{H}\mathbf{d}_b'(x_1)) = \mathbf{0} \quad (96)$$

$$\mathbf{K}_b \mathbf{d}_b''(x_1) - \mathbf{H}^T \mathbf{K}_s (\mathbf{d}_s'(x_1) + \mathbf{H}\mathbf{d}_b(x_1)) = \mathbf{0} \quad (97)$$

where the following notation has been introduced:

$$\mathbf{H} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (98)$$

$$\mathbf{K}_b = E \begin{bmatrix} I_{33} & 0 \\ 0 & I_{22} \end{bmatrix} \quad (99)$$

$$\mathbf{K}_s = G \begin{bmatrix} A_{22} & A_{23} \\ A_{23} & A_{33} \end{bmatrix} \quad (100)$$

$$\mathbf{d}_b(x_1) = \{\theta_2(x_1) \quad \theta_3(x_1)\} \quad (101)$$

$$\mathbf{d}_s(x_1) = \{u_2(x_1) \quad u_3(x_1)\} \quad (102)$$

The equilibrium equations in Eqs. (96) and (97) show the coupling between the bending and shear modes. When the section has at least one axis of symmetry, the coupling term  $A_{23}$  vanishes and the bending and shear modes don't interact. Combining Eqs. (96) and (97) and observing that neither  $\mathbf{K}_b$  nor  $\mathbf{K}_s$  are singular, the following simplified form of the equilibrium equations can be obtained:

$$\mathbf{d}_b'''(x_1) = \mathbf{0} \quad (103)$$

$$\mathbf{d}_s^{iv}(x_1) = \mathbf{0} \quad (104)$$

From Eqs. (94) and (95), the axial displacement  $u_1$  and rotation  $\theta_1$  vary linearly along the beam axis:

$$u_1(x_1) = \left(1 - \frac{x_1}{L}\right) u_1^1 + \left(\frac{x_1}{L}\right) u_1^2 \quad (105)$$

$$\theta_1(x_1) = \left(1 - \frac{x_1}{L}\right) \theta_1^1 + \left(\frac{x_1}{L}\right) \theta_1^2 \quad (106)$$

Therefore, the axial  $\varepsilon_1(x_1)$  and torsional  $\kappa_1(x_1)$  deformations, obtained with Eqs. (63 - 65), are constant:

$$\varepsilon_1(x_1) = \frac{u_1^2 - u_1^1}{L} \quad (107)$$

$$\kappa_1(x_1) = \frac{\theta_1^2 - \theta_1^1}{L} \quad (108)$$

From Eqs. (103) and (104), the bending  $\mathbf{d}_b$  and shear  $\mathbf{d}_s$  fields have a quadratic and cubic variation along the beam axis, respectively. Introducing the constant parameters  $\mathbf{a}_k$  and  $\mathbf{b}_k$ , the bending and shear fields may be written as:

$$\mathbf{d}_b(x_1) = \mathbf{a}_0 + x_1 \mathbf{a}_1 + x_1^2 \mathbf{a}_2 \quad (109)$$

$$\mathbf{d}_s(x_1) = \mathbf{b}_0 + x_1 \mathbf{b}_1 + x_1^2 \mathbf{b}_2 + x_1^3 \mathbf{b}_3 \quad (110)$$

Introducing the solution form above in the equilibrium equations (Eqs. 96 and 97), the following relations between the constant parameters can be obtained:

$$\mathbf{b}_2 = -\frac{1}{2} \mathbf{H} \mathbf{a}_1 \quad (111)$$

$$\mathbf{b}_3 = -\frac{1}{3} \mathbf{H} \mathbf{a}_2 \quad (112)$$

$$\mathbf{a}_2 = \frac{1}{2} \overline{\mathbf{K}} (\mathbf{b}_1 + \mathbf{H} \mathbf{a}_0) \quad (113)$$

where:

$$\overline{\mathbf{K}} = \mathbf{K}_b^{-1} \mathbf{H}^T \mathbf{K}_s \quad (114)$$

Combining Eqs. (109 - 110) with Eqs. (111 - 113), the displacement fields can be written as:

$$\mathbf{d}(x_1) = \mathbf{A}(x_1) \mathbf{p} \quad (115)$$

where the following notation has been introduced:

$$\mathbf{p} = \{\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{b}_0 \quad \mathbf{b}_1\} \quad (116)$$

$$\mathbf{d}(x_1) = \{\mathbf{d}_b(x_1) \quad \mathbf{d}_s(x_1)\} \quad (117)$$

$$\mathbf{A}(x_1) = \begin{bmatrix} \mathbf{I} + \frac{x_1^2}{2} \overline{\mathbf{K}}\mathbf{H} & x_1 \mathbf{I} & \mathbf{0} & \frac{x_1^2}{2} \overline{\mathbf{K}} \\ -\frac{x_1^3}{6} \overline{\mathbf{H}}\mathbf{K}\mathbf{H} & -\frac{x_1^2}{2} \mathbf{H} & \mathbf{I} & x_1 \mathbf{I} - \frac{x_1^3}{6} \overline{\mathbf{H}}\mathbf{K} \end{bmatrix} \quad (118)$$

The reduced set of constant parameters  $\mathbf{p}$  can then be linked to the nodal degrees of freedom  $\mathbf{d}_c$  as:

$$\mathbf{p} = \mathbf{A}_c^{-1} \mathbf{d}_c \quad (119)$$

where:

$$\mathbf{d}_c = \{\mathbf{d}(0) \quad \mathbf{d}(L)\} \quad (120)$$

$$\mathbf{A}_c = [\mathbf{A}^T(0) \quad \mathbf{A}^T(L)]^T \quad (121)$$

The final form of the exact solution, holding the exact interpolation functions, is given by:

$$\mathbf{d}(x_1) = \mathbf{A}(x_1) \mathbf{A}_c^{-1} \mathbf{d}_c \quad (122)$$

From Eqs. (8 - 10), the bending  $\mathbf{e}_b = \{\kappa_2, \kappa_3\}$  and shear  $\mathbf{e}_s = \{\gamma_2, \gamma_3\}$  strain fields can then be obtained as:

$$\mathbf{e}_b(x_1) = \mathbf{d}'_b(x_1) = \mathbf{a}_1 + 2x_1 \mathbf{a}_2 \quad (123)$$

$$\mathbf{e}_s(x_1) = \mathbf{d}'_s(x_1) + \mathbf{H} \mathbf{d}_b(x_1) = \mathbf{b}_1 + \mathbf{H} \mathbf{a}_0 \quad (124)$$

The strain fields can be written in matrix form as:

$$\mathbf{e}(x_1) = \mathbf{E}(x_1) \mathbf{A}_c^{-1} \mathbf{d}_c \quad (125)$$

where:

$$\mathbf{e}(x_1) = \{\mathbf{e}_b(x_1) \quad \mathbf{e}_s(x_1)\} \quad (126)$$

$$\mathbf{E}(x_1) = \begin{bmatrix} x_1 \overline{\mathbf{K}}\mathbf{H} & \mathbf{I} & \mathbf{0} & x_1 \overline{\mathbf{K}} \\ \mathbf{H} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad (127)$$

It can be easily shown that the exact interpolation functions for the Timoshenko's beam theory [15] are a particular case of the present solution when the coupling term  $A_{23}$  vanishes. It's also noteworthy that, in this case, the bending deformations ( $\kappa_2, \kappa_3$ ) and resultants ( $M_2, M_3$ ) vary linearly along the beam axis, while the shear deformations ( $\gamma_2, \gamma_3$ ) and resultants ( $F_2, F_3$ ) are constant, as in the classical Bernoulli's and Timoshenko's beam theories.

### 3.2. Case 2

In the second case, the contributions of the bi-shear  $V_k$  components to the internal virtual work are neglected, while the components of the bi-moment  $B_k$  are considered. From the constitutive relations in Eqs. (81 - 83), the warping related deformations are equal to the warping displacement fields  $\mu_k$ , i.e.  $\lambda_k(x_1) = 0$ . Therefore, each node of the beam has as degrees of freedom  $\{u_1, u_2, u_3, \theta_1, \theta_2, \theta_3, \mu_1, \mu_2, \mu_3\}$ . Combining the elastic constitutive relations and the equilibrium equations we have:

$$u_1''(x_1) = 0 \quad (128)$$

$$\mathbf{K}_s \mathbf{d}'_q(x_1) - \mathbf{K}_w \mathbf{d}'''_q(x_1) = \mathbf{0} \quad (129)$$

$$\mathbf{K}_b \mathbf{d}''_b(x_1) - \mathbf{H}^T (\mathbf{K}_s \mathbf{d}_q(x_1) - \mathbf{K}_w \mathbf{d}''_q(x_1)) = \mathbf{0} \quad (130)$$

where the following notation has been introduced:

$$\mathbf{H} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (131)$$

$$\mathbf{K}_s = G \begin{bmatrix} J & 0 & 0 \\ 0 & A_{22} & A_{32} \\ 0 & A_{23} & A_{33} \end{bmatrix} \quad (132)$$

$$\mathbf{K}_w = E \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} \quad (133)$$

$$\mathbf{d}_b(x_1) = \{\theta_2(x_1) \quad \theta_3(x_1)\} \quad (134)$$

$$\mathbf{d}_s(x_1) = \{\theta_1(x_1) \quad u_2(x_1) \quad u_3(x_1)\} \quad (135)$$

$$\mathbf{d}_q(x_1) = \mathbf{d}'_s(x_1) + \mathbf{H}\mathbf{d}_b(x_1) \quad (136)$$

As in case 1, the axial displacement varies linearly and is not coupled with the other fields. In this case, however, the torsion angle  $\theta_1$  is coupled with all the remaining fields via the warping inertia matrix  $\mathbf{K}_w$  and, hence, the matrices  $\mathbf{H}$  and  $\mathbf{K}_s$  and the vector  $\mathbf{d}_s$  are redefined to take into account its contribution. When the section has two axes of symmetry, the warping related matrices  $\mathbf{K}_s$  and  $\mathbf{K}_w$  become diagonal and the torsion and shear/bending modes don't interact. When the section has only one axis of symmetry, the matrix  $\mathbf{K}_s$  is diagonal but the matrix  $\mathbf{K}_w$  is not, preserving the coupling between the torsion mode and one of the shear/bending modes. Combining Eqs. (129) and (130) and observing that  $\mathbf{K}_b$  is not singular, the following simplified form of the equilibrium equations can be obtained:

$$\mathbf{d}_b'''(x_1) = \mathbf{0} \quad (137)$$

$$\mathbf{K}_s \mathbf{d}_s^{iv}(x_1) - \mathbf{K}_w \mathbf{d}_s^{vi}(x_1) = \mathbf{0} \quad (138)$$

From Eq. (137), the bending fields  $\mathbf{d}_b$  have a quadratic variation along the beam axis, as in case 1 of Sec. 3.1. However, due to the coupling between the torsion and shear/bending modes and the contribution of the warping stiffness  $\mathbf{K}_w$ , the shear fields  $\mathbf{d}_s$  vary according to a more elaborate shape. In fact, the solution of differential equation in Eq. (138) consists of a cubic polynomial plus hyperbolic terms related to the coupling between the matrices  $\mathbf{K}_s$  and  $\mathbf{K}_w$ . Introducing the constant parameters  $\mathbf{a}_k$  and  $\mathbf{b}_k$ , the bending and shear fields may be written as:

$$\mathbf{d}_b(x_1) = \mathbf{a}_0 + x_1 \mathbf{a}_1 + x_1^2 \mathbf{a}_2 \quad (139)$$

$$\mathbf{d}_s(x_1) = \mathbf{b}_0 + x_1 \mathbf{b}_1 + x_1^2 \mathbf{b}_2 + x_1^3 \mathbf{b}_3 + \mathbf{P} [\mathbf{S}(x_1) \mathbf{b}_4 + \mathbf{C}(x_1) \mathbf{b}_5] \quad (140)$$

where:

$$\mathbf{P}^T \mathbf{K}_w \mathbf{P} = \mathbf{I} \quad (141)$$

$$\mathbf{P}^T \mathbf{K}_s \mathbf{P} = \mathbf{D}^2 \quad (142)$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (143)$$

$$\mathbf{S}(x_1) = \begin{bmatrix} \sinh(\lambda_1 x_1) & 0 & 0 \\ 0 & \sinh(\lambda_2 x_1) & 0 \\ 0 & 0 & \sinh(\lambda_3 x_1) \end{bmatrix} \quad (144)$$

$$\mathbf{C}(x_1) = \begin{bmatrix} \cosh(\lambda_1 x_1) & 0 & 0 \\ 0 & \cosh(\lambda_2 x_1) & 0 \\ 0 & 0 & \cosh(\lambda_3 x_1) \end{bmatrix} \quad (145)$$

In the equations above,  $\mathbf{P}$  and  $\mathbf{D}$  represents the eigenvector and eigenvalue matrices, respectively, of the generalized symmetric eigenvalue problem  $\mathbf{K}_s \mathbf{v} - \lambda^2 \mathbf{K}_w \mathbf{v} = \mathbf{0}$ . Introducing the solution from Eqs. (137) and (138) in the equilibrium equations (Eqs. 129 and 130), the following relations between the constant parameters can be obtained:

$$\mathbf{b}_2 = -\frac{1}{2} \mathbf{H} \mathbf{a}_1 \quad (146)$$

$$\mathbf{b}_3 = -\frac{1}{3}\mathbf{H}\mathbf{a}_2 \quad (147)$$

$$\mathbf{a}_2 = \frac{1}{2}\overline{\mathbf{K}}(\mathbf{b}_1 + \mathbf{H}\mathbf{a}_0) \quad (148)$$

With respect to case 1 in Sec. 3.1, the solution for the shear fields has two additional  $3 \times 1$  vector parameters ( $\mathbf{b}_4$  and  $\mathbf{b}_5$ ) and, hence, two additional boundary conditions must be considered. From the virtual work in Eq. (72), the bi-moment contribution imply boundary terms related to the warping fields  $\mathbf{d}_\mu = \{\mu_1, \mu_2, \mu_3\}$  that, in this case, are equal to the shear strains  $\mathbf{e}_s$ . The bending  $\mathbf{e}_b$  and shear  $\mathbf{e}_s$  strains can be computed as:

$$\mathbf{e}_b(x_1) = \mathbf{d}'_b(x_1) = \mathbf{a}_1 + 2x_1\mathbf{a}_2 \quad (149)$$

$$\mathbf{e}_s(x_1) = \mathbf{d}'_s(x_1) + \mathbf{H}\mathbf{d}_b(x_1) = \mathbf{b}_1 + \mathbf{H}\mathbf{a}_0 + \mathbf{PD}[\mathbf{C}(x_1)\mathbf{b}_4 + \mathbf{S}(x_1)\mathbf{b}_5] \quad (150)$$

Also, the derivative of the warping fields  $\mathbf{d}'_\mu(x_1)$  is related to the bi-moments, via the elastic constitutive relations (Eqs. 79), and can be obtained as:

$$\mathbf{d}'_\mu(x_1) = \mathbf{d}''_s(x_1) + \mathbf{H}\mathbf{d}'_b(x_1) = \mathbf{PD}^2[\mathbf{S}(x_1)\mathbf{b}_4 + \mathbf{C}(x_1)\mathbf{b}_5] \quad (151)$$

Combining Eqs. (139 - 140) with Eqs. (146 - 148), the unknown fields can be written as:

$$\mathbf{d}(x_1) = \mathbf{A}(x_1)\mathbf{p} \quad (152)$$

where:

$$\mathbf{p} = \{\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{b}_0 \quad \mathbf{b}_1 \quad \mathbf{b}_4 \quad \mathbf{b}_5\} \quad (153)$$

$$\mathbf{d}(x_1) = \{\mathbf{d}_b(x_1) \quad \mathbf{d}_s(x_1) \quad \mathbf{d}_\mu(x_1)\} \quad (154)$$

$$\mathbf{A}(x_1) = \begin{bmatrix} \mathbf{I} + \frac{x_1^2}{2}\overline{\mathbf{K}}\mathbf{H} & x_1\mathbf{I} & \mathbf{0} & \frac{x_1^2}{2}\overline{\mathbf{K}} & \mathbf{0} & \mathbf{0} \\ -\frac{x_1^3}{6}\overline{\mathbf{H}}\overline{\mathbf{K}}\mathbf{H} & -\frac{x_1^2}{2}\mathbf{H} & \mathbf{I} & x_1\mathbf{I} - \frac{x_1^3}{6}\mathbf{H}\overline{\mathbf{K}} & \mathbf{PS}(x_1) & \mathbf{PC}(x_1) \\ \mathbf{H} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{PDC}(x_1) & \mathbf{PDS}(x_1) \end{bmatrix} \quad (155)$$

The reduced set of constant parameters  $\mathbf{p}$  can then be linked to the nodal degrees of freedom  $\mathbf{d}_c$  as:

$$\mathbf{p} = \mathbf{A}_c^{-1}\mathbf{d}_c \quad (156)$$

where:

$$\mathbf{d}_c = \{\mathbf{d}(0) \quad \mathbf{d}(L)\} \quad (157)$$

$$\mathbf{A}_c = [\mathbf{A}^T(0) \quad \mathbf{A}^T(L)]^T \quad (158)$$

The final form of the exact solution, holding the exact interpolation functions, is given by:

$$\mathbf{d}(x_1) = \mathbf{A}(x_1)\mathbf{A}_c^{-1}\mathbf{d}_c \quad (159)$$

Finally, from Eqs. (149 - 150), the strains can be written in matrix form as:

$$\mathbf{e}(x_1) = \mathbf{E}(x_1)\mathbf{A}_c^{-1}\mathbf{d}_c \quad (160)$$

where:

$$\mathbf{e}(x_1) = \{\mathbf{e}_b(x_1) \quad \mathbf{e}_s(x_1) \quad \mathbf{d}'_\mu(x_1)\} \quad (161)$$

$$\mathbf{E}(x_1) = \begin{bmatrix} x_1\overline{\mathbf{K}}\mathbf{H} & \mathbf{I} & \mathbf{0} & x_1\overline{\mathbf{K}} & \mathbf{0} & \mathbf{0} \\ \mathbf{H} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{PDC}(x_1) & \mathbf{PDS}(x_1) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{PD}^2\mathbf{S}(x_1) & \mathbf{PD}^2\mathbf{C}(x_1) \end{bmatrix} \quad (162)$$

It can easily be shown that the exact interpolation functions for the Vlasov's beam theory [16] are a particular case of the present solution when the terms coupling the shear and torsion modes vanishes, for example in the case of a doubly symmetric cross-section.

### 3.3. Case 3

In the third and last case, the contributions of the bi-shear  $V_k$  and bi-moment  $B_k$  components to the internal virtual work are considered. The warping displacement fields  $\mu_k$  are then completely independent of the other fields. Therefore, each node of the beam has as degrees of freedom  $\{u_1, u_2, u_3, \theta_1, \theta_2, \theta_3, \mu_1, \mu_2, \mu_3\}$ . Combining the elastic constitutive relations and the equilibrium equations we have:

$$u_1'' = 0 \quad (163)$$

$$\mathbf{K}_s(\mathbf{d}_s''(x_1) + \mathbf{H}\mathbf{d}_b'(x_1)) - \mathbf{K}_w\mathbf{d}_\mu'''(x_1) = \mathbf{0} \quad (164)$$

$$\mathbf{K}_w\mathbf{d}_\mu''(x_1) - (\mathbf{K}_0 - \mathbf{K}_s)(\mathbf{d}_\mu(x_1) - \mathbf{d}_s'(x_1) - \mathbf{H}\mathbf{d}_b(x_1)) = \mathbf{0} \quad (165)$$

$$\mathbf{K}_b\mathbf{d}_b''(x_1) - \mathbf{H}^T [\mathbf{K}_s(\mathbf{d}_s'(x_1) + \mathbf{H}\mathbf{d}_b(x_1)) - \mathbf{K}_w\mathbf{d}_\mu''(x_1)] = \mathbf{0} \quad (166)$$

where:

$$\mathbf{K}_0 = G \begin{bmatrix} I_p & c_3A & -c_2A \\ c_3A & A & 0 \\ -c_2A & 0 & A \end{bmatrix} \quad (167)$$

As in the cases 1 and 2 of Sec. 3.1 and 3.2, the axial displacement varies linearly and is uncoupled from the other fields. In this case, however, the torsion angle  $\theta_1$  and the warping fields  $\mu_k$  are coupled with all the remaining fields via the warping inertia matrix  $\mathbf{K}_w$ . When the section has two axes of symmetry, the warping related matrices  $\mathbf{K}_0$ ,  $\mathbf{K}_s$  and  $\mathbf{K}_w$  become diagonal and the torsion and the shear/bending modes do not interact. When the section has only one axis of symmetry, the matrix  $\mathbf{K}_s$  is diagonal but the matrices  $\mathbf{K}_0$  and  $\mathbf{K}_w$  are not, preserving the coupling between the torsion mode and one of the shear/bending modes. Combining Eqs. (164 - 166) and observing that  $\mathbf{K}_b$  is not singular, the following simplified form of the equilibrium equations can be obtained:

$$\mathbf{d}_b'''(x_1) = \mathbf{0} \quad (168)$$

$$\mathbf{K}_w\mathbf{d}_\mu'''(x_1) - \mathbf{K}_s^*\mathbf{d}_\mu'(x_1) = \mathbf{0} \quad (169)$$

$$\mathbf{d}_s''(x_1) = \mathbf{K}_s^{-1}\mathbf{K}_s^*\mathbf{d}_\mu'(x_1) - \mathbf{H}\mathbf{d}_b'(x_1) \quad (170)$$

where:

$$\mathbf{K}_s^* = \mathbf{K}_s - \mathbf{K}_s\mathbf{K}_0^{-1}\mathbf{K}_s \quad (171)$$

From Eq. (168), the bending fields  $\mathbf{d}_b$  vary linearly along the beam axis, as in cases 1 and 2 of Sec. 3.1 and 3.2. Here, as in case 2 of Sec. 3.2, due to the coupling between the shear, torsion, bending and warping modes and the contribution of the warping stiffnesses  $\mathbf{K}_0$  and  $\mathbf{K}_w$ , the shear  $\mathbf{d}_s$  and warping  $\mathbf{d}_\mu$  fields vary in a more elaborate shape. In fact, the solution of differential equations in Eq. (169) consists of a constant term plus hyperbolic terms related to the coupling between the matrices  $\mathbf{K}_s$  and  $\mathbf{K}_w$ . Introducing the constant parameters  $\mathbf{a}_k$ ,  $\mathbf{b}_k$  and  $\mathbf{q}_k$ , the bending, shear and warping fields may be written as:

$$\mathbf{d}_b(x_1) = \mathbf{a}_0 + x_1\mathbf{a}_1 + x_1^2\mathbf{a}_2 \quad (172)$$

$$\mathbf{d}_\mu(x_1) = \mathbf{b}_0 + \mathbf{P}\mathbf{D}(\mathbf{S}_h(x_1)\mathbf{b}_1 + \mathbf{C}_h(x_1)\mathbf{b}_2) \quad (173)$$

$$\mathbf{d}_s(x_1) = \mathbf{q}_0 + x_1\mathbf{q}_1 + x_1^2\mathbf{q}_2 + x_1^3\mathbf{q}_3 + \mathbf{K}_s^{-1}\mathbf{K}_s^*\mathbf{P}(\mathbf{C}_h(x_1)\mathbf{b}_1 + \mathbf{S}_h(x_1)\mathbf{b}_2) \quad (174)$$

where:

$$\mathbf{P}^T\mathbf{K}_w\mathbf{P} = \mathbf{I} \quad (175)$$

$$\mathbf{P}^T\mathbf{K}_s^*\mathbf{P} = \mathbf{D}^2 \quad (176)$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (177)$$

$$\mathbf{S}(x_1) = \begin{bmatrix} \sinh(\lambda_1 x_1) & 0 & 0 \\ 0 & \sinh(\lambda_2 x_1) & 0 \\ 0 & 0 & \sinh(\lambda_3 x_1) \end{bmatrix} \quad (178)$$

$$\mathbf{C}(x_1) = \begin{bmatrix} \cosh(\lambda_1 x_1) & 0 & 0 \\ 0 & \cosh(\lambda_2 x_1) & 0 \\ 0 & 0 & \cosh(\lambda_3 x_1) \end{bmatrix} \quad (179)$$

As in the case 2 of Sec. 3.2, in the equations above,  $\mathbf{P}$  and  $\mathbf{D}$  represents the eigenvector and eigenvalue matrices, respectively, of the generalized symmetric eigenvalue problem  $\mathbf{K}_s^* \mathbf{v} - \lambda^2 \mathbf{K}_w \mathbf{v} = \mathbf{0}$ . Introducing the solution from Eqs. (168) and (169) in the equilibrium equations (Eqs. 164 and 165), the following relations between the constant parameters can be obtained:

$$\mathbf{q}_2 = -\frac{1}{2} \mathbf{H} \mathbf{a}_1 \quad (180)$$

$$\mathbf{q}_3 = -\frac{1}{3} \mathbf{H} \mathbf{a}_2 \quad (181)$$

$$\mathbf{q}_1 = \mathbf{b}_0 - \mathbf{H} \mathbf{a}_0 \quad (182)$$

$$\mathbf{a}_2 = \frac{1}{2} \overline{\mathbf{K}} \mathbf{b}_0 \quad (183)$$

The bending  $\mathbf{e}_b$ , shear  $\mathbf{e}_s$  and warping  $\mathbf{e}_\lambda$  strains can be computed as:

$$\mathbf{e}_b(x_1) = \mathbf{d}'_b(x_1) = \mathbf{a}_1 + 2x_1 \mathbf{a}_2 \quad (184)$$

$$\mathbf{e}_s(x_1) = \mathbf{d}'_s(x_1) + \mathbf{H} \mathbf{d}_b(x_1) = \mathbf{b}_0 + \mathbf{K}_s^{-1} \mathbf{K}_s^* \mathbf{P} \mathbf{D} (\mathbf{S}(x_1) \mathbf{b}_1 + \mathbf{C}(x_1) \mathbf{b}_2) \quad (185)$$

$$\mathbf{e}_\lambda(x_1) = \mathbf{d}_\mu(x_1) - \mathbf{d}'_s(x_1) - \mathbf{H} \mathbf{d}_b(x_1) = \mathbf{K}_0^{-1} \mathbf{K}_s \mathbf{P} \mathbf{D} (\mathbf{S}(x_1) \mathbf{b}_1 + \mathbf{C}(x_1) \mathbf{b}_2) \quad (186)$$

Combining Eqs. (172 - 174) with Eqs. (180 - 182), the displacement fields can be written as:

$$\mathbf{d}(x_1) = \mathbf{A}(x_1) \mathbf{p} \quad (187)$$

where:

$$\mathbf{P}^* = \mathbf{K}_s^{-1} \mathbf{K}_s^* \mathbf{P} \quad (188)$$

$$\mathbf{p} = \{\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{b}_0 \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{q}_0\} \quad (189)$$

$$\mathbf{d}(x_1) = \{\mathbf{d}_b(x_1) \quad \mathbf{d}_s(x_1) \quad \mathbf{d}_\mu(x_1)\} \quad (190)$$

$$\mathbf{A}(x_1) = \begin{bmatrix} \mathbf{I} & x_1 \mathbf{I} & \frac{x_1^2}{2} \overline{\mathbf{K}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -x_1 \mathbf{H} & -\frac{x_1^2}{2} \mathbf{H} & x_1 \mathbf{I} - \frac{x_1^3}{6} \mathbf{H} \overline{\mathbf{K}} & \mathbf{P}^* \mathbf{C}_h(x_1) & \mathbf{P}^* \mathbf{S}_h(x_1) & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{P} \mathbf{D} \mathbf{S}_h(x_1) & \mathbf{P} \mathbf{D} \mathbf{C}_h(x_1) & \mathbf{0} \end{bmatrix} \quad (191)$$

The reduced set of constant parameters  $\mathbf{p}$  can then be linked to the nodal degrees of freedom  $\mathbf{d}_c$  as:

$$\mathbf{p} = \mathbf{A}_c^{-1} \mathbf{d}_c \quad (192)$$

where:

$$\mathbf{d}_c = \{\mathbf{d}(0) \quad \mathbf{d}(L)\} \quad (193)$$

$$\mathbf{A}_c = [\mathbf{A}^T(0) \quad \mathbf{A}^T(L)]^T \quad (194)$$

The final form of the exact solution, holding the exact interpolation functions, is given by:

$$\mathbf{d}(x_1) = \mathbf{A}(x_1) \mathbf{A}_c^{-1} \mathbf{d}_c \quad (195)$$

Finally, from Eqs. (184 - 186), the strains can be written in matrix form as:

$$\mathbf{e}(x_1) = \mathbf{E}(x_1)\mathbf{A}_c^{-1}\mathbf{d}_c \quad (196)$$

where:

$$\mathbf{e}(x_1) = \{\mathbf{e}_b(x_1) \quad \mathbf{e}_s(x_1) \quad \mathbf{e}_\lambda(x_1) \quad \mathbf{d}'_\mu(x_1)\} \quad (197)$$

$$\mathbf{E}(x_1) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & x_1\bar{\mathbf{K}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{K}_s^{-1}\mathbf{K}_s^*\mathbf{PDS}_h(x_1) & \mathbf{K}_s^{-1}\mathbf{K}_s^*\mathbf{PDC}_h(x_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_0^{-1}\mathbf{K}_s\mathbf{PDS}_h(x_1) & \mathbf{K}_0^{-1}\mathbf{K}_s\mathbf{PDC}_h(x_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{PD}^2\mathbf{C}_h(x_1) & \mathbf{PD}^2\mathbf{S}_h(x_1) & \mathbf{0} \end{bmatrix} \quad (198)$$

It can easily be shown that the exact interpolation functions for the Benscoter's beam theory [5, 6] are a particular case of the present solution when the terms coupling the shear and torsion modes vanishes, for example in the case of a section with two axes of symmetry.

#### 4. Numeric examples

In this section, the exact solutions developed in Sec. 3 are discussed for each of the three cases studied. The beams used in the numeric models are made of steel with elastic modulus  $E = 200$  GPa and Poisson's ratio  $\nu = 0.30$  and have thin-walled sections in order to better study the influence of the warping. The length  $L = 500$  mm is chosen to be only five times the cross-sections height in order better investigate the influence of the shear deformations.

The thin-walled sections used in the numerical models consists of the I-shaped, C-shaped and L-shaped profiles shown in Fig. 2 with dimensions  $h = 100$  mm,  $w = 100$  mm and  $t = 10$  mm. These sections are chosen in order to illustrate the effects of the symmetries on the coupling between the bending, shear, torsion and warping fields. In all figures, the blue, red and black lines represents the exact solutions obtained with the interpolation function of cases 1 (Sec. 3.1), 2 (Sec. 3.2) and 3 (Sec. 3.3), respectively.

##### 4.1. I-shaped profile

First the I-shaped profile is considered. Due to the cross-section double symmetry, the shear, torsion and warping modes are uncoupled in all cases. This implies that the field sets  $\{u_2, \theta_3\}$ ,  $\{u_3, \theta_2\}$  and  $\{\theta_1\}$  are uncoupled from one another. The obtained results are shown in Figs. 3, 4 and 5. Although the exact interpolation functions for the shear fields differ in nature (cubic and hyperbolic) for case 1 and cases 2 and 3, they are very similar, even for a short thin-walled beam, where the effects of shear and warping are more prominent. The differences between the three cases is better observed when comparing the torsion  $\theta_1$  and warping  $\mu_k$  fields (Fig. 4).

In the context of finite element analysis, one is usually more concerned with the interpolation of the strains rather than the fields, since they have a direct impact on the internal force vector and stiffness matrix. The obtained results are shown in Figs. 6, 7 and 8. The bending strains are very similar in all three cases. However, the distribution of the shear and torsion strains vary significantly as a consequence of the exact hyperbolic interpolation functions.

##### 4.2. C-shaped profile

Next the C-shaped profile is considered. Due to its symmetry with respect to the  $x_2$  axes, the mixed shear area  $A_{23}$ , the torsion center coordinate  $c_3$  and the warping inertia  $W_{12}$  and  $W_{23}$  are null and, hence, the fields  $\{u_2, \theta_3, \mu_2\}$  are uncoupled from the remaining ones. However, all other fields are coupled. The obtained results are shown in Figs. 9 - 13. As for the I-shaped profile, the shear mode is very similar for all cases and the differences between the three cases is better observed when comparing the torsion  $\theta_1$  and warping  $\mu_k$  fields (Fig. 4). It's noteworthy that the coupled interpolation functions are significantly smaller than the uncoupled ones as they are generated by the off diagonal terms of the warping stiffness matrix, whom are necessarily smaller than the diagonal terms, given the positive definite property of the matrix. The obtained strains are shown in Figs. 14, 15 and 16. For the C-shaped profile, the bending, shear and torsion strains vary considerably in the three cases as a consequence of the modes coupling.



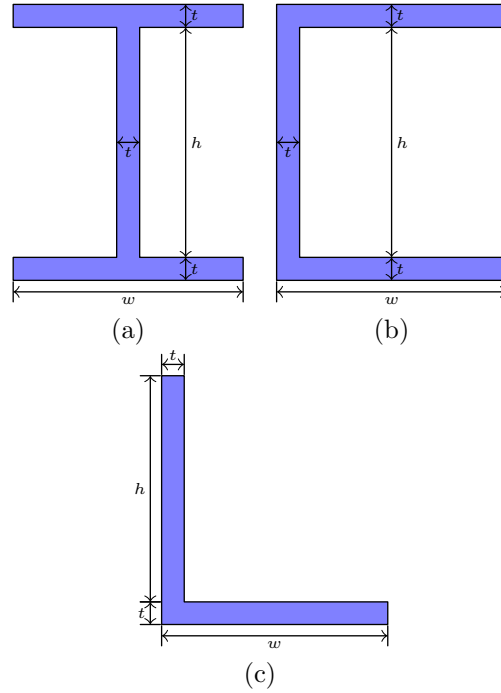


Figure 2: Thin-walled sections used in the numeric models.

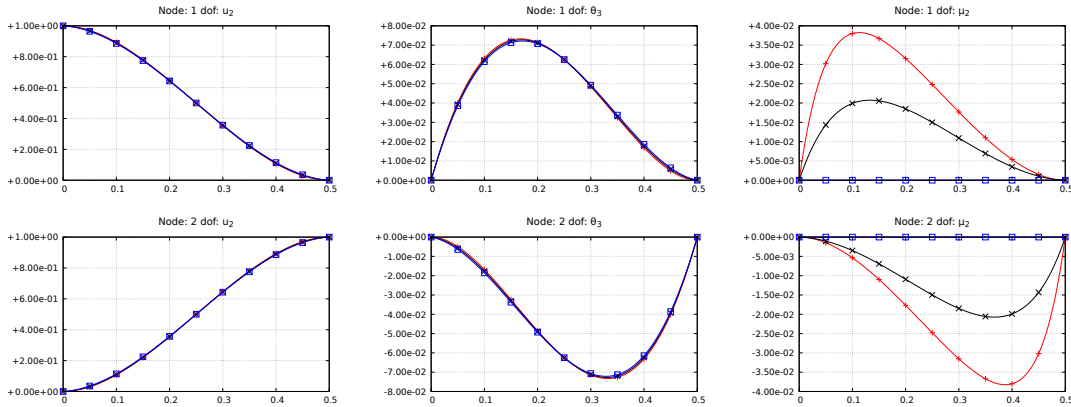


Figure 3: Exact interpolation functions for the  $u_2$  field of the Profile I.

#### 4.3. L-shaped profile

Next the L-shaped profile is considered. Due to its lack of symmetry, the shear  $\mathbf{K}_s$  and warping  $\mathbf{K}_w$  stiffness matrices are not diagonal and, hence, all fields are coupled. The obtained results are shown in Figs. 17 - 18. As for the I-shaped and C-shaped profiles, the shear mode is very similar for all cases and the differences between the three cases is better observed when comparing the torsion  $\theta_1$  and warping  $\mu_k$  fields (Fig. 4). It's noteworthy that the coupling interpolation functions are significantly smaller than the main ones as they are generated by the off diagonal terms of the warping stiffness matrix, whom are necessarily smaller than the diagonal terms, given the positive definite property of the matrix. The obtained strains are shown in Figs. 19 and 20. For the L-shaped profile, the bending, shear and torsion strains vary considerably in the three cases as a consequence of the modes coupling.

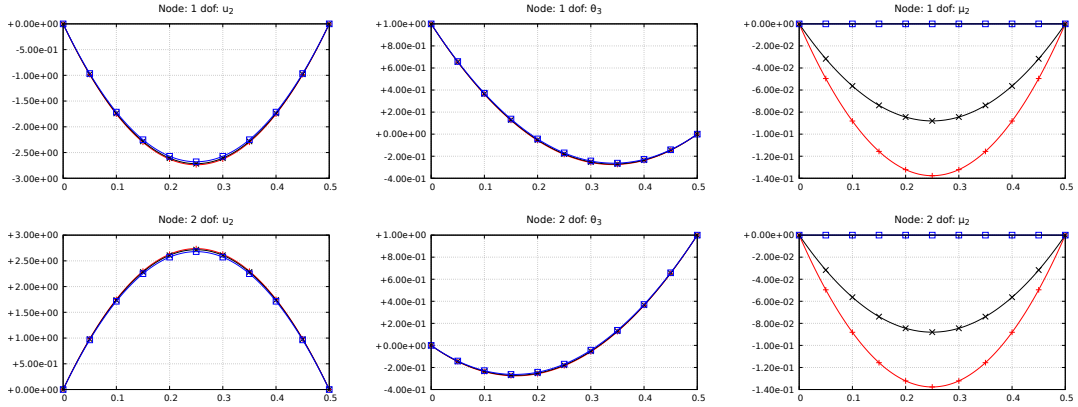


Figure 4: Exact interpolation functions for the  $\theta_3$  field of the Profile I.

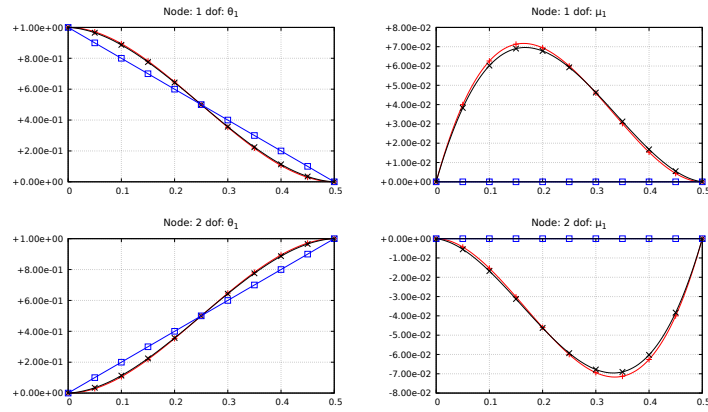


Figure 5: Exact interpolation functions for the  $\theta_1$  field of the Profile I.

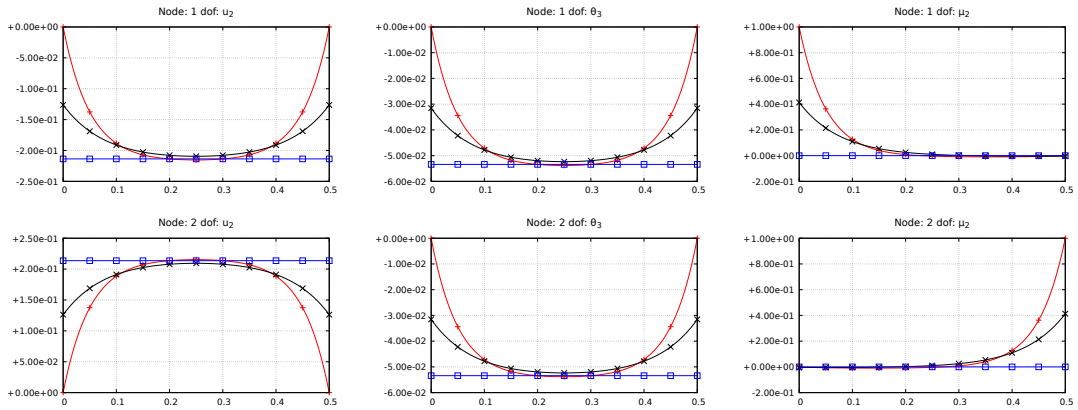


Figure 6: Exact interpolation functions for the  $\gamma_2$  strain field of the Profile I.

## 5. Conclusions and Outlook

In the present contribution, exact solutions for coupled warping beam problem were developed. The torsion and shear warping problems were taken into account and three beam formulations were considered,

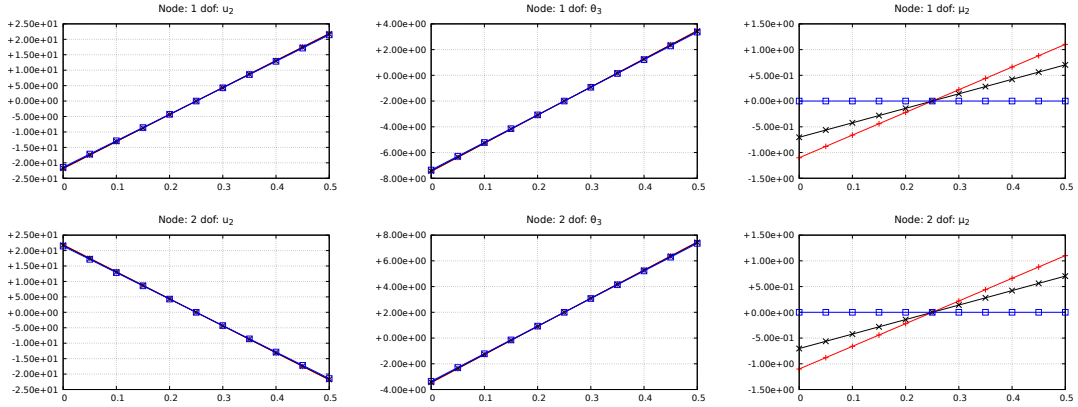


Figure 7: Exact interpolation functions for the  $\kappa_3$  strain field of the Profile I.

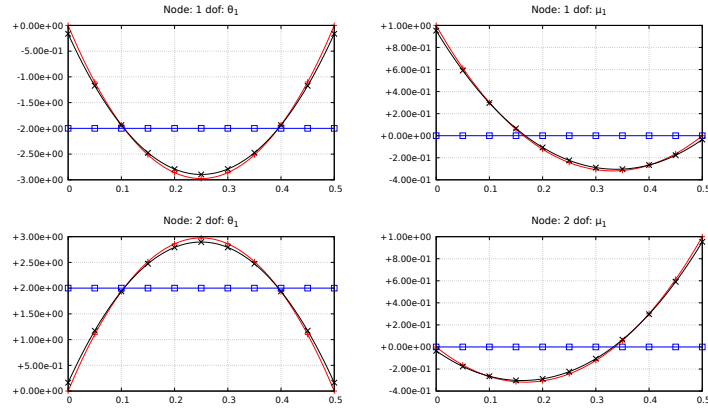


Figure 8: Exact interpolation functions for the  $\kappa_1$  strain field of the Profile I.

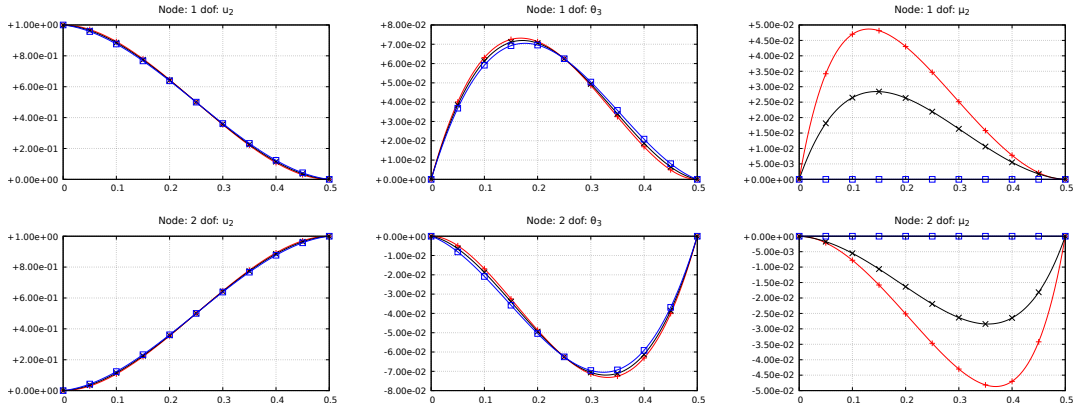


Figure 9: Exact interpolation functions for the  $u_2$  field of the Profile C.

regarding the influence of the bi-moment and bi-shear terms. In the first case, where the bi-moment and bi-shear terms are neglected, the exact solutions have a simple polynomial form. In the second and third cases, the exact solutions have a more elaborate form, involving hyperbolic terms. The coupling between the modes is related to the lack of symmetry of the cross-section and is captured in the exact solution for

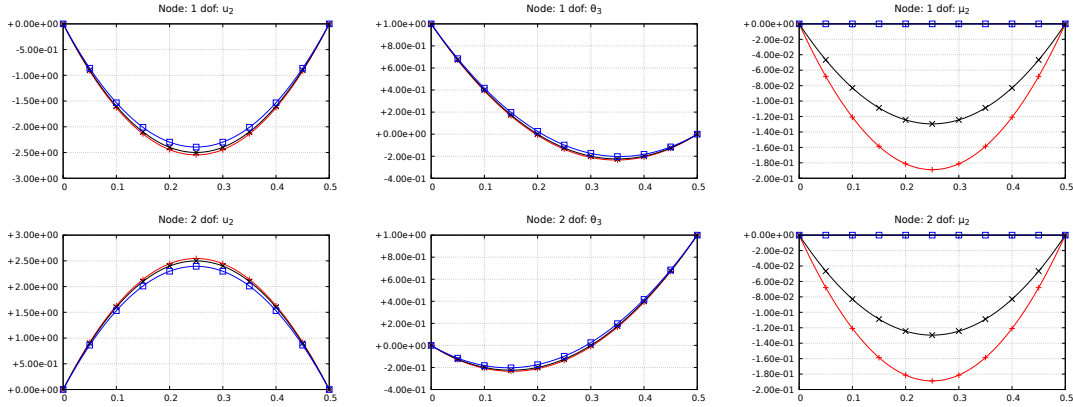


Figure 10: Exact interpolation functions for the  $\theta_3$  field of the Profile C.

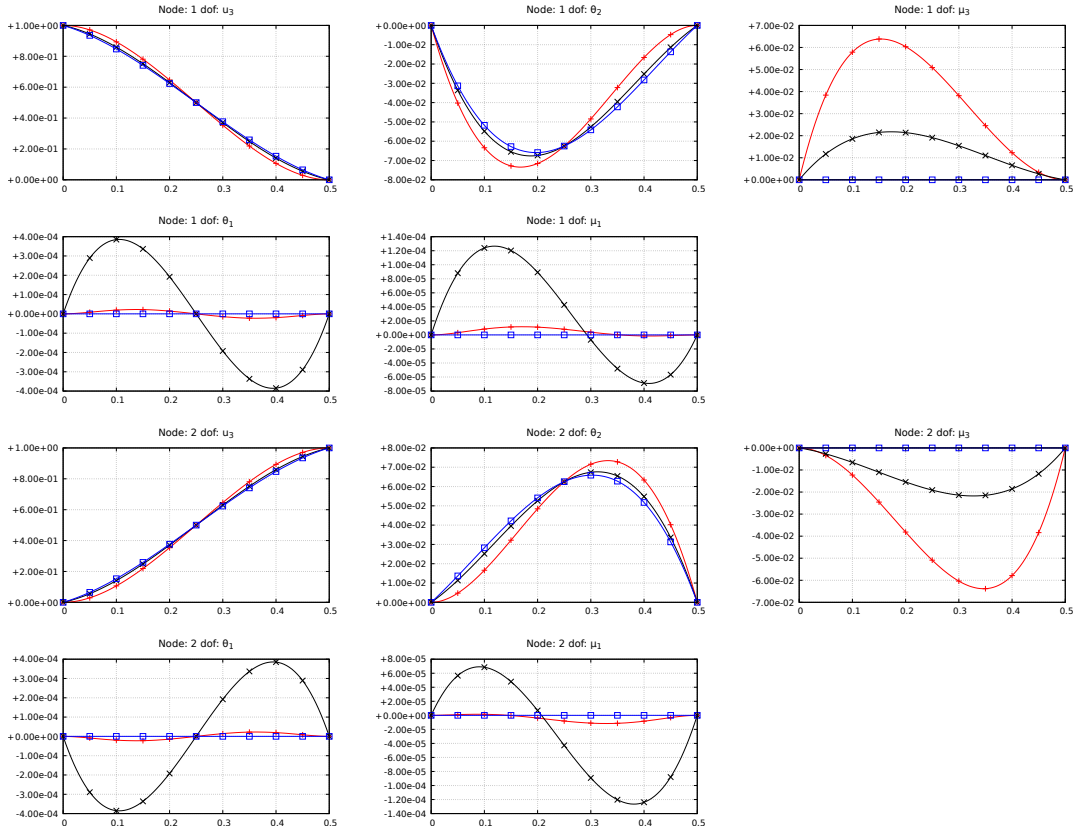


Figure 11: Exact interpolation functions for the  $u_3$  field of the Profile C.

all modes.

The development of exact solutions offers advantages in the structural analysis of beams since it allows for an exact representation of the displacement and rotation fields and, consequently, of the section strains and stress-resultants. Perhaps most importantly, the use of exact interpolation functions for the strain fields completely suppresses problems of numerical locking of the stiffness matrix in the context of finite element analysis. Also, the exact interpolation of the displacement and rotation fields completely avoids problems

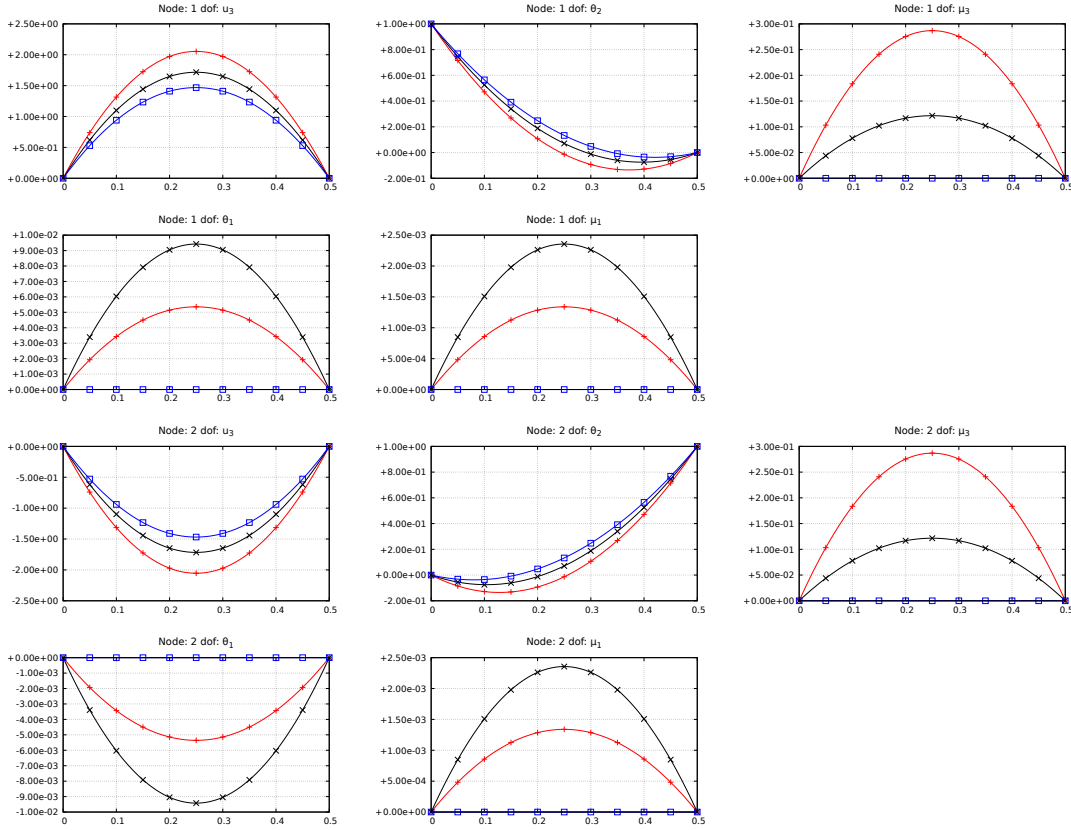


Figure 12: Exact interpolation functions for the  $\theta_2$  field of the Profile C.

of numerical locking of the inertia (mass) matrix when performing dynamic and modal analysis.

Depending on the support boundary conditions, higher-order warping modes may play a significant role in the description of the kinematics of the beam, specially when computing the strains and stresses in thin-walled beams. In a future work, the extension of the exact solutions for these high-order modes can be investigated. The basic principle of writing the resulting differential equations in a suitable matrix form and reducing the equations to a simplified form can be applied.

## Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflict of interest.

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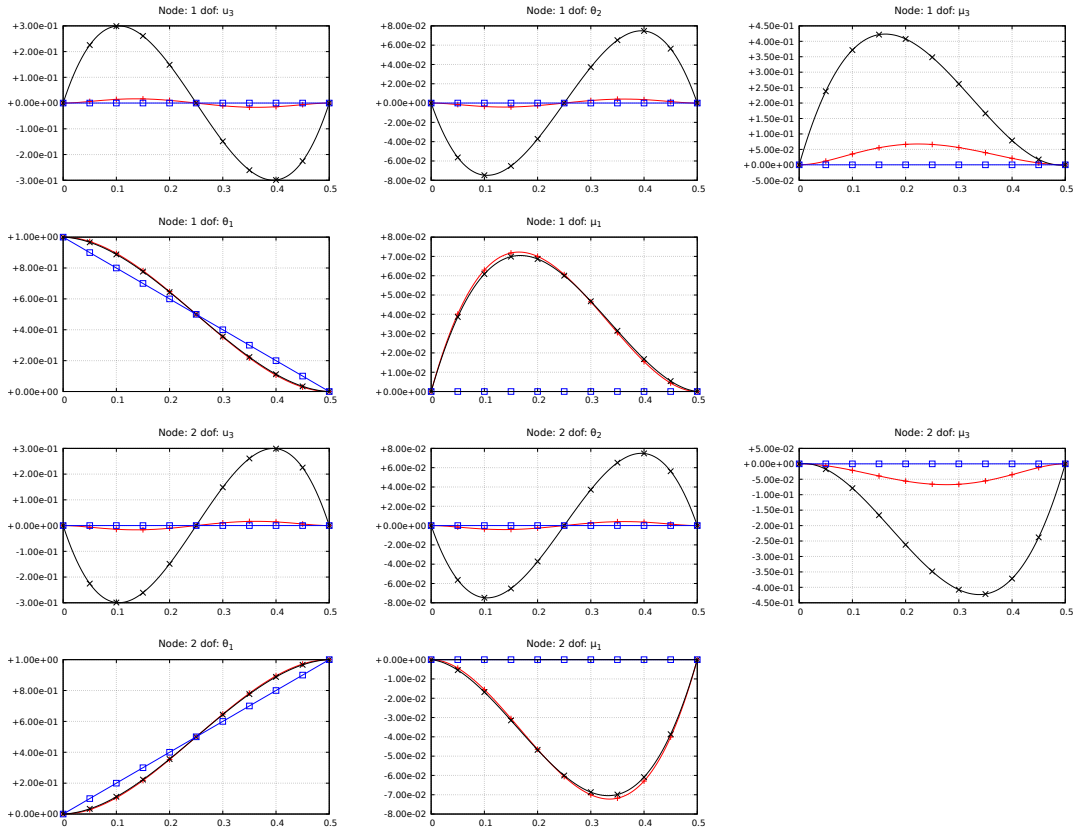


Figure 13: Exact interpolation functions for the  $\theta_1$  field of the Profile C.

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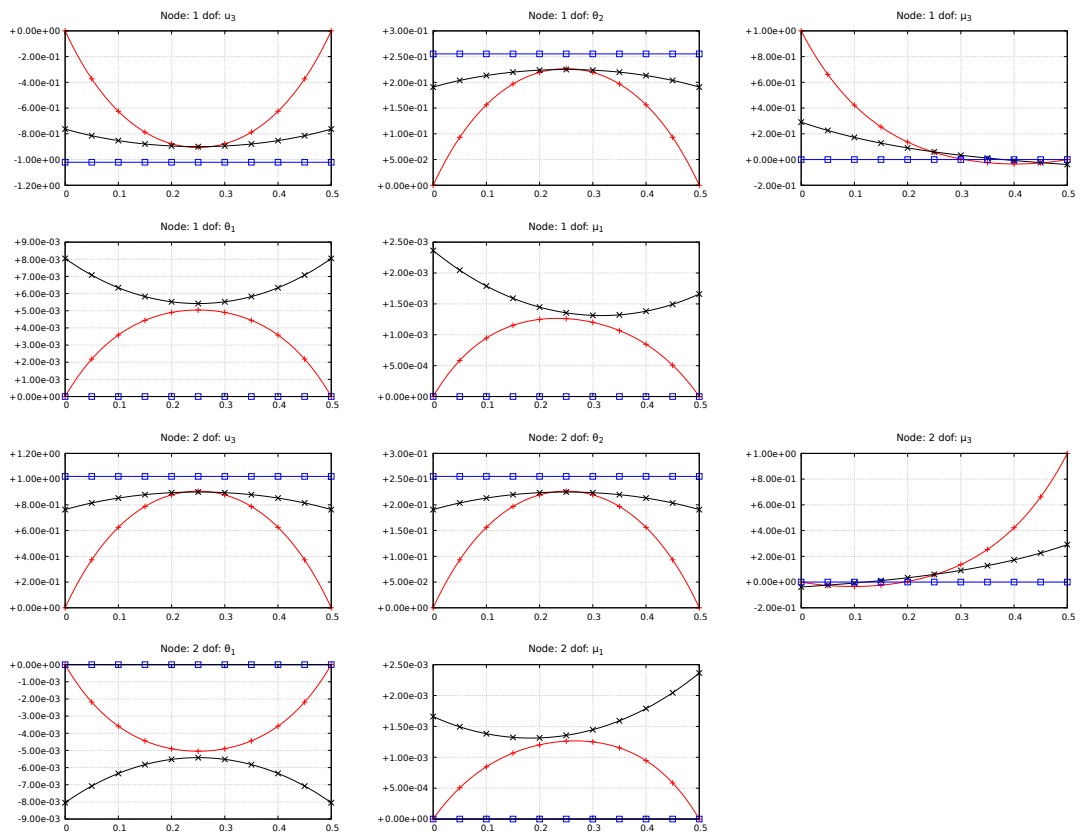


Figure 14: Exact interpolation functions for the  $\gamma_3$  strain field of the Profile C.

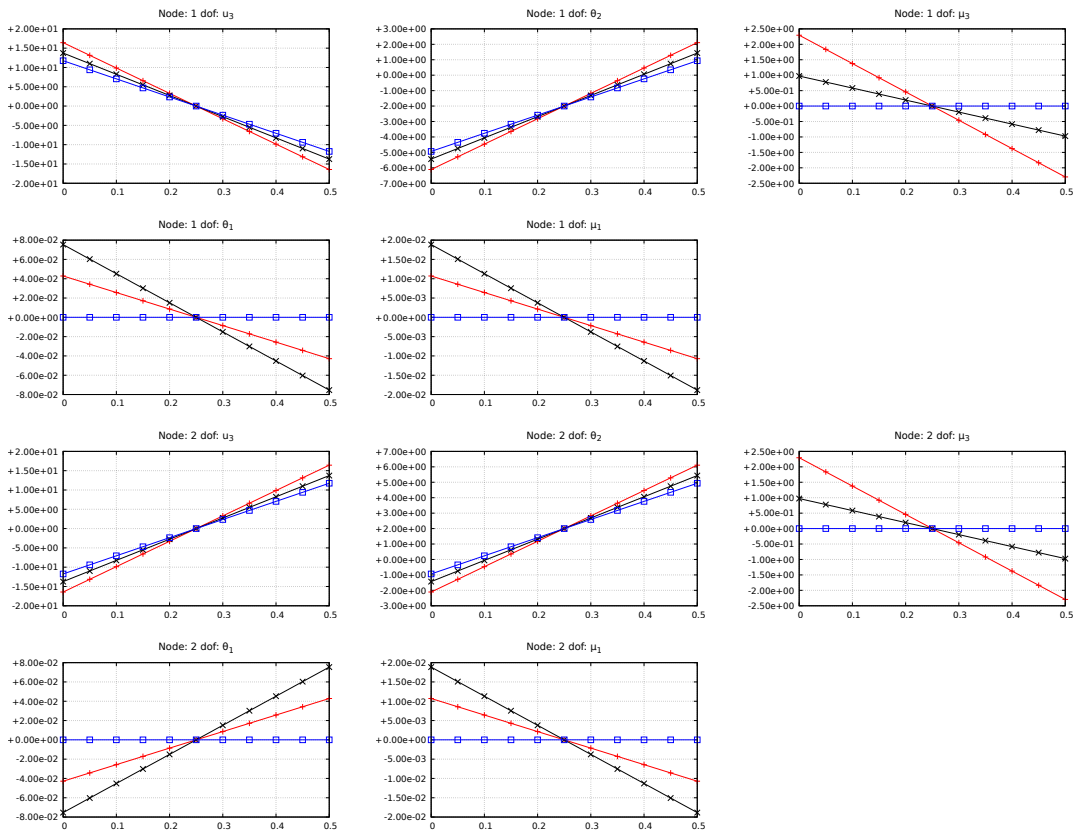


Figure 15: Exact interpolation functions for the  $\kappa_2$  strain field of the Profile C.



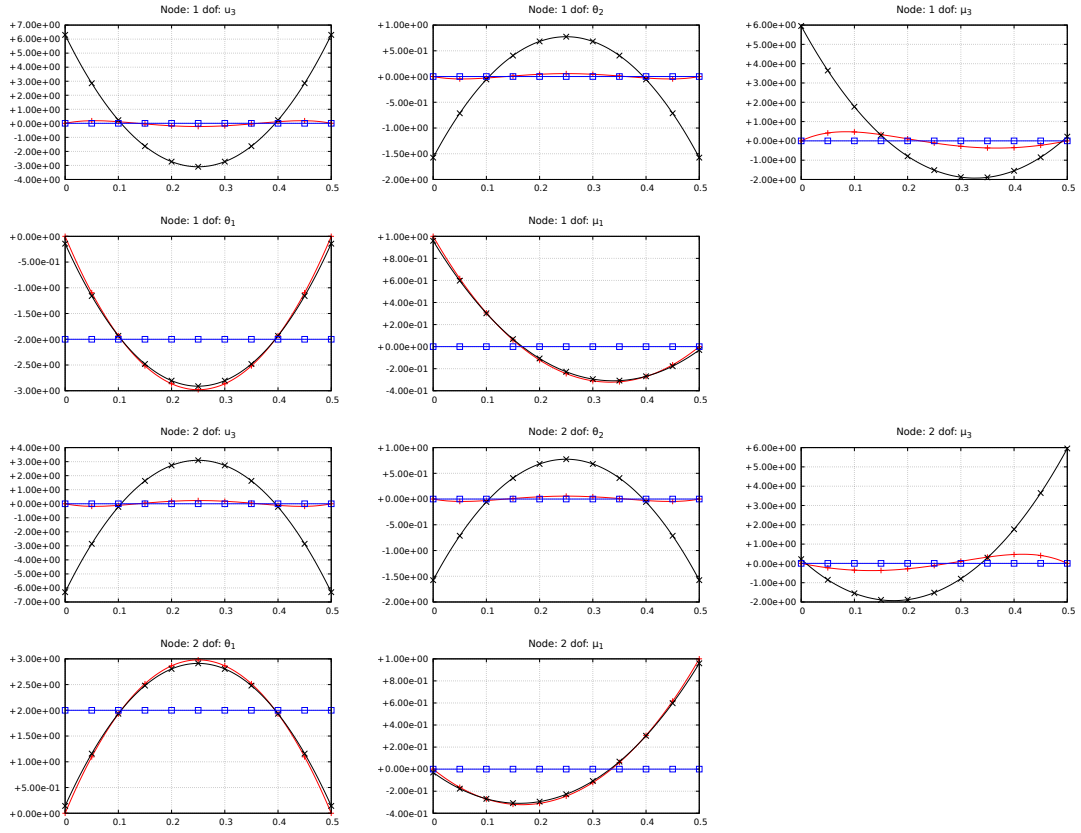


Figure 16: Exact interpolation functions for the  $\kappa_1$  strain field of the Profile C.

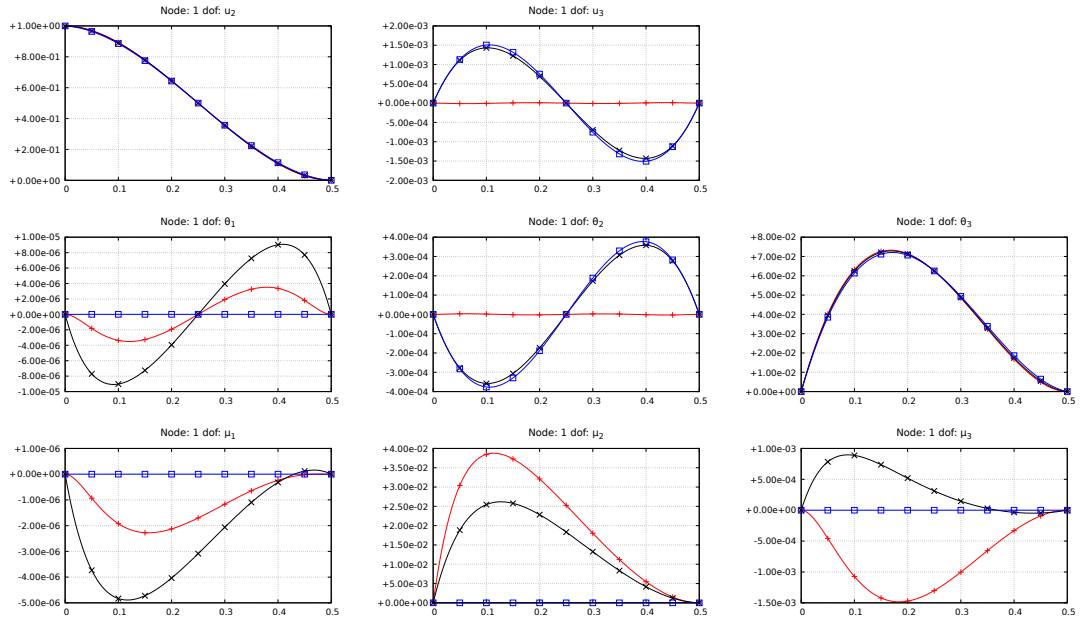


Figure 17: Exact interpolation functions for the  $u_2$  field of the Profile L.

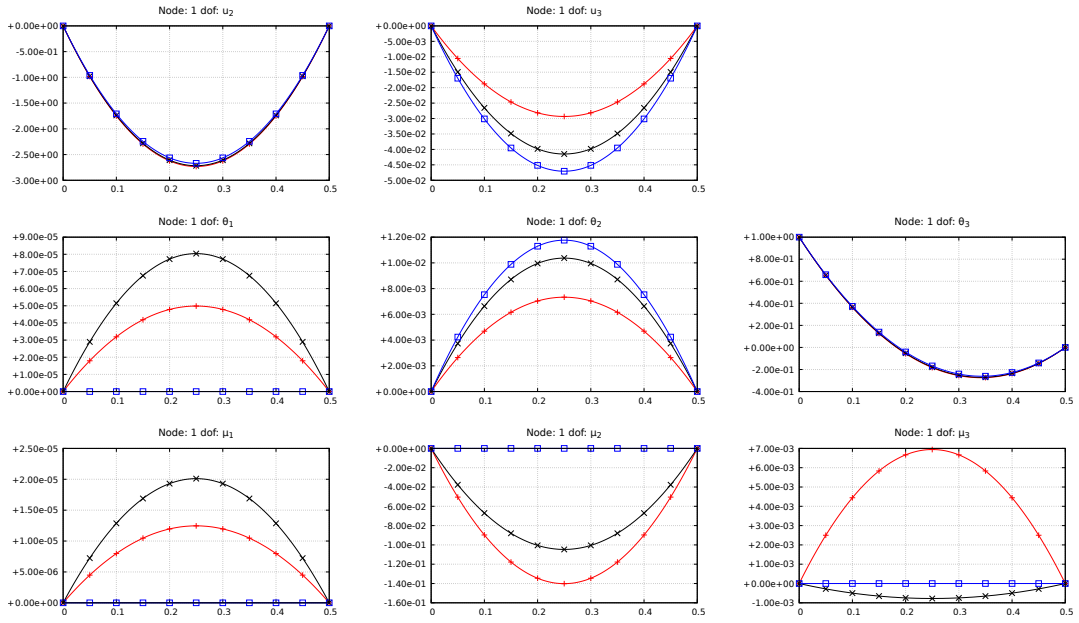


Figure 18: Exact interpolation functions for the  $\theta_3$  field of the Profile L.

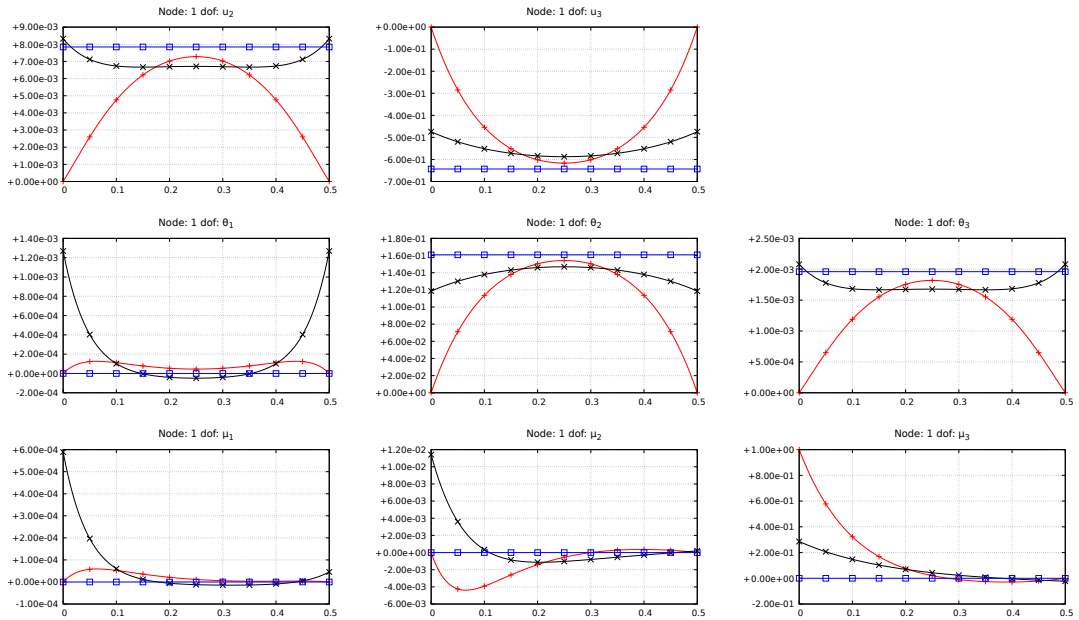


Figure 19: Exact interpolation functions for the  $\gamma_2$  strain field of the Profile L.

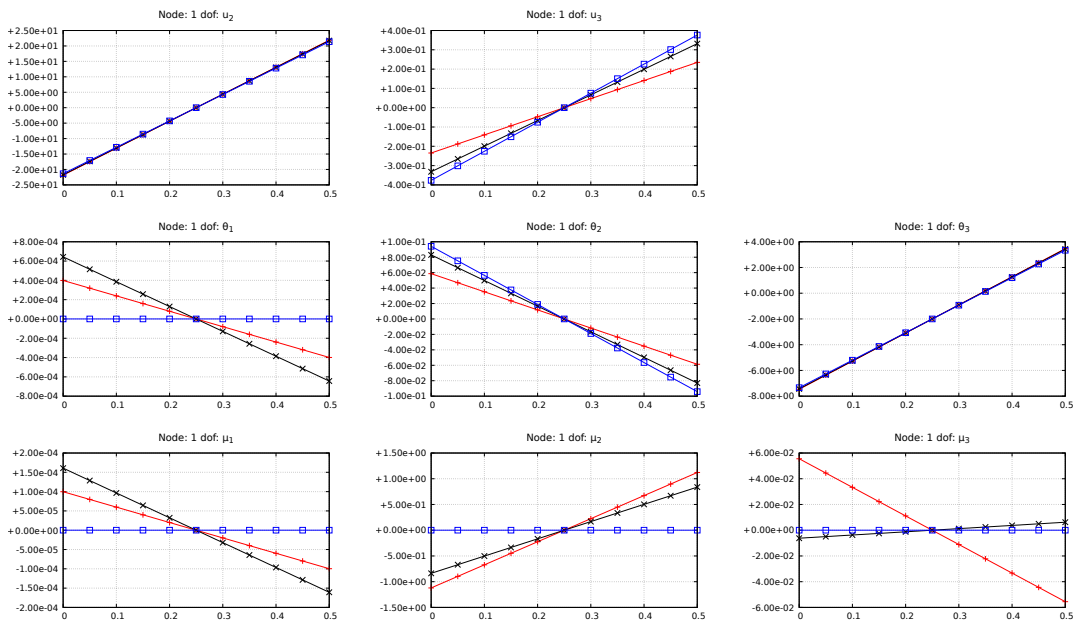


Figure 20: Exact interpolation functions for the  $\kappa_3$  strain field of the Profile L.