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CONJUGATE POINTS ALONG KOLMOGOROV FLOWS ON THE TORUS

ALICE LE BRIGANT AND STEPHEN C. PRESTON

ABSTRACT. The geodesics in the group of volume-preserving diffeomorphisms (volumorphisms) of a manifold M , for a Riemannian metric defined by the kinetic energy, can be used to model the movement of ideal fluids in that manifold. The existence of conjugate points along such geodesics reveal that these cease to be infinitesimally length-minimizing between their endpoints. In this work, we focus on the case of the torus $M = \mathbb{T}^2$ and on geodesics corresponding to steady solutions of the Euler equation generated by stream functions $\psi = -\cos(mx)\cos(ny)$ for positive integers m and n , called Kolmogorov flows. We show the existence of conjugate points along these geodesics for all (m, n) , with the sole exception of $m = n = 1$. We also discuss the unusual features of this special case and conjecture that there are no conjugate points in this case.

1. INTRODUCTION

1.1. Motivation. Since Arnold in 1966 [1], the Euler equations for ideal fluids have had a well-known geometric interpretation as geodesics on the group $\text{Diff}_\mu(M)$ of volume-preserving diffeomorphisms, or volumorphisms, of a manifold M , under a right-invariant Riemannian metric defined by the kinetic energy. Since this approach can be made rigorous as in Ebin-Marsden [5] and establishes that the geodesic equation is actually a smooth ODE on the group of Sobolev H^s volumorphisms on M for $s > \frac{1}{2} \dim(M) + 1$, it can be shown that the Riemannian exponential map is C^∞ , and is invertible near zero. This shows that volumorphisms sufficiently close in H^s can be joined to the identity by a unique minimizing geodesic. Since then similar geometric interpretations have been found for a variety of other PDEs of continuum mechanics; see Arnold-Khesin [2] for a survey and Misiołek-Preston [11] for an overview of the ODE approach.

On a sufficiently long time interval, a geodesic may cease to minimize the length between its endpoints (or equivalently, the energy for a constant-speed parameterization). When this happens, the endpoint is called a *cut point*, and if in addition the geodesic ceases to be even infinitesimally minimizing, it is called a *conjugate point*. The existence of conjugate points is intimately connected to the existence of positive-curvature sections along a geodesic. On the volumorphism group this existence was unclear since curvature computations have seemed to suggest $\text{Diff}_\mu(M)$ has mostly negative curvature. The question of whether conjugate points exist on $\text{Diff}_\mu(M)$ was posed already by Arnold [1] in 1966, but was not solved until

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Misiólek [9] in 1993 found them along rotations on the space $\text{Diff}_\mu(M)$ of volumorphisms of the 2- and 3-spheres $M = S^2$ and $M = S^3$.

The first example on a flat M was also found by Misiólek [10], who showed that along the Kolmogorov flow with stream function $\psi = -\cos 6x \cos 2y$, there is eventually a conjugate point. To do so he devised what is now known as the Misiólek criterion, giving a sufficient condition for existence of a conjugate point. This criterion has been successfully used to find conjugate points along other steady flows on other manifolds, such as the 2D ellipsoid [18], the 3D ellipsoid [7], and the sphere [3]. In three dimensions, conjugate points have a substantially different nature [6] and are much more common than in two dimensions, and can be found using a necessary and sufficient local criterion along any particle path [13, 14], as shown by the second author. This technique was used in [16] to find conjugate points along axisymmetric 3D flows.

It is known that the Misiólek criterion cannot capture *all* conjugate points. For example Tauchi-Yoneda [18] observed that the Misiólek index never detects conjugate points along the spherical rotation, in spite of the fact that infinitely many of them exist. More generally Tauchi-Yoneda [17] showed that the Misiólek criterion cannot detect conjugate points along an Arnold stable flow. A condition of the second author [15] is more suitable for detecting conjugate points in such cases (and particularly along rotational flows).

Specifically on the torus, the problem of finding conjugate points along Kolmogorov flows $\psi = -\cos mx \cos ny$ for all positive integer pairs (m, n) was posed by Drivas et al. [4], generalizing from Misiólek's original example of $m = 6, n = 2$. (We will assume throughout that $m \geq n$ without loss of generality due to symmetry.) They found many additional examples using Misiólek's criterion, for pairs (m, n) satisfying a condition equivalent to $n \geq 2$ and $m > \frac{3n^2+6}{\sqrt{3n}}$. Soon after the second author [15] found via brute force search several more examples, including $(m, 1)$ for $m \geq 2$, along with $(2, 2)$, $(3, 2)$, and $(3, 3)$, and conjectured that conjugate points existed for all (m, n) except possibly for $(1, 1)$.

1.2. Contributions. In this paper we prove this conjecture, that conjugate points exist for all (m, n) with $m, n \geq 1$ different from $(1, 1)$, giving an explicit form of a test function for $m > n$, and a slightly different test function for $m = n \geq 2$. In the case $n = 0$ with $m > 0$, it is known [9] that there are no conjugate points. Hence we have obtained a nearly complete characterization of all pairs $(m, n) \in \mathbb{Z}_{\geq 0}^2$ such that the Kolmogorov flow $\psi = -\cos mx \cos ny$ generates a geodesic in the volume-preserving diffeomorphism group that infinitesimally minimizes length between its endpoints for all time: if m and n are both nonzero and not both equal to 1, the geodesic cannot minimize; if m or n is zero then the geodesic infinitesimally minimizes. The sole exception is $(1, 1)$ where it still remains unknown.

1.3. Outline. In Section 2 we describe the geometric approach to the Euler equation of ideal fluids as originally pioneered by Arnold. We also review Kolmogorov flows, which may be generally defined as those steady solutions of the Euler equation with stream function ψ satisfying $\Delta\psi = -\lambda^2\psi$, and specifically on the torus of the form $\psi = -\cos mx \cos ny$. Finally we recall the index form for detecting conjugate points and the Misiólek criterion which greatly simplifies this computation. In Section 3 we prove Theorem 2 (the $m > n$ case) and Theorem 3 (the $m = n$ case) showing the existence of conjugate points. In Section 4 we discuss the relationship

between the Misiólek index and the Rayleigh quotient restricted to a closed subspace. Section 5 describes the algorithm we devised to exploit this relationship in order to construct an optimal variation field numerically; those given in Section 3 are simply truncated versions of these that still work. Finally in Section 6 we discuss why the $(1, 1)$ case is substantially more difficult, and suggest some other problems about conjugate points that one can tackle using the same methods as those presented here. In an Appendix we present a simple formula that is useful numerically when writing the Poisson bracket in a Fourier basis, necessary for using standard matrix algorithms to optimize the Misiólek index.

1.4. Computations. Symbolic computations were performed in Maple 2021, while numerical computations were performed using Python. Maple code for computing the index form as in Theorems 2 and 3, and Python code for numerically minimizing the index form and obtaining the form of the candidate minimizers, are both available on github: <https://github.com/alebrigant/conjugate-points>.

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2. BACKGROUND

2.1. Volumorphisms and the Euler-Arnold equation. Suppose M is a 2-dimensional manifold equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ inducing an area form μ . We are interested in diffeomorphisms $\varphi : M \rightarrow M$ that preserve the area form: $\varphi^*\mu = \mu$, called volume-preserving diffeomorphisms of M or volumorphisms for short. The space $\text{Diff}_\mu(M)$ of volumorphisms of M is (formally) a submanifold of the space of diffeomorphisms $\text{Diff}(M)$. The tangent vectors at $\varphi \in \text{Diff}_\mu(M)$ are right translations $X \circ \varphi$ of divergence-free vector fields X tangent to the boundary. We refer the interested reader to [5] for more details on these manifold structures in the context of Sobolev spaces.

A volumorphism φ can be seen as describing the positions, at a given time, of the particles of an ideal fluid (incompressible and inviscid) moving inside of M : the value $\varphi(p)$ gives the position of the particle that was at position p at $t = 0$. The volume-preserving property of φ is a consequence of the incompressibility of the fluid. With this interpretation, a tangent vector $X \circ \varphi$ is the velocity field of the fluid at that time, and the kinetic energy of the fluid defines a Riemannian metric on $\text{Diff}_\mu(M)$

$$(1) \quad g_\varphi(X \circ \varphi, Y \circ \varphi) = \int_M \langle X \circ \varphi, Y \circ \varphi \rangle \mu.$$

Since for any volume-preserving $\varphi \in \text{Diff}_\mu(M)$, the pullback measure is $\varphi^*\mu = \mu$, we see that

$$g_\varphi(X \circ \varphi, Y \circ \varphi) = \int_M \langle X, Y \rangle \mu = g_{\text{id}}(X, Y),$$

and so the kinetic metric is right-invariant on $\text{Diff}_\mu(M)$.

The result shown by Arnold [1] is the following: the geodesics in $\text{Diff}_\mu(M)$ for the kinetic metric (1) describe the motion of an ideal fluid in M . To see this, we

can consider the geodesic equation on the larger space $\text{Diff}(M)$, which is simply $\partial^2\gamma/\partial t^2 = 0$, and then orthogonally project it on the submanifold $\text{Diff}_\mu(M)$. In terms of the velocity field $X(t, \cdot)$ associated the geodesic $\gamma(t, \cdot)$

$$\frac{\partial\gamma}{\partial t}(t, x) = X(t, \gamma(t, x)),$$

this gives

$$P\left(\frac{\partial^2\gamma}{\partial t^2}\right) = P\left(\frac{\partial X}{\partial t} + \nabla_X X\right) \circ \gamma = 0,$$

where ∇ is the Levi-Civita connection associated to the Riemannian metric on M . The orthogonal projection P on the space of divergence-free vector fields tangent to the boundary is obtained by the Hodge decomposition of vector fields

$$(2) \quad P(X) = X - \text{grad } f,$$

where f is a function verifying $\Delta f = \text{div } X$ and whose normal component along the boundary equals that of the vector field X . The geodesic equation for the kinetic metric (1) on $\text{Diff}_\mu(M)$ is therefore given by

$$(3) \quad \frac{\partial X}{\partial t} + \nabla_X X = -\text{grad } p,$$

where p is called the pressure function, defined up to a constant by $\Delta p = -\text{div}(\nabla_X X)$ and its component along the normal ν : $\langle \text{grad } p, \nu \rangle = -\langle X, \nu \rangle$. Equation (3) is the Euler equation for incompressible fluids, also called the Euler-Arnold equation in virtue of its interpretation by Arnold as a geodesic equation on the space of volumorphisms.

2.2. Kolmogorov flows on the torus. In the two-dimensional case, it can be shown that the Euler-Arnold equation (3) can be rewritten in terms of the curl

$$\frac{\partial}{\partial t}(\text{curl } X) + X(\text{curl } X) = 0.$$

Now divergence-free vector fields on surfaces can be written as $X = \text{sgrad } \psi + W$ where $\psi : M \rightarrow \mathbb{R}$ is defined by $\Delta\psi = \text{curl } X$ and $\psi|_{\partial M} = 0$, while the vector field W satisfies $\text{curl } W = 0$ and $\text{div } W = 0$. For such a vector field X , the Euler-Arnold equation becomes

$$\frac{\partial}{\partial t}\Delta\psi + \{\psi, \Delta\psi\} + W(\Delta\psi) = 0.$$

Here $\{\cdot, \cdot\}$ is the Poisson bracket, which can be defined by the following property of Hamiltonian vector fields: if $\text{sgrad } f$ and $\text{sgrad } g$ are Hamiltonian vector fields generated by mean-zero functions $f, g : M \rightarrow \mathbb{R}$, then $\{f, g\}$ is the unique mean-zero function generating their Lie bracket

$$[\text{sgrad } f, \text{sgrad } g] = \text{sgrad}\{f, g\}.$$

In dimension 2, the Poisson bracket is simply given by

$$\{f, g\}\mu = df \wedge dg,$$

which reduces to $\{f, g\} = \partial_x f \partial_y g - \partial_y f \partial_x g$ on the torus. We then see that any vector field generated by an eigenvector of the Laplacian

$$X = \text{sgrad } \psi \quad \text{with} \quad \Delta\psi = -\lambda^2\psi, \quad \lambda \in \mathbb{R},$$

generates a steady solution of the Euler-Arnold equation, i.e. a geodesic γ with associated velocity field X satisfying $\frac{\partial X}{\partial t} = 0$. The function ψ is called the *stream*

function. Here we will focus on geodesics on $\text{Diff}_\mu(\mathbb{T}^2)$ whose velocity field is generated by the stream functions

$$(4) \quad \psi(x, y) = -\cos(mx)\cos(ny), \quad (m, n) \in \mathbb{N}^2.$$

These are eigenvectors of the Laplacian for the eigenvalues $-\lambda^2 = -(m^2 + n^2)$, and generate the so-called Kolmogorov flows on the torus $M = \mathbb{T}^2$, which are particular steady solutions of the Euler-Arnold equation. The question that we ask is: can we find conjugate points along any Kolmogorov flow on the torus? The existence of conjugate points is related to the question of the uniqueness of the geodesic between its endpoints, and whether a perturbation of the initial condition of the corresponding flow can lead to the same result as no perturbation at all.

2.3. Conjugate points and the Misiołek criterion. We are interested in finding conjugate points along Kolmogorov flows, i.e., geodesics of $\text{Diff}_\mu(\mathbb{T}^2)$ with velocity field $X = \text{sgrad } \psi$ generated by a stream function of the form (4). Two volumorphisms φ_1 and φ_2 connected by a geodesic $\gamma(t)$ in $\text{Diff}_\mu(\mathbb{T}^2)$ are conjugate if there exists a non-zero Jacobi field $J(t)$ along γ that vanishes at φ_1 and φ_2 ; more precisely, if there exists a family of geodesics $\gamma(s, t)$ in $\text{Diff}_\mu(M)$ with $(s, t) \in (-\epsilon, \epsilon) \times [0, 1]$ such that $\gamma(0, t) = \gamma(t)$ and the corresponding Jacobi field $J(t) = \frac{\partial \gamma}{\partial s}(0, t)$ satisfies $J(0) = J(1) = 0$. In other words, φ_1 and φ_2 are conjugate if there exists a family of geodesics that start at φ_1 and end at φ_2 up to first order. We say that there is a conjugate point along a geodesic γ if there exists a time $T > 0$ such that $\gamma(0)$ and $\gamma(T)$ are conjugate points. Conjugate points are particular cases of cut points. However, they can only happen in manifolds with some positive curvature, while cut points can occur even in flat manifolds due to non trivial topology.

Conjugate points can be detected using the so-called index form, defined for any vector field $Y(t)$ along the geodesic $\gamma(t)$ by

$$I(Y, Y) = \int_0^T \left\langle \frac{DY}{dt}, \frac{DY}{dt} \right\rangle - \langle R(Y, \dot{\gamma})\dot{\gamma}, Y \rangle dt.$$

It can be shown [9] that $I(Y, Y) = 0$ for some Y satisfying $Y(0) = Y(T) = 0$ if and only if Y is a Jacobi field. Furthermore if $I(Y, Y) < 0$ for some such field Y , then there is a Jacobi field J along γ vanishing at $t = 0$ and for $t = \tau$ for some $\tau < T$. Establishing negativity of the index form for some vector field Y is thus an effective way to show there must be a conjugate point without actually having to find it. In our setting of volume-preserving diffeomorphisms on a surface, the index form can be written in the following way.

Proposition 1 (Corollary 9 in [15]). *Suppose M is a surface and that $X = \text{sgrad } \psi$ solves the Euler equation (3) on $[0, T]$. For a time-dependent family of functions $g(t)$ on M , vanishing on the boundary of M and at $t = 0$ and $t = T$, the index form for $Y = \text{sgrad } g$ along the geodesic with velocity field X becomes*

$$(5) \quad I(Y, Y) = \int_0^T \int_M |\text{grad } h|^2 + \Delta \psi \{g, h\} d\mu dt, \quad h := \frac{\partial g}{\partial t} + \{\psi, g\}.$$

The Misiołek criterion consists in computing the index form for a particular family of deformations, namely $g(t) = \sin(\frac{\pi t}{T})f$ for some function $f : M \rightarrow \mathbb{R}$, and then letting time T go to infinity. If the result is negative for some function f , then

for sufficiently large T the index form will be negative and hence there will be a conjugate point occurring at some time $\tau < T$.

3. MAIN RESULTS

Consider a geodesic in $\text{Diff}_\mu(M)$ generated by a stream function $\psi : M \rightarrow \mathbb{R}$ satisfying $\Delta\psi = -\lambda^2\psi$. The index form (5) at a vector field $Y = \text{sgrad } g$ generated by a family of deformations $g(t) = \sin(\frac{\pi t}{T})f$ for a given function f can be written in terms of the Poisson bracket $\phi = \{\psi, f\}$ as

$$I(g) = \frac{\pi^2}{2T} \int_M |f|^2 d\mu + \frac{T}{2} \int_M (|\text{grad } \phi|^2 - \lambda^2 \phi^2) d\mu.$$

The Misiołek criterion states that there is a conjugate point eventually occurring along the geodesic if $\lim_{T \rightarrow \infty} \frac{2}{T} I(g)$ is negative for some function f , i.e.

$$(6) \quad MI(\phi) := \int_M (|\text{grad } \phi|^2 - \lambda^2 \phi^2) d\mu < 0, \quad \phi = \{\psi, f\}.$$

For $M = \mathbb{T}^2$, we will take

$$(7) \quad \psi = -\cos mx \cos ny, \quad \text{with } \lambda^2 = m^2 + n^2.$$

We will show that for every pair (m, n) with $m \geq n$, the Misiołek criterion (6) detects a conjugate point along the geodesic in $\text{Diff}_\mu(\mathbb{T}^2)$, except in the case $(1, 1)$. It is easier to handle the cases $m > n$ and $m = n$ separately.

Theorem 2. *If m and n are positive integers with $m > n$, and $\psi(x, y) = -\cos mx \cos ny$, then there is a function $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ such that $\phi = \{\psi, f\}$ satisfies the Misiołek criterion (6). Hence if the geodesic γ in $\text{Diff}_\mu(\mathbb{T}^2)$ has initial conditions $\gamma(0) = \text{id}$ and $\gamma'(0) = \text{sgrad } \psi$, then $\gamma(T)$ is conjugate to $\gamma(0)$ for some $T > 0$, and γ is not minimizing beyond T .*

Proof. See Maple file in <https://github.com/alebrigant/conjugate-points> for details of the computations here.

For numbers $a, b \in \mathbb{R}$, define

$$(8) \quad f(x, y) = \cos x (1 + a \cos(2mx) + b \cos(2ny)).$$

We compute the Poisson bracket

$$\begin{aligned} \phi(x, y) &= \psi_x(x, y) f_y(x, y) - \psi_y(x, y) f_x(x, y) \\ &= 2mn \cos x (a \cos(mx) \sin(ny) \sin(2mx) - b \sin(mx) \cos(ny) \sin(2ny)) \\ &\quad + n \cos(mx) \sin(ny) \sin x (1 + a \cos(2mx) + b \cos(2ny)). \end{aligned}$$

The quantity MI from (6) is then given by

$$MI = \frac{\pi^2 n^2}{4} H(a, b, m, n),$$

where

$$(9) \quad H(a, b, m, n) = 16a^2 m^4 + 16b^2 m^2 n^2 + 24a^2 m^2 - 12abm^2 + 4b^2 m^2 + 4b^2 n^2 \\ + 8am^2 - 8bm^2 + a^2 - ab + b^2 + 2a - 2b + 2.$$

This is quadratic in both a and b , with positive leading-order coefficients, and thus the critical point must be a global minimizer. However the formula is slightly complicated, and it is easier to use the points

$$(10) \quad a_0 = -\frac{4m^2 + 1}{16m^4 + 24m^2 + 1} \quad \text{and} \quad b_0 = \frac{1}{4n^2 + 1};$$

here a_0 is the minimizer of $H(a, 0, m, n)$ and b_0 is the minimizer of $H(0, b, m, n)$. With these choices we obtain

$$(11) \quad H(a_0, b_0, m, n) = \frac{J(m, n)}{(16m^4 + 24m^2 + 1)(4n^2 + 1)}, \quad \text{where}$$

$$J(m, n) = 4n^2(16m^4 + 40m^2 + 1) - 64m^6 - 48m^4 + 28m^2 + 1.$$

We want to show that $J(m, n) < 0$ for every choice of naturals m and n with $m > n$. Clearly J is increasing as a function of n , so we have

$$J(m, n) \leq J(m, m-1) = -128m^5 + 176m^4 - 320m^3 + 192m^2 - 8m + 5 \quad \text{for } m \geq 2.$$

Writing $m = k + 1$ for $k \geq 1$ this becomes

$$J(m, n) \leq J(k+1, k) = -128k^5 - 464k^4 - 896k^3 - 992k^2 - 520k - 83,$$

which is obviously negative.

We conclude that $J(m, n) < 0$ for all positive integers m, n with $m > n$. This ensures that $H(a_0, b_0, m, n) < 0$, and thus that $MI < 0$ for these choices of a_0 and b_0 . By the Misiolek criterion, there is eventually a conjugate point along the corresponding geodesic. \square

Observe in the proof above that the worst-case scenario is when $n = m - 1$: when n is smaller than $m - 1$ the index form is even more negative. This corresponds to conjugate points being easier to find when m and n are farther apart, as was found by Drivas et al. [4]. It thus stands to reason that the hardest case is when $n = m$. Indeed we need two extra terms in our formula for the variation to make it work in this case. Note also that we require that $n \geq 2$; the proof does not work if $m = n = 1$.

Theorem 3. *Suppose $n \geq 2$ and $\psi(x, y) = -\cos(nx)\cos(ny)$. Then there is a function $f: \mathbb{T}^2 \rightarrow \mathbb{R}$ such that $\phi = \{\psi, f\}$ satisfies (6). Thus there is a time $T > 0$ such that $\gamma(T)$ is conjugate to $\gamma(0)$ along the corresponding geodesic γ , and γ is not minimizing past T .*

Proof. See Maple file in <https://github.com/alebrigant/conjugate-points> for details of the computations here.

For coefficients $a, b, c, d \in \mathbb{R}$, define

$$(12) \quad f(x, y) = \cos x (1 + a \cos(2ny) + b \cos(4ny) + c \cos(2nx)) + d \sin x \sin(2nx).$$

We again compute

$$(13) \quad \phi(x, y) = n \sin(nx) \cos(ny) \cos x \left(-2an \sin(2ny) - 4bn \sin(4ny) \right)$$

$$- n \cos(nx) \sin(ny) \left(-\sin x [(1 + a \cos(2ny) + b \cos(4ny) + c \cos(2nx))] \right)$$

$$- 2cn \cos x \sin(2nx) + d \cos x \sin(2nx) + 2dn \sin x \cos(2nx)$$

Computing the Misiołek index (6) now gives

$$MI = \frac{\pi^2 n^2}{4} H(a, b, c, d, n),$$

where

$$(14) \quad H(a, b, c, d, n) = 16a^2n^4 + 64abn^4 + 256b^2n^4 + 16c^2n^4 + 16d^2n^4 + 8adn^3 \\ - 64cdn^3 + 8a^2n^2 - 4abn^2 - 12acn^2 + 32b^2n^2 + 24c^2n^2 + 24d^2n^2 + 6adn - 8an^2 \\ - 16cdn + 8cn^2 + a^2 - ab - ac + b^2 + c^2 + d^2 - 8dn - 2a + 2c + 2.$$

Again, this is quadratic in the four unknown coefficients a, b, c, d with positive leading-order coefficients, so the unique critical point is a global minimum.

In this case there is no simpler formula which works, so we find the critical point by solving the linear system

$$\frac{\partial H}{\partial a} = \frac{\partial H}{\partial b} = \frac{\partial H}{\partial c} = \frac{\partial H}{\partial d} = 0.$$

Explicitly this system looks like

$$(32a + 64b)n^4 + 8dn^3 + (16a - 4b - 12c - 8)n^2 + 6dn + 2a - b - c - 2 = 0, \\ (64a + 512b)n^4 + (-4a + 64b)n^2 - a + 2b = 0, \\ 32cn^4 - 64dn^3 + (-12a + 48c + 8)n^2 - 16dn - a + 2c + 2 = 0, \\ 32dn^4 + (8a - 64c)n^3 + 48dn^2 + (6a - 16c - 8)n + 2d = 0.$$

There is a unique solution of the system above, given by

$$a_0 = \frac{(8n^2 + 1)(256n^4 + 32n^2 + 1)}{6144n^8 + 4864n^6 + 920n^4 + 58n^2 + 1}, \\ b_0 = -\frac{512n^6 + 32n^4 - 12n^2 - 1}{2(6144n^8 + 4864n^6 + 920n^4 + 58n^2 + 1)}, \\ c_0 = -\frac{49152n^{10} + 59392n^8 + 17088n^6 + 1952n^4 + 88n^2 + 1}{2(98304n^{12} + 28672n^{10} - 18048n^8 - 1568n^6 + 472n^4 + 50n^2 + 1)}, \\ d_0 = -\frac{n(32768n^8 + 19456n^6 + 3328n^4 + 196n^2 + 3)}{(16n^4 - 8n^2 + 1)(6144n^8 + 4864n^6 + 920n^4 + 58n^2 + 1)}.$$

In spite of how complicated these formulas appear, the value of H at the critical point is relatively simple in n : we get

$$(15) \quad H(a_0, b_0, c_0, d_0, n) = \frac{-4096n^8 + 3584n^6 + 1008n^4 + 68n^2 + 1}{12288n^8 + 9728n^6 + 1840n^4 + 116n^2 + 2}.$$

Replacing n with $\sqrt{4+k}$ for some $k \geq 0$ (since $n \geq 2$ by assumption), this becomes

$$H(a_0, b_0, c_0, d_0, \sqrt{4+k}) = \frac{-4096k^4 - 61952k^3 - 349200k^2 - 868412k - 802799}{12288k^4 + 206336k^3 + 1298224k^2 + 3627508k + 3798226},$$

and this is obviously negative for all real $k \geq 0$.

We conclude that with $n \geq 2$, the choice $(a, b, c, d) = (a_0, b_0, c_0, d_0)$ leads to a variation f given by (12) such that $\phi = \{\psi, f\}$ satisfies the Misiołek criterion (6). Hence there is eventually a conjugate point along the geodesic. \square

Example 1. In the case $m = 3$ and $n = 2$, the formula (8) with a and b given by (10) becomes

$$(16) \quad f(x, y) = \cos x \left(1 - \frac{37}{1513} \cos 6x + \frac{1}{17} \cos 4y \right).$$

This is plotted on the left side of Figure 1 below.

On the other hand if $m = n = 2$, the variation f from (12) becomes

$$(17) \quad f(x, y) = \cos x \left(1 + \frac{139425}{1899113} \cos 4y - \frac{33231}{3798226} \cos 8y - \frac{66661217}{854600850} \cos 4x \right) - \frac{19375654}{427300425} \sin x \sin 4x.$$

The plot is shown on the right side of Figure 1.

Note that especially in the off-diagonal case, both the formula and graph are substantially simpler than the one given in the second author's paper [15]. Also note that the graphs are basically indistinguishable to the naked eye, although none of the terms in (17) can be omitted.

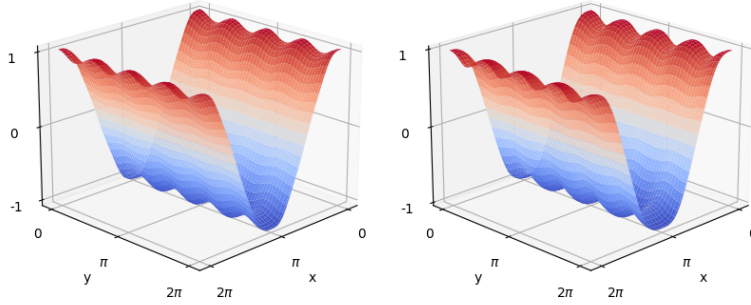


FIGURE 1. On the left, the optimal perturbation f of the form (8) in the case $m = 3$ and $n = 2$, given explicitly by (16). On the right, the optimal perturbation f of the form (12), given explicitly by (17), in the case $m = n = 2$. These perturbations generate a variation ϕ satisfying the Misiólek criterion (6) for conjugate points. Note that they appear quite similar at this level.

4. THE MINIMIZATION PRINCIPLE

Recall that the Misiólek criterion (6) is given by

$$(18) \quad MI(\phi) := \int_M |\text{grad } \phi|^2 - \lambda^2 \phi^2 d\mu < 0,$$

where $\phi = \{\psi, f\}$ for some function f .

Theorem 4. *If $\psi: M \rightarrow \mathbb{R}$ is a fixed smooth function satisfying $\Delta\psi = -\lambda^2\psi$ and L denotes the operator $L(f) = \{\psi, f\}$, then $\Delta^{-1}L$ is continuous from $\dot{H}^1(M)$ to itself. The image $\text{Im}[L]$ is a closed subspace of $\dot{H}^1(M)$, and Δ^{-1} restricts to a continuous*

operator on it. The Misiolek criterion is satisfied if and only if the operator norm of Δ^{-1} on $\text{Im}[L]$ satisfies

$$(19) \quad \|\Delta^{-1}|_{\text{Im}[L]}\|_{op} \geq \frac{1}{\lambda^2}.$$

This happens if and only if there is a number $c \geq 1/\lambda^2$ and a function $f: M \rightarrow \mathbb{R}$ satisfying

$$\Delta^{-1}L(f) = cL(f).$$

Proof. Recall that the first-order homogeneous Sobolev metric is given by

$$\langle f, g \rangle_{\dot{H}^1} = - \int_M f \Delta g \, d\mu = \int_M \langle \nabla f, \nabla g \rangle \, d\mu.$$

To establish continuity it is sufficient to establish boundedness, which follows from

$$\|L(f)\|_{L^2} = \|\text{grad } \psi \times \text{grad } f\|_{L^2} \leq \|\psi\|_{C^1} \|\text{grad } f\|_{L^2} = \|\psi\|_{C^1} \|f\|_{\dot{H}^1}.$$

As a result the image of L is a closed subspace in $\dot{H}^1(M)$. Since L is antisymmetric in the L^2 inner product we see that the operator $\Delta^{-1}L$ is bounded and antisymmetric in \dot{H}^1 , via

$$\begin{aligned} \langle \Delta^{-1}L(f), g \rangle_{\dot{H}^1(M)} &= - \int_M L(f)g \, d\mu = - \int_M \{\psi, f\}g \, d\mu \\ &= \int_M \{\psi, g\}f \, d\mu = - \langle \Delta^{-1}L(g), f \rangle_{\dot{H}^1(M)}. \end{aligned}$$

Now the Misiolek index from (18) rescaled by the \dot{H}^1 norm is given by

$$\frac{MI(\phi)}{\|\phi\|_{\dot{H}^1(M)}^2} = 1 - \lambda^2 \frac{\int_M \phi^2 \, d\mu}{\int_M |\text{grad } \phi|^2 \, d\mu} = 1 + \lambda^2 \frac{\langle \phi, \Delta^{-1}\phi \rangle_{\dot{H}^1(M)}}{\|\phi\|_{\dot{H}^1(M)}^2},$$

and this can be made negative for some $\phi \in \text{Im}[L]$ if and only if

$$\sup_{\phi \in \text{Im}[L]} \frac{\langle \phi, (-\Delta)^{-1}\phi \rangle_{\dot{H}^1(M)}}{\|\phi\|_{\dot{H}^1(M)}^2} \geq \frac{1}{\lambda^2},$$

and the left-hand side is precisely the operator norm of $(-\Delta)^{-1}$ restricted to $\text{Im}[L]$. Since $(-\Delta)^{-1}$ is positive, compact, and self-adjoint on $\dot{H}^1(M)$, it is also positive, compact, and self-adjoint on the closed subspace $\text{Im}[L]$. Hence it has a sequence of positive eigenvalues converging to zero, and the operator norm of it is the largest one. So the operator norm is larger than $1/\lambda^2$ if and only if there is a $c \geq 1/\lambda^2$ such that

$$(20) \quad (-\Delta)^{-1}L(f) = cL(f).$$

□

Solving the eigenfunction problem (20) is rather difficult since L is quite far from invertible—its kernel consists of all functions constant on the level sets of ψ , and in particular any function $\Phi \circ \psi$ for $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ will be in that kernel. In principle one could integrate the Green function for Δ^{-1} along the level sets to try to solve directly for f , or apply the operator $L^{-1}\Delta^{-1}L$ repeatedly to a Fourier basis in hopes that it converges to an eigenfunction, but computationally this becomes rather difficult. As such we take a more indirect approach in the next section, using higher-order Sobolev inner products.

5. NUMERICS

Here we give some details on the numerics that helped provide the results of Theorems 2 and 3. First we discuss the general setup for any surface M possibly with boundary, then we specialize to $M = \mathbb{T}^2$. The implementation described in this section and used to generate Figures 2 and 3 is available on github: <https://github.com/alebrignant/conjugate-points>.

Let $C_0^\infty(M)$ denote the space of C^∞ -functions $f : M \rightarrow \mathbb{R}$ that vanish on the boundary ∂M . We define the following operators on $C_0^\infty(M)$

$$(21) \quad Lf := \{\psi, f\}, \quad \Gamma f := -\Delta f, \quad \Omega f := -L(\Lambda - \lambda^2 I)Lf.$$

The goal is to find a function $f \in C_0^\infty$ such that the corresponding Poisson bracket $\phi = \{\psi, f\}$ minimizes some normalized version of the Misiolek index (18). This index can be rewritten in terms of the previously defined operators (21) as

$$MI(\phi) = - \int_M \phi(\Delta + \lambda^2 I)\phi \, d\mu = \int_M Lf(\Lambda - \lambda^2 I)Lf \, d\mu = \langle f, \Omega f \rangle_{L^2},$$

where we have used the antisymmetry of the operator L with respect to the L^2 inner product on $C_0^\infty(M)$. To minimize $MI(\phi)$ we need to constrain f , otherwise the minimum is either zero or negative infinity. The easiest way to do this is to require that some norm of f be constrained to be 1, which is equivalent to choosing f so that it minimizes the Rayleigh quotient

$$RQ(f) = \frac{\langle f, \Omega f \rangle_{L^2}}{\langle f, \Gamma f \rangle_{L^2}},$$

where Γ is some positive-definite symmetric operator on $C_0^\infty(M)$. If Γ has some relatively high Sobolev order, the minimizer we find will be smoother. By the usual calculus of variations method, minimizers of this must satisfy

$$\Omega f = c\Gamma f \quad \text{for some } c \in \mathbb{R}.$$

Equivalently, f is an eigenfunction of the operator $\Gamma^{-1}\Omega$, and if we can find an eigenvalue c which is negative, it proves the index form can be made negative, and thus that there is eventually a conjugate point along the geodesic. The idea is therefore to make Γ strong enough as a differential operator that it more than cancels all the differential operators in Ω . Choosing $\Gamma = \Lambda^p$ is equivalent to changing the inner product to the homogeneous Sobolev metric of order p

$$\langle f, g \rangle_{\dot{H}^p} = (-1)^p \int_M f \Delta^p g \, d\mu = \langle f, \Lambda^p g \rangle_{L^2}$$

so that the Rayleigh quotient becomes

$$(22) \quad RQ(f) = \frac{\langle f, \Omega_p f \rangle_{\dot{H}^p}}{\langle f, f \rangle_{\dot{H}^p}},$$

where

$$\Omega_p = \Gamma^{-1}\Omega = -\Lambda^{-p}L(\Lambda - \lambda^2 I)L.$$

Replacing the L^2 -inner product by the \dot{H}^p inner product yields an operator Ω_p that is bounded for $p \geq 2$, and compact for $p \geq 3$.

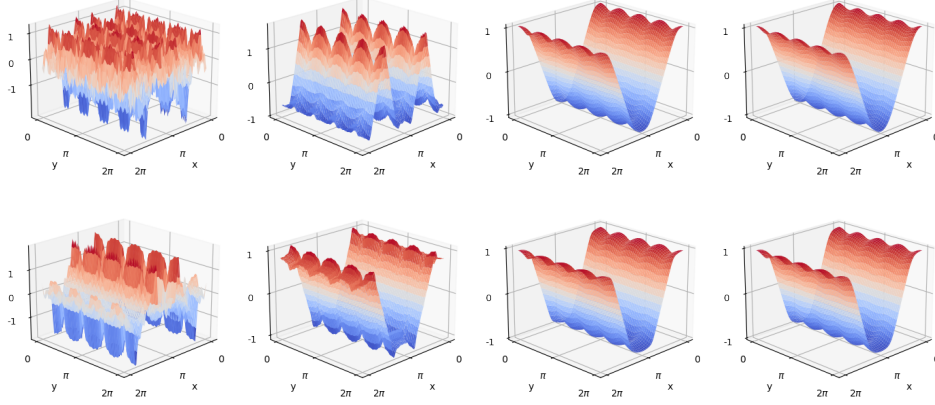


FIGURE 2. On the top row, minimizers $f : M \rightarrow \mathbb{R}$ of (22) in the case $m = 3$ and $n = 2$ and for different values of the order p of the homogeneous Sobolev norm: 0, 1, 2, 3 from left to right. On the bottom row, minimizers in the case $m = 2$ and $n = 2$ for the same values of p .

The problem of minimizing the Rayleigh quotient (22) can be made finite-dimensional by decomposing the unknown f in the following spatial Fourier basis

$$(23) \quad f(x, y) = \sum_{j=0}^N \sum_{k=-N}^N a_{jk} \cos(jx + ky).$$

Due to the evenness of the cosine function, it is enough to take only positive k indices for $j = 0$ in the above double sum. Then each function f can be represented by a $(N(2N + 2) + 1)$ -size vector of real coordinates $(a_{jk})_{j,k}$, and the operator Ω_p by a square matrix of same size. The matrix representation of Ω_p is obtained by multiplying matrices that are all diagonal except for the Poisson bracket operator L matrix (for which we give a formula in the Appendix, for the aid of the reader). Then the problem of minimizing (22) boils down to finding the minimal eigenvalue of the Ω_p matrix, and the function f whose Poisson bracket minimizes the Misiolek criterion is given by the corresponding eigenvector.

Figure 2 shows the minimizers found in the case $m = 3$ and $n = 2$ (top row) and the case $m = 2$ and $n = 2$ (bottom row), for different values of the Sobolev order p . The deformations $g(t, x, y) = \sin(\frac{\pi t}{T})f(x, y)$ associated to all the spatial deformations f shown in this figure induce vector fields $Y = \text{sgrad } g$ that make the Misiolek criterion, and thus the index form, negative. However the “shape” of this minimizing deformation stabilizes only for $p \geq 2$, and for these values we see a characteristic “V-shape” appearing, i.e. a perturbation of $\cos(x)$, which we used to obtain formulas (8) and (12).

In the diagonal case $m = n$, due to the symmetry in the variables x and y , another minimizer is given by a perturbation of $\cos(y)$, the symmetric perturbation obtained by exchanging x and y . To obtain the figures of the bottom row of Figure 2, we constrained the solution to have Fourier coefficient $a_{01} = 0$. Without this constraint, we find the results of Figure 3, where both minimizers appear

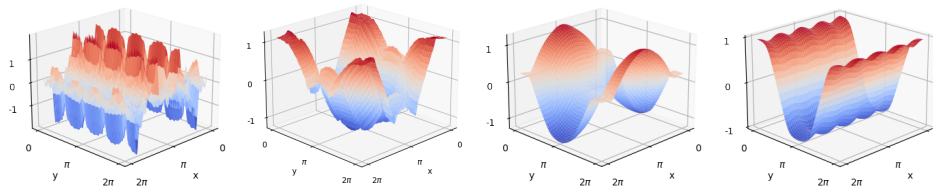


FIGURE 3. Minimizers $f : M \rightarrow \mathbb{R}$ of (22) in the case $m = 2$ and $n = 2$ and for different values of the order p of the homogeneous Sobolev norm: 0, 1, 2, 3 from left to right.

separately as well as mixed, as in the case $p = 1$. This is similar to the minimizer found by the second author in [15], displayed in Figure 7.

6. THE (1, 1) CASE AND OTHER OPEN QUESTIONS

Although we have been able to discover variation fields that reduce the energy for almost every pair (m, n) of Kolmogorov stream functions $\psi = -\cos mx \cos ny$, we were not able to find one for the (1, 1) case, regardless of what choice we make in the Misiolek criterion. We conjecture that in fact the Misiolek criterion cannot be used at all in this case, which by Theorem 4 implies that

$$\left\| \Delta^{-1} \Big|_{\text{Im}[L]} \right\|_{\text{op}} < \frac{1}{2}, \quad L(f) := -\{\cos x \cos y, f\}.$$

In principle one should be able to compute this, but it is not trivial. We therefore conjecture that

Conjecture 1. If $\psi = -\cos x \cos y$, then for every smooth function $f : \mathbb{T}^2 \rightarrow \mathbb{R}$ we have

$$\int_{\mathbb{T}^2} |\text{grad } \phi|^2 d\mu \geq 2 \int_{\mathbb{T}^2} \phi^2 d\mu$$

for $\phi = \{\psi, f\}$.

Obviously, $\phi = \cos x$ or anything else with Laplacian eigenvalue 1 on the torus would violate this inequality; however no such ϕ is in the image of L since the integral of such a ϕ is not zero over the level sets of ψ .

The Misiolek criterion is not the only way to detect conjugate points. For example in [15] a variety of conjugate points along steady flows are found that cannot be detected by the Misiolek criterion, but can be found using slightly more complicated time dependence: instead of using variations of the form $g(t, x, y) = \sin \frac{\pi t}{T} f(x, y)$, we can instead use $g(t, x, y) = \sin \frac{\pi t}{T} f(\eta_{\alpha t}(x, y))$ for some drift parameter α , where η is the flow of the velocity field, and nonzero choices of α can sometimes produce new examples. This happens for example on the sphere, where the analogue of Conjecture (1) is actually true but there are still many conjugate points; see Tauchi-Yoneda [18].

We thus attempted to find flows with possibly more complicated time dependence, considering the full index form (5) for any function $g(t, x, y)$ periodic in the (x, y) variables and satisfying $g(0, x, y) = g(T, x, y) = 0$. Obviously one can expand this in Fourier components and attempt to minimize the resulting quadratic form on the Fourier coefficients, possibly after dividing by a strong Sobolev norm in

both time and space (analogous to the technique of the last section using a strong Sobolev norm in space only). And while one can get the index form to be fairly small, it does not seem to become negative regardless of how many terms one takes. We thus also conjecture a stronger version of Conjecture 1, that the full index form (5) is never negative (which would ensure the absence of conjugate points):

Conjecture 2. If $\psi = -\cos x \cos y$, then for every smooth function $g: [0, T] \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with $g(0, x, y) = g(T, x, y) = 0$, we have

$$\int_0^T \int_{\mathbb{T}^2} \left(|\text{grad } h(t, x, y)|^2 - 2\{\psi(x, y), g(t, x, y)\}h(t, x, y) \right) d\mu dt \geq 0,$$

where $h = \frac{\partial g}{\partial t} + \{\psi, g\}$.

The only known way to prove absence of conjugate points along the geodesic corresponding to a steady fluid flow X on a manifold M is to show that the sectional curvature satisfies

$$(24) \quad \langle R(X, Y)Y, X \rangle \leq 0 \quad \text{for all divergence-free } Y \text{ on } M.$$

This is sometimes true; for example Lukatskii [8] and more generally Misiołek [9] showed that (24) is true whenever X is a pressure-constant flow satisfying $\nabla_X X = 0$ (such as when $\psi = \cos x$ on the torus), and the second author showed that for many rotational flows (24) is also satisfied (such as when $\psi = \frac{1}{2}r^2$ on the disc). However it can never be satisfied for a steady flow on a surface without rotational symmetry such as the (1, 1) case. Indeed whenever $X = \text{sgrad } \psi$ and $Y = \text{sgrad}(G \circ \psi)$ for a function $G: \mathbb{R} \rightarrow \mathbb{R}$, we will have $\langle R(X, Y)Y, X \rangle = |P(\nabla_Y X)|^2$ where P is the projection (2) onto divergence-free vector fields, which is strictly positive if $|X|$ is not constant along X ; see [12].

These are rather “trivial” directions of positive curvature however: they cannot be used to make the index form (5) negative, as the following proposition shows.

Proposition 5. *Suppose $X = \text{sgrad } \psi$ is a steady solution of the Euler equation, and consider time-dependent variations $Y = \text{sgrad } g$ of the form $g(t, x, y) = F(t)G(\psi(x, y))$ with $F(0) = F(T) = 0$ and $G: \mathbb{R} \rightarrow \mathbb{R}$. Then the index form (5) satisfies $I(Y, Y) \geq 0$.*

Proof. Equation (5) says that

$$I(Y, Y) = \int_0^T \int_M |\text{grad } h|^2 + \Delta\psi\{g, h\}d\mu dt$$

where $\Delta\psi$ is the vorticity and $h = g_t + \{\psi, g\}$. For g of the given form however, we will always have $\{g, h\} = 0$, since $h = F'(t)G(\psi) + F(t)\{\psi, G(\psi)\} = F'(t)G(\psi)$. Hence the index is given just by the nonnegative first term. \square

To find conjugate points, one must therefore find fields Y which are not of the trivial form $Y = \text{sgrad}(G \circ \psi)$ but which still satisfy $\langle R(X, Y)Y, X \rangle > 0$. It might be expected that the absence of conjugate points along the (1, 1) direction is a result of not having any such directions, but this is also not true.

Proposition 6. *Suppose $\psi = -\cos x \cos y$ with $X = \text{sgrad } \psi$. Then there is a divergence-free field Y such that $[X, Y] \neq 0$ and $\langle R(X, Y)Y, X \rangle > 0$.*

Proof. We use the curvature formula derived by Misiölek in [9], based on the Gauss-Codazzi formula on $\text{Diff}_\mu(\mathbb{T}^2)$. If p is the pressure function satisfying $\nabla_X X = -\text{grad } p$ and q is the function satisfying $\Delta q = -\text{div}(\nabla_Y X)$, then the curvature is given by

$$\langle R(Y, X)X, Y \rangle = \int_{\mathbb{T}^2} \text{Hess } p(Y, Y) - |\text{grad } q|^2 d\mu.$$

In this case we have $p = \frac{1}{2}(\sin^2 x + \sin^2 y)$. Choose

$$g = \cos 2x + \cos 2y + 2 \cos 2x \cos 2y,$$

and set $Y = \text{sgrad } g$. We then verify that

$$q = 4 \cos x \cos y (\cos^2 y - \sin^2 x)$$

is the unique (up to a constant) solution of $\Delta q = -\text{div}(\nabla_Y X)$, and it is straightforward to compute from the formula that

$$\langle R(Y, X)X, Y \rangle = 4\pi^2.$$

Obviously g is not a function of ψ ; in particular $\{\psi, g\} = 4 \sin x \sin y (\cos^2 y - \cos^2 x) \neq 0$. \square

Hence the (1, 1) case is surprisingly resistant to either finding conjugate points or proving their absence. There is positive sectional curvature along nontrivial sections containing the geodesic tangent vector, but seemingly not enough to ensure a conjugate point.

We conclude with some possible other directions to pursue these techniques. An obvious one is to apply this to other steady flows on the torus that come from Laplacian eigenfunctions; the Kolmogorov flows we have discussed have rectangular cells, but one could also consider things of the form

$$\psi = \cos(mx + ny) + c \cos(mx - ny)$$

which for $c \neq 1$ have skew quadrilateral cells. One could also consider stretched tori where the eigenfunctions look like $\cos(mx + \alpha ny)$ for some positive parameter α , as considered in [4].

It is also easy to see how the same techniques could generate conjugate points on other surfaces. It would be interesting to try the same methods to find conjugate points along other eigenfunctions of the 2-sphere: the smallest-eigenvalue function generates rigid rotations and its geodesic is known to have many conjugate points [9]. It seems likely that every eigenfunction of the Laplacian on S^2 generates a geodesic with conjugate points, but no others are actually known concretely. This is connected with work of Benn [3] on conjugate points along *nonsteady* Rossby-Haurwitz waves on the sphere.

APPENDIX

Proposition 7. *Let $f : M \rightarrow M$ be a function and a_{jk} the coefficients of its Fourier decomposition (23), for $(j, k) \in \{0\} \times \{0, \dots, N\} \cup \{1, \dots, N\} \times \{-N, \dots, N\}$. Then*

$$\begin{aligned} Lf = \{\psi, f\} = \frac{1}{8} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} & ((mk - nj)(A_{j-m, k-n} - A_{j+m, k+n}) \\ & + (mk + nj)(A_{j-m, k+n} - A_{j+m, k-n})) \cos(jx + ky) \end{aligned}$$

where

$$A_{jk} = \begin{cases} a_{jk} & \text{if } 0 < j \leq N, |k| \leq N, \\ a_{-j,-k} & \text{if } -N \leq j < 0, |k| \leq N, \\ a_{j|k|} & \text{if } j = 0, |k| \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

REFERENCES

- [1] V. I. Arnold. On the differential geometry of infinite-dimensional Lie groups and its application to the hydrodynamics of perfect fluids. *Vladimir I. Arnold-Collected Works: Hydrodynamics, Bifurcation Theory, and Algebraic Geometry 1965-1972*, pages 33–69, 2014.
- [2] V. I. Arnold and B. A. Khesin. *Topological methods in hydrodynamics*, volume 125 of *Applied Mathematical Sciences*. Springer, Cham, [2021] ©2021. Second edition [of 1612569].
- [3] J. Benn. Conjugate points in $\mathcal{D}_\mu(S^2)$. *Journal of Geometry and Physics*, 170:104369, 2021.
- [4] T. D. Drivas, G. Misiołek, B. Shi, and T. Yoneda. Conjugate and cut points in ideal fluid motion. *Annales mathématiques du Québec*, pages 1–19, 2021.
- [5] D. G. Ebin and J. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Annals of Mathematics*, pages 102–163, 1970.
- [6] D. G. Ebin, G. Misiołek, and S. C. Preston. Singularities of the exponential map on the volume-preserving diffeomorphism group. *Geometric and Functional Analysis*, 16(4):850–868, 2006.
- [7] L. Lichtenfelz, T. Tauchi, and T. Yoneda. Existence of a conjugate point in the incompressible Euler flow on a three-dimensional ellipsoid. *arXiv preprint arXiv:2204.00732*, 2022.
- [8] A. M. Lukatskii. Structure of the curvature tensor of the group of measure-preserving diffeomorphisms of a compact two-dimensional manifold. *Siberian Mathematical Journal*, 29(6):947–951, 1988.
- [9] G. Misiołek. Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms. *Indiana University mathematics journal*, pages 215–235, 1993.
- [10] G. Misiołek. Conjugate points in $\mathcal{D}_\mu(\mathbb{T}^2)$. *Proceedings of the American Mathematical Society*, pages 977–982, 1996.
- [11] G. Misiołek and S. C. Preston. Fredholm properties of Riemannian exponential maps on diffeomorphism groups. *Inventiones mathematicae*, 179(1):191, 2010.
- [12] S. C. Preston. Nonpositive curvature on the area-preserving diffeomorphism group. *Journal of Geometry and Physics*, 53(2):226–248, 2005.
- [13] S. C. Preston. On the volumorphism group, the first conjugate point is always the hardest. *Communications in mathematical physics*, 267:493–513, 2006.
- [14] S. C. Preston. The WKB method for conjugate points in the volumorphism group. *Indiana University Mathematics Journal*, pages 3303–3327, 2008.
- [15] S. C. Preston. Conjugate point criteria on the area-preserving diffeomorphism group. *Journal of Geometry and Physics*, 183:104680, 2023.
- [16] S. C. Preston and P. Washabaugh. The geometry of axisymmetric ideal fluid flows with swirl. *Arnold Mathematical Journal*, 3:175–185, 2017.
- [17] T. Tauchi and T. Yoneda. Arnold stability and Misiołek curvature. *Monatshefte für Mathematik*, 199(2):411–429, 2022.
- [18] T. Tauchi and T. Yoneda. Existence of a conjugate point in the incompressible Euler flow on an ellipsoid. *Journal of the Mathematical Society of Japan*, 74(2):629–653, 2022.