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The calculus of fuzzy restrictions as a basis for flexible constraint satisfaction

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Abstract—This paper proposes a unified treatment of prioritized and flexible constraints, both of them being represented by possibility distributions. Arc- and path-consistency-based methods for constraint satisfaction problems are extended to this possibility theory framework. An illustrative example is given.

1. INTRODUCTION

Many engineering problems such as design or scheduling may be modelled as Constraint Satisfaction Problems (CSP) [12], [10]: a CSP consists in finding an assignment of values to a set of variables ranging on discrete domains, so as to satisfy prescribed constraints. Most studies in this framework try to tackle the highly combinatorial nature (NP hard) of such problems, considering crisp constraints only. However, the classical CSP framework does not suit many practical situations, in which flexibility is an intrinsic characteristic of the information to be taken into account.

The notion of soft constraint, whereby feasibility is a matter of degree, is not new [1] but it is only recently that this concept becomes attractive in Artificial Intelligence [2], [13], [6]. In [14] and [7] the flexible aspect of CSPs is stressed, and theoretical frameworks for expressing such constraints are proposed. General Branch and Bound techniques are developed in [7] for dealing with these constraints without extending classical propagation methods to them. In [14] it is pointed out that prioritized constraints modelled by logical formulas induce preference among interpretations. This is also the case with possibilistic logic [9]. Other papers suggest that possibility theory [18] may be a suitable framework for the representation of prioritized constraints [15] and soft constraints [11]. Making a step further, we claim that both prioritized constraints and soft constraints expressing preferences between values can be regarded as local criteria inducing total orderings over tuples of values that can be represented by fuzzy restrictions [17]. Hence, we propose an approach based on possibility theory for representing and solving such "Fuzzy Constraint Satisfaction Problems" (FCSP) involving both types of constraints.

The main difficulty is to compute the global ordering induced over solutions - and to choose the best one(s). Finding the (ordered) set of all the solutions becomes a problem of fuzzy restriction calculus. This problem is highly combinatorial when solved by global combination techniques, but local propagation algorithms may be applied when the problem presents an hypertree structure [16], [5], [8]. We claim that local propagation can also be applied to hypergraphs in order to provide a good approximation of the solution. We have developed efficient algorithms inspired by both Zadeh's

principle of combination-projection [17], and existing techniques in constraint propagation.

The next section deals with representational issues concerning flexible constraints. Section 3 then explains how a CSP involving such constraints may be defined in the framework of possibility theory. In Sections 4 and 5, we present some computational schemes. These techniques are finally illustrated in Section 6 by a simple example.

2. REPRESENTATIONAL ISSUES

A hard constraint C between a set of variables $\{x_1, \dots, x_k\}$ is traditionally represented by means of a crisp relation R on $U_1 \times \dots \times U_k$, U_j being the range of variable x_j ; R is the crisp subset of $U_1 \times \dots \times U_k$ of k -tuples of values (u_1, \dots, u_k) (also called "instanciations" of $\{x_1, \dots, x_k\}$) which satisfy C . The set $\{x_1, \dots, x_k\}$ is denoted $V(R)$.

A soft constraint expresses preferences between values (k -tuples) that can be assigned to a set of variables. In this paper we assume that these preferences can be modelled by means of a total order and that this total order is encoded by means of a fuzzy relation R , that assigns to each k -tuple (u_1, \dots, u_k) its level of preference $\mu_R(u_1, \dots, u_k)$ in a totally ordered set L . This set is usually chosen as the unit interval $[0, 1]$. $\mu_R(u_1, \dots, u_k) > \mu_R(u'_1, \dots, u'_k)$ means that (u_1, \dots, u_k) is preferred to (u'_1, \dots, u'_k) as a value of $V(R)$, $\mu_R(u_1, \dots, u_k) = 0$ that (u_1, \dots, u_k) is a forbidden k -tuple and $\mu_R(u_1, \dots, u_k) = 1$ that this k -tuple totally satisfies the constraint. More generally, $\mu_R(u_1, \dots, u_k)$ can also be interpreted as degrees of satisfaction of the soft constraint. $\mu_R(u_1, \dots, u_k) > 0$ indicates a feasible assignment, even if it satisfies C only partially.

A fuzzy relation restricts the possible values that can be assigned to the variables according to a preference criterion C . Hence the concept of fuzzy restriction, that is "a fuzzy relation which acts as an elastic constraint on the values that may be assigned to variables" [17], seems to be suitable for capturing such a kind of information. In the terminology of possibility theory, μ_R is a possibility distribution on $U_1 \times \dots \times U_k$.

Fuzzy restrictions also offer a suitable formalism for the expression of prioritized constraints. When it is possible to a priori exhibit a total order over the respective priorities of the constraints, these priorities will be represented by means of levels in the scale $[0, 1]$: a coefficient α_C is attached to each constraint C (its priority degree) and indicates the degree to which C must be satisfied. If $\alpha_C = 1$, C is an imperative constraint; if $\alpha_C = 0$, it is completely possible to violate C .

(C has no incidence in the problem). Given two constraints C and C', $\alpha_C > \alpha_{C'}$ means that the satisfaction of C is more necessary than the satisfaction of C': if C and C' cannot be satisfied simultaneously, solutions compatible with C are preferable to solutions compatible with C' only.

Priorities on constraints can be transformed, without any loss of information, into satisfaction degrees on values. Indeed, since α_C represents to what extent it is necessary to satisfy C, $1 - \alpha_C$ indicates to what extent it is possible to violate C. In other words, any instantiation (u_1, \dots, u_k) satisfies C to a degree greater than or equal to $1 - \alpha_C$. Hence, the pair (C, α_C) , where C is a crisp constraint, can be modeled as a soft constraint represented by the fuzzy relation S on $U_1 \times \dots \times U_k$: $\mu_S(u_1, \dots, u_k) = 1$ if (u_1, \dots, u_k) satisfies C

$$= 1 - \alpha_C \text{ if } (u_1, \dots, u_k) \text{ violates C.}$$

If C is a soft constraint modeled by the fuzzy relation R, (C, α_C) is represented by the fuzzy relation S: $\mu_S(u_1, \dots, u_k) = \max(1 - \alpha_C, \mu_R(u_1, \dots, u_k))$. These definitions are in accordance with the treatment of certainty-qualified assertions in possibility theory, and the principle of minimum specificity, interpreting priority levels as necessity degrees. When R is crisp, they are in full agreement with possibilistic logic [4].

3. FUZZY CONSTRAINT SATISFACTION PROBLEM

A FCSP is a set of m fuzzy (soft or prioritized) constraints $P = \{R_1, \dots, R_m\}$ restricting the possible values of n variables, say $\{x_1, \dots, x_n\}$, each x_j ranging on a finite domain U_j . The use of fuzzy relations enables the usual operations of conjunctive combination and projection to be applied [17].

Given $W = \{x_{w1}, \dots, x_{wk}\}$ and $Y = \{x_{y1}, \dots, x_{yh}\}$ two subsets of $\{x_1, \dots, x_n\}$ such that $W \subseteq Y$, and a fuzzy relation T restricting the possible values of Y, the *projection* (or the *marginalisation*) of T on W is a fuzzy relation R = $(T) \downarrow W$ restricting the possible values of W. It is defined on $U_{w1} \times \dots \times U_{wk}$ by: $\mu_R(u_{w1}, \dots, u_{wk}) =$

$\sup\{(\mu_T(u_{w1}, \dots, u_{wk}, u_{y1}, \dots, u_{yh})) \downarrow W = (u_{w1}, \dots, u_{wk})\} \mu_T(u_{y1}, \dots, u_{yh})$ where $(u_{y1}, \dots, u_{yh}) \downarrow W$ denotes the restriction of (u_{y1}, \dots, u_{yh}) to W. $\mu_R(u_{w1}, \dots, u_{wk})$ estimates to what extent the instantiation (u_{w1}, \dots, u_{wk}) which is a partial instantiation of Y can be extended to a complete instantiation of Y that satisfies T.

The *conjunctive combination* of two fuzzy restrictions R and S restricting the possible values of two sets of variables X and Y is a fuzzy restriction T = $R \otimes S$ over the possible values of $W = X \cup Y$. It is defined by: $\mu_T(u_{w1}, \dots, u_{wk}) =$

$$\min(\mu_R((u_{w1}, \dots, u_{wk}) \downarrow X), \mu_S((u_{w1}, \dots, u_{wk}) \downarrow Y)).$$

$\mu_T(u_{w1}, \dots, u_{wk})$ estimates to which extent the instantiation (u_{w1}, \dots, u_{wk}) of W satisfies both R and S.

Note that the use of the combination rule underlies an assumption of *commensurability* between preference levels pertaining to different constraints, i. e. the user who specifies the constraints must describe them by means of a unique preference scale L. Moreover, priority levels must be chosen on the dual scale $\{c(\ell), \ell \in L\}$ where c is an order-reversing

mapping) so as to acknowledge the transformation of priority levels on constraints into preference degrees on solutions. Although natural and often implicit, these assumptions must be made clear. Note that soft constraints differ from criteria (or objective functions) by the fact that no compensation is allowed among them, as modelled by the min operation.

The degree $\mu_{R_1 \otimes \dots \otimes R_m}(u_1, \dots, u_n)$ estimates to what extent the instantiation (u_1, \dots, u_n) satisfies all the constraints $\{R_1, \dots, R_m\}$. In other words, $\mu_{R_1 \otimes \dots \otimes R_m}(u_1, \dots, u_n)$ is the global satisfaction degree of problem P by (u_1, \dots, u_n) . The set of fuzzy restrictions $\{R_1, \dots, R_m\}$ can be regarded as a *decomposition* [17] or a *factorization* [16] of a global fuzzy relation $\rho = R_1 \otimes \dots \otimes R_m$ restricting the combinations of values that may be assigned to (x_1, \dots, x_n) .

Even if each constraint is often normalized in practice, ρ may be subnormalized if some constraints are conflicting, i.e. the problem can be partially or even totally inconsistent. The height of ρ is in fact the consistency degree of the FCSP:

$$\text{Cons}(P) = \text{Height}(\rho) = \sup_{(u_1, \dots, u_n) \in U_1 \times \dots \times U_n} \mu_\rho(u_1, \dots, u_n)$$

Since $R_1 \otimes \dots \otimes R_i \otimes \dots \otimes R_m \subseteq R_i$ (where \subseteq is the fuzzy set inclusion defined by the inequality \leq between the membership functions), it holds $\rho \downarrow V(R_i) \subseteq R_i$, which means that R_i does not explicitly account for all the dependencies between the variables it relates: further restrictions may be induced by other constraints. For instance, the problem $P = (R_1, R_2)$, where R_1 : " $x_1 = x_2$ " and R_2 : " $x_2 = x_3$ ", induces a restriction R_3 : " $x_1 = x_3$ " which is not expressed in P. The induced constraints may be computed according to the combination/projection principle: the constraint R induced by a set of constraints $\{R_1, \dots, R_m\}$ on a subset of variables $\{x_{j1}, \dots, x_{jh}\}$ is defined by: $R = (R_1 \otimes \dots \otimes R_m) \downarrow \{x_{j1}, \dots, x_{jh}\}$. It should be pointed out that:

$(R_1 \otimes \dots \otimes R_m) \downarrow \{x_{j1}, \dots, x_{jh}\} \otimes (R_1 \otimes \dots \otimes R_m) = (R_1 \otimes \dots \otimes R_m)$ according to the properties of projection and combination, especially the idempotence of combination. Hence, problems $P = \{R_1, \dots, R_m\}$ and $P' = \{R_1, \dots, R_m\} \cup \{R\}$ have the same fuzzy set of solutions: they are said to be *equivalent*.

In other words, there may be several decompositions of the same restriction ρ corresponding to equivalent FCSP in which the constraints are more or less explicit. Expliciting induced constraints will transform a problem into an equivalent one.

Even if not stated explicitly in $\{R_1, \dots, R_m\}$, ρ also induces a restriction on each variable x_1, \dots, x_n : $\rho \downarrow \{x_i\}$ is the (explicit) restriction of possible values that can be assigned to x_i in accordance with P no matter what values are assigned to the other variables. Most of the time values than can be assigned to one variable are not independent of the values that can be assigned to the others: $\rho \downarrow \{x_i, x_j\} \subseteq \rho \downarrow \{x_i\} \otimes \rho \downarrow \{x_j\}$. In case of equality, both variables are said to be non-interactive [17]. If all the variables are non-interactive, $\rho = \otimes_{i=1, \dots, n} \rho \downarrow \{x_i\}$ and ρ is said to be *separable*. FCSP are seldom separable since constraints express dependencies between variables. In the general case we have: $\rho \subseteq \otimes_{i=1, \dots, n} \rho \downarrow \{x_i\}$.

In summary, a FCSP is a set of m fuzzy relations $P = \{R_1, \dots, R_m\}$ restricting the possible values of n variables $\{x_1, \dots, x_n\}$, and represents a global restriction $\rho = \bigotimes_{i=1, \dots, m} R_i$ on the values that can be assigned to the variables. The following propositions are equivalent:

Determine the existence of a solution \Leftrightarrow compute $\text{Cons}(P) = \text{Height}(\rho)$.

Find one of the best solutions \Leftrightarrow determine any instantiation (u_1, \dots, u_n) s.t. $\mu_\rho(u_1, \dots, u_n) = \text{Cons}(P)$

Determine the relation restricting $\{x_{i1}, \dots, x_{ik}\}$ \Leftrightarrow compute $\rho \downarrow \{x_{i1}, \dots, x_{ik}\}$

Find all the possible values for variable x_i \Leftrightarrow compute $\rho \downarrow \{x_i\}$.

Thus, FCSP reduces to a fuzzy restrictions calculus problem.

4. LOCAL COMPUTATION SCHEMES

Computing $\rho = \bigotimes_{i=1, \dots, m} R_i$ or some of its projections is a NP-hard problem when using a global combination technique. These tasks can be approximated using local computation techniques, whose aim is to transform the problem into an equivalent one in which some induced constraints have been made explicit (in the best case, they are equal to the appropriate projection of ρ).

In order to graphically represent the structure of the problem, an hypergraph is often defined whose nodes are the variables $\{x_1, \dots, x_n\}$ and hyperedges represent constraints R_i relating the subset of variables $V(R_i)$. A unary restriction A_j is supposed to be associated with each variable x_j ; by default, A_j stands for " $x_j \in U_j$ ". A_j is an upper approximation of the fuzzy set of possible values of x_j ($A_j \supseteq \rho \downarrow \{x_j\}$) which is improved by the following local computation schemes.

4.1. Making Unary Restrictions Explicit by Arc-Consistency

The fuzzy unary restriction A_j of the possible values of a variable x_j is said to be consistent with the hyperedge R_i relating x_j to variables $V(R_i) - \{x_j\} = \{x_1, \dots, x_k\} - \{x_j\}$ if:

$$A_j \subseteq (R_i \otimes_{\ell=1, \dots, k, \ell \neq j} A_\ell) \downarrow \{x_j\}.$$

A FCSP is said to be arc-consistent if, for each variable x_j , the associated unary restriction is consistent with all the restrictions relating this variable to some others; formally, if

$$\forall x_j, \forall R_i \in \{R_1, \dots, R_m\} \text{ s.t. } x_j \in V(R_i) = \{x_1, \dots, x_{k_i}\}, \\ A_j \subseteq (R_i \otimes_{\ell=1, \dots, k_i, \ell \neq j} A_\ell) \downarrow \{x_j\} \quad (1)$$

In other terms, a problem is arc-consistent if the neighbour of each x_j does not induce on x_j other restrictions than those already described by the unary constraint A_j associated to x_j , i.e. if these induced restrictions are explicit in A_j . We now describe a local propagation scheme which transforms a FCSP into an equivalent one verifying arc-consistency.

Arc-consistency of the FCSP is ensured by changing all A_j 's into $A'_j = A_j \otimes R_i \rightarrow x_j$ for each R_i , where $R_i \rightarrow x_j = R_i \otimes (\bigotimes_{\ell \in V(R_i), \ell \neq j} A_\ell) \downarrow \{x_j\}$. Note that $A'_j = (R_i \otimes (\bigotimes_{\ell \in V(R_i), \ell \neq j} A_\ell)) \downarrow \{x_j\}$.

Since making a constraint explicit does not change the set of solutions, we can update the problem by replacing A_j by A'_j . The modified problem is equivalent to the previous one, but x_j is now consistent with the hyperedge R_i . After being updated, A'_j is a better approximation of $\rho \downarrow \{x_j\}$ than A_j .

For instance, let A_1 (resp. A_2) be a fuzzy set restricting the values of x_1 (resp. x_2) and R models the fuzzy rule "if x_1 is C , then x_2 is D ": $\mu_R(u_1, u_2) = \mu_C(x_1) \rightarrow \mu_D(x_2)$ for some implication function \rightarrow . The restriction induced via arc-consistency on x_2 is $B' = (R \otimes A_1 \otimes A_2) \downarrow \{x_2\}$; clearly, $(R \otimes A_1) \downarrow \{x_2\}$ computes the generalized modus ponens with fuzzy fact A_1 and fuzzy rule R , while $(R \otimes A_2) \downarrow \{x_1\}$ computes the generalized modus tollens.

Combination being associative, commutative and idempotent, we can compute an equivalent problem which is arc-consistent using a local propagation scheme. First consider each couple (R_i, x_j) and update A_j ; if A_j has been modified, propagate the modification on all the variables depending on x_j , i.e. consider each couple (R_k, x_h) such as R_k relates x_h to x_j . If A_h has been modified, propagate the modification similarly. The process is repeated until no modification occurs. This procedure has been implemented by means of an algorithm (F-AC3) which is an extended fuzzy version of the classical AC3 algorithm [10]. The algorithm is not given here for the sake of brevity.

Since each variable x_j ranges on a discrete set of values U_j , this algorithm stops in a finite number of steps. F-AC3 increases the worst-case complexity of AC3, $O(mrd^{r+1})$ (m being the number of constraints, r the maximal arity of the constraints and d the maximum cardinality of sets of values U_j), by a factor p only, which is the number of distinct satisfaction degrees effectively used to describe the preferences: the complexity of F-AC3 is $O(pmrdr^{r+1})$. The result does not depend on the order in which constraints are examined. However, the choice of this order may enhance the efficiency.

Our hypergraph representation is similar to those proposed in [16], [5]. Note that edges are not required to be directed and may represent other dependencies than causal ones. Contrary to algorithms proposed in [5], [16], [8] our propagation scheme is not limited to hypertrees. However, it just computes, for each variable, an upper approximation of the marginalization of ρ . Moreover, the approximation is exact if the hypergraph is in fact an hypertree (proof is similar to the one given in [8]). Another difference with these algorithms is that F-AC3 does not control the propagation of the fuzzy unary restrictions. As a consequence, a piece of information A_j or R_i may be combined with itself, for instance after being propagated via a circle. Since the combination is idempotent propagation does not generate meaningless information. Moreover, the result of conjunctive combination can only decrease in the sense of fuzzy set inclusion and no new level of satisfaction can appear. These properties ensure the termination of the algorithm. In fact, other formalisms which use combination operations that do not verify these important

properties (for instance, Dempster rule of combination), can not be dealt with as simply by local propagation algorithms in structures other than hypertrees.

4.2. Making Induced Restrictions Explicit by Ensuring Path-Consistency

None of the previous local computation schemes in hypertrees addresses the computation of the restrictions induced over pairs of variables by the global problem. For instance, consider a problem $P = \{R_{12}, R_{23}, R_{13}\}$, restricting the possible values of $\{x_1, x_2, x_3\}$: $R_{12}: x_1 \leq x_2$ $R_{23}: x_2 \leq x_3$ $R_{13}: x_1 \geq x_3$. Constraints R_{12} and R_{23} induce a restriction on x_1 and x_3 : $x_1 \leq x_3$; therefore, the constraint between x_1 and x_3 becomes " $x_1 \leq x_3$ and $x_1 \geq x_3$ ", i.e. $R'_{13}: "x_1 = x_3"$.

The concept of path-consistency introduced by classical CSP literature [12], captures this notion of transitivity in constraint graphs where restrictions are unary or binary. Let us denote R_{ij} the fuzzy restriction linking x_i and x_j , and $R_{ii} = A_i$ the unary restriction on x_i . In terms of fuzzy constraint networks, the original definition can be extended as follows: a problem P is "path-consistent" if and only if every pair of values (u_i, u_j) allowed by a direct constraint R_{ij} for the pair of variables (x_i, x_j) with the satisfaction degree $\mu_{R_{ij}}(u_i, u_j)$ is also allowed with a degree at least equal by every path from x_i to x_j . That is all the restrictions induced by these paths are subsumed by the direct constraints.

Moreover, the graph can be considered as complete, adding non-informative restrictions between pairs of variables which are not linked. Under this assumption, the consistency of each direct constraint with every path of length 2 implies path consistency. This condition which is due to Mackworth [10] when relation are crisp is one of transitive closure. It may be easily extended to fuzzy binary relations (by induction on the length of the path) and is formally expressed by:

$$\forall \{x_i, x_j\} \subseteq \{x_1, \dots, x_n\}, \forall x_k \in \{x_1, \dots, x_n\} - \{x_i, x_j\}, \\ A_i \otimes R_{ij} \otimes A_j \subseteq [R_{ik} \otimes A_k \otimes R_{kj}] \downarrow \{x_i, x_j\} \quad (2)$$

Hence, a procedure which considers the graph as complete and transforms the problem into an equivalent one which verifies property (2) ensures path consistency. The following algorithm is a generalisation to FCSP of PC-2 [10]. It ensures both arc and path consistency in graphs of binary fuzzy restrictions. Since it takes more induced restrictions into account than F-AC3, each unary constraint A_j is a better approximation of $\rho \downarrow \{x_j\}$. Since the variables range on finite domains, F-PC-2 is of complexity $O(d^5 n^3 p)$. Like for F-AC3, some heuristics on the ordering of nodes may be used.

F-PC-2 ((R_1, \dots, R_m)):

<number all the nodes from 1 to n >

$Q \leftarrow \{(i, k, j) / i \leq j, \text{not}(i=j=k)\}$

While $Q \neq \emptyset$ do:

<choose and remove (i, k, j) in Q >

Modified := false

$R'_{ij} := R_{ij} \otimes [R_{ik} \otimes A_k \otimes R_{kj}] \downarrow \{x_i, x_j\}$

If $\text{height}(R'_{ij}) = 0$, stop /* $\text{Cons}(P)=0$ */

else, if $R'_{ij} \neq R_{ij}$ do $R_{ij} := R'_{ij}$ Modified := true.

If Modified, then $Q \leftarrow Q \cup \text{related-paths}(i, k, j)$.

Related-paths (i, k, j) :

if $i < j$, $\{(i, j, m) / i \leq m, m \neq j\} \cup \{(m, i, j) / m \leq j, m \neq i\} \cup$
 $\{(j, i, m) / j < m\} \cup \{(m, j, i) / m < i\}$

else if $i = j$ $\{(p, i, m) / p \leq m, \text{not}(p=i=m), \text{not}(p=m=k)\}$

Note that the updating pattern $R'_{ij} := R_{ij} \otimes [R_{ik} \otimes A_k \otimes R_{kj}] \downarrow \{x_i, x_j\}$ encodes modus ponens, modus tollens and a kind of resolution principle (i.e. involving only three variables). Indeed, if $i = j$, the updating pattern becomes $A'_i := A_i \otimes [R_{ik} \otimes A_k] \downarrow \{x_i\}$, which is nothing but the updating pattern of arc-consistency which encodes both modus ponens and modus tollens. Moreover, suppose that p, q and r are three Boolean variables restricted by the following constraints:

$A_p: p \in \{t, f\}$; $A_q: q \in \{t, f\}$; $A_r: r \in \{t, f\}$

$R_{pq}: "p \text{ or } \neg q \text{ (priority a)"} \text{ represented by } \mu_{R_{pq}}(p, q) = 1 - a \text{ if } p = f \text{ and } q = t, \mu_{R_{pq}}(p, q) = 1 \text{ otherwise;}$

$R_{qr}: "q \text{ or } r \text{ (priority b)"} \text{ represented by } \mu_{R_{qr}}(q, r) = 1 - b \text{ if } q = f \text{ and } r = f, \mu_{R_{qr}}(q, r) = 1 \text{ otherwise;}$

$[R_{ik} \otimes A_k \otimes R_{kj}] \downarrow \{x_i, x_j\}$ is the fuzzy restriction corresponding to the fuzzy set $\mu_{R_{pr}}(p, r) = \max(1 - a, 1 - b)$ if $p = f$ and $r = f$, $\mu_{R_{pr}}(p, r) = 1$ otherwise. In other words, path-consistency computes the pattern of resolution of possibilistic logic: $(p \vee \neg q \ a); (q \vee r \ b) \vdash (p \vee r \ \min(a, b))$, a, b being lower bounds of necessity degrees.

5. FINDING THE BEST SOLUTION

The previous algorithms calculate approximations of $\rho \downarrow \{x_j\}$ which can be exact under some structural conditions. If the approximation is exact, the minimal height of all the constraints is equal to the consistency of the problem: $\text{Cons}(P) = \text{height}(\rho)$. Otherwise, it is only an upper bound of $\text{Cons}(P)$ and computing the problem consistency requires the determination of a best solution, i.e. an instantiation (u_1, \dots, u_n) such as $\mu_\rho(u_1, \dots, u_n) = \text{height}(\rho)$. In fact, the problem of computing $\text{Cons}(P)$ can be formulated as a sup/min optimisation problem (some R_i may be unary):

$$\sup_{(u_1, \dots, u_n)} \min_{R_i \in \{R_1, \dots, R_m\}} \mu_{R_i}((u_1, \dots, u_n) \downarrow \{x_{i1}, \dots, x_{ik_i}\})$$

Following Lang [9] or Schiex [15], such a problem may be solved in the finite case using classical tree-search algorithms, as Depth first, Branch&Bound, α - β , SSS*, beam search,...

Using a classical tree search algorithm, variables are instantiated in a prescribed order say (x_1, \dots, x_n) . The root of the tree is the empty assignment. Intermediary nodes denote partial instantiations. Leaves represent complete instantiations of (x_1, \dots, x_n) , i.e. potential solutions. Following a depth-first exploration of the tree, we search for the leaves that maximize μ_ρ .

The use of flexible constraints makes it possible to prune each branch that will necessarily lead to complete instantiations whose degree of satisfaction is worse than the one of the best already evaluated solutions. In other terms, it

is useless to extend intermediary nodes (u_1, \dots, u_k) such that $\mu_P \downarrow \{x_1, \dots, x_k\}(u_1, \dots, u_k) \leq \alpha$, α being the current lower bound of $\text{Cons}(P)$. Threshold α may be initialized by 0 and updated when a solution (u_1, \dots, u_n) such as $\mu_P(u_1, \dots, u_n) > \alpha$ is reached.

The computation of $\mu_P \downarrow \{x_1, \dots, x_k\}(u_1, \dots, u_k)$ is too costly (since it requires the computation of all the extensions of (u_1, \dots, u_k) to $\{x_1, \dots, x_k\}$). So in practice, we compute an upper bound of $\mu_P \downarrow \{x_1, \dots, x_k\}(u_1, \dots, u_k)$ instead. We call this upper bound the "consistency" of the partial instantiation (u_1, \dots, u_k) . The consistency of (u_1, \dots, u_k) only considers the restrictions pertaining to the variables instantiated in (u_1, \dots, u_k) , i.e. constraints R_i such that $V(R_i) \subseteq \{x_1, \dots, x_k\}$:

$$\text{Cons}(u_1, \dots, u_k) =$$

$$\min_{R_i \text{ s.t. } V(R_i) \subseteq \{x_1, \dots, x_k\}} \mu_{R_i}((u_1, \dots, u_k) \downarrow V(R_i))$$

This bound decreases when extending the nodes and becomes exact for complete instantiation. Moreover, it may be incrementally computed as the tree is explored downward:

$$\text{Cons}(u_1, \dots, u_k, u_{k+1}) = \min(\text{Cons}(u_1, \dots, u_k), \min_{\{R_i/x_{k+1} \in V(R_i)\}} \mu_{R_i}((u_1, \dots, u_k, u_{k+1}) \downarrow V(R_i)))$$

$$\text{and } V(R_i) \subseteq \{x_1, \dots, x_{k+1}\} \} \mu_{R_i}((u_1, \dots, u_{k+1}) \downarrow V(R_i)))$$

It is interesting to notice that the incremental computation of $\text{Cons}(u_1, \dots, u_n)$ considers each constraint only once.

Let's (u_1, \dots, u_k) denotes the current partial instantiation, α the current lower bound of $\text{Cons}(P)$. For all u_{k+1} , $\text{Cons}(u_1, \dots, u_k, u_{k+1})$ is computed. Only the nodes $(u_1, \dots, u_k, u_{k+1})$ such that $\text{Cons}(u_1, \dots, u_k, u_{k+1}) \geq \alpha$ may be created, and the next to be developed is the one such that $\text{Cons}(u_1, \dots, u_k)$ is maximal.

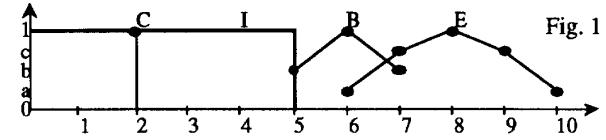
The search stops when no more node can be created. It is successful if a complete instance has been reached, and the best among those which have been reached is optimal ($\alpha = \text{Cons}(P)$). If no complete instance has been reached, $\text{Cons}(P) = 0$ and the problem is unfeasible.

It is possible to stop the search as soon as a solution of a degree greater than or equal to some threshold β is obtained. Therefore, if a satisfaction level b is considered as sufficient for the solution you look for, fix $\beta = b$. Moreover, some heuristics may be applied to this tree search, namely on the order of instantiation. In fact, it is possible to develop a large class of tree search algorithms (e.g., beam search) based on the same principle and integrating different heuristics or variants.

6. EXAMPLE

A tutorial is supposed to be made of lectures on topics P and Q, plus training sessions and computer sessions for topic P. Each type of activity must not involve more than 5 sessions. There should be approximately 8 lectures overall, and the person in charge of lectures and computer sessions on topic P proposes about 6 sessions overall. There should be exactly two more computer sessions than exercise sessions, and exactly as many lectures on P as exercise sessions. The FCSP is defined in Figure 1. A 5-level satisfaction scale is used $L = \{0, a, b, c, 1\}$. Representing fuzzy sets by pairs

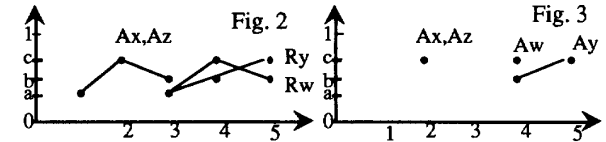
$(u, \mu_F(u))$, we have the following representations of 'about 8' (E) and 'about 6' (B), and 'exactly 2' (C): $E = \{(6, a), (7, c), (8, 1), (9, c), (10, a)\}$; $B = \{(5, b), (6, 1), (7, b)\}$; $C = \{(2, 1)\}$.



With the following understanding of the variables, x (resp. y): number of lectures on P (resp. Q); w : number of computer sessions on P; z : number of exercise sessions, the constraints write $Ax: x \in I$; $Ay: y \in I$; $Az: z \in I$; $Aw: w \in I$; $Rxy: x + y \in E$; $Rxw: x + w \in B$; $Rwz: w - z \in C$; $Rxz: x = z$.

Arc consistency performed on x via Rxy leads to compute $\mu_{Rxy \rightarrow x}(u) = \max_{y=1,5} \mu_E(u + y)$, $\forall u$ which gives $I \cap Rxy \rightarrow x = \{(1, a), (2, c), (3, 1), (4, 1), (5, 1)\} = A'_x$. Clearly more precision has been obtained on x since only 3, 4, 5 remain totally possible values. Note that the computation of $Rxy \rightarrow x$ could be done via fuzzy arithmetics: the restriction $Rxy \rightarrow x = (Ry \otimes Rxy) \downarrow \{x\}$ induced on x is in fact: $x \in E \ominus I$, \ominus denoting the subtraction of fuzzy quantities.

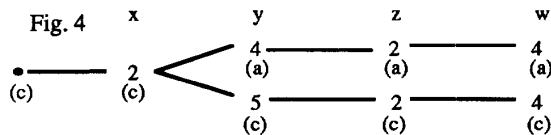
Applying FAC-3 to this example (see Figure 2), we discover that the height of each unary restriction is c , i.e. that constraints are conflicting. Note that the unary restrictions are only upper approximations of the marginals of the global relation. For instance, $x = 1$ is allowed with degree b , although there is in fact no assignment of x, y, z, w with $x = 1$ satisfying all the constraints even partially.



Consider now arc+path consistency instead of arc-consistency only. Rxz states that $x = z$. The path x, w, z states that $Rxw: x + w \in B$ and $Rwz: w - z \in C$. Combining these restrictions, we obtain the induced restriction $(Rxw \otimes Rwz \otimes Aw) \downarrow \{xz\}$. Because of Aw , this restriction is more restrictive than $x + z \in B \ominus C = (Rxw \otimes Rwz) \downarrow \{xz\}$. In this example, path consistency performs nothing but variable elimination. Intersecting this restriction with the set Rxz of possible pairs of values for x and z , we get $R'xz: x = z = 2$. The new restriction is then propagated on x by arc-consistency: $\mu_{R'x}(2) = c$, $\mu_{R'x}(x) = 0$ otherwise (notice that value $x = 1$ is ruled out). Applying F-PC2 instead of FAC-3 to the example (see Figure 3), the restrictions Rx, Ry , and Rz are tighter. Moreover, they are equal to the marginals of the global relation.

The search algorithm is then applied, with $\beta = c$: we take advantage of the results of FPC2 to enhance the search giving an upper-bound of $\text{Cons}(P)$. 8 nodes are generated (see Fig. 4) and the best solution is: $x = 2, y = 5, z = 2, w = 4$. In fact, there is no assignment satisfying completely all the constraints; thus, constraint Rxy is "relaxed" according to the

preferences: one accepts to reduce the number of lectures to 7 since no plan with 8 lectures is feasible.



Notice that if the filtering algorithm is not applied before the search, reaching the best solution requires 18 nodes. The first reason is that the upper bound of $\text{Cons}(P)$ provided by FPC2 allows the search to stop immediately after the discovery of the solution. Moreover, the problem obtained by FPC2 contains more explicit constraints than the original one; thus, the consistency of a partial instantiation is a better upper bound of the satisfaction level of its best extension: more useless branches are pruned.

Our example involves imperative constraints only. Consider a problem where R_{xz} has now a priority $\alpha_{R_{xz}} = d \leq 1 - c$. This conflicting FCSP may be solved by relaxing R_{xz} : the consistency of the problem is $1 - d$ and a solution satisfying all the constraints but R_{xz} is got ($x = 3, y = 5, z = 1, w = 3$).

7. CONCLUSION

Flexible constraints have two advantages over crisp ones: the different solutions are rank-ordered in accordance to their degrees of feasibility and the search algorithm takes advantage of this flexibility to efficiently focus on the best solutions. Moreover, the FCSP approach may handle partial inconsistencies between constraints and provides a solution (the best one) as long as the problem is not totally inconsistent.

Thus the FCSP framework seems to be a suitable approach to handle flexible constraints, i.e. constraints involving preferences as well as prioritized constraints - and more generally problems involving fuzzy relations. It marries possibility theory and classical CSP theory. Possibility theory offers its rich and powerful setting for the representation of soft and prioritized constraints; the classical operations of combination and projection of fuzzy restrictions present a low computational cost, and properties (especially idempotence) allow the use of local computational schemes on general structures. Moreover, the CSP literature provides theoretical results as well as efficient algorithms which can be easily adapted to the treatment of fuzzy restrictions.

In terms of applications, FCSP seems particularly promising in areas such as scheduling, planning, design which often make use of classical CSP techniques mixed with informal approaches to flexibility (see for instance [3]). Note that the framework we have described only handles variables ranging on discrete domains; in order to handle some of these applications, it must be extended to continuous domains.

Lastly, the FCSP approach may be regarded as a problem solving paradigm for every application involving fuzzy restrictions, e.g. problems of combination of vague information, fuzzy relational databases, or qualitative reasoning.

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