Norm inflation for solutions of semi-linear one dimensional Klein-Gordon equations

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Abstract

In space dimension larger or equal to two, the non-linear Klein-Gordon equation with small, smooth, decaying initial data has global in time solutions. This no longer holds true in one space dimension, where examples of blowing up solutions are known. On the other hand, it has been proved that if the nonlinearity satisfies a convenient compatibility condition, the "null condition", one recovers global existence and that the solutions satisfy the same dispersive bounds as linear solutions. The goal of this paper is to show that, in the case of cubic semi-linear nonlinearities, this null condition is optimal, in the sense that, when it does not hold, one may construct small, smooth, decaying initial data giving rise to solutions that display inflation of their L^∞ and L^2 norms in finite time.

0 Introduction

It is well-known that quasi-linear Klein-Gordon equations with smooth, small, decaying initial data have global in time solutions, in space dimension larger or equal to 3, as it has been proved independently by Klainerman [21] and Shatah [28]. The same holds true in 2 space dimension, according to Simon-Taflin [29] and Ozawa, Tsutaya and Tsutsumi [27]. On the other hand, in one space dimension, finite time blow-up may occur. Examples of non-linearities for which this happens have been obtained by Yordanov [32] and Keel and Tao [18]. In [10], we introduced for a general quasi-linear non-linearity a "null condition", expressed explicitly in terms of the quadratic and cubic parts of the nonlinearity, and we conjectured that, under that null condition, small data that are smooth and have some decay at infinity should give rise to global solutions. We showed in [11, 12] that this conjecture holds true for C_0^∞ initial data. We refer to Lindblad and Soffer [23, 24, 25] for nonlinearities depending only on u, to Hayashi and Naumkin [17] and to Stingo [31] for more general data, and to the bibliography of [11] for references about the state of the art at the time of publication of that paper.
The goal of the present paper is to show that, in the case of cubic semi-linear nonlinearities, i.e. for the equation

$$(\partial_t^2 - \partial_x^2 + 1)u = P(u, \partial_t u, \partial_x u) \tag{1}$$

where P is a polynomial homogeneous of degree three, our null condition is optimal, in that sense that if it is not satisfied, one can construct solutions, with small and decaying initial data, that

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do not enjoy the same dispersive bounds as the ones that hold true for linear solutions (or non-linear global solutions when the null-condition is satisfied). More precisely, the null condition was obtained in [10] extracting from the PDE an ODE which has global solutions for small data if and only if the null condition holds. When this is the case, the asymptotics of the solution of this ODE give the asymptotic behavior of the global solution of the PDE. When the null condition is *not* satisfied, this ODE blows-up at some finite time, depending on the parameter $y = \frac{x}{t} \in]-1, 1[$. The minimal blow-up time for y describing $]-1, 1[$ is of the form $e^{\frac{T_*}{\epsilon^2}}$ for some $T_* > 0$ (when blowing-up occurs in the future), $\epsilon \ll 1$ being the size of the initial condition. In [10] it was shown that the solution exists and has L^∞ norm at time t which is $O(\frac{\epsilon}{\sqrt{t}})$ for $t < e^{\frac{A}{\epsilon^2}}$, for any constant $A < T_*$. The main result of this paper (see Theorem 1.2.1 below) asserts that one may construct initial data so that for $t = T(\epsilon)$ close enough to $e^{\frac{T_*}{\epsilon^2}}$, one has inflation of norms in the sense that

$$\sqrt{T(\epsilon)}(\|u(T(\epsilon))\|_{L^\infty} + \|\partial_t u(T(\epsilon))\|_{L^\infty}) \geq cT(\epsilon)^{\frac{1}{2}-c} \sim e^{\frac{c'}{\epsilon^2}}, \epsilon \to 0$$

for positive constants c, c'. In other words, the solution is still small at time $T(\epsilon)$, but exponentially large when compared to the size of linear solutions. Of course, this norm inflation result does not mean that the solution does blow-up, but we explain in the remarks that follow the statement of Theorem 1.2.1 that this is the best we may expect, if we want to single out a property of the solution that follows *only* from the violation of the null condition, and that is in contrast with the kind of estimates that hold true under the null condition.

The proof of the main theorem relies on the construction of an approximate blowing-up solution, that was inspired to us by the papers of Cazenave, Martel and Zhao [6] and Cazenave, Han and Martel [5]. In these references, the authors construct blowing-up solutions for Schrödinger equations of the form

$$(i\partial_t - \partial_x^2)u = \alpha|u|^2 u, \alpha \in \mathbb{C} - \mathbb{R}. \tag{2}$$

(Actually, their result is not limited to one space dimension nor to cubic nonlinearities). They first look for an approximate solution given in terms of a profile that satisfies some ODE and blows-up at time $t = 1$. Next they write the equation satisfied by the difference between this approximate solution and the exact one. They prove that this equation has a global backwards solution with zero initial condition at (or close to) the blow-up time. The sum of this solution and of the approximate one brings thus an exact solution to (2) that blows-up at time $t = 1$. See also Liu and Zhang [26] and for blowing-up solutions of Schrödinger equations with small data, the preprint of Kita [19].
Our general strategy is the same, except that we have to cope with some difficulties inherent to the Klein-Gordon equation. To describe it, let us write equation (1) as a first order system on $(u_+, \bar{u}_+)$, where u_+ is a new complex valued unknown deduced from u, with first equation

$$\begin{aligned}(D_t - \sqrt{1 + D_x^2})u_+ = M^{(1)}(u_+, u_+, u_+) + M^{(2)}(u_+, u_+, \bar{u}_+) \\ + M^{(3)}(u_+, \bar{u}_+, \bar{u}_+) + M^{(4)}(\bar{u}_+, \bar{u}_+, \bar{u}_+),\end{aligned} \tag{3}$$

$M^{(j)}$ being non local expression of their arguments homogeneous of degree 3. The difference with (2) comes from $M^{(1)}, M^{(3)}, M^{(4)}$ which are not invariant under $u_+ \to zu_+$ for $z \in U(1)$. On the other hand these terms are "non characteristic" ones, since when computed on a linear solution, they oscillate along a non characteristic phase for the linear part of (3). Our proof has thus two steps, as in [6, 5]. First, we construct an approximate solution starting from small initial data $(\epsilon f_0, \epsilon g_0)$ with f_0, g_0 in $\mathcal{S}(\mathbb{R})$. If the null condition is not satisfied, choosing f_0, g_0 conveniently,

we had constructed in [10] an approximate solution defined on some interval $[1, e^{\frac{T_*}{\epsilon^2}}[$ as

$$u^1_{\rm app}(t,x) = 2{\rm Re}\Big[\frac{\epsilon}{\sqrt{t}}a_{1,1}(\epsilon^2\log t, \frac{x}{t})e^{i\sqrt{t^2-x^2}} + \frac{\epsilon^3}{t^{\frac{3}{2}}}a_{3,3}(\epsilon^2\log t, \frac{x}{t}, \epsilon)e^{3i\sqrt{t^2-x^2}} + O(\epsilon t^{-\frac{5}{2}})\Big], \tag{4}$$

where $a_{1,1}(s,y), a_{3,3}(s,y,\epsilon)$ are functions supported for $|y| \leq 1$, smooth in (s,y) for $s < T_*$. As a consequence of the violation of the null condition, one may construct $a_{1,1}(s,y), a_{3,3}(s,y,\epsilon)$ that blow-up if $s \to T_*-$, so that (4) provides a useful approximate solution only for $t < e^{\frac{A}{\epsilon^2}}$ with $A < T_*$. If one wants to study what happens for t close to $e^{\frac{T_*}{\epsilon^2}}$, one has to construct a more accurate approximate solution, gluing (4) for say $t < e^{\frac{3T_*}{4\epsilon^2}}$ to another approximate solution, defined on $e^{\frac{T_*}{2\epsilon^2}} < t < e^{\frac{T_*}{\epsilon^2}}$, given by an Ansatz of the form

$$\begin{aligned} u^2_{\rm app}(t,x) = 2{\rm Re}\Big[&\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{N} \epsilon^{2-\ell} t^{-\frac{\ell}{2}} e^{i\sqrt{t^2-x^2}} a_{\ell,1}(\epsilon^2\log t, \frac{x}{t}, \epsilon) \\ &+ \sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N} \sum_{\substack{3\leq q\leq \ell\\ q \text{ odd}}} \epsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{iq\sqrt{t^2-x^2}} a_{\ell,q}(\epsilon^2\log t, \frac{x}{t}, \epsilon)\Big], \end{aligned} \tag{5}$$

where $a_{\ell,q}(s,y,\epsilon)$ are functions that blow-up at $s = T_*$ like $(T_*-s)^{-\frac{\ell}{2}-0}$. If $s = \epsilon^2\log t$ is close to T_*, the $a_{\ell,q}$ terms in the two sums in (5) are thus larger and larger, so that (5) cannot provide an approximate solution. But we may exploit the dispersive decay factor $t^{-\frac{\ell}{2}}$ and limit ourselves to times $t < T(\epsilon)$, where $T(\epsilon)$ is such that $T(\epsilon)^{-1}(T_* - \epsilon^2\log T(\epsilon))^{-1} \ll 1$. Under this restriction, (5) provides a function satisfying (1) up to a small remainder. Moreover, $T(\epsilon)$ is close enough to $e^{\frac{T_*}{\epsilon^2}}$ so that $u^2_{\rm app}(T(\epsilon),x)\sqrt{T(\epsilon)}$ will be large (actually of size $e^{\frac{c'}{\epsilon^2}}$) in L^∞.

The second step of the proof is to look for an exact solution $u(t,x) = u_{\rm app}(t,x) + r(t,x)$, where $u_{\rm app}$ is the approximate solution obtained gluing together $u^1_{\rm app}$ and $u^2_{\rm app}$ above, and r a remainder that will be zero as well as its time derivative at $t = T(\epsilon)$. Then r solves the backwards equation with force term deduced from (1) replacing u by $u_{\rm app} + r$. One has to show that if the approximate solution has been constructed in an accurate enough way, the remainder r exists down to time $t \sim 1$ and that at this initial time, it perturbs the initial condition $(\epsilon f_0, \epsilon g_0)$ used to construct the approximate solution only at order $o(\epsilon)$. The general strategy employed to prove such properties is to use the methods that are useful in the study of global existence (normal forms, energy estimates for the action of $x \pm t\frac{x}{\langle x\rangle}$ on the solution of the reduced system obtained by normal forms for the remainder). A difference with problems of global existence is that the equation satisfied by the remainder contains linear terms (coming from the linearization on the approximate solution). The coefficients of these linear term being expressions on the approximate solution, they are relatively large close to $T(\epsilon)$, and thus cannot be treated as perturbations. In order to overcome this difficulty, we use an idea of Cazenave, Han and Martel [5]: we remark that in a Gronwall inequality, the growth of the amplifying factor coming from this large coefficient is more than compensated by the fact that the source term against which it is integrated – that comes from the error in the equation applied to the approximate solution – may be made as small as we want.

The plan of the paper is as follows: in section 1, we recall the definition of the null condition and state the main theorem. Section 2 is devoted to the construction of the approximate solution. In section 3, we study the remainder given by the difference between the exact and the approximate solution. We express it as a solution of a 2×2-system with source term, and obtain energy estimates for the Sobolev norm of the remainder and for the L^2 norm of the action of $L_+ = x + t\frac{D_x}{\langle D_x\rangle}$ on it. Finally, in section 4, we conclude the proof using a bootstrap argument and a

Klainerman-Sobolev estimate to control L^∞ norms. The appendix is devoted so some technical results used in the proof.

To conclude this introduction, let us give some references to other works concerning the construction of blowing-up solutions for non-linear *wave* equations instead of Klein-Gordon ones. In the quasi-linear case, recall that in three space dimension, the null condition has been introduced by Christodoulou [9] and by Klainerman [20, 22] who proved that global existence with small decaying initial data holds true under that assumption. In two space dimensions, Alinhac [3] defined the (more complicated) corresponding version of the null condition and proved also global existence when it holds.
When the null condition is *not* satisfied, the study of blowing-up solutions and of their asymptotic behavior had been undertaken by Alinhac in a series of papers [2, 1, 4]. For more recent references on that and further results, we refer to the book of Speck [30] and especially its preface and introduction. We notice also that the situation considered in all these papers is pretty different from the one we encounter in the present work, as in these quasi-linear models, the singularities that form are of shock type, i.e. the quantities that blow-up are *second* order derivatives, while in our setting, the function itself (or its time derivative) will display norm inflation. For the construction of blowing-up solutions for semi-linear wave equations with a nonlinearity depending only on the function itself, and not on its derivatives, we refer to the papers of Cazenave, Martel and Zhao [8, 7] and their bibliographies.

1 Statement of the main theorem

1.1 Semi-linear Klein-Gordon equation and null condition

We consider the cubic semi-linear Klein-Gordon equation in one space dimension

$$(\partial_t^2 - \partial_x^2 + 1)u = P(u, \partial_t u, \partial_x u) \tag{1.1.1}$$

where P is a polynomial homogeneous of degree 3, with real coefficients, that we write under the form

$$P(u, \partial_t u, \partial_x u) = \sum_{k=0}^{3} P_k(u; \partial_t u, \partial_x u) \tag{1.1.2}$$

where $P_k(T; Z_1, Z_2)$ is homogeneous of degree k in (Z_1, Z_2) and $3-k$ in T, with real coefficients. We define for $y \in]-1, 1[$

$$\omega_0(y) = \frac{1}{\sqrt{1-y^2}},\ \omega_1(y) = -\frac{y}{\sqrt{1-y^2}} \tag{1.1.3}$$

and we set

$$\begin{aligned} p_k(\omega_0(y), \omega_1(y)) &= P_k(1; \omega_0(y), \omega_1(y)) \\ \phi(y) &= (p_1 + 3p_3)(\omega_0(y), \omega_1(y)) \\ \psi(y) &= -(3p_0 + p_2)(\omega_0(y), \omega_1(y)). \end{aligned} \tag{1.1.4}$$

We recall the following definition from [10]:

Definition 1.1.1 *One says that the nonlinearity in (1.1.1) satisfies the null condition if* $\phi \equiv 0$.

Assume that the null condition is satisfied and take in (1.1.1) initial conditions of the form $u(1,x) = \epsilon f(x)$, $\partial_t u(1,x) = \epsilon g(x)$ with $f, g \in C_0^\infty(\mathbb{R})$. Then, it has been proved in [11, 12] (see also Stingo [31]), including in the case of quasi-linear equations with quadratic and cubic nonlinearities (for which one has to modify the expression of ϕ in (1.1.4)) that, if the null condition is satisfied, for $\epsilon > 0$ small enough, the solution to (1.1.1) is globally defined for $t \geq 1$ and satisfies L^∞ bounds of the form $\|\partial_x^k u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}})$ when t goes to $+\infty$. The solution thus decays like a solution of the linear Klein-Gordon equation in one space dimension. Of course, a similar statement holds when t goes to $-\infty$. On the other hand, it was proved as well that scattering does not hold (one has only modified scattering).
We are interested here in the case when the null condition is *not* satisfied, and we want to construct initial data that generate inflation of the norms of the solution in finite time i.e. we want to show for instance that the L^∞ norm will *not* satisfy the dispersive bounds that hold true under the null condition. Consequently, in order to ensure that the null condition does not hold, we assume

$$\sup_{y\in]-1,1[} \phi(y) > 0. \tag{1.1.5}$$

This will allow us to construct solutions that display norm inflation at some positive time. If in (1.1.5) ϕ was replaced by $-\phi$, we would in the same way get inflation of the norms at some negative time.

1.2 Main theorem and norm inflation

Let f_0, g_0 be two real valued functions in $\mathcal{S}(\mathbb{R})$. We associate to them

$$\Gamma(y) = \frac{1}{8\pi}(1-y^2)^{-1}\big|\hat{f}_0(\omega_1(y)) - i\sqrt{1-y^2}\hat{g}_0(\omega_1(y))\big|^2 \tag{1.2.1}$$

which is a smooth function on $]-1,1[$ that, extended by zero outside this interval, gives a smooth function on $\mathbb{R}$. (This function was introduced in [10], formula (1.18), but the expression given there is correct only if f_0, g_0 satisfy some evenness or oddness conditions. In general, the correct expression is (1.2.1)). By (1.1.5), we may choose f_0, g_0 in $\mathcal{S}(\mathbb{R})$ such that $\sup_{y\in]-1,1[}(\Gamma(y)\phi(y))$ is positive, and we define $T_* > 0$ by

$$\frac{1}{T_*} = \sup_{y\in]-1,1[}(\Gamma(y)\phi(y)). \tag{1.2.2}$$

As $\Gamma(y)$ vanishes at infinite order at $y = \pm 1$, and ϕ grows at most polynomially at these points, the supremum is reached at some points in $]-1,1[$. We shall assume

$$\begin{aligned} &y \to \Gamma(y)\phi(y) \text{ reaches its maximum at a unique point } y_0 \in]-1,1[. \\ &\text{Moreover, there is } \kappa_0 \in \mathbb{N}^* \text{ such that } \partial_y^\alpha(\Gamma(y)\phi(y))|_{y=y_0} = 0 \\ &\text{for } \alpha = 0,\dots,2\kappa_0-1 \text{ and } \partial_y^{2\kappa_0}(\Gamma(y)\phi(y))|_{y=y_0} < 0. \end{aligned} \tag{1.2.3}$$

On course, one may always choose functions $\hat{f}_0, \hat{g}_0$ in $\mathcal{S}(\mathbb{R})$ such that (1.2.3) holds, because of (1.1.5).
Let $\gamma > 0$, $\delta' > 0$ be fixed positive numbers. For $\epsilon > 0$ small, define

$$\epsilon' = \epsilon^{-\frac{2+\gamma+2\delta'}{1+2\delta'}} \exp\Big(-\frac{T_*}{\epsilon^2(1+2\delta')}\Big) \ll 1. \tag{1.2.4}$$

Let $u(\epsilon')$ be the unique small solution satisfying $u(0) = 0$ of the equation

$$u = \epsilon' \exp\Big(\frac{u}{1+2\delta'}\Big)$$

so that $u(\epsilon') = \epsilon' + O(\epsilon'^2)$, $\epsilon' \to 0$. We define

$$T(\epsilon) = \exp\Big(\frac{T_*}{\epsilon^2} - u(\epsilon')\Big) = e^{\frac{T_*}{\epsilon^2}}(1 - \epsilon' + O(\epsilon'^2)), \epsilon' \to 0. \tag{1.2.5}$$

Our main theorem is the following one:

Theorem 1.2.1 *Let $f_0, g_0 \in \mathcal{S}(\mathbb{R})$ be given such that assumption (1.2.3) holds. Let $c > 0, \theta > 0$ be given small numbers. Let $s_0 \in \mathbb{N}$ be a large enough integer. There is $\delta_0' > 0$ and for any $\delta' \in]0, \delta_0']$, any $\gamma \geq 2(\delta' + 2)$, there are $\epsilon_0 > 0$, $C > 0$ such that for any $\epsilon \in]0, \epsilon_0[$, there are functions $x \to (f(x,\epsilon), g(x,\epsilon))$ in $H^{s_0+1}(\mathbb{R}) \times H^{s_0}(\mathbb{R})$, small in the sense that*

$$\begin{aligned} \|f(\cdot,\epsilon)\|_{H^{s_0+1}} + \|g(\cdot,\epsilon)\|_{H^{s_0}} &\leq C\epsilon^{1-\theta} \\ \|xf(\cdot,\epsilon)\|_{H^1} + \|xg(\cdot,\epsilon)\|_{L^2} &\leq C\epsilon^{1-\theta}, \end{aligned} \tag{1.2.6}$$

so that the unique solution u of (1.1.1) with initial data

$$u(1,x) = \epsilon(f_0(x) + f(x,\epsilon)),\ \partial_t u(1,x) = \epsilon(g_0(x) + g(x,\epsilon)) \tag{1.2.7}$$

is defined for $t \in [1, T(\epsilon)]$ and satisfies

$$\begin{aligned} \|u(T(\epsilon),\cdot)\|_{L^\infty} + \|\partial_t u(T(\epsilon),\cdot)\|_{L^\infty} &= \frac{\epsilon}{\sqrt{T(\epsilon)}} I(\epsilon) \\ \|u(T(\epsilon),\cdot)\|_{L^2} + \|\partial_t u(T(\epsilon),\cdot)\|_{L^2} &= \epsilon J(\epsilon), \end{aligned} \tag{1.2.8}$$

where

$$I(\epsilon) \geq cT(\epsilon)^{\frac{1}{2}-c},\ J(\epsilon) \geq cT(\epsilon)^{\frac{1}{2}-\frac{1}{4\kappa_0}-c}. \tag{1.2.9}$$

Remarks: • By (1.2.5), $T(\epsilon)$ is exponentially large when $\epsilon \to 0+$. Then (1.2.8) and the first inequality (1.2.9) show that one has inflation of the estimate of the L^∞ norm of the solution by a factor $I(\epsilon)$ in comparison with the $O(\epsilon/\sqrt{T(\epsilon)})$ bound that holds when the null condition is satisfied. In the same way, if $\kappa_0 \geq 2$, (1.2.8) and the lower bound for $J(\epsilon)$ in (1.2.9) imply inflation of the L^2 norms in comparison with the $O(T(\epsilon)^\alpha)$ ($\alpha > 0$ arbitrary) bound that holds under the null condition.

• The solution u will be written as the sum of an approximate solution and of a remainder. The lower bounds (1.2.8) are those of this approximate solution (constructed from f_0, g_0) at time $T(\epsilon)$.

• The exact solution will be given by the sum of the approximate solution and of an error obtained solving a backwards Klein-Gordon equation with zero data at $t = T(\epsilon)$ and source term determined by the approximate solution. This error generates in the initial conditions (1.2.7) the $O(\epsilon^{2-\theta})$ perturbation of $(\epsilon f_0, \epsilon g_0)$.

• As mentioned in the introduction, our method of proof is inspired by the construction of blowing-up solutions for nonlinear Schrödinger equations by Cazenave, Martel and Zhao [6] and Cazenave, Han and Martel [5]. For Klein-Gordon equations that do not satisfy the null condition, we cannot expect to get in general blowing-up solutions, but only norm inflation. Actually, the null condition provides a global existence criterium only in the framework of *small data*: in order to uncover it, one has to make some reductions (through normal forms) in order to eliminate some non-characteristic contributions to the nonlinearity. These reductions bring new terms in the nonlinearity, vanishing at order five at the origin. As long as data are small, these quintic corrections are negligible, but they could play a prominent role for larger solutions. As a toy example, consider the ODE $\dot{y} = \frac{1}{2}y^3$, with data $y(0) = \epsilon$, whose solution $y(t) = \frac{\epsilon}{\sqrt{1-t\epsilon^2}}$ blows-up

at time $t = \frac{1}{\epsilon^2}$. The perturbed equation $\dot{y} = \frac{1}{2}y^3(1-y^2)$ with the same initial condition has solutions that are globally defined for $t \geq 0$. If we set

$$a(\epsilon) = \Big(1 - \epsilon^2 \log \frac{\epsilon^2}{1-\epsilon^2}\Big)^{-\frac{1}{2}}$$

the solution satisfies

$$y(t)\Big(1 - y(t)^2 \log \frac{y(t)^2}{1-y(t)^2}\Big)^{-\frac{1}{2}} = \frac{\epsilon a(\epsilon)}{(1 - t\epsilon^2 a(\epsilon)^2)^{\frac{1}{2}}}. \tag{1.2.10}$$

At time $t_\epsilon = \epsilon^{-2}a(\epsilon)^{-2}(1-\epsilon^{2-2\delta})$ with $\delta > 0$ small, we deduce from (1.2.10) that $y(t_\epsilon)$ will be of size essentially ϵ^δ, much larger than the size ϵ of the initial data (though still small). This is the same phenomenon as the one that happens in the theorem.

2 Construction of approximate solution

2.1 Construction for moderate time

Our first goal is to construct an approximate solution for equation (1.1.1) with initial condition

$$u(1,\epsilon) = \epsilon f_0(x),\ \partial_t u(1,\epsilon) = \epsilon g_0(x) \tag{2.1.1}$$

where (f_0, g_0) are functions in $\mathcal{S}(\mathbb{R})$ chosen so that (1.2.2) and (1.2.3) hold true. In this subsection, we construct the solution up to time $e^{\frac{3T_*}{4\epsilon^2}}$, following essentially [10]. We introduce the notation

$$p(\xi) = \sqrt{1+\xi^2},\ L_\pm = x \pm tp'(D_x). \tag{2.1.2}$$

We first take as an approximate solution over an interval $[1, \epsilon^{-1+\theta}]$, where $\theta > 0$ is small, the solution u_0 of the linear equation

$$\begin{aligned} &(\partial_t^2 - \partial_x^2 + 1)u_0 = 0 \\ &u_0(1,\cdot) = \epsilon f_0,\ \partial_t u_0(1,\cdot) = \epsilon g_0. \end{aligned} \tag{2.1.3}$$

Proposition 2.1.1 *Set*

$$r_0(t,x) = (\partial_t^2 - \partial_x^2 + 1)u_0 - P(u_0, \partial_t u_0, \partial_x u_0). \tag{2.1.4}$$

Then for any $s_0 \in \mathbb{N}$, $\theta > 0, c > 0$, there is $C > 0$ such that

$$\begin{aligned} &\int_1^{c\epsilon^{-1+\theta}} \|r_0(\tau,\cdot)\|_{H^{s_0}}\, d\tau \leq C\epsilon^{3-0} \\ &\int_1^{c\epsilon^{-1+\theta}} \|L_\pm r_0(\tau,\cdot)\|_{H^1}\, d\tau \leq C\epsilon^{2+\theta}. \end{aligned} \tag{2.1.5}$$

Moreover, if for $|y| < 1$, we denote

$$\varphi(y) = \sqrt{1-y^2}, \tag{2.1.6}$$

then for $t \geq 1$, we may write u_0 under the form

$$u_0(t,x) = 2\mathrm{Re}\Big[\frac{\epsilon}{\sqrt{t}} e^{it\varphi(x/t)}\Big(a_1^0\Big(\frac{x}{t}\Big) + \frac{1}{t}b_1^0\Big(\frac{x}{t}\Big) + \frac{1}{t^2}c_1^0\Big(t, \frac{x}{t}\Big)\Big)\Big] + \epsilon e(t,x) \tag{2.1.7}$$

where $a_1^0(y), b_1^0(y)$, (resp. $c_1^0(t,y)$) are smooth functions on $\mathbb{R}$ (resp. $[1,+\infty[\times\mathbb{R}$), supported for $|y|\leq 1$, with

$$a_1^0(y)=\frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi}}(1-y^2)^{-\frac{3}{4}}\big[\hat{f}_0(\omega_1(y))-i\sqrt{1-y^2}\hat{g}_0(\omega_1(y))\big] \tag{2.1.8}$$

for $|y|<1$, with c_1^0 satisfying for any α,β,N in $\mathbb{N}$

$$|\partial_t^\alpha\partial_y^\beta c_1^0(t,y)|\leq C_{\alpha,\beta,N}t^{-\alpha}(1-|y|)^N, \tag{2.1.9}$$

and where $e(t,x)$ is a real valued function in $\mathcal{S}([1,+\infty]\times\mathbb{R})$.

Proof: Expansion (2.1.7) is given in Proposition 2.1.1 of [10]. To get estimates (2.1.5), we just notice that $r_0(t,x)=-P(u_0,\partial_t u_0,\partial_x u_0)$ may be written according to (2.1.7) as the sum of an element of $\mathcal{S}([1,+\infty[\times\mathbb{R})$ that is $O(\epsilon^3)$ in that space, that trivially satisfies (2.1.5), and of expressions of the form

$$\frac{\epsilon^3}{t^{\frac{3}{2}}}e^{iqt\varphi(x/t)}c\Big(t,\frac{x}{t}\Big)$$

for some function c of the same form as c_1^0 in (2.1.9) and some q in $\mathbb{Z}$. Such terms satisfy (2.1.5). This concludes the proof □

Our next step is to construct the approximate solution for t up to $e^{\frac{3T_*}{4\epsilon^2}}$. We first introduce the solution $s\to a_{1,1}(s,y)$ of the differential equation

$$\begin{aligned}\omega_0(y)\partial_s a_{1,1}(s,y)&=\frac{1}{2}(\phi(y)+i\psi(y))|a_{1,1}(s,y)|^2a_{1,1}(s,y)\\ a_{1,1}(0,y)&=a_1^0(y),\end{aligned} \tag{2.1.10}$$

where $a_1^0(y)\in C_0^\infty(\mathbb{R})$ with support in $[-1,1]$ is defined in (2.1.8) and where $\omega_0(y),\phi(y),\psi(y)$ have been introduced in (1.1.3), (1.1.4). It follows from (2.1.10) that

$$\partial_s|a_{1,1}(s,y)|^2=\phi(y)\omega_0(y)^{-1}|a_{1,1}(s,y)|^4$$

when $|y|<1$, so that

$$|a_{1,1}(s,y)|^2=\frac{|a_1^0(y)|^2}{1-|a_1^0(y)|^2\phi(y)\sqrt{1-y^2}s}=\frac{\Gamma(y)\omega_0(y)}{1-\Gamma(y)\phi(y)s} \tag{2.1.11}$$

using the definition (2.1.8) of a_1^0 and notation (1.2.1). By (1.2.2), $a_{1,1}$ is thus defined for $s\in[1,T_*[$ and plugging (2.1.11) in (2.1.10), we get the explicit expression

$$a_{1,1}(s,y)=a_1^0(y)(1-\Gamma(y)\phi(y)s)^{-\frac{1}{2}}\exp\Big[-\frac{i}{2}\frac{\psi(y)}{\phi(y)}\log(1-\Gamma(y)\phi(y)s)\Big]. \tag{2.1.12}$$

In particular, $a_{1,1}$ is a smooth function of $(s,y)\in[0,T_*[\times]-1,1[$ that extended by zero for $|y|\geq 1$ is smooth on $[0,T_*[\times\mathbb{R}$, since a_1^0 and Γ are C^∞ on $\mathbb{R}$, supported in $[-1,1]$.
We shall construct an approximate solution of (1.1.1) defined for $t\in[c\epsilon^{-1+\theta},e^{\frac{3T_*}{4\epsilon^2}}]$ that will match with u_0 defined in Proposition 2.1.1.

Proposition 2.1.2 *There are, in addition to function $a_{1,1}$ introduced in (2.1.12), smooth functions $(s,y)\to a_{3,3}^1(s,y)$ (resp. $(s,y,\epsilon)\to a_{5,3}^1(s,y,\epsilon)$, (resp. $(s,y,\epsilon)\to a_{5,5}^1(s,y,\epsilon)$) defined on*

$[0,\frac{3T_*}{4}]\times\mathbb{R}$ *(resp. on* $[0,\frac{3T_*}{4}]\times\mathbb{R}\times[0,1]$*), supported for* $|y|\leq 1$*, such that if we define for* $c\epsilon^{-1+\theta}\leq t\leq e^{\frac{3T_*}{4\epsilon^2}}$

$$
\begin{aligned}
u^1_{\rm app}(t,x)=2{\rm Re}\,\Big[&\frac{\epsilon}{\sqrt{t}}a_{1,1}(\epsilon^2\log t,\frac{x}{t})e^{it\varphi(x/t)}\\
&+\frac{\epsilon^3}{t^{\frac{3}{2}}}a^1_{3,3}(\epsilon^2\log t,\frac{x}{t})e^{3it\varphi(x/t)}\\
&+\frac{\epsilon}{t^{\frac{5}{2}}}a^1_{5,3}(\epsilon^2\log t,\frac{x}{t},\epsilon)e^{3it\varphi(x/t)}\\
&+\frac{\epsilon^5}{t^{\frac{5}{2}}}a^1_{5,5}(\epsilon^2\log t,\frac{x}{t},\epsilon)e^{5it\varphi(x/t)}\Big]
\end{aligned}
\tag{2.1.13}
$$

the following holds true: The remainder

$$
r^1_{\rm app}(t,x)=(\partial_t^2-\partial_x^2+1)u^1_{\rm app}-P(u^1_{\rm app},\partial_t u^1_{\rm app},\partial_x u^1_{\rm app})
\tag{2.1.14}
$$

may be written as

$$
r^1_{\rm app}(t,x)=2{\rm Re}\,\Big[\epsilon t^{-\frac{5}{2}}e^{it\varphi(x/t)}\tilde{c}_{5,1}\big(\epsilon^2\log t,\frac{x}{t},\frac{1}{t},\epsilon\big)\Big]+F^1_{\rm app}
\tag{2.1.15}
$$

where $\tilde{c}_{5,1}(s,y,h,\epsilon)$ *is continuous on* $[0,\frac{3T_*}{4}]\times\mathbb{R}\times]0,1]\times[0,1]$*, with uniform estimates for the function and all its* $\partial_s,\partial_y,h\partial_h$*-derivatives, supported for* $|y|\leq 1$*, and* $F^1_{\rm app}$ *satisfies for any* $s\in\mathbb{N}$ *estimates*

$$
\begin{aligned}
&\int_{c\epsilon^{-1+\theta}}^{\exp(3T_*/4\epsilon^2)}\|F^1_{\rm app}(t,\cdot)\|_{H^s}\,dt\leq C\epsilon^{2-\theta}\\
&\int_{c\epsilon^{-1+\theta}}^{\exp(3T_*/4\epsilon^2)}\|L_\pm F^1_{\rm app}(t,\cdot)\|_{H^1}\,dt\leq C\epsilon^{2-\theta}.
\end{aligned}
\tag{2.1.16}
$$

Before starting the proof of the proposition, we introduce some notation. We shall denote by $\mathcal{P}$ the ring of continuous functions $(y,h,\epsilon)\to\omega(y,h,\epsilon)$ defined on $]-1,1[\times]0,1]\times[0,1]$, such that for any α,α' in $\mathbb{N}^2$, there is $K_{\alpha,\alpha'}$ in $\mathbb{N}$ such that $(1-y^2)^{K_{\alpha,\alpha'}}\partial_y^\alpha(h\partial_h)^{\alpha'}\omega$ is uniformly bounded. Then the space of functions of (y,h,ϵ) defined and continuous on $\mathbb{R}\times]0,1]\times[0,1]$, bounded as well as their $\partial_y,h\partial_h$ derivatives on that domain, and supported for $|y|\leq 1$, is a $\mathcal{P}$-module. If $(s,y)\to a(s,y)$ is a smooth function on $[0,T_*[\times\mathbb{R}$, supported for $|y|\leq 1$, and if q,ℓ are (odd) integers with $1\leq|q|\leq\ell$, we notice that

$$
\begin{aligned}
(\partial_t^2-\partial_x^2+1)&\big[e^{itq\varphi(x/t)}t^{-\frac{\ell}{2}}a(\epsilon^2\log t,\frac{x}{t})\big]\\
&=(1-q^2)t^{-\frac{\ell}{2}}e^{itq\varphi(y)}a(s,y)|_{s=\epsilon^2\log t,y=x/t}\\
&+2iqt^{-\frac{\ell}{2}-1}e^{itq\varphi(y)}\omega_0(y)[\epsilon^2\partial_s a-\frac{1}{2}(\ell-1)a](s,y)|_{s=\epsilon^2\log t,y=x/t}\\
&+t^{-\frac{\ell}{2}-2}e^{itq\varphi(y)}R_2(a)(s,y,\epsilon)|_{s=\epsilon^2\log t,y=x/t},
\end{aligned}
\tag{2.1.17}
$$

where ω_0 has been defined in (1.1.3) and $R_2(a)$ belongs to the $\mathcal{P}$-module generated by $\partial_s^\alpha\partial_y^{\alpha'}a(s,y)$, $\alpha+\alpha'\leq 2$.
Let us compute first the linear part in the expression (2.1.14) of $r^1_{\rm app}$.

Lemma 2.1.3 *There is a smooth function* $(s,y,\epsilon)\to b_{5,3}(s,y,\epsilon)$ *(resp.* $(s,y,\epsilon)\to b_{5,1}(s,y,\epsilon)$*), defined on* $[0,\frac{3T_*}{4}]\times\mathbb{R}\times[0,1]$*, supported for* $|y|\leq 1$*, which is fully determined by* $a^1_{3,3}$ *(resp.* $a_{1,1}$*) in (2.1.13), there are continuous functions* $(s,y,h,\epsilon)\to b_{7,q}(s,y,h,\epsilon)$*, defined on* $[0,\frac{3T_*}{4}]\times$

$\mathbb{R}\times]0,1]\times[0,1]$, *supported for* $|y|\leq 1$, *bounded as well as their* $\partial_s,\partial_y,(h\partial_h)$*-derivatives, for* $q=3,5,7$, *fully determined by* $a_{1,1},a^1_{3,3},a^1_{5,3},a^1_{5,5}$ *such that the following equality holds true:*

$$
\begin{aligned}
(\partial_t^2-\partial_x^2+1)u^1_{\rm app} = 2{\rm Re}\,\Big[&2i\omega_0(y)\frac{\epsilon^3}{t^{\frac{3}{2}}}e^{it\varphi(y)}\partial_s a(s,y)\\
&-8\frac{\epsilon^3}{t^{\frac{3}{2}}}e^{3it\varphi(y)}a^1_{3,3}(s,y)\\
&+\frac{\epsilon}{t^{\frac{5}{2}}}e^{it\varphi(y)}b_{5,1}(s,y,\epsilon)\\
&-\frac{\epsilon}{t^{\frac{5}{2}}}e^{3it\varphi(y)}[8a^1_{5,3}(s,y,\epsilon)-b_{5,3}(s,y,\epsilon)]\\
&-24\frac{\epsilon^5}{t^{\frac{5}{2}}}e^{5it\varphi(y)}a^1_{5,5}(s,y,\epsilon)\\
&+\frac{\epsilon}{t^{\frac{7}{2}}}\sum_{q=1}^{5}e^{itq\varphi(y)}b^1_{7,q}(s,y,\frac{1}{t},\epsilon)\Big]_{|s=\epsilon^2\log t,y=x/t}.
\end{aligned}
\tag{2.1.18}
$$

Proof: We apply (2.1.17) to each term in the definition (2.1.13) of $u^1_{\rm app}$. The $a_{1,1}$-term in (2.1.13) brings the first term in the right hand side of (2.1.18) and the third one. If we apply (2.1.17) to the $t^{-\frac{3}{2}}\epsilon^3e^{3i\varphi(y)}a^1_{3,3}(s,y)$ term in (2.1.13), we get the second term in the right hand side of (2.1.18), the $b_{5,3}$ term (that depends only on $a^1_{3,3}$) and contributions to the last sum. In the same way, applying (2.1.17) to the $\epsilon t^{-\frac{5}{2}}a^1_{5,3}e^{3i\varphi}$-term in (2.1.13), we get the $a^1_{5,3}$-term in (2.1.18) and contributions to the last sum. Finally, the last term of (2.1.13) brings the $a^1_{5,5}$-term in (2.1.18) and contributions to the last sum. □

Next we compute the nonlinear part in the definition (2.1.15) of $r^1_{\rm app}$.

Lemma 2.1.4 *There are continuous functions* $(s,y)\to c_{3,q}(s,y)$, $q=1,3$ *(resp.* $(s,y,\epsilon)\to c_{5,q}(s,y,\epsilon)$, $1\leq q\leq 5$, q *odd, resp.* $(s,y,h,\epsilon)\to c_{7,q}(s,y,h,\epsilon)$, $1\leq q\leq 15$, q *odd) defined on* $[0,\frac{3T_*}{4}]\times\mathbb{R}$ *(resp.* $[0,\frac{3T_*}{4}]\times\mathbb{R}\times[0,1]$, *resp.* $[0,\frac{3T_*}{4}]\times\mathbb{R}\times]0,1]\times[0,1]$*), supported for* $|y|\leq 1$, *with all their* $\partial_s,\partial_y,h\partial_h$*-derivatives bounded, such that* $P(u^1_{\rm app},\partial_t u^1_{\rm app},\partial_x u^1_{\rm app})$ *may be written under the form*

$$
\begin{aligned}
2{\rm Re}\,\Big[&\epsilon^3t^{-\frac{3}{2}}\sum_{q=1,3}e^{itq\varphi(y)}c_{3,q}(s,y)\\
&+\epsilon^3t^{-\frac{5}{2}}\sum_{q=1,3}e^{itq\varphi(y)}c_{5,q}(s,y,\epsilon)+\epsilon^5t^{-\frac{5}{2}}e^{it5\varphi(y)}c_{5,5}(s,y,\epsilon)\\
&+\epsilon^3t^{-\frac{7}{2}}\sum_{\substack{q\ \text{odd}\\1\leq q\leq 15}}e^{itq\varphi(y)}c_{7,q}(s,y,\frac{1}{t},\epsilon)\Big]_{|s=\epsilon^2\log t,y=x/t}.
\end{aligned}
\tag{2.1.19}
$$

Moreover, $c_{3,1}$ *is given by*

$$
c_{3,1}(s,y)=i(\phi(y)+i\psi(y))|a_{1,1}(s,y)|^2a_{1,1}(s,y)
\tag{2.1.20}
$$

with ϕ,ψ *defined in (1.1.4) and*

$$
\begin{aligned}
&c_{3,3}\ \textit{depends only on}\ a_{1,1}\\
&c_{5,q},q=1,3,5,\ \textit{depends only on}\ a_{1,1},a^1_{3,3}\\
&c_{7,q},q\ \textit{odd},1\leq q\leq 15,\ \textit{depends only on}\ a_{1,1},a^1_{3,3},a^1_{5,q'},q'=3,5.
\end{aligned}
\tag{2.1.21}
$$

Proof: For the proof, we introduce the notation

$$\begin{aligned} U_1(t,x) &= \frac{\epsilon}{\sqrt{t}} e^{it\varphi(x/t)} a_{1,1}\big(\epsilon^2 \log t, \frac{x}{t}\big) \\ U_3(t,x) &= \frac{\epsilon^3}{t^{\frac{3}{2}}} e^{3it\varphi(x/t)} a^1_{3,3}\big(\epsilon^2 \log t, \frac{x}{t}\big). \end{aligned} \tag{2.1.22}$$

Then, by (2.1.13), we may write

$$P(u^1_{\rm app}, \partial_t u^1_{\rm app}, \partial_x u^1_{\rm app}) - P(2{\rm Re}\,(U_1+U_3, \partial_t(U_1+U_3), \partial_x(U_1+U_3))) \tag{2.1.23}$$

as a linear combination of expressions of the form

$$\epsilon^p t^{-\frac{\ell}{2}} e^{iqt\varphi(x/t)} c\big(\epsilon^2 \log t, \frac{x}{t}, \epsilon\big)$$

with $p \geq 3$, $\ell \geq 7$, $1 \leq |q| \leq 15$, q odd and $c(s,y,\epsilon)$ smooth on $[0, \frac{3T_*}{4}] \times \mathbb{R} \times [0,1]$, supported for $|y| \leq 1$, i.e. (2.1.23) contributes to the last term in (2.1.19). We are thus reduced to the study of $P(\mathcal{U}_1 + \mathcal{U}_3)$ where

$$\mathcal{U}_j = (U_j + \bar{U}_j, \partial_t(U_j + \bar{U}_j), \partial_x(U_j + \bar{U}_j)).$$

By Taylor expansion,

$$P(\mathcal{U}_1 + \mathcal{U}_3) = P(\mathcal{U}_1) + DP(\mathcal{U}_1) \cdot \mathcal{U}_3 \tag{2.1.24}$$

modulo terms that contribute again to the last term in (2.1.19). The last term $DP(\mathcal{U}_1) \cdot \mathcal{U}_3$ may be written as contributions to the $t^{-\frac{5}{2}}$-expression in (2.1.19) with coefficients $c_{5,q}$ satisfying (2.1.21) and as contributions to the last term in (2.1.19). It remains to study

$$P(\mathcal{U}_1) = P(U_1 + \bar{U}_1, \partial_t(U_1 + \bar{U}_1), \partial_x(U_1 + \bar{U}_1)). \tag{2.1.25}$$

When computing $\partial_t U_1, \partial_x U_1$, if the derivative does not fall on the exponential, we get an extra t^{-1} factor, so that (2.1.25) may be written as new contributions to the last two sums in (2.1.19) and as the expression

$$\frac{\epsilon^3}{t^{\frac{3}{2}}} P\big(e^{it\varphi(y)} a_{1,1}(s,y)\Omega(y) + e^{-it\varphi(y)} \bar{a}_{1,1}(s,y)\overline{\Omega(y)}\big) \tag{2.1.26}$$

with the notation

$$\Omega(y) = (1, i\omega_0(y), i\omega_1(y)). \tag{2.1.27}$$

Then (2.1.26) provides the $t^{-\frac{3}{2}}$-term in (2.1.19), and to prove (2.1.20) we have to compute explicitly the $e^{it\varphi(y)}$-term in (2.1.26), which gives

$$\begin{aligned} c_{3,1}(s,y) &= DP(a_{1,1}(s,y)\Omega(y)) \cdot \bar{a}_{11}(s,y)\overline{\Omega(y)} \\ &= |a_{1,1}(s,y)|^2 a_{1,1}(s,y) DP(\Omega(y)) \cdot \overline{\Omega(y)} \end{aligned} \tag{2.1.28}$$

since P is homogeneous of degree 3. The explicit expression of $c_{3,1}$ given by (2.1.20) follows from the following lemma. □

Lemma 2.1.5 *Let P be the cubic polynomial given by (1.1.2) and let Ω be given by (2.1.27). Then*

$$DP(\Omega(y)) \cdot \overline{\Omega(y)} = i(\phi(y) + i\psi(y)) \tag{2.1.29}$$

with ϕ, ψ defined in (1.1.4). Moreover,

$$D^2 P(\Omega(y)) \cdot (\overline{\Omega(y)}, \Omega(y)) = 2i(\phi(y) + i\psi(y)). \tag{2.1.30}$$

Proof: Since P is homogeneous of order three, $DP(X)X = 3P(X)$ whence $D^2P(X)(Y,X) = 2DP(X)\cdot Y$, so that (2.1.29) implies (2.1.30). Let us show (2.1.29). Writing (T, Z_1, Z_2) the variables of P, we have by (2.1.27)

$$\begin{aligned} DP(\Omega)\cdot\overline{\Omega} &= \frac{\partial P}{\partial T}(1, i\omega_0, i\omega_1) - i\omega_0\frac{\partial P}{\partial Z_1}(1, i\omega_0, i\omega_1) - i\omega_1\frac{\partial P}{\partial Z_2}(1, i\omega_0, i\omega_1) \\ &= (T\partial_T - Z_1\partial_{Z_1} - Z_2\partial_{Z_2})P(1, i\omega_0, i\omega_1). \end{aligned} \tag{2.1.31}$$

Write the decomposition (1.1.2) $P(T, Z_1, Z_2) = \sum_{k=0}^{3} P_k(T; Z_1, Z_2)$ where P_k is homogeneous of degree k in (Z_1, Z_2) and $3-k$ in T. We get that (2.1.31) is given by

$$\begin{aligned} (3P_0 - P_2)(1, i\omega_0, i\omega_1) + (P_1 - 3P_3)(1, i\omega_0, i\omega_1) \\ = (3P_0 + P_2)(1, \omega_0, \omega_1) + i(P_1 + 3P_3)(1, \omega_0, \omega_1). \end{aligned}$$

Going back to the definition (1.1.4) of ϕ, ψ we obtain (2.1.29). □

Proof of Proposition 2.1.2: By (2.1.14), $r^1_{\rm app}$ is the difference of (2.1.18) and (2.1.19). We choose first $a_{1,1}$ as the solution to equation (2.1.10). By (2.1.20) this implies that the first term in the right hand side of (2.1.18) cancels out the $c_{3,1}$-term in (2.1.19). By (2.1.21), the $c_{3,3}$-term in (2.1.19) is now determined, and we may eliminate it from (2.1.19) choosing $a^1_{3,3}(s,y) = -\frac{1}{8}c_{3,3}(s,y)$ in (2.1.18). By (2.1.21), the $c_{5,q}$'s are now determined, as are $b_{5,1}$ and $b_{5,3}$ in (2.1.18), according to the statement of Lemma 2.1.3. The $b_{5,1}$ contribution to (2.1.18) and the $c_{5,1}$ one in (2.1.19) will form part of the $\tilde{c}_{5,1}$-term in (2.1.15). On the other hand, if we set

$$\begin{aligned} a^1_{5,3} &= -\frac{1}{8}(c_{5,3}\epsilon^2 - b_{5,3}) \\ a^1_{5,5} &= -\frac{1}{24}c_{5,5}, \end{aligned}$$

we cancel out the $t^{-\frac{5}{2}}$-terms in (2.1.18) and (2.1.19).
We are thus left with only the $t^{-\frac{7}{2}}$ contributions coming from (2.1.18), (2.1.19), that are all of the form

$$\epsilon e^{itq\varphi(y)}t^{-\frac{7}{2}}c(s, y, \frac{1}{t}, \epsilon)|_{s=\epsilon^2\log t, y=x/t} \tag{2.1.32}$$

for continuous functions on $[0, \frac{3Y_*}{4}]\times\mathbb{R}\times]0,1]\times[0,1]$, supported for $|y|\le 1$, bounded as well as their $\partial_s, \partial_y, (h\partial_h)$-derivatives. The Sobolev norms of (2.1.32) integrated for $t \ge c\epsilon^{-1+\theta}$ is thus $O(\epsilon^{3-2\theta})$ which is better than the first inequality (2.1.16). If we make act $L_\pm$ on (2.1.32) before computing the H^1 norm, we lose an extra power of t and get instead after integration a $O(\epsilon^{2-\theta})$ bound that brings the second estimate (2.1.16). This concludes the proof. □

Next we glue together the function u_0 solution to (2.1.3), which is an approximate solution of (2.1.1) for small times according to Proposition 2.1.1, and the function $u^1_{\rm app}$ defined by (2.1.13), that is also an approximate solution for intermediate times.

Proposition 2.1.6 *Let χ_0 in $C^\infty(\mathbb{R})$ be equal to one close to zero. Define for $1 \le t \le e^{\frac{3T_*}{4\epsilon^2}}$*

$$u^{\rm M}_{\rm app}(t,x) = \chi_0(\epsilon^{1-\theta}(t-1))u_0(t,x) + (1-\chi_0)(\epsilon^{1-\theta}(t-1))u^1_{\rm app}(t,x) \tag{2.1.33}$$

and

$$r^{\rm M}_{\rm app}(t,x) = (\partial_t^2 - \partial_x^2 + 1)u^{\rm M}_{\rm app} - P(u^{\rm M}_{\rm app}, \partial_t u^{\rm M}_{\rm app}, \partial_x u^{\rm M}_{\rm app}). \tag{2.1.34}$$

One may write

$$r^{\rm M}_{\rm app}(t,x) = 2{\rm Re}\left[\frac{\epsilon}{t^{\frac{5}{2}}}e^{it\varphi(y)}\chi_1(\epsilon^{1-\theta}t)c_{5,1}(s,y,\frac{1}{t},\epsilon)\right]|_{s=\epsilon^2\log t, y=x/t} + F^{\rm M}_{\rm app}(t,x) \tag{2.1.35}$$

where $\chi_1 \in C^\infty(\mathbb{R})$ *is equal to zero close to zero and equal to one outside a neighborhood of zero,* $c_{5,1}(s,y,h,\epsilon)$ *is a continuous function, bounded as well as its* $\partial_s, \partial_y, (h\partial_h)$*-derivatives on* $[0,\frac{3T_*}{4}]\times\mathbb{R}\times]0,1]\times[0,1]$*, supported for* $|y|\leq 1$*, and where* $F^{\rm M}_{\rm app}$ *satisfies*

$$\int_1^{\exp(3T_*/4\epsilon^2)} \|F^{\rm M}_{\rm app}(t,\cdot)\|_{H^s}\,dt \leq C_s\epsilon^{2-\theta} \tag{2.1.36}$$

for any $s\in\mathbb{N}$ *and*

$$\int_1^{\exp(3T_*/4\epsilon^2)} \|L_\pm F^{\rm M}_{\rm app}(t,\cdot)\|_{H^1}\,dt \leq C\epsilon^{2-\theta}. \tag{2.1.37}$$

Proof: We decompose $r^{\rm M}_{\rm app}$ using notation (2.1.4), (2.1.14) as

$$r^{\rm M}_{\rm app} = r^{\rm M}_{\rm app,L} + r^{\rm M}_{\rm app,NL} + \chi_0(\epsilon^{1-\theta}(t-1))r_0 + (1-\chi_0)(\epsilon^{1-\theta}(t-1))r^1_{\rm app} \tag{2.1.38}$$

with

$$\begin{aligned} r^{\rm M}_{\rm app,L} = (\partial_t^2-\partial_x^2+1)u^{\rm M}_{\rm app} - \chi_0(\epsilon^{1-\theta}(t-1))(\partial_t^2-\partial_x^2+1)u_0 \\ -(1-\chi_0)(\epsilon^{1-\theta}(t-1))(\partial_t^2-\partial_x^2+1)u^1_{\rm app} \end{aligned} \tag{2.1.39}$$

and

$$\begin{aligned} r^{\rm M}_{\rm app,NL} = -P(u^{\rm M}_{\rm app},\partial_t u^{\rm M}_{\rm app},\partial_x u^{\rm M}_{\rm app}) + \chi_0(\epsilon^{1-\theta}(t-1))P(u_0,\partial_t u_0,\partial_x u_0) \\ +(1-\chi_0)(\epsilon^{1-\theta}(t-1))P(u^1_{\rm app},\partial_t u^1_{\rm app},\partial_x u^1_{\rm app}). \end{aligned} \tag{2.1.40}$$

Let us study successively (2.1.39) and (2.1.40).

• **Study of (2.1.39)**

By the definition (2.1.33) of $u^{\rm M}_{\rm app}$, we may write (2.1.39) as

$$2\epsilon^{1-\theta}\chi_0'(\epsilon^{1-\theta}(t-1))(\partial_t u_0 - \partial_t u^1_{\rm app}) + \epsilon^{2-2\theta}\chi_0''(\epsilon^{1-\theta}(t-1))(u_0-u^1_{\rm app}). \tag{2.1.41}$$

By (2.1.7) and (2.1.13), we have

$$\begin{aligned} u_0(t,x) - u^1_{\rm app}(t,x) = 2{\rm Re}\Big[&\frac{\epsilon}{\sqrt{t}}e^{it\varphi(y)}(a^0_1(y)-a_{1,1}(s,y)) \\ &+\frac{\epsilon}{t^{\frac{3}{2}}}e^{it\varphi(y)}c_{3,1}(y,\frac{1}{t}) \\ &+\frac{\epsilon^3}{t^{\frac{3}{2}}}e^{3it\varphi(y)}c_{3,3}(s,y) \\ &+\frac{\epsilon}{t^{\frac{5}{2}}}e^{3it\varphi(y)}c_{5,3}(s,y,\epsilon) \\ &+\frac{\epsilon}{t^{\frac{5}{2}}}e^{5it\varphi(y)}c_{5,5}(s,y,\epsilon)\Big]|_{s=\epsilon^2\log t,y=x/t} \\ &+\epsilon e(t,x) \end{aligned} \tag{2.1.42}$$

where the functions $c_{\ell,q}(s,y,h,\epsilon)$ are continuous functions of their arguments $s\in[0,\frac{3T_*}{4}]$, $y\in\mathbb{R}$, $h\in]0,1]$, $\epsilon\in[0,1]$, supported for $|y|\leq 1$, bounded as well as their $\partial_s,\partial_y,(h\partial_h)$-derivatives on their domain of definition, and where $e(t,x)$ is in $\mathcal{S}([1,+\infty[\times\mathbb{R})$.

Denote by $\tilde{\chi}_0$ some function in $C_0^\infty(]0,+\infty[)$. If we take eventually a time derivative of the second term in the right hand side of (2.1.42) and multiply it by $\epsilon^{1-\theta}\tilde{\chi}_0(\epsilon^{1-\theta}(t-1)) = t^{-1}t\epsilon^{1-\theta}\tilde{\chi}_0(\epsilon^{1-\theta}(t-1))$ we get an expression of the form of the first term in the right hand side of (2.1.35). This shows that the contribution of the $c_{3,1}$-term in (2.1.42) to (2.1.41) has such a form.
We need thus to prove that all other terms in (2.1.42) give, when plugged in (2.1.41), contributions to $F^{\rm M}_{\rm app}$ in (2.1.35). By (2.1.10), $a_{1,1}(0,y) = a_1^0(y)$, so that the product of the first term in the right hand side of (2.1.42) by $\epsilon^{1-\theta}\tilde{\chi}_0(\epsilon^{1-\theta}(t-1))$ is bounded in modulus by

$$\frac{\epsilon}{t^{\frac{5}{2}}}\epsilon^{1-\theta}t|\tilde{\chi}_0(\epsilon^{1-\theta}(t-1))|\epsilon^2 t\log t\mathbb{1}_{|x/t|\leq 1}. \tag{2.1.43}$$

Similar or better estimates hold if we take ∂_t or ∂_x-derivatives, so that the contribution of the first term in the right hand side of (2.1.42) to (2.1.41) satisfies, as well as its derivatives, bound (2.1.43). As the $L^2(dx)$-norm of (2.1.43) is $O(\epsilon^{2+\theta-0}t^{-2}\mathbb{1}_{t\sim\epsilon^{-1+\theta}})$, we see that a bound of the form (2.1.36) holds. If we make act $L_\pm$ on the corresponding term before computing a H^1 norm, we get a bound in $O(\epsilon^{2+\theta-0}t^{-1}\mathbb{1}_{t\sim\epsilon^{-1+\theta}})$ which implies that (2.1.37) holds as well.
We are thus reduced to showing that the third to the last terms in the right hand side of (2.1.42) give also contributions satisfying (2.1.36), (2.1.37) when plugged inside (2.1.41). This is evident for the last term. The other ones bring to (2.1.41) expressions of the form

$$\frac{\epsilon^a}{t^{\frac{\ell}{2}+1}}\tilde{\chi}(\epsilon^{1-\theta}(t-1))e^{iqt\varphi(y)}c(s,y,\epsilon)|_{s=\epsilon^2\log t, y=x/t} \tag{2.1.44}$$

with either $a=3,\ell=3$ or $a=1,\ell\geq 5$. The L^2 norm of (2.1.44) and its derivatives is $O(\epsilon^a t^{-\frac{\ell+1}{2}}\mathbb{1}_{t\sim\epsilon^{-1+\theta}})$ whose integral largely satisfies (2.1.36). To obtain (2.1.37), one has to make act $L_\pm$ on (2.1.44), which makes appear one factor t, so that the H^1 norm is $O(\epsilon^a t^{-\frac{\ell-1}{2}}\mathbb{1}_{t\sim\epsilon^{-1+\theta}})$. Because of the conditions on a, ℓ one gets a $O(\epsilon^{2-\theta})$ bound as in (2.1.37). This concludes the estimate of (2.1.39).

- **Estimate of (2.1.40)**

From the definition (2.1.33) of $u^{\rm M}_{\rm app}$, we may write (2.1.40) as the sum of expressions

$$P(u^{\rm M}_{\rm app},\chi_0\partial_t u_0+(1-\chi_0)\partial_t u^1_{\rm app},\partial_x u^{\rm M}_{\rm app})-P(u^{\rm M}_{\rm app},\partial_t(\chi_0 u_0+(1-\chi_0)u^1_{\rm app}),\partial_x u^{\rm M}_{\rm app}) \tag{2.1.45}$$

and of

$$\begin{aligned}&\chi_0 P(u_0,\partial_t u_0,\partial_x u_0)+(1-\chi_0)P(u^1_{\rm app},\partial_t u^1_{\rm app},\partial_x u^1_{\rm app})\\&\quad-P(\chi_0 u_0+(1-\chi_0)u^1_{\rm app},\chi_0\partial_t u_0+(1-\chi_0)\partial_t u^1_{\rm app},\chi_0\partial_x u_0+(1-\chi_0)\partial_x u^1_{\rm app}).\end{aligned} \tag{2.1.46}$$

Difference (2.1.45) may be bounded point-wise by

$$C\epsilon^{1-\theta}|\chi_0'(\epsilon^{1-\theta}(t-1))||u_0-u^1_{\rm app}|\Big(\sum_{\alpha+\beta\leq 1}\big(|\partial_t^\alpha\partial_x^\beta u^1_{\rm app}|+|\partial_t^\alpha\partial_x^\beta u_0|\big)\Big)^2. \tag{2.1.47}$$

The difference $u_0-u^1_{\rm app}$ is given by (2.1.42), so that its modulus is bounded from above on the support of $\chi_0'(\epsilon^{1-\theta}(t-1))$ by $C\epsilon t^{-\frac{3}{2}}$ (using that the first term in (2.1.42) is $O\big(\frac{\epsilon}{\sqrt{t}}s\big)$ with $s=\epsilon^2\log t=O\big(\frac{1}{t}\big)$ if $t\sim\epsilon^{-1+\theta}$). In addition, $u^1_{\rm app}$, u_0 are $O(\epsilon t^{-\frac{1}{2}})$, as well as their derivatives. Then (2.1.47) is bounded from above by

$$C\epsilon^{4-\theta}|\chi'(\epsilon^{1-\theta}(t-1))|t^{-\frac{5}{2}}(\mathbb{1}_{|x|\leq t}+O(\langle x\rangle^{-N})). \tag{2.1.48}$$

A similar bound holds for the derivatives of (2.1.45), so that (2.1.36) is largely satisfied. To get (2.1.37), one has to bound the L^2 norm of (2.1.48) multiplied by t, so that the conclusion follows as well.

It remains to study (2.1.46), that may be written as

$$\chi_0(1-\chi_0)M(u_0,\partial_t u_0,\partial_x u_0,u^1_{\rm app},\partial_t u^1_{\rm app},\partial_x u^1_{\rm app}) \tag{2.1.49}$$

for some cubic expression M. Since by (2.1.7), (2.1.13), $u_0, u^1_{\rm app}$ and their derivatives are $O(\epsilon t^{-\frac12}\mathbb{1}_{|x|\le t})+O(\epsilon t^{-N}\langle x\rangle^{-N})$, we get that the Sobolev norm of (2.1.49) is $O(\epsilon^3 t^{-1}\mathbb{1}_{t\sim\epsilon^{-1+\theta}})$, which brings an estimate of the form (2.1.36). In the same way, the integrand in (2.1.37) is $O(\epsilon^3\mathbb{1}_{t\sim\epsilon^{-1+\theta}})$ which gives a $O(\epsilon^{2+\theta})$ bound for the integral.
This concludes the proof, since we have shown that (2.1.39) may be written as a contribution to the $c_{5,1}$ term in (2.1.35) and as a remainder that may be integrated to $F^{\rm M}_{\rm app}$, since (2.1.40) is also of the form $F^{\rm M}_{\rm app}$ and since the remaining terms $\chi_0 r_0+(1-\chi_0)r^1_{\rm app}$ in (2.1.38) are of the form of the right hand side of (2.1.35) by Proposition 2.1.1 and (2.1.15). □

2.2 Construction for large time

Our next goal is to extend the approximate solution that has been constructed up to time $e^{\frac{3T_*}{4\epsilon^2}}$ in order to almost reach the blow-up time $e^{\frac{T_*}{\epsilon^2}}$. At this time, the main part of the profile (2.1.13) blows-up and we introduce a notation for spaces describing the solution close to the blow-up time.

Definition 2.2.1 *Let $m\in\mathbb{R}$, y_0 be a point in $]-1,1[$ and $\kappa_0\in\mathbb{N}^*$. We denote by Σ^m the space of continuous functions $(s,y,h,\epsilon)\to a(s,y,h,\epsilon)$ defined on $[0,T_*[\times\mathbb{R}\times]0,1]\times[0,1]$, with values in $\mathbb{C}$, smooth in (s,y,h), supported for $|y|\le 1$, that satisfy for any integers α,β,ζ,N, any (s,y,h,ϵ) in the domain of definition, estimates*

$$|\partial_s^\alpha\partial_y^\beta(h\partial_h)^\zeta a(s,y,h,\epsilon)|\le C_{\alpha,\beta,\zeta,N}(T_*-s+|y-y_0|^{2\kappa_0})^{m-\alpha-\frac{\beta}{2\kappa_0}}(1-|y|)^{-N}. \tag{2.2.1}$$

In particular, Σ^m is a $\mathcal{P}$-module (for $\mathcal{P}$ defined after Proposition 2.1.2). Moreover, $\partial_s^\alpha\partial_y^\beta a$ belongs to $\Sigma^{m-\alpha-\frac{\beta}{2\kappa_0}}\subset\Sigma^{m-\alpha-\beta}$. When a does not depend on one of the variables h or ϵ, we remove it from notation.

Example: Consider the function $a_{1,1}(s,y)$ defined in (2.1.12) with a_1^0 smooth on $\mathbb{R}$, supported for $|y|\le 1$. Then $a_{1,1}$ is smooth on $[0,T_*]\times\mathbb{R}-\{(T_*,y_0)\}$ because of (1.2.2), (1.2.3). Moreover, for y close to y_0, (1.2.3) implies that $\Gamma(y)\phi(y)=\frac{1}{T_*}-(y-y_0)^{2\kappa_0}\Theta(y)$ for some smooth positive function Θ, so that we get estimates of the form (2.2.1) with $m=-\frac12$ i.e. $a_{1,1}$ belongs to $\Sigma^{-\frac12}$.

Our goal is to prove the following proposition:

Proposition 2.2.2 *Let $\delta>0$ be a small number, $N\in\mathbb{N}$. One may construct for all odd integers q,ℓ satisfying $1\le q\le\ell\le N$ elements $a_{\ell,q}$ of $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$ with $a_{1,1}$ given by (2.1.12) so that, if we define for $t\in[e^{\frac{T_*}{2\epsilon^2}},e^{\frac{T_*}{\epsilon^2}}[$, $x\in\mathbb{R}$*

$$\begin{aligned}u^2_{\rm app}(t,x)=2{\rm Re}\Big[&\sum_{\substack{\ell=1\\ \ell\ \text{odd}}}^{N}\epsilon^{2-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}a_{\ell,1}(s,y,\epsilon)\\ &+\sum_{\substack{\ell=3\\ \ell\ \text{odd}}}^{N}\sum_{\substack{3\le q\le\ell\\ q\ \text{odd}}}\epsilon^{2q-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}a_{\ell,q}(s,y,\epsilon)\Big]|_{s=\epsilon^2\log t,y=x/t}\end{aligned} \tag{2.2.2}$$

then

$$r^2_{\rm app}=(\partial_t^2-\partial_x^2+1)u^2_{\rm app}-P(u^2_{\rm app},\partial_t u^2_{\rm app},\partial_x u^2_{\rm app}) \tag{2.2.3}$$

may be written as the sum of the non-characteristic expression

$$2\mathrm{Re}\Big[\sum_{\substack{\ell=N+2\\ \ell \text{ odd}}}^{3N}\sum_{\substack{3\le q\le \ell\\ q \text{ odd}}} \epsilon^{2q-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}d_{\ell,q}(s,y,\epsilon)\Big]|_{s=\epsilon^2\log t,y=x/t} \tag{2.2.4}$$

with for $3\le q\le \ell$

$$d_{\ell,q}\in \Sigma^{-\frac{\ell}{2}-\delta(\ell-3)} \tag{2.2.5}$$

and of a characteristic expression

$$2\mathrm{Re}\sum_{\substack{\ell=N+4\\ \ell \text{ odd}}}^{3N} \epsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}d_{\ell,1}(s,y,\epsilon)|_{s=\epsilon^2\log t,y=x/t} \tag{2.2.6}$$

with

$$d_{\ell,1}\in \Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}. \tag{2.2.7}$$

Before proving the proposition, we establish several lemmas.

Lemma 2.2.3 *Assume we are given* N *an odd integer and for any odd integers* ℓ, q *satisfying* $1\le q\le \ell\le N$ *continuous functions* $(s,y,\epsilon)\to a_{\ell,q}(s,y,\epsilon)$ *on* $[0,T_*[\times\mathbb{R}\times[0,1]$*, smooth in* (s,y)*, supported for* $|y|\le 1$*. Let* $\mathcal{P}_0$ *be the ring of functions* $(y,\epsilon)\to\gamma(y,\epsilon)$ *continuous on* $]-1,1[\times[0,1]$*, that are smooth in* y *and have at most algebraic growth, as well as their* ∂_y*-derivatives when* $y^2\to 1$ *(uniformly in* ϵ*). For each* q,ℓ *as above, denote by* $\mathcal{C}_{\ell,q}$ *the* $\mathcal{P}_0$*-module generated by all cubic expressions of the form*

$$\prod_{j=1}^{3}\partial_s^{\alpha_j}\partial_y^{\beta_j}a_{\ell_j,q_j}(s,y,\epsilon) \tag{2.2.8}$$

where $\ell_j\in\mathbb{N}$*,* $q_j\in\mathbb{Z}$ *are odd,* $\alpha_j,\beta_j\in\mathbb{N}$*,* $a_{\ell_j,-q_j}=\bar{a}_{\ell_j,q_j}$*, and where the following inequalities hold true:*

$$\sum_{j=1}^{3}(\ell_j+2\alpha_j+2\beta_j)\le\ell,\ \ q=|q_1+q_2+q_3|. \tag{2.2.9}$$

Introduce

$$U(t,x)=\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{N}\sum_{\substack{1\le q\le \ell\\ q \text{ odd}}} \epsilon^{2q-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}a_{\ell,q}(s,y,\epsilon)|_{s=\epsilon^2\log t,y=x/t}. \tag{2.2.10}$$

Then we may write

$$\begin{aligned}P(2\mathrm{Re}\,(U,\partial_tU,\partial_xU))=2\mathrm{Re}\Big[&\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{3N}\sum_{\substack{3\le q\le \ell\\ q \text{ odd}}} \epsilon^{2q-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}c_{\ell,q}(s,y,\epsilon)\\ &+\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{3N}\epsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}c_{\ell,1}(s,y,\epsilon)\Big]|_{s=\epsilon^2\log t,y=x/t}\end{aligned} \tag{2.2.11}$$

where $c_{\ell,q}$ belongs to $\mathcal{C}_{\ell,q}$. Moreover, for $3 \leq q \leq \ell$,

$$c_{\ell,q} \text{ depends only on } a_{\ell',q'},\ 1 \leq q' \leq \ell' \leq \ell - 2. \tag{2.2.12}$$

In addition, $c_{3,1}$ is given explicitly by

$$c_{3,1}(s,y) = i(\phi(y) + i\psi(y))|a_{1,1}|^2 a_{1,1}(s,y) \tag{2.2.13}$$

and for $\ell \geq 5$, one may decompose

$$c_{\ell,1}(s,y,\epsilon) = c'_{\ell,1}(s,y,\epsilon) + c''_{\ell,1}(s,y,\epsilon) \tag{2.2.14}$$

where $c'_{\ell,1}$ is given explicitly by

$$c'_{\ell,1}(s,y,\epsilon) = 2i(\phi(y) + i\psi(y))(|a_{1,1}|^2 a_{\ell-2,1} + \frac{1}{2} a_{1,1}^2 \bar{a}_{\ell-2,1})(s,y,\epsilon) \tag{2.2.15}$$

and

$$\begin{aligned} &c''_{\ell,1} \text{ depends only on } a_{\ell',q'} \text{ for } 1 \leq q' \leq \ell' \leq \ell - 4 \\ &\qquad \text{or on } a_{\ell-2,q'}, 3 \leq q' \leq \ell - 2 \text{ and } a_{1,1}. \end{aligned} \tag{2.2.16}$$

Proof: We notice first that (2.2.9) implies that

$$\sum_{j=1}^{3} (2|q_j| - \ell_j) \geq 2q - \ell \tag{2.2.17}$$

and that for terms (2.2.8) that are characteristic, i.e. such that $q = |q_1 + q_2 + q_3| = 1$, we have $|q_1| + |q_2| + |q_3| - q \geq 2$, so that

$$\sum_{j=1}^{3} (2|q_j| - \ell_j) \geq 2|q| + 4 - \ell \geq 6 - \ell. \tag{2.2.18}$$

Let us compute (2.2.11). From (2.2.10), and the expressions that may be obtained for $\partial_t U, \partial_x U$ from that formula, we see that the $t^{-\frac{\ell}{2}}$ terms in (2.2.11) are given by the product of $e^{\pm iqt\varphi(x/t)}$ ($q \in \mathbb{N}$, q odd), of an element of $\mathcal{C}_{\ell,q}$ and of a power of ϵ of the form

$$\epsilon^{\sum_{j=1}^{3}(2|q_j| - \ell_j) + a} \tag{2.2.19}$$

for some $a \geq 0$. In the non-characteristic case $q \neq 1$, it follows from (2.2.17) that (2.2.19) is $O(\epsilon^{2q-\ell})$ and in the characteristic case, (2.2.19) will be $O(\epsilon^{6-\ell})$ by (2.2.18). We thus obtain the structure indicated in (2.2.11). Let us check properties (2.2.12) to (2.2.16).
Since in (2.2.9) all ℓ_j are larger or equal to one, and $c_{\ell,q}$ is given by a cubic expression of the form (2.2.8), (2.2.12) holds necessarily.
Let us consider now specifically the characteristic terms $c_{\ell,1}$ in (2.2.11) with $\ell \geq 5$. These terms are given by (2.2.8) with indices satisfying (2.2.9). In this property, either one ℓ_j is equal to $\ell - 2$ and then the other ones are equal to one and $\alpha_j = \beta_j = 0$ for all j, or all ℓ_j are smaller or equal to $\ell - 4$ (recall that they are odd). This last case corresponds to contributions $c''_{\ell,1}$ satisfying the first alternative in (2.2.16). On the other hand, if one ℓ_j is equal to $\ell - 2$, say $\ell_3 = \ell - 2$, then $\ell_1 = \ell_2 = 1$. If the q_3 associated to ℓ_3 satisfies $|q_3| \geq 3$, we get a contribution to $c''_{\ell,1}$ corresponding to the second alternative in (2.2.16). We are thus left with terms of the form (2.2.8) with

$$\alpha_j = \beta_j = 0, \ell_3 = \ell - 2, |q_3| = 1, \ell_1 = \ell_2 = 1, |q_1 + q_2 + q_3| = 1. \tag{2.2.20}$$

These terms give $c'_{\ell,1}$ in (2.2.14) and have to be computed explicitly. Notice also that in the case $\ell = 3$, $c_{3,1}$ is itself of that form. Moreover, we have also $|q_j| \leq \ell_j = 1$, $j = 1, 2$ so that we see that we have to compute those terms of (2.2.11) that oscillate on the phases $\pm t\varphi(y)$ and that come from the contribution to U given by

$$U'(t,x) = \sum_{\ell'=1}^{N} \epsilon^{2-\ell'} t^{-\frac{\ell'}{2}} e^{it\varphi(y)} a_{\ell',1}(s,y)|_{s=\epsilon^2 \log t, y=x/t}. \tag{2.2.21}$$

Denote $U'_{\ell'}$ the general term of that sum and set

$$\mathcal{U}'_{\ell'} = (U'_{\ell'} + \bar{U}'_{\ell'}, \partial_t(U'_{\ell'} + \bar{U}'_{\ell'}), \partial_x(U'_{\ell'} + \bar{U}'_{\ell'})) \tag{2.2.22}$$

We thus have to compute the contribution to (2.2.11) given by those terms in

$$P\Big(\sum_{\substack{\ell'=1 \\ \ell' \text{ odd}}}^{N} (\mathcal{U}'_{\ell'} + \bar{\mathcal{U}}'_{\ell'})\Big) \tag{2.2.23}$$

that oscillate along the phases $e^{\pm it\varphi(x/t)}$ and that come from the terms (2.2.22) where ∂_t and ∂_x act on the oscillatory factors coming from (2.2.21) (since in (2.2.20), $\alpha_j = \beta_j = 0$ for any j). Using notation $\Omega(y) = (1, i\omega_0(y), i\omega_1(y))$ we see that we may reduce (2.2.23) to the expression

$$P\Big(2\mathrm{Re}\Big[\Omega(y)e^{it\varphi(y)} \sum_{\substack{\ell'=1 \\ \ell' \text{ odd}}}^{N} \epsilon^{2-\ell'} t^{-\frac{\ell'}{2}} a_{\ell',1}(s,y)\Big]\Big)|_{s=\epsilon^2 \log t, y=x/t}. \tag{2.2.24}$$

By (2.2.20) we are only interested in contributions to (2.2.24) that are at least quadratic in $a_{1,1}, \bar{a}_{1,1}$ i.e. we may reduce (2.2.24) to

(2.2.25)
$$\epsilon^3 P\big(2\mathrm{Re}\big[\Omega(y)e^{it\varphi(y)}t^{-\frac{1}{2}}a_{1,1}(s,y)\big]\big)$$
$$+ \sum_{\substack{\ell'=1 \\ \ell' \text{ odd}}}^{N} \epsilon^2 DP\big(2\mathrm{Re}\big[\Omega(y)e^{it\varphi(y)}t^{-\frac{1}{2}}a_{1,1}(s,y)\big]\big) \cdot \big(2\mathrm{Re}\big[\Omega(y)e^{it\varphi(y)}t^{-\frac{\ell'}{2}}a_{\ell',1}(s,y)\epsilon^{2-\ell'}\big]\big)|_{s=\epsilon^2 \log t, y=x/t}.$$

The first term in (2.2.25) has been already computed in (2.1.26), (2.1.28) and brings $c_{3,1}$ given by (2.2.13). The term in $t^{-\frac{\ell}{2}}e^{it\varphi(y)}$ coming from the sum in (2.2.25) is obtained when $\ell' = \ell - 2$ and is equal to the $e^{it\varphi(y)}$-term in

$$\epsilon^{6-\ell} t^{-\frac{\ell}{2}} DP(\Omega e^{it\varphi} a_{1,1} + e^{-it\varphi}\bar{\Omega}\bar{a}_{1,1}) \cdot (\Omega e^{it\varphi} a_{\ell-2,1} + e^{-it\varphi}\bar{\Omega}\bar{a}_{\ell-2,1}).$$

Taylor expanding this expression, we see that we have to consider

$$\epsilon^{6-\ell} t^{-\frac{\ell}{2}} e^{it\varphi}[a_{1,1}^2 \bar{a}_{\ell-2,1} DP(\Omega) \cdot \bar{\Omega} + |a_{1,1}|^2 a_{\ell-2,1} D^2 P(\Omega) \cdot (\bar{\Omega}, \Omega)].$$

By (2.1.29), (2.1.30), this gives (2.2.15) and concludes the proof of the lemma. □

We apply the preceding lemma to compute $P(u_{\mathrm{app}}^2, \partial_t u_{\mathrm{app}}^2, \partial_x u_{\mathrm{app}}^2)$.

Corollary 2.2.4 *Assume that $u^2_{\rm app}$ is given by (2.2.2). Then*

$$
\begin{aligned}
P(u^2_{\rm app}, \partial_t u^2_{\rm app}, \partial_x u^2_{\rm app}) = 2{\rm Re}\Big[\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{3N} \epsilon^{6-\ell} t^{-\frac{\ell}{2}} e^{it\varphi(y)} c_{\ell,1}(s,y,\epsilon) \\
+ \sum_{\substack{\ell=3\\ q \text{ odd}}}^{3N} \sum_{\substack{q=3\\ \ell \text{ odd}}}^{\ell} \epsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{itq\varphi(y)} c_{\ell,q}(s,y,\epsilon)\Big]|_{s=\epsilon^2 \log t, y=x/t}
\end{aligned} \tag{2.2.26}
$$

where $c_{\ell,q}$ is an element of $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)+2\delta}$ which, for $q \geq 3$, depends only on $a_{\ell',q'}$, $1 \leq q' \leq \ell' \leq \ell-2$, where $c_{3,1}$ is given by (2.2.13) and for $\ell \geq 5$, $c_{\ell,1}$ may be decomposed under the form (2.2.14) with (2.2.15) and (2.2.16) holding true.

Proof: The left hand side of (2.2.26) is (2.2.11) so that we obtain expression (2.2.26). The coefficients $c_{\ell,q}$ belong to $\mathcal{C}_{\ell,q}$ i.e. are given (up to $\mathcal{P}_0$-multiplicative factors) by expressions of the form (2.2.8) with indices satisfying (2.2.9). Since a_{ℓ_j,q_j} belongs to $\Sigma^{-\frac{\ell_j}{2}-(\ell_j-1)\delta}$, it follows from the definition of these classes and from (2.2.9) that $c_{\ell,q}$ is in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)+2\delta}$. The other assertions of the corollary follow from (2.2.12) to (2.2.16). □

In order to prove Proposition 2.2.2, we also need the following result.

Lemma 2.2.5 *Let $y \to \Theta(y)$ be a complex valued smooth function defined on $]-1,1[$, with at most algebraic growth when $|y| \to 1-$ as well as its derivatives. Let $a(s,y)$ be an element of $\Sigma^{-\frac{1}{2}}$. Assume that there is an open neighborhood V of y_0 in $]-1,1[$ and $c>0$ such that for any y in V, any $s \in [0,T_*[$,*

$$
|{\rm Re}\,\Theta(y)| \geq c \tag{2.2.27}
$$

$$
|a(s,y)| \geq c(T_* - s + |y-y_0|^{2\kappa_0})^{-\frac{1}{2}} \tag{2.2.28}
$$

and that a solves the ODE $\partial_s a(s,y) = \Theta(y)|a(s,y)|^2 a(s,y)$. Let ℓ be an odd integer, $\ell \geq 5$ and let r be an element of $\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}$. Let $(s,y) \to b(s,y)$ be the solution of

$$
\begin{aligned}
\partial_s b(s,y) &= \Theta(y)(2|a(s,y)|^2 b(s,y) + a(s,y)^2 \overline{b(s,y)}) + r(s,y) \\
b(0,y) &= 0.
\end{aligned} \tag{2.2.29}
$$

Then b belongs to $\Sigma^{-\frac{\ell-2}{2}-\delta(\ell-3)}$.

Proof: We notice first that if y stays outside V, then by definition of $\Sigma^{-\frac{1}{2}}$, the coefficients in the right hand side of (2.2.29) are smooth functions on $[0,T_*] \times (\mathbb{R}-V)$, so that the same holds true for the solution b, which is moreover supported for $|y| \leq 1$.
We may thus assume that y stays close to y_0, so that (2.2.27), (2.2.28) hold true. We introduce $B(s,y) = \begin{bmatrix} b(s,y) \\ \overline{b(s,y)} \end{bmatrix}$ that solves the system

$$
\partial_s B(s,y) = M(s,y)B(s,y) + R(s,y) \tag{2.2.30}
$$

with

$$
R(s,y) = \begin{bmatrix} r(s,y) \\ \overline{r(s,y)} \end{bmatrix}, M(s,y) = \begin{bmatrix} 2\Theta|a|^2 & \Theta a^2 \\ \bar\Theta \bar a^2 & 2\bar\Theta |a|^2. \end{bmatrix}
$$

Define the two functions

$$\Phi_1(s,y)=\begin{bmatrix} ia(s,y)\\ -i\overline{a(s,y)}\end{bmatrix},\Phi_2(s,y)=\begin{bmatrix}\partial_s a(s,y)\\ \partial_s\overline{a(s,y)}\end{bmatrix}=|a(s,y)|^2\begin{bmatrix}\Theta a\\ \bar\Theta\bar a\end{bmatrix}. \tag{2.2.31}$$

Then Φ_j solves the homogeneous equation $\partial_s\Phi_j = M(s,y)\Phi_j$ and the wronskian $w(s,y)$ of $\Phi_1(s,y)$, $\Phi_2(s,y)$ is equal to $2i\mathrm{Re}\,\Theta(y)|a(s,y)|^4$, so satisfies for $y\in V$, $|w(s,y)|\geq cA(s,y)^{-4}$ according to (2.2.27), (2.2.28) if we set

$$A(s,y)=(T_*-s+|y-y_0|^{2\kappa_0})^{\frac12}. \tag{2.2.32}$$

The fact that $a\in\Sigma^{-\frac12}$ and that (2.2.27), (2.2.28) hold for y close to y_0 imply that the wronskian matrix $W(s,y)$ and its inverse $W(s',y)$ satisfy

$$W(s,y)=\begin{bmatrix}O(A(s,y)^{-1}) & O(A(s,y)^{-3})\\ O(A(s,y)^{-1}) & O(A(s,y)^{-3})\end{bmatrix},W(s',y)^{-1}=\begin{bmatrix}O(A(s',y)) & O(A(s',y))\\ O(A(s',y)^3) & O(A(s',y)^3)\end{bmatrix}.$$

Since $s\to A(s,y)$ is decreasing, we conclude that for $0\leq s'\leq s$

$$W(s,y)W(s',y)^{-1}=O\Big(\Big(\frac{A(s',y)}{A(s,y)}\Big)^3\Big). \tag{2.2.33}$$

Writing the solution to (2.2.30) with zero initial condition at $s=0$ under the form

$$\int_0^s W(s,y)W(s',y)^{-1}R(s',y)\,ds' \tag{2.2.34}$$

and using that $r\in\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)+2\delta}$, we get from (2.2.33), (2.2.34)

$$|B(s,y)|\leq C\int_0^s A(s',y)^{3-\ell-2\delta(\ell-1)+4\delta}\,ds'A(s,y)^{-3}.$$

Since $\ell\geq 5$ and $\delta>0$, this is $O(A(s,y)^{2-\ell-2\delta(\ell-3)})$, i.e. B satisfies (2.2.1) with $\alpha=\beta=0$, $m=-\frac{\ell-2}{2}-\delta(\ell-3)$ for y close to y_0. If we take ∂_y or ∂_s derivatives in (2.2.34), we get in the same way the estimates (2.2.1) for positive α or β. This concludes the proof. □

Proof of Proposition 2.2.2: We shall compute first the action of $\partial_t^2-\partial_x^2+1$ on $u_{\rm app}^2$ given by (2.2.2) and use the fact that the last term in the expression (2.2.3) of $r_{\rm app}^2$ has been computed in Corollary 2.2.4. We shall then construct the $a_{\ell,q}$'s recursively in order to reduce $r_{\rm app}^2$ to an expression of the form (2.2.4).

• Linear term in (2.2.3)

We apply (2.1.17) to the general term of the sums in (2.2.2). We get on the one hand the characteristic contribution

$$\begin{aligned}&2\mathrm{Re}\Big[2i\sum_{\substack{\ell=3\\ \ell\ \text{odd}}}^{N+2}\epsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}\omega_0(y)\partial_s a_{\ell-2,1}(s,y,\epsilon)\Big]|_{s=\epsilon^2\log t,y=x/t}\\ &+2\mathrm{Re}\Big[\sum_{\substack{\ell=5\\ \ell\ \text{odd}}}^{N+4}\epsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}R_2(a_{\ell-4,1})(s,y,\epsilon)\Big]|_{s=\epsilon^2\log t,y=x/t}\end{aligned} \tag{2.2.35}$$

where $R_2(a_{\ell-4,1})$ belongs to the $\mathcal{P}$-module generated by $\partial_s^\alpha\partial_y^\beta a_{\ell-4,1}$ for $\alpha+\beta\leq 2$, so that

$$R_2(a_{\ell-4,1})\in\Sigma^{-\frac{\ell}{2}-\delta(\ell-5)} \tag{2.2.36}$$

by the definition of this class. On the other hand, the second sum in (2.2.2) provides to the linear term in (2.2.3) the non-characteristic contribution

$$
\begin{aligned}
&2\mathrm{Re}\Big[\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N}\sum_{\substack{3\le q\le \ell\\ q \text{ odd}}} \epsilon^{2q-\ell}t^{-\frac{\ell}{2}}(1-q^2)e^{itq\varphi(y)}a_{\ell,q}(s,y,\epsilon)\Big]|_{s=\epsilon^2\log t, y=x/t}\\
&+2\mathrm{Re}\Big[\sum_{\substack{\ell=5\\ \ell \text{ odd}}}^{N+2}\sum_{\substack{3\le q\le \ell-2\\ q \text{ odd}}} \epsilon^{2q+2-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}R_1(a_{\ell-2,q})\Big]|_{s=\epsilon^2\log t, y=x/t}\\
&+2\mathrm{Re}\Big[\sum_{\substack{\ell=7\\ \ell \text{ odd}}}^{N+4}\sum_{\substack{3\le q\le \ell-4\\ q \text{ odd}}} \epsilon^{2q+4-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}R_2(a_{\ell-4,q})\Big]|_{s=\epsilon^2\log t, y=x/t}
\end{aligned}
\tag{2.2.37}
$$

where $R_1(a_{\ell-2,q})$ (resp. $R_2(a_{\ell-4,q})$) is in the $\mathcal{P}$-module generated by $\partial_s^\alpha\partial_y^\beta a_{\ell-2,q}$ (resp. $\partial_s^\alpha\partial_y^\beta a_{\ell-4,q}$) for $\alpha+\beta\le 1$ (resp. $\alpha+\beta\le 2$). Thus, $R_1(a_{\ell-2,q})$ (resp. $R_2(a_{\ell-4,q})$) is in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}$ (resp. $\Sigma^{-\frac{\ell}{2}-\delta(\ell-5)}$). It follows from (2.2.35) and (2.2.37) that

$$
\begin{aligned}
(\partial_t^2-\partial_x^2+1)u_{\mathrm{app}}^2 = 2\mathrm{Re}\Big[&2i\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N+2}\epsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}\omega_0(y)\partial_s a_{\ell-2,1}(s,y,\epsilon)\\
&+\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N}\sum_{\substack{3\le q\le \ell\\ q \text{ odd}}}\epsilon^{2q-\ell}t^{-\frac{\ell}{2}}(1-q^2)e^{itq\varphi(y)}a_{\ell,q}(s,y,\epsilon)\\
&+\sum_{\substack{\ell=5\\ \ell \text{ odd}}}^{N+4}\epsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}b_{\ell,1}(s,y,\epsilon)\\
&+\sum_{\substack{\ell=5\\ \ell \text{ odd}}}^{N+4}\sum_{\substack{3\le q\le \ell-2\\ q \text{ odd}}}\epsilon^{2q-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}b_{\ell,q}(s,y,\epsilon)\Big]|_{s=\epsilon^2\log t, y=x/t}
\end{aligned}
\tag{2.2.38}
$$

where for $5\le \ell\le N+4$

$$
\begin{aligned}
&b_{\ell,1}\in\Sigma^{-\frac{\ell}{2}-\delta(\ell-5)} \text{ and depends only on } a_{\ell-4,1}\\
&b_{\ell,q}\in\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)} \text{ and depends only on } a_{\ell',q}, \ell'\le \min(\ell-2,N) \text{ when } q\ge 3.
\end{aligned}
\tag{2.2.39}
$$

• **Nonlinear term in (2.2.3)**
This term is given by formula (2.2.26)
• **Determination of the $a_{\ell,q}$'s**
To prove Proposition 2.2.2, we have to choose recursively the $a_{\ell,q}$'s in order to eliminate most terms in the difference (2.2.2) between (2.2.38) and (2.2.26), to be left only with terms of the form (2.2.4) or (2.2.6). We determine first the characteristic coefficient $a_{1,1}$. Equating the $t^{-\frac{3}{2}}e^{it\varphi(y)}$-terms in (2.2.26) and (2.2.38), and using expression (2.2.13) for $c_{3,1}$, we obtain

$$
\omega_0(y)\partial_s a_{1,1} = \frac{1}{2}(\phi(y)+i\psi(y))|a_{1,1}(s,y)|^2 a_{1,1}(s,y).
$$

If we take for $a_{1,1}$ the function (2.1.12), this equality is satisfied by (2.1.10), and the explicit formula (2.1.12) shows that $a_{1,1}$ belongs to $\Sigma^{-\frac{1}{2}}$.

We determine next coefficient $a_{3,3}$, equating the $t^{-\frac{3}{2}}e^{3it\varphi(y)}$ coefficients in (2.2.26) and (2.2.38). We get $-8a_{3,3} = c_{3,3}$ where $c_{3,3}$ is determined by $a_{1,1}$ according to Corollary 2.2.4 and belongs to $\Sigma^{-\frac{3}{2}} \subset \Sigma^{-\frac{3}{2}-2\delta}$.
Assume by induction that we have determined for some $\ell \geq 5$

$$a_{\ell',q'}, 1 \leq q' \leq \ell' \leq \ell - 4 \text{ and } a_{\ell-2,q'}, 3 \leq q' \leq \ell - 2. \tag{2.2.40}$$

Let us determine $a_{\ell-2,1}$. Equating the $t^{-\frac{\ell}{2}}e^{it\varphi(y)}$-terms in (2.2.26) and (2.2.38), we get using also expressions (2.2.14) to (2.2.16)

$$\partial_s a_{\ell-2,1}(s,y) = \Theta(y)[2|a_{1,1}|^2 a_{\ell-2,1} + a_{1,1}^2 \bar{a}_{\ell-2,1}](s,y) + r_{\ell-2,1}(s,y) \tag{2.2.41}$$

where $\Theta(y) = \frac{1}{2}\omega_0(y)^{-1}(\phi(y) + i\psi(y))$ and

$$r_{\ell-2}(s,y) = -\frac{i}{2\omega_0(y)}(c''_{\ell,1}(s,y) - b_{\ell,1}(s,y)). \tag{2.2.42}$$

By Corollary 2.2.4, and decompositions (2.2.14)-(2.2.16), $c''_{\ell,1}$ is in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}$ and depends only on $a_{\ell',q'}$ for $1 \leq q' \leq \ell' \leq \ell - 4$ and on $a_{\ell-2,q'}$, $3 \leq q' \leq \ell - 2$. These coefficients are determined by assumption (2.2.40). Moreover, by (2.2.39), $b_{\ell,1}$ belongs to $\Sigma^{-\frac{\ell}{2}-\delta(\ell-5)}$ and depends only on coefficients already determined. It follows that $r_{\ell-2,1}$ is known and belongs to $\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}$. If we supplement (2.2.41) by the initial condition $a_{\ell-2,1}(0,y) = 0$, we may thus apply Lemma 2.2.5 with $a \equiv a_{1,1}$ to conclude that $a_{\ell-2,1}$ belongs to $\Sigma^{-\frac{\ell-2}{2}-\delta(\ell-3)}$ as wanted in the statement of the proposition, if we check that assumptions (2.2.27), (2.2.28) hold. The first one, that is equivalent to $\frac{1}{2}\omega_0(y_0)^{-1}\phi(y_0) \neq 0$, follows from conditions (1.2.2), (1.2.3). The second one is implied by the explicit expression (2.1.12) of $a_{1,1}$ and the fact that by (2.1.8), (1.2.1) and (1.2.2), $a_1^0(y_0)$ does not vanish.
We have thus determined $a_{\ell',q'}$ for $1 \leq q' \leq \ell' \leq \ell - 2$. To obtain (2.2.40) with ℓ replaced by $\ell + 2$, we are left with finding $a_{\ell,q}$ for $3 \leq q \leq \ell$. Equating terms in $t^{-\frac{\ell}{2}}e^{itq\varphi(y)}$ in (2.2.26) and (2.2.38), we obtain an equation

$$(1 - q^2)a_{\ell,q} = c_{\ell,q} - b_{\ell,q} \in \Sigma^{-\frac{\ell}{2}-\delta(\ell-3)} \subset \Sigma^{-\frac{\ell}{2}-\delta(\ell-1)} \tag{2.2.43}$$

where $c_{\ell,q}, b_{\ell,q}$ depend only on $a_{\ell',q'}$ with $1 \leq q' \leq \ell - 2$ by (2.2.39) and Corollary 2.2.4. Consequently, the right hand side of (2.2.43) is already determined , so that we have defined $a_{\ell-2,q}$ for $3 \leq q \leq \ell$. We have finally recovered (2.2.40) at rank $\ell + 2$.
Consequently, we have eliminated all characteristic terms in (2.2.38) that are $O(t^{-\frac{\ell}{2}})$ for $\ell \leq N+2$ and all non-characteristic terms that are $O(t^{-\frac{\ell}{2}})$ for $\ell \leq N$. We are thus left with the terms in the third (resp. fourth) sum in (2.2.38) corresponding to $\ell = N + 4$ (resp. $\ell = N + 2$ or $N + 4$) and with the terms in the first (resp. second) sum in (2.2.26) corresponding to $N + 4 \leq \ell \leq 3N$ (resp. $N + 2 \leq \ell \leq 3N$). These terms contribute to (2.2.4) and (2.2.6). This concludes the proof. □

We construct now an approximate solution to equation (1.1.1) defied for $t \in [1, e^{\frac{T_*}{\epsilon^2}}[$, gluing together the approximate solution for moderate times $u_{\text{app}}^{\text{M}}$ defined in Proposition 2.1.6 and the approximate solution u_{app}^2 of Proposition 2.2.2.

Corollary 2.2.6 *Let $\tilde{\chi}_0$ be in $C_0^\infty([0, \frac{3T_*}{4}[)$ be equal to one on $[0, \frac{T_*}{2}]$. Define for $t \in [1, e^{\frac{T_*}{\epsilon^2}}[$*

$$u_{\text{app}}(t,x) = \tilde{\chi}_0(\epsilon^2 \log t)u_{\text{app}}^{\text{M}}(t,x) + (1 - \tilde{\chi}_0)(\epsilon^2 \log t)u_{\text{app}}^2(t,x). \tag{2.2.44}$$

Then

$$r_{\rm app}(t,x) = (\partial_t^2 - \partial_x^2 + 1)u_{\rm app} - P(u_{\rm app}, \partial_t u_{\rm app}, \partial_x u_{\rm app}) \tag{2.2.45}$$

may be written as a sum

$$\begin{aligned} 2{\rm Re}\Big[\frac{\epsilon}{t^{\frac52}} e^{it\varphi(y)}\chi_1(\epsilon^{1-\theta}t)c_{5,1}(s,y,\frac1t,\epsilon)\Big]|_{s=\epsilon^2\log t, y=x/t} \\ + (1-\tilde\chi_0)(\epsilon^2\log t) r_{\rm app}^2(t,x) + F_{\rm app}(t,x) \end{aligned} \tag{2.2.46}$$

where χ_1 is smooth, equal to zero close to zero and to 1 outside a neighborhood of zero, where $c_{5,1}(s,y,h,\epsilon)$ is a continuous function on $[0,+\infty[\times\mathbb{R}\times]0,1]\times[0,1]$, supported for $s\le \frac{3T_*}{4}$ and $|y|\le 1$, bounded as well as all its $\partial_s, \partial_y, h\partial_h$ derivatives on that domain, where $r_{\rm app}^2$ given by (2.2.3) is the sum of (2.2.4) and (2.2.6) and where $F_{\rm app}$ is compactly supported for $t\le e^{\frac{3T_*}{4\epsilon^2}}$ and satisfies

$$\begin{aligned} \int_1^{\exp(T_*/\epsilon^2)} \|F_{\rm app}(\tau,\cdot)\|_{H^s}\, d\tau \le C\epsilon^{2-\theta} \\ \int_1^{\exp(T_*/\epsilon^2)} \|L_\pm F_{\rm app}(\tau,\cdot)\|_{H^1}\, d\tau \le C\epsilon^{2-\theta}. \end{aligned} \tag{2.2.47}$$

Proof: By the definition of $u_{\rm app}$ and (2.1.34), (2.2.3), we may write

$$\begin{aligned} r_{\rm app}(t,x) &= \tilde\chi_0(\epsilon^2\log t) r_{\rm app}^{\rm M}(t,x) + (1-\tilde\chi_0)(\epsilon^2\log t) r_{\rm app}^2(t,x) \\ &+ 2\frac{\epsilon^2}{t}\tilde\chi_0'(\epsilon^2\log t)\partial_t(u_{\rm app}^{\rm M} - u_{\rm app}^2)(t,x) \\ &+ \Big(\frac{\epsilon^4}{t^2}\tilde\chi_0''(\epsilon^2\log t) - \frac{\epsilon^2}{t^2}\tilde\chi_0'(\epsilon^2\log t)\Big)(u_{\rm app}^{\rm M} - u_{\rm app}^2)(t,x) \\ &- P(u_{\rm app}, \partial_t u_{\rm app}, \partial_x u_{\rm app}) + \tilde\chi_0(\epsilon^2\log t) P(u_{\rm app}^{\rm M}, \partial_t u_{\rm app}^{\rm M}, \partial_x u_{\rm app}^{\rm M}) \\ &+ (1-\tilde\chi_0)(\epsilon^2\log t) P(u_{\rm app}^2, \partial_t u_{\rm app}^2, \partial_x u_{\rm app}^2) \\ &= {\rm I} + \cdots + {\rm VII}. \end{aligned} \tag{2.2.48}$$

We examine successively terms I to VII.

• **Contribution of term I**: By Proposition 2.1.6, we get contributions to the first term in (2.2.46) and to $F_{\rm app}$.

• **Contribution of term II**: This is the second term in (2.2.46).

• **Contribution of terms III + IV**: On the support of $\tilde\chi_0'(\epsilon^2\log t)$, $u_{\rm app}^{\rm M}$ coincides with $u_{\rm app}^1$ by (2.1.33), so that we have to estimate $u_{\rm app}^1 - u_{\rm app}^2$ and its time derivative. This difference may be computed from (2.1.13) and (2.2.2). The $t^{-\frac12}$ terms cancel out. We are thus reduced to the following terms:

– Characteristic terms in $O(t^{-\frac32})$ coming from the $t^{-\frac32}a_{3,1}$ term in (2.2.2): this provides a contribution to the first term in (2.2.46).

– Non-characteristic terms in $O(t^{-\frac32})$ coming from the $a_{3,3}$ term in (2.2.2) and the $a_{3,3}^1$ term in (2.1.13): when plugged inside III + IV, these terms give contributions

$$\Big[\frac{\epsilon^5}{t^{\frac52}}\underline{\chi}_0(s)e^{3it\varphi(y)}\tilde a_{3,3}(s,y,\frac1t,\epsilon)\Big]|_{s=\epsilon^2\log t, y=x/t} \tag{2.2.49}$$

where $\underline{\chi}_0 \in C_0^\infty(]0,+\infty[)$ and where $\tilde{a}_{3,3}(s,y,h,\epsilon)$ is continuous on $[0,+\infty[\times\mathbb{R}\times]0,1]\times[0,1]$, supported for $s \leq \frac{3T_*}{4}$ and $|y| \leq 1$, bounded as well as all its $\partial_s, \partial_y, h\partial_h$ derivatives on that domain. The Sobolev norm of (2.2.49) is $O(\epsilon^5 t^{-2})$ so that the first estimate (2.2.47) largely holds. If we make act $L_\pm$ on (2.2.49) and bound the H^1 norm, we get a $O(\epsilon^5 t^{-1})$ estimate. Integrating for $1 \leq t \leq e^{\frac{3T_*}{4\epsilon^2}}$ gives a $0(\epsilon^3)$ bound, better than the right hand side of the second inequality (2.2.47). Thus (2.2.49) may be included in $F_{\rm app}$ in (2.2.46).

– Characteristic or non-characteristic terms coming from (2.1.13) or (2.2.2) that are $O(t^{-\frac{5}{2}})$, i.e. terms in $a^1_{5,3}$, $a^1_{5,5}$ in (2.1.13) and $a_{\ell,q}$, $\ell \geq 5$ in (2.2.2): The contributions of all such terms to III + IV may be written under the form

$$\text{Re}\Big[\frac{\epsilon^{6-\ell}}{t^{\frac{\ell}{2}}}\underline{\chi}_0(s)e^{iqt\varphi(y)}\tilde{a}_{\ell,q}(s,y,\frac{1}{t},\epsilon)\Big]|_{s=\epsilon^2\log t, y=x/t} \tag{2.2.50}$$

with $\ell \geq 7$ and $\tilde{a}_{\ell,q}$ satisfying the same estimates as $\tilde{a}_{3,3}$ above. Then the Sobolev norm of (2.2.50) or its H^1 norm after action of $L_\pm$, integrated for t in the support of $\underline{\chi}_0(\epsilon^2\log t)$ is $O(e^{-c/\epsilon^2})$, so that (2.2.47) is largely verified and these terms may be included inside $F_{\rm app}$.

• **Contributions of V + VI + VII:** We write this contribution as the sum of

$$P(u_{\rm app}, \tilde{\chi}_0\partial_t u^{\rm M}_{\rm app} + (1-\tilde{\chi}_0)\partial_t u^2_{\rm app}, \partial_x u_{\rm app}) - P(u_{\rm app}, \partial_t(\tilde{\chi}_0 u^{\rm M}_{\rm app} + (1-\tilde{\chi}_0)u^2_{\rm app}), \partial_x u_{\rm app}) \tag{2.2.51}$$

and of

$$\begin{aligned}&\tilde{\chi}_0 P(u^{\rm M}_{\rm app}, \partial_t u^{\rm M}_{\rm app}, \partial_x u^{\rm M}_{\rm app}) + (1-\tilde{\chi}_0)P(u^2_{\rm app}, \partial_t u^2_{\rm app}, \partial_x u^2_{\rm app})\\ &- P\big(\tilde{\chi}_0 u^{\rm M}_{\rm app} + (1-\tilde{\chi}_0)u^2_{\rm app}, \tilde{\chi}_0\partial_t u^{\rm M}_{\rm app} + (1-\tilde{\chi}_0)\partial_t u^2_{\rm app}, \tilde{\chi}_0\partial_x u^{\rm M}_{\rm app} + (1-\tilde{\chi}_0)\partial_x u^2_{\rm app}\big).\end{aligned} \tag{2.2.52}$$

Consider first (2.2.51). This expression may be bounded point-wise by

$$\frac{\epsilon^2}{t}|\tilde{\chi}_0'(\epsilon^2\log t)||u^{\rm M}_{\rm app} - u^2_{\rm app}|\Big(\sum_{\alpha+\beta\leq 1}|\partial_t^\alpha\partial_x^\beta u^{\rm M}_{\rm app}| + |\partial_t^\alpha\partial_x^\beta u^2_{\rm app}|\Big)^2. \tag{2.2.53}$$

We have seen in the study of III + IV that the $t^{-\frac{1}{2}}$ terms cancel out in $u^{\rm M}_{\rm app} - u^2_{\rm app}$, so that this difference is $O(\epsilon^{-a}t^{-\frac{3}{2}})$ for some a. The squared factor in (2.2.52) is moreover $O(\epsilon^2/t)$ so that we may get for (2.2.53) a bound in $O(\epsilon^{-a}t^{-\frac{7}{2}}\mathbb{1}_{|x|\leq t})$. The same holds for derivatives of (2.2.51) so that, computing its H^s norm or the H^1 norm of the action of $L_\pm$ on it, we shall obtain as at the end of the study of III + IV that the time integral of these quantities is $O(e^{-c/\epsilon^2})$. Thus (2.2.51) largely satisfies (2.2.47).

Finally, consider (2.2.52) that may be written $-\psi(\tilde{\chi}_0(\epsilon^2\log t))$ with

$$\begin{aligned}\psi(\mu) = P(&\mu u^{\rm M}_{\rm app} + (1-\mu)u^2_{\rm app}, \mu\partial_t u^{\rm M}_{\rm app} + (1-\mu)\partial_t u^2_{\rm app}, \mu\partial_x u^{\rm M}_{\rm app} + (1-\mu)\partial_x u^2_{\rm app})\\ &-\mu P(u^{\rm M}_{\rm app}, \partial_t u^{\rm M}_{\rm app}, \partial_x u^{\rm M}_{\rm app}) - (1-\mu)P(u^2_{\rm app}, \partial_t u^2_{\rm app}, \partial_x u^2_{\rm app}).\end{aligned}$$

As $\psi(1) = \psi(0) = 0$, we have

$$\begin{aligned}|\psi(\tilde{\chi}_0(\epsilon^2\log t))| &\leq (1-\tilde{\chi}_0)\tilde{\chi}_0 \sup_{\mu\in[0,1]}|\psi''(\mu)|\\ &\leq C(1-\tilde{\chi}_0)\tilde{\chi}_0\big[|u^{\rm M}_{\rm app} - u^2_{\rm app}| + |\partial_t(u^{\rm M}_{\rm app} - u^2_{\rm app})| + |\partial_x(u^{\rm M}_{\rm app} - u^2_{\rm app})|\big]^2\\ &\qquad\times\sum_{\alpha+\beta\leq 1}\big(|\partial_t^\alpha\partial_x^\beta u^{\rm M}_{\rm app}| + |\partial_t^\alpha\partial_x^\beta u^2_{\rm app}|\big).\end{aligned} \tag{2.2.54}$$

We have seen in the study of (2.2.51) that $u^{\rm M}_{\rm app} - u^2_{\rm app}$ is $O(\epsilon^{-a}t^{-\frac{3}{2}})$, as well as its derivatives, on the support of $(1-\tilde{\chi}_0)\tilde{\chi}_0(\epsilon^2\log t)$. It follows that again (2.2.54) is $O(\epsilon^{-a}t^{-\frac{7}{2}})$ and supported for $|x| \leq t$. As the same bound holds for derivatives of (2.2.52), we conclude that this term satisfies as well (2.2.47). This concludes the proof. □

3 Reduction to a system and normal form

In this section, we shall reduce equation (1.1.1) to a first order system. We shall then look for the solution as the sum of an approximate solution deduced from $u_{\rm app}$ constructed in section 2 and of a remainder. Finally, in subsection 3.2, we shall perform a normal form procedure in order to eliminate part of the cubic nonlinearity.

3.1 Reduction to a system

Let us introduce a notation that will be used in the rest of the paper. We shall denote by $M_0(\xi_1,\dots,\xi_n)$ a smooth positive function on $\mathbb{R}^n$, with values in $\mathbb{R}_+^*$, such that $M_0(\xi_1,\dots,\xi_n)$ is equivalent to $1+\max_2(|\xi_1|,\dots,|\xi_n|)$, where $\max_2$ stands for the second largest among $|\xi_1|,\dots,|\xi_n|$. For instance, we may take

$$M_0(\xi_1,\dots,\xi_n)=\Big(\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ |\alpha|=n\\ \max(\alpha_j)\le n-1}}(\xi^\alpha)^2+1\Big)^{\frac12}\Big(1+\sum_{j=1}^n\xi_j^2\Big)^{-\frac{n-1}{2}}. \tag{3.1.1}$$

Definition 3.1.1 *Let $n\in\mathbb{N}^*,\nu\in\mathbb{R}$, $\kappa\in\mathbb{R}_+,\beta\in\mathbb{R}_+$. We denote by $S_{\kappa,\beta}(M_0^\nu,n)$ the space of smooth functions on $\mathbb{R}\times\mathbb{R}^n$, $(x,\xi_1,\dots,\xi_n)\to m(x,\xi_1,\dots,\xi_n)$, with values in $\mathbb{C}$, satisfying for any $\alpha_0\in\mathbb{N}$, $\alpha\in\mathbb{N}^*$, $N\in\mathbb{N}$ estimates*

$$|\partial_x^{\alpha_0}\partial_\xi^\alpha m(x,\xi_1,\dots,\xi_n)|\le C_{\alpha_0,\alpha,N}M_0(\xi)^{\nu+\kappa(\alpha_0+|\alpha|)}(1+\beta h^\beta|\xi|)^{-N}. \tag{3.1.2}$$

Remark: Most of the time we shall only need the special case $\beta=0$, so that the last factor in the right hand side of (3.1.2) disappears. If m is in $S_{\kappa,0}(M_0^\nu,n)$ and $\chi\in C_0^\infty(\mathbb{R}^n)$, then $m(x,\xi)\chi(h^\beta\xi)$ is in $S_{\kappa,\beta}(M_0^\nu,n)$ for $\beta>0$.

If m is in $S_{\kappa,\beta}(M_0^\nu,n)$ and if $u_1,\dots,u_n$ are in $\mathcal{S}(\mathbb{R}^n)$, we set

$$\mathrm{Op}(m)(u_1,\dots,u_n)=\frac{1}{(2\pi)^n}\int e^{ix(\xi_1+\dots+\xi_n)}m(x,\xi_1,\dots,\xi_n)\prod_{j=1}^n\hat u_j(\xi_j)\,d\xi_1\cdots d\xi_n. \tag{3.1.3}$$

In appendix A.2, we observe that (3.1.3) remains meaningful when u_j belongs to Sobolev spaces of high enough order, so that we may use (3.1.3) for the solution to our problem.

Let $u\to u(t,x)$ be defined on $[1,T[\times\mathbb{R}$ for some $T\in]1,e^{\frac{T_*}{\epsilon^2}}[$ with values in $\mathbb{R}$, which is in $C^0([1,T[,H^s(\mathbb{R}))\cap C^1([1,T[,H^{s-1}(\mathbb{R}))$ for some large enough s, solving equation (1.1.1). We define, with the notation $p(D_x)=\sqrt{1+D_x^2}$,

$$u_\pm=(D_t\pm p(D_x))u \tag{3.1.4}$$

so that

$$u_-=-\bar u_+,\ u=\frac12 p(D_x)^{-1}(u_+-u_-),\ \partial_t u=\frac{i}{2}(u_++u_-). \tag{3.1.5}$$

If $I=(i_1,i_2,i_3)$ is an element of $\{-,+\}^3$, we set $u_I=(u_{i_1},u_{i_2},u_{i_3})$. If we express u and its derivatives from (3.1.5) in (1.1.2), we may write

$$P(u,\partial_t u,\partial_x u)=-\sum_{I\in\{-,+\}^3}\mathrm{Op}(m_I)(u_I) \tag{3.1.6}$$

for some m_I's in $S_{0,0}(1,3)$ (with constant coefficients). Consequently equation (1.1.1) is equivalent to

$$(D_t - p(D_x))u_+ = \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I)(u_I). \tag{3.1.7}$$

Of course, by conjugation, using (3.1.5), we have

$$(D_t + p(D_x))u_- = \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I^-)(u_{\bar I}) \tag{3.1.8}$$

where $\bar I = -I$ and

$$m_I^-(x,\xi_1,\dots,\xi_n) = (-1)^n\overline{m(x,-\xi_1,\dots,-\xi_n)}. \tag{3.1.9}$$

Let us construct from the approximate solution u_{app} of Corollary 2.2.6 an approximate solution u_+^{app} of equation (3.1.7). We shall do that when the time t stays smaller than the time $T(\epsilon)$ defined in (1.2.5). We shall use the following inequality, with $\delta > 0$ introduced in Proposition 2.2.2 and $\delta' > 0, \gamma > 0$ to be chosen:

$$\begin{aligned}&\text{There is } \epsilon_0 \in]0,1] \text{ such that if } 0<\epsilon<\epsilon_0 \text{ and } t\in [e^{\frac{T_*}{2\epsilon^2}}, T(\epsilon)[\text{ then}\\ &\qquad t^{-\frac12}(T_*-\epsilon^2\log t)^{-\frac12-\delta} < \epsilon^{\frac{\gamma}{2}-\delta'}(T_*-\epsilon^2\log t)^{\delta'-\delta}.\end{aligned} \tag{3.1.10}$$

Actually, this inequality is trivial if $T_* - \epsilon^2\log t \geq \epsilon^2$ since, as $t \geq e^{\frac{T_*}{2\epsilon^2}}$, the factor $t^{-\frac12}$ in the left hand side is then exponentially decaying, so that (3.1.10) holds for small enough ϵ. If $u = \frac{T_*-\epsilon^2\log t}{\epsilon^2} \leq 1$, then inequality (3.1.10) is equivalent to

$$ue^{-\frac{u}{1+2\delta'}} > \epsilon^{-\frac{2+\gamma+2\delta'}{1+2\delta'}} e^{-\frac{T_*}{\epsilon^2(1+2\delta')}} \tag{3.1.11}$$

whose right hand side is the quantity ϵ' introduced in (1.2.4). Since $u \to ue^{-\frac{u}{1+2\delta'}}$ is strictly increasing on $[0,1]$ if $\delta' > 0$, inequality (3.1.11) is equivalent to $u > u(\epsilon')$ where $u(\epsilon')$ has been defined before (1.2.5). But by the definitions of u and of $T(\epsilon)$ in (1.2.5), this means $t < T(\epsilon)$. In the sequel, the parameters δ, δ', γ will be chosen positive, with δ and δ' small, satisfying the inequalities

$$\delta' > \delta, \gamma \geq 2(\delta'+2). \tag{3.1.12}$$

We notice for further reference that (3.1.10) implies that $t^{-1}(T_*-\epsilon^2\log t)^{-1} = O(1)$, so that, when $t \in [1, T(\epsilon)[$ the definition (2.2.1) of classes Σ^m shows that

$$a\in\Sigma^m \Rightarrow \partial_t^\alpha\partial_x^\beta[a(\epsilon^2\log t, \frac{x}{t}, \frac{1}{t}, \epsilon)] = b(\epsilon^2\log t, \frac{x}{t}, \frac{1}{t}, \epsilon) \tag{3.1.13}$$

for some b in Σ^m. We define from the approximate solution u_{app} of Corollary 2.2.6

$$\tilde u_+^{\mathrm{app}} = (D_t + p(D_x))u_{\mathrm{app}},\ \tilde u_-^{\mathrm{app}} = -\overline{\tilde u_+^{\mathrm{app}}} \tag{3.1.14}$$

and $\tilde u_I^{\mathrm{app}} = (\tilde u_{i_1}^{\mathrm{app}}, \tilde u_{i_2}^{\mathrm{app}}, \tilde u_{i_3}^{\mathrm{app}})$ if $I = (i_1,i_2,i_3)$. Then, by (2.2.44), (2.2.45), (2.2.46) and (3.1.6)

$$\begin{aligned}&(D_t - p(D_x))\tilde u_+^{\mathrm{app}} - \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I)(\tilde u_I^{\mathrm{app}})\\ &= -\Big(\frac{\epsilon}{t^{\frac52}} e^{it\varphi(y)}\chi_1(\epsilon^{1-\theta}t)c_{5,1}(s,y,\frac1t,\epsilon) + \frac{\epsilon}{t^{\frac52}}e^{-it\varphi(y)}\chi_1(\epsilon^{1-\theta}t)c_{5,-1}(s,y,\frac1t,\epsilon)\Big)|_{s=\epsilon^2\log t, y=x/t}\\ &\qquad - F_{\mathrm{app}}(t,x) - (1-\tilde\chi_0)(\epsilon^2\log t)r_{\mathrm{app}}^2(t,x)\end{aligned} \tag{3.1.15}$$

where $c_{5,-1} = \bar{c}_{5,1}$ is supported for $s \leq \frac{3T_*}{4}$, $|y| \leq 1$ and $F_{\rm app}(t,x)$ for $t \leq e^{\frac{3T_*}{4\epsilon^2}}$, and satisfies (2.2.47). In the right hand side of (3.1.15), we have a $e^{it\varphi(x/t)}$-term that is characteristic for $D_t - p(D_x)$ and a $e^{-it\varphi(x/t)}$-term that is non-characteristic for the same operator. We start by eliminating the non-characteristic term introducing a modification $u_+^{\rm app}$ of $\tilde{u}_+^{\rm app}$. We first define this function and study its structure.

Lemma 3.1.2 *Define*

$$u_+^{\rm app}(t,x) = \tilde{u}_+^{\rm app}(t,x) - \frac{\epsilon}{2t^{\frac{5}{2}}} e^{-it\varphi(y)} \chi_1(\epsilon^{1-\theta}t)\sqrt{1-y^2}\, c_{5,-1}(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2\log t, y=x/t}. \tag{3.1.16}$$

Then we may write for $1 \leq t \leq T(\epsilon)$,

$$\begin{aligned} u_+^{\rm app}(t,x) &= \chi_0(\epsilon^{1-\theta}(t-1))u_{0,+}(t,x) \\ &+ e^{it\varphi(y)}(1-\chi_0)(\epsilon^{1-\theta}(t-1))\frac{\epsilon}{\sqrt{t}} a_{1,1}^+(s,y)|_{s=\epsilon^2\log t, y=x/t} \\ &+ \sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N+1} \sum_{\substack{1\leq|q|\leq\ell\\ q \text{ odd}}} e^{itq\varphi(y)} t^{-\frac{\ell}{2}} a_{\ell,q}^+(s,y,\frac{1}{t},\epsilon) e_{\ell,q}(t,\epsilon)|_{s=\epsilon^2\log t, y=x/t} \\ &+ \epsilon r(t,x) \end{aligned} \tag{3.1.17}$$

where χ_0 *has been introduced in Proposition 2.1.6, where we denoted*

$$u_{0,+}(t,x) = (D_t + p(D_x))u_0, \tag{3.1.18}$$

where $a_{1,1}^+(s,y) = 2(1-y^2)^{-\frac{1}{2}} a_{1,1}(s,y)$, $a_{1,1}$ *being defined in (2.1.12), where* $a_{\ell,q}^+(s,y,h,\epsilon)$ *are elements of* $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$ *and where* $e_{\ell,q}(t,\epsilon)$ *satisfy for any* ζ

$$\begin{aligned} (t\partial_t)^\zeta e_{\ell,q}(t,\epsilon) &= O(\epsilon) \text{ if } t \leq e^{\frac{3T_*}{4\epsilon^2}} \\ (t\partial_t)^\zeta e_{\ell,q}(t,\epsilon) &= O(\epsilon^{2|q|-\ell}) \text{ if } t \geq e^{\frac{T_*}{2\epsilon^2}}, q \neq -1 \\ (t\partial_t)^\zeta e_{\ell,-1}(t,\epsilon) &= O(\epsilon^{4-\ell}) \text{ if } t \geq e^{\frac{T_*}{2\epsilon^2}} \end{aligned} \tag{3.1.19}$$

and where $r(t,x)$ *is a smooth function satisfying for all* α, β, N,

$$|\partial_t^\alpha \partial_x^\beta r(t,x)| \leq C(t+|x|)^{-N}. \tag{3.1.20}$$

Moreover,we may write $u_{0,+}(t,x)$ *under the form*

$$u_{0,+}(t,x) = \frac{\epsilon}{\sqrt{t}} e^{it\varphi(x/t)} a_1^+\Big(\frac{x}{t},\frac{1}{t}\Big) + \frac{\epsilon}{t^{\frac{3}{2}}} e^{-it\varphi(x/t)} a_1^-\Big(\frac{x}{t},\frac{1}{t}\Big) + \epsilon r_0(t,x) \tag{3.1.21}$$

where $a_1^\pm(y,h)$ *are continuous on* $\mathbb{R}\times]0,1]$, *bounded as well as their* ∂_y *and* $h\partial_h$*-derivatives on that domain, supported for* $|y| \leq 1$, *and where* r_0 *satisfies (3.1.20).*

Proof: Consider first an element $a_{\ell,q}$ of $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$, with $1 \leq |q| \leq \ell$. We may apply Corollary A.1.4 of the appendix to compute $p(D_x)[e^{iqt\varphi(x/t)} a_{\ell,q}(\epsilon^2\log t, \frac{x}{t}, \frac{1}{t}, \epsilon)]$ since the assumption $t(T_*-s)^{\frac{1}{2\kappa_0}} \geq c$ of the appendix is satisfied: this is trivial for $s \leq \frac{T_*}{2}$ and holds for $s = \epsilon^2\log t \geq \frac{T_*}{2}$ and $t < T(\epsilon)$ by (3.1.10) (for ϵ small enough). By this corollary and estimates (2.2.1), we have

$$\begin{aligned} &(D_t + p(D_x))\Big[e^{iqt\varphi(y)} a_{\ell,q}(s,y,\frac{1}{t},\epsilon)\Big]|_{s=\epsilon^2\log t, y=x/t} \\ &= \Big(\frac{q+\sqrt{1+(q^2-1)y^2}}{\sqrt{1-y^2}}\Big) e^{itq\varphi(y)} a_{\ell,q}(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2\log t, y=x/t} \\ &\qquad + \frac{1}{t} e^{itq\varphi(y)} a_{\ell,q}^1(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2\log t, y=x/t} + r(t,x) \end{aligned} \tag{3.1.22}$$

where $a^1_{\ell,q}$ is in $\Sigma^{-\frac{\ell}{2}-1-\delta(\ell-1)}$ and r satisfies (3.1.20).
We compute first $\tilde{u}^{\rm app}_+$ in (3.1.16) from its definition (3.1.14) making act $(D_t + p(D_x))$ on the definition (2.2.44) of $u_{\rm app}$. Using expression (2.1.33) of $u^{\rm M}_{\rm app}$, we get

$$\begin{aligned}\tilde{u}^{\rm app}_+ &= \chi_0(\epsilon^{1-\theta}(t-1))(D_t+p(D_x))u_0\\ &\quad + (1-\chi_0)(\epsilon^{1-\theta}(t-1))\tilde{\chi}_0(\epsilon^2\log t)(D_t+p(D_x))u^1_{\rm app}\\ &\quad + (1-\tilde{\chi}_0)(\epsilon^2\log t)(D_t+p(D_x))u^2_{\rm app}\\ &\quad - i\epsilon^{1-\theta}\chi_0'(\epsilon^{1-\theta}(t-1))u_0(t,x)\\ &\quad + i\big(\epsilon^{1-\theta}\chi_0'(\epsilon^{1-\theta}(t-1)) - \epsilon^2 t^{-1}\tilde{\chi}_0'(\epsilon^2\log t)\big)u^1_{\rm app}(t,x)\\ &\quad + i\epsilon^2 t^{-1}\tilde{\chi}_0'(\epsilon^2\log t)u^2_{\rm app}(t,x)\\ &= {\rm I}+\cdots+{\rm VI}.\end{aligned} \tag{3.1.23}$$

We study successively terms I to VI above in order to obtain expressions (3.1.17) from (3.1.16).
• **Term I:** This provides the first term in the right hand side of (3.1.17) by (3.1.18).
• **Term II:** Recall that $u^1_{\rm app}$ is given by (2.1.13). We may apply (3.1.22) to all terms in that sum. Since each of these terms is supported for $s=\epsilon^2\log t \le \frac{3T_*}{4}$ and since the first term in the right hand side of (3.1.22) vanishes if $q=-1$, we shall get that

$$\begin{aligned}{\rm II} &= (1-\chi_0)(\epsilon^{1-\theta}(t-1))\tilde{\chi}_0(s)\frac{\epsilon}{\sqrt{t}}e^{it\varphi(y)}a^+_{1,1}(s,y,\epsilon)|_{s=\epsilon^2\log t,y=x/t}\\ &\quad + \sum_{\substack{\ell=3\\ \ell\ \text{odd}}}^{5}\ \sum_{\substack{1\le|q|\le\ell\\ q\ \text{odd}}}(1-\chi_0)(\epsilon^{1-\theta}(t-1))\frac{\epsilon}{t^{\frac{\ell}{2}}}e^{itq\varphi(y)}a^{+,1}_{\ell,q}(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2\log t,y=x/t}\\ &\quad + \epsilon r(t,x)\end{aligned} \tag{3.1.24}$$

with $a^{+,1}_{1,1}(s,y)$ defined in the statement of the lemma, and where the $a^{+,1}_{\ell,q}$, $\ell=3,5$ are elements of $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$ and are supported for $0\le s\le\frac{3T_*}{4}$ and $|y|\le 1$. Thus, (3.1.24) provides a contribution to the last sum in (3.1.17) and to ϵr. Notice also that the last term in (3.1.16) may be written also as a contribution to the sum in (3.1.17) with $\ell=5$, $q=-1$.
• **Term III:** We make act $D_t+p(D_x)$ on the sums (2.2.2). Consider first the contributions coming from the second sum

$$(D_t+p(D_x))\big[\epsilon^{2|q|-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}a_{\ell,q}(s,y,\epsilon)\big]|_{s=\epsilon^2\log t,y=x/t}. \tag{3.1.25}$$

According to (3.1.22), we get a first term which is of the form of the (ℓ,q)-term in the sum (3.1.17) with $3\le|q|\le\ell$. The second term in the right hand side of (3.1.22) may be also be written under this form: actually, we may write it from

$$t^{-\frac{\ell}{2}}e^{itq\varphi(y)}(T_*-s)a^1_{\ell,q}(s,y,\frac{1}{t},\epsilon)e_{\ell,q}(t,\epsilon)|_{s=\epsilon^2\log t,y=x/t}$$

with $e_{\ell,q}(t,\epsilon)=\epsilon^{2|q|-\ell}t^{-1}(T_*-\epsilon^2\log t)^{-1}\underline{\chi}_1(\epsilon^2\log t)$ for some function $\underline{\chi}_1$ supported for $s\ge\frac{T_*}{2}$. Then property (3.1.10) shows that for $t<T(\epsilon)$, $e_{\ell,q}$ satisfies the second inequality (3.1.19) when $3\le|q|\le\ell$.
We consider next the contributions coming from the first sum in (2.2.2). We have to study

$$(D_t+p(D_x))\big[\epsilon^{2-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}a_{\ell,1}(s,y,\epsilon)\big]|_{s=\epsilon^2\log t,y=x/t} \tag{3.1.26}$$

$$(D_t+p(D_x))\big[\epsilon^{2-\ell}t^{-\frac{\ell}{2}}e^{-it\varphi(y)}a_{\ell,-1}(s,y,\epsilon)\big]|_{s=\epsilon^2\log t,y=x/t} \tag{3.1.27}$$

with $a_{\ell,-1} = \overline{a_{\ell,1}}$. We apply (3.1.22) to (3.1.26). We get a first term that may be written

$$\epsilon^{2-\ell} t^{-\frac{\ell}{2}} 2(1-y^2)^{-\frac{1}{2}} e^{it\varphi(y)} a_{\ell,1}(s,y,\epsilon)|_{s=\epsilon^2 \log t, y=x/t}. \tag{3.1.28}$$

For $\ell = 1$, this brings the second term in the right hand side of (3.1.17), when we combine it with the first term in the right hand side of II in (3.1.24), since we defined $a^+_{1,1}(s,y,\epsilon) = 2(1-y^2)^{-\frac{1}{2}} a_{1,1}(s,y,\epsilon)$. Terms (3.1.28) with $\ell \geq 3$ contribute to the last sum in (3.1.17) with $q = 1$. On the other hand, the second term in the right hand side of (3.1.22) applied to (3.1.26) is of the form

$$\epsilon^{2-\ell} t^{-\frac{\ell+2}{2}} e^{it\varphi(y)} a^1_{\ell,1}(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2 \log t, y=x/t} \tag{3.1.29}$$

with $a^1_{\ell,1}$ in $\Sigma^{-\frac{\ell+2}{2}-\delta(\ell-1)} \subset \Sigma^{-\frac{\ell+2}{2}-\delta(\ell+2-1)}$. For any $\ell \geq 1$, we may incorporate that term to the sum in (3.1.17) with coefficients $e_{\ell+2,1}$ satisfying (3.1.19) with $q = 1$. Notice also that the remainder in (3.1.22) may be incorporated to the one in (3.1.17), in spite of the negative powers of ϵ that may appear, since term III is supported for $t \geq e^{\frac{T_*}{\epsilon^2}}$, so that the rapid decay in (3.1.20) brings also smallness in ϵ.
We still have to cope with (3.1.27). Because the oscillatory term is $e^{-it\varphi(x/t)}$, when we apply (3.1.22) with $q = -1$, the first term disappears, and we are left only with a term of the form (3.1.29) with $e^{it\varphi}$ replaced by $e^{-it\varphi}$. Such a term may be rewritten as

$$\epsilon^{4-\ell} e^{-it\varphi} t^{-\frac{\ell}{2}} a^+_{\ell,-1}(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2 \log t, y=x/t}$$

for some $a^+_{\ell,-1}$ in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$ and $\ell \geq 3$ i.e. brings a contribution to the sum in (3.1.17) with $q = -1$ and a coefficient $e_{\ell,-1}$ which is $O(\epsilon^{4-\ell})$ as in the last equality (3.1.19) and not just $O(\epsilon^{2-\ell})$.
This concludes the treatment of term III in (3.1.23).
• **Term IV:** If we use expansion (2.1.7) of u_0 and again (3.1.22), we see that this term may be rewritten as a contribution to the $t^{-\frac{3}{2}} e^{\pm it\varphi}$ term in the sum (3.1.17), with a coefficient $e_{3,1}$ satisfying the first bound (3.1.19) and to the remainder ϵr.
• **Term V:** Using (2.1.13), we see in the same way that this term may be written as a contribution to the sum in (3.1.17), with coefficients $e_{\ell,q}$ satisfying the first bound (3.1.19).
• **Term VI:** We use (2.2.2), which implies that VI may be written as a contribution to the last sum in (3.1.17) with coefficients satisfying the second or third equality in (3.1.19). This concludes the proof of equality (3.1.17).
To obtain (3.1.21), we notice that we may apply Corollary A.1.4 in the special case when functions $a(s,y,h,\epsilon)$ of that corollary are replaced by smooth functions of the sole variable y supported for $|y| \leq 1$ and use again (3.1.22) in that context. Using expansion (2.1.7) of u_0, we thus get (3.1.21). □

Next we shall check that the function $u^{\rm app}_+$ defined in (3.1.16) will provide an approximate solution for the nonlinear equation given by the left hand side of (3.1.15).

Proposition 3.1.3 *Let N_0 be an integer. Then if we define the approximate solution $u^{\rm app}_+$ by (3.1.16), (3.1.17), with N large enough relatively to N_0, $u^{\rm app}_+$ solves an equation*

$$(D_t - p(D_x))u^{\rm app}_+ - \sum_{I \in \{-,+\}^3} {\rm Op}(m_I)(u^{\rm app}_I) = -(F + r_{\rm app}) \tag{3.1.30}$$

where $u_I^{\rm app} = (u_{i_1}^{\rm app}, u_{i_2}^{\rm app}, u_{i_3}^{\rm app})$, *and where the source term is given from a function* $F(t,x)$ *supported for* $1 \leq t \leq e^{\frac{3T_*}{4\epsilon^2}}$, *that satisfies for any* s_0 *in* $\mathbb{R}$

$$\begin{aligned} &\int_1^{+\infty} \|F(t,\cdot)\|_{H^{s_0}}\, dt \leq C_{s_0}\epsilon^{2-\theta} \\ &\int_1^{+\infty} \|L_+F(t,\cdot)\|_{H^1}\, dt \leq C\epsilon^{2-\theta} \end{aligned} \tag{3.1.31}$$

and from a function $(t,x) \to r_{\rm app}(t,x)$ *supported for* $t \geq e^{\frac{T_*}{2\epsilon^2}}$ *that satisfies for any* $t < T(\epsilon)$, *any* s_0

$$\begin{aligned} \|r_{\rm app}(t,\cdot)\|_{H^{s_0}} &\leq C_{s_0} t^{-2}\epsilon^{N_0}(T_* - \epsilon^2\log t)^{N_0} \\ \|L_+ r_{\rm app}(t,\cdot)\|_{H^1} &\leq Ct^{-1}\epsilon^{N_0}(T_* - \epsilon^2\log t)^{N_0}. \end{aligned} \tag{3.1.32}$$

Proof: To compute the left hand side of (3.1.30), we use the definition (3.1.16) of $u_+^{\rm app}$, (3.1.15) and the fact that we may apply Corollary A.1.4 with $\psi = -\varphi$ in order to compute the action of $D_t - p(D_x)$ on the last term in (3.1.16). (Notice that the assumption $t(T_* - s)^{\frac{1}{2\kappa_0}} \geq c$ holds on the support of that function). By (A.1.27), the action of that operator on this last term is equal to

$$\frac{\epsilon}{t^{\frac{5}{2}}} e^{-it\varphi(y)} \chi_1(\epsilon^{1-\theta}t) c_{5,-1}(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2\log t, y=x/t} \tag{3.1.33}$$

modulo a term of the same form where $t^{-\frac{5}{2}}$ is replaced by $t^{-\frac{7}{2}}$ and $c_{5,-1}$ by a function $c_{7,-1}$ satisfying the same conditions, and modulo a remainder satisfying (A.1.28) and supported for $s \leq \frac{3T_*}{4}$. Since (3.1.33) compensates the second term in the right hand side of (3.1.15), we get more precisely

$$\begin{aligned} (D_t - p(D_x))u_+^{\rm app} - \sum_{I\in\{-,+\}^3} {\rm Op}(m_I)(\tilde u_I^{\rm app}) \\ = -\Big[\frac{\epsilon}{t^{\frac{5}{2}}} e^{it\varphi(y)} \chi_1(\epsilon^{1-\theta}t) c_{5,1}(s,y,\frac{1}{t},\epsilon) \\ + \frac{\epsilon}{t^{\frac{7}{2}}} e^{-it\varphi(y)} \chi_1(\epsilon^{1-\theta}t) c_{7,-1}(s,y,\frac{1}{t},\epsilon) \\ + \chi_1(\epsilon^{1-\theta}t)\epsilon r(s,y,\frac{1}{t},\epsilon)\Big]|_{s=\epsilon^2\log t, y=x/t} \\ - (1-\tilde\chi_0)(\epsilon^2\log t) r_{\rm app}^2 - F_{\rm app} = {\rm I} + \cdots + {\rm V} \end{aligned} \tag{3.1.34}$$

where $c_{7,-1}(s,y,h,\epsilon)$ is continuous on $[0,+\infty[\times\mathbb{R}\times]0,1] \times [0,1]$, supported for $s \leq \frac{3T_*}{4}$ and $|y| \leq 1$, bounded as well as all its $\partial_s, \partial_y, h\partial_h$-derivatives, where χ_1 is a new function supported inside a neighborhood of zero (that may vary from line to line), and where r satisfies (A.1.28) and is supported for $s \leq \frac{3T_*}{4}$. If in the cubic term in the left hand side of (3.1.34), we replace $\tilde u_+^{\rm app}$ by $u_+^{\rm app}$ using (3.1.16), we generate in the right hand side a perturbation

$$\sum_{I\in\{-,+\}^3} \big({\rm Op}(m_I)(u_I^{\rm app} + \Delta_I^{\rm app}) - {\rm Op}(m_I)(u_I^{\rm app})\big) \tag{3.1.35}$$

where

$$\Delta_+^{\rm app} = \frac{\epsilon}{2t^{\frac{5}{2}}} e^{-it\varphi(y)} \chi_1(\epsilon^{1-\theta}t)\sqrt{1-y^2}\, c_{5,-1}(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2\log t, y=x/t}. \tag{3.1.36}$$

It remains to show that terms I to V in (3.1.34) and (3.1.36) may be written as contributions to $F + r_{\rm app}$ in the right hand side of (3.1.30). We start with the terms supported for $s \leq \frac{3T_*}{4}$ i.e. I to III and V in (3.1.34) and (3.1.35).

• **Term I in (3.1.34):** Since $c_{5,1}$ is bounded, as well as its ∂_s and ∂_y derivatives, and supported for $|y| \leq 1$, the Sobolev norm of I is $O(\epsilon t^{-2})$, so that its integral for $t \geq \epsilon^{-1+\theta}$ is $O(\epsilon^{2-\theta})$ as in (3.1.31). If we make act L_+ on I and use (A.1.32) with $q = 1$ (and a symbol a supported for $s \leq \frac{3T_*}{4}$), we obtain the same estimate for the H^1 norm of L_+I integrated for $t \geq \epsilon^{-1+\theta}$, so that the second inequality (3.1.31) holds as well.

• **Term II in (3.1.34):** The reasoning is the same, except that we use (A.1.32) with $q = -1$, so that the first term in the right hand side of this equality remains. We thus get a $O(|x|) = O(t)$ factor, which is compensated by the fact that $c_{7,1}$ is $O(t^{-\frac{7}{2}})$ instead of $O(t^{-\frac{5}{2}})$.

• **Term III in (3.1.34):** By (A.1.28), this term is rapidly decaying in t and $|\frac{x}{t}|$, so that the bounds (3.1.31) are trivial when integrating for $t \geq \epsilon^{-1+\theta}$.

• **Term V in (3.1.34):** This term is $F_{\rm app}$ coming from (3.1.15) which by (2.2.47) satisfies (3.1.31).

• **Term (3.1.35):** Note that $\Delta_+^{\rm app}$ in (3.1.36) is supported for $s \leq \frac{3T_*}{4}$, as the same holds for $c_{5,1}$. We have to study terms of the form

$$\text{(3.1.37)} \qquad {\rm Op}(m_I)(u_{i_1}^{\rm app}, u_{i_2}^{\rm app}, \Delta_{i_3}^{\rm app}),\ {\rm Op}(m_I)(u_{i_1}^{\rm app}, \Delta_{i_2}^{\rm app}, \Delta_{i_3}^{\rm app}),\ {\rm Op}(m_I)(\Delta_{i_1}^{\rm app}, \Delta_{i_2}^{\rm app}, \Delta_{i_3}^{\rm app})$$

with $I = (i_1, i_2, i_3) \in \{-,+\}^3$, $\Delta_-^{\rm app} = -\overline{\Delta_+^{\rm app}}$ and m_I in $S_{0,0}(1,3)$. By inequality (A.2.3) of Appendix A.2, there is $\rho_0 \in \mathbb{R}^+$ such that for any $s_0 \in \mathbb{N}$, the H^{s_0} norm of any term in (3.1.37) is bounded from above by

$$\begin{aligned}\text{(3.1.38)} \quad & C\big(\|u_+^{\rm app}\|_{W^{\rho_0,\infty}} + \|\Delta_+^{\rm app}\|_{W^{\rho_0,\infty}}\big)^2 \|\Delta_+^{\rm app}\|_{H^{s_0}} \\ & \qquad + \big(\|u_+^{\rm app}\|_{W^{\rho_0,\infty}} + \|\Delta_+^{\rm app}\|_{W^{\rho_0,\infty}}\big)\|\Delta_+^{\rm app}\|_{W^{\rho_0,\infty}}\|u_+^{\rm app}\|_{H^{s_0}}.\end{aligned}$$

Notice that for t in the support of (3.1.17) i.e. $\epsilon^{-1+\theta} \leq t \leq e^{\frac{3T_*}{4\epsilon^2}}$, we have

$$\text{(3.1.39)} \qquad \|u_+^{\rm app}(t,\cdot)\|_{H^{s_0}} = O(\epsilon),\ \|u_+^{\rm app}(t,\cdot)\|_{W^{\rho_0,\infty}} = O\Big(\frac{\epsilon}{\sqrt{t}}\Big)$$

$$\text{(3.1.40)} \qquad \|\Delta_+^{\rm app}(t,\cdot)\|_{H^{s_0}} = O(\epsilon t^{-2}),\ \|\Delta_+^{\rm app}(t,\cdot)\|_{W^{\rho_0,\infty}} = O(\epsilon t^{-\frac{5}{2}}).$$

Actually, $u_+^{\rm app}$ is given by (3.1.17), with the $e_{\ell,q}$'s bounded by the first inequality (3.1.19) by our assumption on t, and with s in (3.1.17) smaller than $\frac{3T_*}{4}$, so that the functions $a_{\ell,q}^+(s,y,h,\epsilon)$ are uniformly bounded. Then (3.1.39) follows. On the other hand, (3.1.40) follows from (3.1.36).
If we plug these estimates inside (3.1.38), we get a bound in $O(\epsilon^3 t^{-3})$, whose time integral largely satisfies the first inequality (3.1.31). If we make act on (3.1.35) L_+ before computing the L^2 norm, we get an $O(\epsilon^3 t^{-2})$ estimate that is still sufficient to obtain (3.1.31).

• **Term IV in (3.1.34):** Term IV is supported for $t \geq e^{\frac{T_*}{2\epsilon^2}}$ and is expressed in terms of $r_{\rm app}^2$ coming from (3.1.15), i.e. from (2.2.46), and is given by (2.2.3) i.e. by the sum of (2.2.4) and (2.2.6). The general term in these sums is bounded from above if $t < T(\epsilon)$, by

$$\begin{aligned}\text{(3.1.41)} \quad & C\epsilon^{6-\ell} t^{-\frac{\ell}{2}} (T_* - \epsilon^2 \log t)^{-\ell\left(\frac{1}{2}+\delta\right)} \mathbb{1}_{|x|\leq t} \\ & \qquad \leq C t^{-\frac{5}{2}} \epsilon^{6-\ell} (T_* - \epsilon^2 \log t)^{-5\left(\frac{1}{2}+\delta\right)} \big[\epsilon^{\frac{\gamma}{2}-\delta'} (T_* - \epsilon^2 \log t)^{\delta'-\delta}\big]^{\ell-5} \mathbb{1}_{|x|\leq t},\end{aligned}$$

where we have used (3.1.10) and that $\ell \geq N + 2 \geq 5$. As we assumed that (3.1.12) holds, if N is so large that

$$(\delta' - \delta)(N-3) \geq N_0 + 5\big(\frac{1}{2} + \delta\big),\ \big(\frac{\gamma}{2} - \delta' - 1\big)(N-3) \geq N_0 - 1,$$

we get a bound in $\epsilon^{N_0} t^{-\frac{5}{2}} (T_* - \epsilon^2 \log t)^{N_0}$.
If we take ∂_x derivatives of the general sum in (2.2.4), (2.2.6), we may use (3.1.13) to see that we still get expressions of the same type so that (3.1.41) will still hold true. This implies that for any s_0 in $\mathbb{N}$

$$\|(1-\tilde{\chi}_0)(\epsilon^2 \log t) r^2_{\rm app}(t,\cdot)\|_{H^{s_0}} \leq C t^{-2} \epsilon^{N_0} (T_* - \epsilon^2 \log t)^{N_0}$$

i.e. the first estimate (3.1.32) holds. The second one holds in the same way, since the action of L_+ makes lose at most $O(t)$. This concludes the proof of the proposition. □

To finish this subsection, we introduce the equation satisfied by the difference $v_+ = u_+ - u_+^{\rm app}$ between the solution of (3.1.7) and the approximate solution $u_+^{\rm app}$ of Lemma 3.1.2.

Proposition 3.1.4 *The function $v_+ = u_+ - u_+^{\rm app}$ satisfies, with symbols $m_I^{(j)}$ in $S_{0,0}(1,3)$,*

$$\begin{aligned}(D_t - p(D_x))v_+ = &\sum_{I\in\{-,+\}^3} {\rm Op}(m_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3}) \\ &+ \sum_{I\in\{-,+\}^3} {\rm Op}(m_I^{(2)})(v_{i_1}, v_{i_2}, u^{\rm app}_{i_3}) \\ &+ \sum_{I\in\{-,+\}^3} {\rm Op}(m_I^{(3)})(v_{i_1}, u^{\rm app}_{i_2}, u^{\rm app}_{i_3}) \\ &+ F + r_{\rm app}\end{aligned} \tag{3.1.42}$$

where F is supported for $t \leq e^{\frac{3T_*}{4\epsilon^2}}$, $r_{\rm app}$ is supported for $t \geq e^{\frac{T_*}{2\epsilon^2}}$ and F (resp. $r_{\rm app}$) satisfies (3.1.31) (resp. (3.1.32)).

Proof: One has just to make the difference between (3.1.7) and (3.1.30). □

3.2 Normal forms

We shall denote, identifying $\{-,+\}$ to $\{-1,1\}$,

$$\begin{aligned}\mathcal{I}_{\rm c} &= \{I=(i_1,i_2,i_3)\in\{-,+\}^3; \sum_{\ell=1}^3 i_\ell = 1\} \\ \mathcal{I}_{\rm nc} &= \{I=(i_1,i_2,i_3)\in\{-,+\}^3; \sum_{\ell=1}^3 i_\ell \neq 1\}\end{aligned} \tag{3.2.1}$$

the set of characteristic and non characteristic indices. We shall eliminate by normal forms all non-characteristic terms in the right hand side of (3.1.42). We recall that normal forms for Klein-Gordon equations have been introduced by Shatah [28] and for further results on these methods, we refer to the review paper of Germain [16] and references therein.
Consider $I=(i_1,i_2,i_3)\in\mathcal{I}_{\rm nc}$. Up to permutations, we have thus either $(i_1,i_2,i_3)=(1,1,1)$, or $(i_1,i_2,i_3)=(1,-1,-1)$, or $(i_1,i_2,i_3)=(-1,-1,-1)$. We set

$$D_I(\xi_1,\xi_2,\xi_3) = i_1\sqrt{1+\xi_1^2} + i_2\sqrt{1+\xi_2^2} + i_3\sqrt{1+\xi_3^2} - \sqrt{1+(\xi_1+\xi_2+\xi_3)^2}. \tag{3.2.2}$$

Since $i_2 = i_3$, we may write with some $c>0$

$$\begin{aligned}|D_I(\xi_1,\xi_2,\xi_3)| &\geq \sqrt{1+\xi_2^2} + \sqrt{1+\xi_3^2} - |\xi_2+\xi_3| \\ &\geq c(1+\min(|\xi_2|,|\xi_3|)^{-1} \geq c M_0(\xi_1,\xi_2,\xi_3)^{-1}\end{aligned}$$

if $M_0(\xi_1,\xi_2,\xi_3)$ is defined by (3.1.1) and so is equivalent to the second largest among $1+|\xi_1|$, $1+|\xi_2|$, $1+|\xi_3|$. This implies that for any non-characteristic index I, $D_I(\xi_1,\xi_2,\xi_3)^{-1}$ belongs to the class $S_{1,0}(M_0,3)$ of Definition 3.1.1. Consider the symbols $m_I^{(\ell)}$ in the right hand side of (3.1.42) and define when $I\in\mathcal{I}_{\rm nc}$

$$\hat{m}_I^{(\ell)}(\xi_1,\xi_2,\xi_3)=m_I^{(\ell)}(\xi_1,\xi_2,\xi_3)D_I(\xi_1,\xi_2,\xi_3)^{-1}\in S_{1,0}(M_0,3). \tag{3.2.3}$$

We shall prove:

Proposition 3.2.1 *Define from the solution v_+ of (3.1.42)*

$$\begin{aligned} w_+ = v_+ - \sum_{\substack{I=(i_1,i_2,i_3)\\ I\in\mathcal{I}_{\rm nc}}} \Big[&{\rm Op}(\hat{m}_I^{(1)})(v_{i_1},v_{i_2},v_{i_3})+{\rm Op}(\hat{m}_I^{(2)})(v_{i_1},v_{i_2},u_{i_3}^{\rm app})\\ &+{\rm Op}(\hat{m}_I^{(3)})(v_{i_1},u_{i_2}^{\rm app},u_{i_3}^{\rm app})\Big]. \end{aligned} \tag{3.2.4}$$

Then w_+ solves for $t<T(\epsilon)$ an equation of the form

$$\begin{aligned} (D_t-p(D_x))w_+ = \sum_{\substack{I=(i_1,i_2,i_3)\\ I\in\mathcal{I}_{\rm c}}} \Big[&{\rm Op}(m_I^{(1)})(v_{i_1},v_{i_2},v_{i_3})+{\rm Op}(m_I^{(2)})(v_{i_1},v_{i_2},u_{i_3}^{\rm app})\\ &+{\rm Op}(m_I^{(3)})(v_{i_1},u_{i_2}^{\rm app},u_{i_3}^{\rm app})\Big]+\mathcal{R} \end{aligned} \tag{3.2.5}$$

where $\mathcal{R}$ is a sum of terms of the following form:

- *A contribution $F(t,x)$, supported for $t\le e^{\frac{3T_*}{4\epsilon^2}}$, satisfying (3.1.31).*
- *A term $r_{\rm app}(t,x)$, supported for $e^{\frac{T_*}{2\epsilon^2}}\le t$, satisfying (3.1.32).*
- *"Quintic" terms of the form*

$$\begin{aligned} &{\rm Op}(\tilde{m})(v_{J_1},u_{J_2}^{\rm app}),\ |J_1|+|J_2|=5,|J_1|\ge 1\\ &{\rm Op}(\tilde{m})(v_{J_1},u_{J_2}^{\rm app},F_{i_3}),\ |J_1|+|J_2|=2,\\ &{\rm Op}(\tilde{m})(v_{J_1},u_{J_2}^{\rm app},r_{{\rm app},i_3}),\ |J_1|+|J_2|=2, \end{aligned} \tag{3.2.6}$$

for different symbols $\tilde{m}$ belonging to $S_{1,0}(M_0^\nu,5)$ (resp. $S_{1,0}(M_0^\nu,3)$) for the first line (resp. the second and third lines) for some $\nu\in\mathbb{N}$, where we denoted $F_+=F,F_-=-\bar{F}$, $r_{{\rm app},+}=r_{\rm app},r_{{\rm app},-}=-\overline{r_{\rm app}}$.

Proof: We make act $D_t-p(D_x)$ on (3.2.4). We get using (3.2.3) and (3.2.2)

$$\begin{aligned} (D_t-p(D_x))w_+ = (D_t-p(D_x))v_+ - \sum_{\substack{I=(i_1,i_2,i_3)\\ I\in\mathcal{I}_{\rm nc}}} \Big[&{\rm Op}(m_I^{(1)})(v_{i_1},v_{i_2},v_{i_3})\\ &+{\rm Op}(m_I^{(2)})(v_{i_1},v_{i_2},u_{i_3}^{\rm app})+{\rm Op}(m_I^{(3)})(v_{i_1},u_{i_2}^{\rm app},u_{i_3}^{\rm app})\Big]+\mathcal{R}' \end{aligned} \tag{3.2.7}$$

where $\mathcal{R}'$ is a sum of expressions of the following form, up to permutation of factors:

$${\rm Op}(m_I^{(\ell)})((D_t-i_1p(D_x))v_{i_1},v_{i_2},v_{i_3}) \tag{3.2.8}$$

$${\rm Op}(m_I^{(\ell)})((D_t-i_1p(D_x))v_{i_1},v_{i_2},u_{i_3}^{\rm app}) \tag{3.2.9}$$

$${\rm Op}(m_I^{(\ell)})((D_t-i_1p(D_x))v_{i_1},u_{i_2}^{\rm app},u_{i_3}^{\rm app}) \tag{3.2.10}$$

$$\mathrm{Op}(m_I^{(\ell)})(v_{i_1}, v_{i_2}, (D_t - i_3 p(D_x))u_{i_3}^{\mathrm{app}}) \tag{3.2.11}$$

$$\mathrm{Op}(m_I^{(\ell)})(v_{i_1}, u_{i_2}^{\mathrm{app}}, (D_t - i_3 p(D_x))u_{i_3}^{\mathrm{app}}) \tag{3.2.12}$$

where $I = (i_1, i_2, i_3)$ is in $\mathcal{I}_{\mathrm{nc}}$.
In (3.2.8), we replace $(D_t - i_1 p(D_x))v_{i_1}$ by the right hand side of (3.1.42) if $i_1 = 1$, and by the opposite of the conjugate of this right hand side if $i_1 = -1$. Using Lemma A.2.1, we get expressions of the form (3.2.6). The same conclusion holds for (3.2.9), (3.2.10). Using (3.1.30) instead of (3.1.42), we see in the same way that (3.2.11), (3.2.12) may be written as contributions to (3.2.6). Thus $\mathcal{R}'$ contributes to $\mathcal{R}$ in (3.2.5). Finally, if in the right hand side of (3.2.7), we replace $(D_t - p(D_x))v_+$ by its expression coming from (3.1.42), the non-characteristic contributions cancel each other, and we are left only with the characteristic ones, as in the right hand side of (3.2.5), and the contribution $F + r_{\mathrm{app}}$ to $\mathcal{R}$. This concludes the proof. □

To prepare the energy estimates of next section, we notice that getting bounds on w_+ or v_+ will be essentially equivalent, up to small errors.

Lemma 3.2.2 *There is $\rho_0 \in \mathbb{N}$ such that for any s_0 in $\mathbb{N}$*

$$\|w_+ - v_+\|_{H^{s_0}} \leq C[\|v_+\|^2_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|^2_{W^{\rho_0+s_0,\infty}}]\|v_+\|_{H^{s_0}} \tag{3.2.13}$$

(3.2.14)
$$\begin{aligned}\|L_+(w_+ - v_+)\|_{L^2} &\leq C\|v_+\|^2_{W^{\rho_0,\infty}}[\|L_+v_+\|_{L^2} + t\|v_+\|_{L^2}]\\ &\quad + C[\|L_+u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + t\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}](\|v_+\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|v_+\|_{L^2}.\end{aligned}$$

Proof: To get (3.2.13), we express $w_+ - v_+$ from (3.2.4). We apply (A.2.3) to the first term in the sum in the right and side. To treat the two remaining ones, we use (A.2.8) with $\ell = 2$ or $\ell = 1$ respectively. We obtain a bound by the right hand side of (3.2.13).
Let us prove (3.2.14). We may write for any functions f_1, f_2, f_3 and any symbol m in $S_{1,0}(M_0, 3)$

$$L_+\mathrm{Op}(m)(f_1, f_2, f_3) = \mathrm{Op}(\tilde{m})(f_1, f_2, f_3) + \mathrm{Op}(m)(f_1, f_2, xf_3) + tp'(D_x)\mathrm{Op}(m)(f_1, f_2, f_3)$$

for some $\tilde{m}$ in $S_{1,0}(M_0^2, 3)$. Writing then $xf_3 = (x + i_3 tp'(D_x))f_3 - i_3 tp'(D_x)f_3$, we obtain

$$\begin{aligned}L_+\mathrm{Op}(m)(f_1, f_2, f_3) &= \mathrm{Op}(m)(f_1, f_2, L_{i_3}f_3) + \mathrm{Op}(\tilde{m})(f_1, f_2, f_3)\\ &\quad - i_3 t\mathrm{Op}(m)(f_1, f_2, p'(D_x)f_3) + tp'(D_x)(f_1, f_2, f_3).\end{aligned} \tag{3.2.15}$$

We write $L_+(w_+ - v_+)$ from (3.2.4) on which we make act L_+. We apply to the $\mathrm{Op}(\hat{m}_I^{(1)})$-term in (3.2.4) equality (3.2.15) with $(f_1, f_2, f_3) = (v_{i_1}, v_{i_2}, v_{i_3})$. By (A.2.4) applied with $j = 3$, we get that the L^2 norm of the action of L_+ on the first term in the sum (3.2.4) is estimated from

$$(\|L_+v_+\|_{L^2} + t\|v_+\|_{L^2})\|v_+\|^2_{W^{\rho_0,\infty}}. \tag{3.2.16}$$

In the same way, applying (3.2.15) to $(f_1, f_2, f_3) = (v_{i_1}, v_{i_2}, u_{i_3}^{\mathrm{app}})$, and using (A.2.4) with $j = 1$, we estimate the L^2-norm of the action of L_+ on the $\mathrm{Op}(\hat{m}_I^{(2)})$-term in (3.2.4) by

$$(\|L_+u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + t\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|v_+\|_{W^{\rho_0,\infty}}\|v_+\|_{L^2}. \tag{3.2.17}$$

Finally, doing the same for the $\mathrm{Op}(\hat{m}_I^{(3)})$-term, we get a bound in

$$(\|L_+u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + t\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\|v_+\|_{L^2}.$$

Together with (3.2.16) and (3.2.17), this gives (3.2.14). □

4 Construction of the solution and proof of the main theorem

Recall that we want to construct a solution u to equation (1.1.1) that displays inflation of its norms and that we have rewritten that equation as a first order system (3.1.7)-(3.1.8). We look next for the solution $(u_+, u_- = -\bar{u}_+)$ of that system under the form $u_+ = u_+^{\rm app} + v_+$, where $u_+^{\rm app}$ is the approximate solution defined in (3.1.16), that solves (3.1.30), and that blows-up at time $e^{\frac{T_*}{\epsilon^2}}$, and where v_+ is the perturbation introduced in Proposition 3.1.4, that solves equation (3.1.42). We shall construct v_+ solving that equation backwards, starting at time $T(\epsilon)$ introduced in (1.2.5), with initial condition $v_+|_{t=T(\epsilon)} = 0$. In order to show that v_+ exists up to time $t = 1$, and remains under control down to that time, we shall prove in this section a priori estimates for $\|v_+(t,\cdot)\|_{H^{s_0}}$ for s_0 large enough and $\|L_+v_+(t,\cdot)\|_{L^2}$. In order to do so, we shall exploit the fact that $\|v_+\|_{W^{\rho_0,\infty}}$ remains small, so that Lemma 3.2.2 will imply that the H^s (resp. L^2) norm of v_+ (resp. L_+v_+) is equivalent to the H^s (resp. L^2) norm of w_+ (resp. L_+w_+), where w_+ solves equation (3.2.5), in which the explicit cubic terms in the right hand side are all *characteristic*. In the following subsections, we shall successively prove estimates for $\|v_+(t,\cdot)\|_{H^{s_0}}$, $\|L_+v_+(t,\cdot)\|_{L^2}$ and then perform the bootstrap argument that gives the proof of the main theorem.

4.1 Sobolev estimates

In the estimates of this subsection and the following ones, it will be important to track the dependence of some constants on other ones. We shall fix indices of smoothness ρ_0, s_0 (that will be taken large enough), as well as the parameters δ, δ', γ that satisfy (3.1.12). A universal constant will be a constant that depends eventually on these parameters, but on no other quantity. Next, we shall have constants like N (the order at which we construct the approximate solutions (3.1.17)) or N_0 in (3.1.32), as well as the constants A_0, A_1, B that we introduce below in the estimates of v_+. It will be important to track how other constants depend on them. Because of that, when we introduce a constant like $K(A_0, A_1, B, \dots)$, we mean that K depends only on the quantities explicitly mentioned in the argument.

Proposition 4.1.1 *Let $\rho_0 \in \mathbb{N}$ be fixed such that the estimates of Proposition A.2.2 of the appendix hold. Let $s_0 \in \mathbb{N}$ be given. There is an integer $N_{0,\min} > 0$ such that for any $N_0 \geq N_{0,\min}$, the following holds:*
The choice of ρ_0, s_0, N_0 determines the constants in the right hand side of (3.1.31), (3.1.32). For any couple of constants (A_0, B) with A_0 large enough relatively to N_0, there is $\epsilon_0 \in]0,1]$ such that, for any $\epsilon \in]0, \epsilon_0]$, the following bootstrap holds: Denote by v_+ the backwards solution of (3.1.42), with initial condition $v_+(T(\epsilon),\cdot) = 0$, and source term $F + r_{\rm app}$ (with $F, r_{\rm app}$ satisfying (3.1.31), (3.1.32)). Assume that this solution is defined on an interval $[T, T(\epsilon)]$ for some $T \in [1, T(\epsilon)[$ and that the following a priori estimates hold true for any $t \in [T, T(\epsilon)]$:

$$\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \leq \frac{B}{\sqrt{t}}\epsilon^{2-\theta}. \tag{4.1.1}$$

Then, for any $t \in [T, T(\epsilon)]$ one has

$$\|v_+(t,\cdot)\|_{H^{s_0}} \leq \frac{A_0}{2}\epsilon^{2-\theta}(T_* - \epsilon^2 \log t)^{N_0}. \tag{4.1.2}$$

Before starting the proof, we introduce a notation for the cubic terms in the right hand side of (3.1.42), namely

$$\begin{aligned}\mathcal{F}_3(v_+, u_+^{\rm app}) = &\sum_{I\in\{-,+\}^3} {\rm Op}(m_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3})\\ &+ \sum_{I\in\{-,+\}^3} {\rm Op}(m_I^{(2)})(v_{i_1}, v_{i_2}, u_{i_3}^{\rm app})\\ &+ \sum_{I\in\{-,+\}^3} {\rm Op}(m_I^{(3)})(v_{i_1}, u_{i_2}^{\rm app}, u_{i_3}^{\rm app}).\end{aligned} \tag{4.1.3}$$

As $m_I^{(j)}$ is in $S_{0,0}(1,3)$, independent of x, we may apply (A.2.3) to the first sum in (4.1.3) and (A.2.8) to the second and third ones, with $\ell = 2$ and $\ell = 1$ respectively. We get for any $s_0 \in \mathbb{N}$

$$\|\mathcal{F}_3(v_+, u_+^{\rm app})\|_{H^{s_0}} \leq C(\|v_+\|^2_{W^{\rho_0,\infty}} + \|u_+^{\rm app}\|^2_{W^{\rho_0+s_0,\infty}})\|v_+\|_{H^{s_0}}\,. \tag{4.1.4}$$

To prove Proposition 4.1.1, we shall need a bound for $\|u_+^{\rm app}\|_{W^{\rho_0+s_0,\infty}}$ in the right hand side of (4.1.4).

Lemma 4.1.2 *For any $\rho > 0$, there are $C_0(\rho)$, $\theta' > 0$ and for any $N \in \mathbb{N}$, there is a constant $K(N)$ such that the approximate solution $u_+^{\rm app}$ given by (3.1.17) with that value of N satisfies*

$$\begin{aligned}\|u_+^{\rm app}\|_{W^{\rho,\infty}} \leq\ & C_0(\rho)\frac{\epsilon}{\sqrt{t}}(T_* - \epsilon^2\log t)^{-\frac12}\\ &+ K(N)\frac{\epsilon^{1+\theta'}}{\sqrt{t}}(T_* - \epsilon^2\log t)^{-\frac12} + K(N)\frac{\epsilon}{\sqrt{t}}.\end{aligned} \tag{4.1.5}$$

Remark: We shall use (4.1.5) to estimate $\|u_+^{\rm app}\|_{W^{\rho_0+s_0,\infty}}$ in the right hand side of (4.1.4), so that the first multiplicative constant in the right hand side $C_0(\rho_0 + s_0) = C_0$ will be a universal constant with the terminology introduced at the beginning of this section. In particular, it is independent of N. The two other constants in (4.1.5) do depend on N, but they are either multiplied by a small factor $\epsilon^{\theta'}$ or are not affected by the large factor $(T_* - \epsilon^2\log t)^{-\frac12}$.

Proof: We bound the $W^{\rho,\infty}$-norm of each term in the right hand side of (3.1.17).
• By (3.1.18) and (3.1.21), the first term in the right hand side of (3.1.17) has $W^{\rho,\infty}$-norm bounded by $C(\rho)\frac{\epsilon}{\sqrt{t}}$ for some constant $C(\rho)$ depending only on ρ.
• In the second term in the right hand side of (3.1.17), $a^+_{1,1}(s,y)$ is equal $2(1-y^2)^{-\frac12}a_{1,1}(s,y)$, where $a_{1,1}$ is the element of $\Sigma^{-\frac12}$ given explicitly by (2.1.12). It depends only on the initial data of (2.1.10). By the definition (2.2.1) of class $\Sigma^{-\frac12}$

$$\partial_x^\ell\Big(a^+_{1,1}\Big(\epsilon^2\log t, \frac{x}{t}\Big)\Big) = t^{-\ell}b_\ell\Big(\epsilon^2\log t, \frac{x}{t}\Big) \tag{4.1.6}$$

for some $b_\ell \in \Sigma^{-\frac12-\ell}$. Since by (3.1.10), $t^{-\ell}(T_* - \epsilon^2\log t)^{-\ell} = O(1)$, the $W^{\rho,\infty}$ norm of the second term in the right hand side of (3.1.17) is bounded by $C_0(\rho)\frac{\epsilon}{\sqrt{t}}(T_* - \epsilon^2\log t)^{-\frac12}$ for some constant $C_0(\rho)$ depending only on ρ.
• Consider next the $W^{\rho,\infty}$ norm of each term in the last sum in (3.1.17). Since as above in (4.1.6), any ∂_x derivative may be written as an expression of the same form as the general term in that sum, it is enough to bound the L^∞ norm, that is smaller than

$$C_\ell t^{-\frac{\ell}{2}}(T_* - \epsilon^2\log t)^{-\frac{\ell}{2}-\delta(\ell-1)}e_\ell(t,\epsilon) \tag{4.1.7}$$

with a factor $e_\ell(t,\epsilon)$ that satisfies according to (3.1.19)

$$
\begin{aligned}
&e_\ell(t,\epsilon) = O(\epsilon) \text{ if } t \le e^{\frac{3T_*}{4\epsilon^2}} \\
&e_\ell(t,\epsilon) = O(\epsilon^{2-\ell}) \text{ if } t \ge e^{\frac{T_*}{2\epsilon^2}}.
\end{aligned} \tag{4.1.8}
$$

Using (3.1.10), we estimate (4.1.7) when $t \ge e^{\frac{T_*}{2\epsilon^2}}$ by

$$
C_\ell t^{-\frac{1}{2}} (T_* - \epsilon^2 \log t)^{-\frac{1}{2}} e_\ell(t,\epsilon) \epsilon^{(\ell-1)\left(\frac{\gamma}{2}-\delta'\right)}. \tag{4.1.9}
$$

Using (4.1.8), (3.1.12) and the fact that $\ell \ge 3$, we get that (4.1.9) is estimated by the second term in the right hand side of (4.1.5) with $\theta' \ge 2$. The sum of all these terms for $3 \le \ell \le N+1$, ℓ odd, is thus also controlled by this quantity.
For $t \le e^{\frac{3T_*}{4\epsilon^2}}$ (4.1.8) implies that the sum of expressions (4.1.7) is smaller than the last term in (4.1.5). This concludes the proof of the lemma. □

We show next

Lemma 4.1.3 *Assume the a priori inequality (4.1.1) and that the source term in (3.1.42) satisfies (3.1.31), (3.1.32). Let N_0 be given in $\mathbb{N}$. Then there are a universal constant C_0 and a constant $K(N_0)$ depending on N_0, a constant $K(B)$ depending on B in (4.1.1), such that*

$$
\begin{aligned}
\|v_+(t,\cdot)\|_{H^s_0} \le & \int_t^{T(\epsilon)} [\|F(\tau,\cdot)\|_{H^{s_0}} + \|r_{\rm app}(\tau,\cdot)\|_{H^{s_0}}]\, d\tau \\
& + K(N_0)\epsilon^2 \int_t^{T(\epsilon)} [\epsilon^{2\theta'}(T_* - \epsilon^2 \log\tau)^{-1} + 1] \|v_+(\tau,\cdot)\|_{H^{s_0}} \frac{d\tau}{\tau} \\
& + C_0 \epsilon^2 \int_t^{T(\epsilon)} (T_* - \epsilon^2 \log\tau)^{-1} \|v_+(\tau,\cdot)\|_{H^{s_0}} \frac{d\tau}{\tau} \\
& + K(B)\epsilon^{4-2\theta} \int_t^{T(\epsilon)} \|v_+(\tau,\cdot)\|_{H^{s_0}} \frac{d\tau}{\tau}
\end{aligned} \tag{4.1.10}
$$

for any $t \in [T, T(\epsilon)]$.

Proof: We write the backwards energy inequality for the solution to (3.1.42) with zero initial condition at $t = T(\epsilon)$ using notation (4.1.3). We obtain

$$
\|v_+(t,\cdot)\|_{H^{s_0}} \le \int_t^{T(\epsilon)} \|\mathcal{F}_3(v_+, u_+^{\rm app})(\tau,\cdot)\|_{H^{s_0}}\, d\tau + \int_t^{T(\epsilon)} \|F(\tau,\cdot)\|_{H^{s_0}}\, d\tau + \int_t^{T(\epsilon)} \|r_{\rm app}(\tau,\cdot)\|_{H^{s_0}}\, d\tau. \tag{4.1.11}
$$

Under the first integral in the right hand side, we plug (4.1.4). By estimate (4.1.1), the $\|v_+\|^2_{W^{\rho_0,\infty}}$ term in the right hand side of (4.1.4) brings the last term in (4.1.10). To study the contribution of the $\|u_+^{\rm app}\|^2_{W^{\rho_0+s_0,\infty}}$ term of (4.1.4), we apply (4.1.5) with $\rho = \rho_0 + s_0$ for a N taken large enough relatively to N_0 so that Proposition 3.1.3 holds. We obtain thus from the right hand side of (4.1.5) the second and third terms in the right hand side of (4.1.10), with a constant K that depends on N, and thus on N_0. This concludes the proof. □

Proof of Proposition 4.1.1: We make the change of variable $t = e^{\frac{s}{\epsilon^2}}$ with $s \in [0, S(\epsilon)]$, $S(\epsilon) = \epsilon^2 \log T(\epsilon)$, and rewrite (4.1.10) as

$$
f(s) \le \int_s^{S(\epsilon)} g(\sigma)\, d\sigma + \int_s^{S(\epsilon)} \psi(\sigma) f(\sigma)\, d\sigma \tag{4.1.12}
$$

where

$$
\begin{aligned}
f(s) &= \|v_+(e^{\frac{s}{\epsilon^2}},\cdot)\|_{H^{s_0}} \\
g(\sigma) &= e^{\frac{\sigma}{\epsilon^2}}\epsilon^{-2}[\|F(e^{\frac{\sigma}{\epsilon^2}},\cdot)\|_{H^{s_0}} + \|r(e^{\frac{\sigma}{\epsilon^2}},\cdot)\|_{H^{s_0}}] \\
\psi(\sigma) &= K(N_0)(\epsilon^{2\theta'}(T_*-\sigma)^{-1}+1) + C_0(T_*-\sigma)^{-1} + K(B)\epsilon^{2-2\theta}.
\end{aligned}
\tag{4.1.13}
$$

Denote

$$
\Phi(s) = -\int_s^{S(\epsilon)} \psi(\sigma)\, d\sigma \tag{4.1.14}
$$

so that (4.1.12) implies by Grownwall inequality

$$
f(s) \leq \int_s^{S(\epsilon)} e^{\Phi(s')-\Phi(s)} g(s')\, ds'. \tag{4.1.15}
$$

Assume that ϵ is small enough so that in the expression of $\psi(\sigma)$ in (4.1.13), $K(N_0)\epsilon^{2\theta'} + K(B)\epsilon^{2-2\theta} \leq 1$. Then (4.1.14) implies that for $T \leq s \leq s' \leq T(\epsilon) < T_*$

$$
e^{\Phi(s')-\Phi(s)} \leq e^{(K(N_0)+1)T_*}\Big(\frac{T_*-s}{T_*-s'}\Big)^{C_0+1}.
$$

We thus get from (4.1.15)

(4.1.16)

$$
\|v_+(e^{\frac{s}{\epsilon^2}},\cdot)\|_{H^{s_0}} \leq K(N_0)\int_s^{S(\epsilon)} \Big(\frac{T_*-s}{T_*-s'}\Big)^{C_0+1} e^{\frac{s'}{\epsilon^2}}\epsilon^{-2}\big[\|F(e^{\frac{s}{\epsilon^2}},\cdot)\|_{H^{s_0}} + \|r_{\rm app}(e^{\frac{s}{\epsilon^2}},\cdot)\|_{H^{s_0}}\big]\, ds'
$$

for a new constant $K(N_0)$. We use next (3.1.31), (3.1.32) to estimate the right hand side. If $s \geq \frac{3T_*}{4}$, then the $F(e^{\frac{s}{\epsilon^2}},\cdot)$-contribution in the right hand side of (4.1.16) vanishes, so that by (3.1.32), we get an estimate by

$$
C_{s_0}K(N_0)\int_s^{S(\epsilon)} (T_*-s)^{C_0+1}(T_*-s')^{-C_0-1+N_0} e^{-\frac{s'}{\epsilon^2}}\epsilon^{N_0-2}\, ds'. \tag{4.1.17}
$$

If $N_0 > C_0$, which may be imposed since C_0 is a universal constant, we get a bound in $K(N_0)e^{-\frac{s}{\epsilon^2}}(T_*-s)^{N_0+1}\epsilon^{N_0-2}$ (for a new $K(N_0)$) that largely implies an estimate of the form (4.1.2), if we assume $N_0 \geq 4$ and $\epsilon < \epsilon_0$ small enough.

If on the other hand in (4.1.16), $s < \frac{3T_*}{4}$, the integral in the right hand side of (4.1.16) for $s \in [\frac{3T_*}{4}, S(\epsilon)]$ is estimated as above and the remaining one by

$$
\begin{aligned}
K(N_0)\int_s^{3T_*/4} (T_*-s)^{C_0+1}(T_*-s')^{-C_0-1} e^{\frac{s'}{\epsilon^2}}\epsilon^{-2}\|r_{\rm app}(e^{\frac{s'}{\epsilon^2}},\cdot)\|_{H^{s_0}}\, ds' \\
+4^{C_0+1}K(N_0)\int_1^{e^{3T_*/4}} \|F(\tau,\cdot)\|_{H^{s_0}}\, d\tau.
\end{aligned}
\tag{4.1.18}
$$

The first term may be bounded again by (4.1.17) and then by $K(N_0)\epsilon^{2-\theta}$ if N_0 is large enough. By (3.1.31), the last contribution to (4.1.18) is also in $K(N_0)\epsilon^{2-\theta}$ for a new constant depending on N_0. If the constant A_0 is chosen large enough in function of N_0, we may ensure that (4.1.2) holds. This concludes the proof. □

4.2 Estimates for the action of L_+

We want to prove estimates for the L^2 norm of L_+v_+ analogous to those of Proposition 4.1.1 in the case of Sobolev norms. To do so, we shall have to use the auxiliary unknown w_+ of Proposition 3.2.1.

Proposition 4.2.1 *Assume given large enough integers ρ_0, s_0. Assume also given a large enough integer N_1 and an integer N_0 satisfying $N_0 \geq N_1+1+2\delta$. For any constant $A_0 > 0$ (that depends on the preceding ones), there is $A_1 > 0$ and, for any constant $B > 0$ (that may depend on A_0, A_1), there is $\epsilon_0 \in]0,1]$, such that the following holds true for any $\epsilon \in]0,\epsilon_0]$:*
Let $T \in]0, T(\epsilon)[$ and let v_+ be a solution of equation (3.1.42) defined on $[T, T(\epsilon)]$, with the initial condition $v_+(T(\epsilon),\cdot) = 0$, such that v_+ satisfies for any $t \in [T, T(\epsilon)]$ the following estimates:

$$\begin{aligned} \|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} &\leq \frac{B}{\sqrt{t}}\epsilon^{2-\theta} \\ \|v_+(t,\cdot)\|_{H^{s_0}} &\leq A_0\epsilon^{2-\theta}(T_* - \epsilon^2\log t)^{N_0}. \end{aligned} \tag{4.2.1}$$

Then for any $t \in [T, T(\epsilon)]$, we have the estimate

$$\|L_+v_+(t,\cdot)\|_{L^2} \leq \frac{A_1}{2}\epsilon^{2-\theta}(T_* - \epsilon^2\log t)^{N_1}. \tag{4.2.2}$$

To prove the proposition, we first need an estimate for $\|L_+u_+^{\rm app}\|_{W^{\rho,\infty}}$ for any ρ.

Lemma 4.2.2 *For any $\rho > 0$, any $N \in \mathbb{N}^*$, there is a constant $K(\rho, N)$ such that if $u_+^{\rm app}$ is defined by (3.1.17), one has for any $t \leq T(\epsilon)$ the bound*

$$\|L_+u_+^{\rm app}(t,\cdot)\|_{W^{\rho,\infty}} \leq K(N,\rho)\frac{\epsilon}{\sqrt{t}}(T_* - \epsilon\log t)^{-\frac{3}{2}-2\delta}. \tag{4.2.3}$$

Proof: We make act L_+ on (3.1.17). We get a first term

$$\chi_0(\epsilon^{1-\theta}(t-1))L_+u_{0,+}. \tag{4.2.4}$$

If we apply to expression (3.1.21) of $u_{0,+}$ Corollary A.1.5 with $q = 1$ or $q = -1$, we conclude that the $W^{\rho,\infty}$-norm of (4.2.4) is $O(\epsilon/\sqrt{t})$. Applying again Corollary A.1.5 with $q = 1$ to the second term in the right hand side of (3.1.17), we get that the action of L_+ on it gives an expression

$$e^{it\varphi(y)}(1-\chi_0)(\epsilon^{1-\theta}(t-1))\frac{\epsilon}{\sqrt{t}}(\tilde{a}^+_{1,1}(s,y,\frac{1}{t},\epsilon) + \tilde{r}(s,y,\frac{1}{t},\epsilon)|_{s=\epsilon^2\log t, y=x/t} \tag{4.2.5}$$

with $\tilde{a}^+_{1,1}$ in $\Sigma^{-\frac{1}{2}-\frac{1}{2\kappa_0}} \subset \Sigma^{-\frac{3}{2}}$ and $\tilde{r}$ with all its $\partial_s, \partial_y, h\partial_h$-derivatives smaller than $h^N\langle y\rangle^{-N}$ for any N. Then the $W^{\rho,\infty}$ norm of (4.2.5) is $O\big(\frac{\epsilon}{\sqrt{t}}(T_* - \epsilon^2\log t)^{-\frac{3}{2}}\big)$ since the action of each ∂_x-derivations makes lose at most $1 + t^{-1}(T_* - \epsilon^2\log t)^{-1}$, which is $O(1)$ (using (3.1.10) when t satisfies $e^{T_*/2\epsilon^2} < t < T(\epsilon)$).
We consider next the action of L_+ on the last sum in (3.1.17). We have on the one hand the characteristic terms corresponding to $q = 1, \ell \geq 3$. We apply Corollary A.1.5 with $q = 1$ to see that the action of L_+ on these terms is given by a sum for $3 \leq \ell \leq N+1$ of expressions

$$e^{it\varphi(y)}t^{-\frac{\ell}{2}}a^{+,2}_{\ell,1}(s,y,\frac{1}{t},\epsilon)e_{\ell,1}(t,\epsilon)|_{s=\epsilon^2\log t, y=x/t} \tag{4.2.6}$$

with $a^{+,2}_{\ell,1}$ in $\Sigma^{-\frac{\ell}{2}-1-\delta(\ell-1)}$ and $e_{\ell,1}(t,\epsilon)$ given by (3.1.19), modulo a remainder $\epsilon\tilde{r}$, with $\tilde{r}$ as in (4.2.5), so that it will trivially satisfy a bound of the form (4.2.3). In (4.2.6), $e_{\ell,1}(t,\epsilon) = O(\epsilon)$ if

$t \le e^{T_*/2\epsilon^2}$, so that in this case bounds (4.2.3) hold immediately. If $t > e^{T_*/2\epsilon^2}$, we bound the modulus of (4.2.6) by

$$t^{-\frac{\ell}{2}}(T_* - \epsilon^2 \log t)^{-\frac{\ell}{2}-1-\delta(\ell-1)}\epsilon^{2-\ell} = t^{-\frac{1}{2}}(t^{-\frac{1}{2}}(T_*-s)^{-\frac{1}{2}-\delta})^{\ell-1}\epsilon^{2-\ell}(T_*-s)^{-\frac{3}{2}}|_{s=\epsilon^2\log t}.$$

By (3.1.10) this is $O\Big(\frac{\epsilon}{\sqrt{t}}(T_* - \epsilon^2\log t)^{-\frac{3}{2}}\Big)$ since $(\ell-1)(\frac{\gamma}{2}-\delta') + 2 - \ell \ge 1$ by (3.1.12). Since the same estimates hold for ∂_x-derivatives of (4.2.6), we get for the $W^{\rho,\infty}$ norm of the action of L_+ on the characteristic terms in the sum in (3.1.17) a bound by the right hand side of (4.2.3).
We still have to study the non-characteristic terms in that sum i.e. those for which $q \neq 1$. By Corollary A.1.5 and (3.1.10), we get that the action of L_+ on these terms gives

$$e^{itq\varphi(y)}t^{-\frac{\ell}{2}+1}a^{+,2}_{\ell,q}(s,y,\frac{1}{t},\epsilon)e_{\ell,q}(t,\epsilon)|_{s=\epsilon^2\log t, y=x/t} \tag{4.2.7}$$

with $a^{+,2}_{\ell,q}$ in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$, $\ell \ge 3$, modulo again a remainder that is again like $\epsilon\tilde{r}$ in (4.2.5). The modulus of (4.2.7) is bounded by

$$\begin{aligned} &t^{-\frac{\ell}{2}+1}(T_*-\epsilon^2\log t)^{-\frac{\ell}{2}-\delta(\ell-1)}|e_{\ell,q}(t,\epsilon)| \\ &\qquad = t^{-\frac{1}{2}}(t^{-\frac{1}{2}}(T_*-s)^{-\frac{1}{2}-\delta})^{\ell-3}(T_*-s)^{-\frac{3}{2}-2\delta}|e_{\ell,q}(t,\epsilon)||_{s=\epsilon^2\log t}. \end{aligned} \tag{4.2.8}$$

By (3.1.10), we estimate that from

$$t^{-\frac{1}{2}}\epsilon^{(\ell-3)\left(\frac{\gamma}{2}-\delta'\right)}|e_{\ell,q}(t,\epsilon)|(T_*-\epsilon^2\log t)^{-\frac{3}{2}-2\delta} \tag{4.2.9}$$

if $t \ge e^{T_*/2\epsilon^2}$. If $\ell = 3, q = -1$, the last inequality (3.1.19) gives a bound of the form (4.2.3). If $\ell = 3, |q| = 3$, the second estimate (3.1.19) shows that $e_{\ell,q}(t,\epsilon) = O(\epsilon^3)$, so that we obtain again the wanted bound. If $\ell \ge 5$, using that $e_{\ell,q}(t,\epsilon) = O(\epsilon^{2-\ell})$ and (3.1.12), we obtain that (4.2.9) is controlled by the right hand side of (4.2.3). When $t \le e^{T_*/2\epsilon^2}$, the first estimate (3.1.19) shows that the bound by (4.2.3) holds trivially. Finally, since similar bounds are satisfied by ∂_x-derivatives, we get that the non-characteristic terms in the sum in (3.1.17) are controlled as in (4.2.3). This concludes the proof. □

We prove next a lemma relating estimates for L_+v_+ and L_+w_+

Lemma 4.2.3 *Let v_+ be a function defined on some interval $[T, T(\epsilon)]$ satisfying for any $t \in [T, T(\epsilon)]$ estimates*

$$\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \le \frac{B}{\sqrt{t}}\epsilon^{2-\theta} \tag{4.2.10}$$

for some constant B. There is $\epsilon_0 > 0$, depending on B, such that if $\epsilon \in]0,\epsilon_0[$ and (4.2.10) holds, then w_+ defined by (3.2.4) from v_+ and $u^{\rm app}_+$ given by (3.1.16) satisfies

$$\|L_+v_+(t,\cdot)\|_{L^2} \le 2\|L_+w_+(t,\cdot)\|_{L^2} + K(N, B\epsilon^{1-\theta})\epsilon^2(T_*-\epsilon^2\log t)^{-1-2\delta}\|v_+(t,\cdot)\|_{L^2} \tag{4.2.11}$$

(with N equal to the order at which $u^{\rm app}_+$ has been constructed in (3.2.15)).

Proof: By (4.1.5) and (4.2.3) we have

$$\begin{aligned} &\|L_+u^{\rm app}_+(t,\cdot)\|_{W^{\rho_0,\infty}} + t\|u^{\rm app}_+(t,\cdot)\|_{W^{\rho_0,\infty}} \\ &\qquad \le \epsilon K(N)\sqrt{t}(T_*-\epsilon^2\log t)^{-\frac{1}{2}-2\delta}[1 + t^{-1}(T_*-\epsilon^2\log t)^{-1}]. \end{aligned} \tag{4.2.12}$$

On the other hand, still by (4.1.5) and the a priori estimate of $\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}}$ in (4.2.10), we have

$$\|u_+^{\rm app}(t,\cdot)\|_{W^{\rho_0,\infty}} + \|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \leq \frac{\epsilon K(N, B\epsilon^{1-\theta})}{\sqrt{t}}(T_* - \epsilon^2\log t)^{-\frac{1}{2}}. \tag{4.2.13}$$

Using (3.1.10), we see that the product of (4.2.12) by (4.2.13) is smaller than

$$\epsilon^2 K(N, B\epsilon^{1-\theta})(T_* - \epsilon^2\log t)^{-1-2\delta}.$$

Plugging this in (3.2.14), and using also the a priori estimate (4.2.10) of $\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}}$, we get

$$\begin{aligned}\|L_+(w_+ - v_+)(t,\cdot)\|_{L^2} \leq K(B\epsilon^{1-\theta})\frac{\epsilon^2}{t}\|L_+v_+(t,\cdot)\|_{L^2} + K(B\epsilon^{1-\theta})\epsilon^2\|v_+(t,\cdot)\|_{L^2}\\ +\epsilon^2 K(N, B\epsilon^{1-\theta})(T_* - \epsilon^2\log t)^{-1-2\delta}\|v_+(t,\cdot)\|_{L^2}\end{aligned} \tag{4.2.14}$$

which implies (4.2.11). This concludes the proof. □

We prove next an energy inequality for $\|L_+w_+(t,\cdot)\|_{L^2}$.

Lemma 4.2.4 *Assume that for t in some interval $[T, T(\epsilon)]$ the following a priori estimate holds true*

$$\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \leq \frac{B}{\sqrt{t}}\epsilon^{2-\theta}. \tag{4.2.15}$$

Then for t in the same interval, one has an inequality

(4.2.16)

$$\begin{aligned}\|(D_t - p(D_x))L_+w_+(t,\cdot)\|_{L^2} \leq \frac{\epsilon^2}{t}\big[(T_* - \epsilon^2\log t)^{-1}(C_0 + K(N,B)\epsilon^{\theta''}) + K(N)\big]\|L_+v_+(t,\cdot)\|_{L^2}\\ + \frac{\epsilon^2}{t}(T_* - \epsilon^2\log t)^{-2-2\delta}(K(N) + K(N,B)\epsilon^{\theta''})\|v_+(t,\cdot)\|_{H^{s_0}} + \mathcal{R}_L(t) + \mathcal{R}_H(t),\end{aligned}$$

where C_0 is a universal constant, $\theta'' > 0$ and $\mathcal{R}_H, \mathcal{R}_L$ satisfy

$$\begin{aligned}\int_1^{+\infty}\|\mathcal{R}_L(t)\|_{L^2}\,dt \leq (K(N) + K(N,B)\epsilon^{\theta''})\epsilon^{2-\theta}\\ \|\mathcal{R}_H(t)\|_{L^2} \leq K(N,B)t^{-1}\epsilon^{N_0}(T_* - \epsilon^2\log t)^{N_0-1}\end{aligned} \tag{4.2.17}$$

where N_0 is the integer introduced in Proposition 3.1.3, and $\mathcal{R}_L$ is supported for $t \leq e^{3T_*/4\epsilon^2}$.

Proof: We make act L_+ on equation (3.2.5) to get

$$\begin{aligned}(D_t - p(D_x))L_+w_+ = \sum_{\substack{I=(i_1,i_2,i_3)\\ I\in\mathcal{I}_{\rm c}}}\Big[L_+{\rm Op}(m_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3}) + L_+{\rm Op}(m_I^{(2)})(v_{i_1}, v_{i_2}, u_{i_3}^{\rm app})\\ +L_+{\rm Op}(m_I^{(3)})(v_{i_1}, u_{i_2}^{\rm app}, u_{i_3}^{\rm app})\Big] + L_+\mathcal{R}\\ = {\rm I} + \cdots + {\rm IV}.\end{aligned} \tag{4.2.18}$$

We estimate the L^2 norm of the terms in the right hand side. Since the index I is characteristic, we may use Proposition A.3.1 in order to estimate I + II + III.

• **Estimate of I, II, III**

By (A.3.1), we get

$$\|\mathrm{I}\|_{L^2} \leq C\|v_+\|^2_{W^{\rho_0,\infty}}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}) \leq CB^2\frac{\epsilon^{4-2\theta}}{t}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}) \tag{4.2.19}$$

by (4.2.15). We estimate II using (A.3.2). We get

$$\begin{aligned}\|\mathrm{II}\|_{L^2} \leq 2C\|v_+\|_{W^{\rho_0,\infty}}\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}) \\ + C\|v_+\|_{W^{\rho_0,\infty}}(\|L_+u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|v_+\|_{L^2}.\end{aligned}$$

Using (4.2.15), bound (4.1.5) of u_+^{app} which implies

$$\|u_+^{\mathrm{app}}(t,\cdot)\|_{W^{\rho_0,\infty}} \leq K(N)\frac{\epsilon}{\sqrt{t}}(T_* - \epsilon^2\log t)^{-\frac{1}{2}}, \tag{4.2.20}$$

and (4.2.3), we get

$$\begin{aligned}\|\mathrm{II}\|_{L^2} \leq K(N,B)\frac{\epsilon^{3-\theta}}{t}(T_* - \epsilon^2\log t)^{-\frac{1}{2}}\|L_+v_+(t,\cdot)\|_{L^2} \\ + K(N,B)\frac{\epsilon^{3-\theta}}{t}(T_* - \epsilon^2\log t)^{-\frac{3}{2}-2\delta}\|v_+(t,\cdot)\|_{H^{s_0}}.\end{aligned} \tag{4.2.21}$$

To estimate III, we use (A.3.3). We obtain

$$\begin{aligned}\|\mathrm{III}\|_{L^2} \leq 2C\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}(\|Lu_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|v\|_{L^2} \\ + C\|u_+^{\mathrm{app}}\|^2_{W^{\rho_0,\infty}}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}).\end{aligned}$$

Using (4.1.5) to estimate $\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}$ and (4.2.3), we obtain a bound

$$\begin{aligned}\|\mathrm{III}\|_{L^2} \leq C_0\frac{\epsilon^2}{t}(T_* - \epsilon^2\log t)^{-1}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}) \\ + K(N)\frac{\epsilon^2}{t}(\epsilon^{2\theta'}(T_* - \epsilon^2\log t)^{-1} + 1)(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}) \\ + K(N)\frac{\epsilon^2}{t}(T_* - \epsilon^2\log t)^{-2-2\delta}\|v_+(t,\cdot)\|_{L^2}.\end{aligned} \tag{4.2.22}$$

Summing (4.2.19), (4.2.21) and (4.2.22), we deduce

(4.2.23)

$$\begin{aligned}\|\mathrm{I}+\mathrm{II}+\mathrm{III}\|_{L^2} \leq \frac{\epsilon^2}{t}[C_0(T_* - \epsilon^2\log t)^{-1} + K(N,B)\epsilon^{\theta''}(T_* - \epsilon^2\log t)^{-1} + K(N)]\|L_+v_+\|_{L^2} \\ +(K(N) + K(N,B)\epsilon^{\theta''})\frac{\epsilon^2}{t}(T_* - \epsilon^2\log t)^{-2-2\delta}\|v_+(t,\cdot)\|_{H^{s_0}}\end{aligned}$$

for some $\theta'' > 0$, which is controlled by the right hand side of (4.2.16).

• **Estimate of IV**

We estimate now the L^2-norm of the last term $L_+\mathcal{R}$ in (4.2.18), where $\mathcal{R}$ is the last term in (3.2.5) and has the structure described in the statement of Proposition 3.2.1. The contribution L_+F to $L_+\mathcal{R}$ satisfies (3.1.31) and is supported for $t \leq e^{\frac{3T_*}{4\epsilon^2}}$, so may be incorporated to $\mathcal{R}_L$ in (4.2.16), with $\mathcal{R}_L$ satisfying (4.2.17). The contribution L_+r_{app} to $L_+\mathcal{R}$ is supported for $t \geq e^{\frac{T_*}{2\epsilon^2}}$ and satisfies (3.1.32), so that we may incorporate it to $\mathcal{R}_H$ in (4.2.16), with $\mathcal{R}_H$ satisfying (4.2.17). We are left with studying the quintic terms obtained making act L_+ on (3.2.6). We consider first the action of L_+ on the first term in (3.2.6). Since $|J_1| \geq 1$, the first argument in $\mathrm{Op}(\tilde{m})(\cdots)$ is

equal to $v_\pm$. When we make act $L_+ = x + tp'(D_x)$ on it, we argue as in (3.2.15), and rewrite the resulting expression as a sum of terms of the following form

$$\begin{aligned} &\mathrm{Op}(m)(L_\pm v_\pm, v_{J_1'}, u^{\mathrm{app}}_{J_2'}) \\ &\mathrm{Op}(m)(v_\pm, v_{J_1'}, u^{\mathrm{app}}_{J_2'}) \\ &tp'(D_x)\mathrm{Op}(m)(v_\pm, v_{J_1'}, u^{\mathrm{app}}_{J_2'}) \\ &\pm t\mathrm{Op}(m)(p'(D_x)v_\pm, v_{J_1'}, u^{\mathrm{app}}_{J_2'}) \end{aligned} \tag{4.2.24}$$

whee $|J_1'| + |J_2'| = 4$ and m is a new symbol in the class $S_{1,0}(M_0^\nu, 3)$ for some ν, with constant coefficients. We apply (A.2.4) with $j = 1$ to all these expressions. We get an estimate of their L^2 norms by

$$(\|v_+\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})^4(\|L_+v_+\|_{L^2} + t\|v_+\|_{L^2}). \tag{4.2.25}$$

By (4.2.20) and the a priori assumption (4.2.15), we get a bound

$$K(N,B)\frac{\epsilon^4}{t}(T_* - \epsilon^2\log t)^{-2}[t^{-1}\|L_+v_+\|_{L^2} + \|v_+\|_{L^2}].$$

Using again that by (3.1.10) $t^{-1}(T_* - \epsilon^2\log t)^{-1} = O(1)$, we get finally the upper bound

$$K(N,B)\frac{\epsilon^4}{t}(T_* - \epsilon^2\log t)^{-1}\|L_+v_+\|_{L^2} + K(N,B)\frac{\epsilon^4}{t}(T_* - \epsilon^2\log t)^{-2}\|v_+\|_{L^2} \tag{4.2.26}$$

which is better than the right hand side of (4.2.16). Finally, we have to estimate the L^2 norm of the action of L_+ on the last two terms in (3.2.6). Arguing again as in (3.2.15), we have to study

$$\begin{aligned} &\mathrm{Op}(m)(v_{J_1}, u^{\mathrm{app}}_{J_2}, L_{i_3}G_{i_3}) \\ &\mathrm{Op}(m)(v_{J_1}, u^{\mathrm{app}}_{J_2}, G_{i_3}) \\ &tp'(D_x)\mathrm{Op}(m)(v_{J_1}, u^{\mathrm{app}}_{J_2}, G_{i_3}) \\ &t\mathrm{Op}(m)(v_{J_1}, u^{\mathrm{app}}_{J_2}, p'(D_x)G_{i_3}) \end{aligned} \tag{4.2.27}$$

for symbols m in $S_{1,0}(M_0^\nu, 3)$ with constant coefficients, $|J_1| + |J_2| = 2$, $G_+ = F$ or r_{app}, $G_- = -\bar{F}$ or $-\overline{r_{\mathrm{app}}}$. Using (A.2.4) with $j = 3$, we bound the L^2 norm of all these terms by

$$(\|v_+\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})^2(\|L_+G_+\|_{L^2} + t\|G_+\|_{L^2}). \tag{4.2.28}$$

When $G_+ = F$, since this term is supported for $t \le e^{3T_*/4\epsilon^2}$, it follows from (4.2.20) and (4.2.15) that this is bounded by

$$K(B,N)\epsilon^2(\|L_+F(t,\cdot)\|_{L^2} + \|F(t,\cdot)\|_{L^2}).$$

By (3.1.31), the integral in t of that quantity is $O(K(B,N)\epsilon^{4-\theta})$, so may be incorporated to $\mathcal{R}_L$ satisfying (4.2.17).
When $G_+ = r_{\mathrm{app}}$, we use again (4.2.20) and (4.2.15) to bound (4.2.28) by

$$\begin{aligned} \big(K(N)\frac{\epsilon^2}{t}(T_* - \epsilon^2\log t)^{-1} + K(B)\frac{\epsilon^2}{t}\big)\|L_+r_{\mathrm{app}}(t,\cdot)\|_{L^2} \\ + K(N,B)\epsilon^2(T_* - \epsilon^2\log t)^{-1}\|r_{\mathrm{app}}(t,\cdot)\|_{L^2}. \end{aligned}$$

If we plug (3.1.32) in this inequality, we largely get an estimate in

$$\frac{\epsilon^2}{t}K(N,B)\epsilon^{N_0}(T_* - \epsilon^2\log t)^{N_0-1}$$

so that we obtain a contribution to $\mathcal{R}_H$ satisfying (4.2.17). Combining this to (4.2.23) and (4.2.26) we get (4.2.16). □

Proof of Proposition 4.2.1: We assume a priori inequalities (4.2.1). For $\epsilon_0 > 0$ small enough, if $\epsilon < \epsilon_0$, inequality (4.2.11) holds. Plugging this inequality in the right hand side of (4.2.16), and assuming also ϵ_0 small enough so that $K(N,B)\epsilon^{\theta''} \leq 1$ and $B\epsilon^{1-\theta} \leq 1$, we get

$$\begin{aligned}(4.2.29)\quad \|(D_t - p(D_x))L_+w_+(t,\cdot)\|_{L^2} \leq \frac{\epsilon^2}{t}2\big[(C_0+1)(T_* - \epsilon^2\log t)^{-1} + K(N)\big]\|L_+w_+(t,\cdot)\|_{L^2} \\ + \frac{\epsilon^2}{t}K(N)(T_* - \epsilon^2\log t)^{-2-2\delta}\|v_+(t,\cdot)\|_{H^{s_0}} + \mathcal{R}_L(t) + \mathcal{R}_H(t).\end{aligned}$$

In the right hand side of (4.2.29), we plug the second a priori estimate (4.2.1) and we write the energy inequality associated to (4.2.29), starting from time $t = T(\epsilon)$ at which L_+w_+ vanishes. We get for $T \leq t \leq T(\epsilon)$, using also (4.2.17),

$$\begin{aligned}\|L_+w_+(t,\cdot)\|_{L^2} \leq \int_t^{T(\epsilon)} 2\big[(C_0+1)(T_* - \epsilon^2\log\tau)^{-1} + K(N)\big]\|L_+w_+(\tau,\cdot)\|_{L^2}\epsilon^2\,\frac{d\tau}{\tau} \\ + K(N,A_0)\epsilon^{2-\theta}\int_t^{T(\epsilon)}(T_* - \epsilon^2\log\tau)^{N_0-2-2\delta}\epsilon^2\,\frac{d\tau}{\tau} \\ + \int_t^{T(\epsilon)}\|\mathcal{R}_L(\tau)\|_{L^2}\,d\tau \\ + K(N,B)\epsilon^{N_0-2}\int_t^{T(\epsilon)}(T_* - \epsilon^2\log\tau)^{N_0-1}\epsilon^2\,\frac{d\tau}{\tau}.\end{aligned} \tag{4.2.30}$$

We set $t = e^{\frac{s}{\epsilon^2}}$, $\tau = e^{\frac{s'}{\epsilon^2}}$, $S(\epsilon) = \epsilon^2\log T(\epsilon)$

$$\begin{aligned}f(s) &= \|L_+w_+(e^{\frac{s}{\epsilon^2}},\cdot)\|_{L^2} \\ g(s) &= \epsilon^{2-\theta}(K(N,A_0) + \epsilon^\theta K(N,B))(T_* - s)^{N_1-1} + \|\mathcal{R}_L(e^{\frac{s}{\epsilon^2}})\|_{L^2}\epsilon^{-2}e^{\frac{s}{\epsilon^2}}\end{aligned} \tag{4.2.31}$$

with $N_1 \leq N_0 - 1 - 2\delta$ and N_1 large enough so that $N_0 \geq 4$. We may thus rewrite (4.2.30) under the form

$$f(s) \leq \int_s^{S(\epsilon)}\psi(s')f(s')\,ds' + \int_s^{S(\epsilon)} g(s')\,ds' \tag{4.2.32}$$

with

$$\psi(s) = 2(C_0+1)(T_* - s)^{-1} + 2K(N). \tag{4.2.33}$$

We may apply estimate (4.1.15), with notation (4.1.14). We obtain, with new constants,

(4.2.34)

$$\begin{aligned}\|L_+w_+(e^{\frac{s}{\epsilon^2}},\cdot)\|_{L^2} \leq \int_s^{S(\epsilon)}\Big(\frac{T_* - s}{T_* - s'}\Big)^{2(C_0+1)}[\epsilon^{2-\theta}\big(K(N,A_0) + \epsilon^\theta K(N,B)\big)(T_* - s')^{N_1-1} \\ + K(N)\|\mathcal{R}_L(e^{\frac{s'}{\epsilon^2}},\cdot)\|_{L^2}\epsilon^{-2}e^{\frac{s'}{\epsilon^2}}]\,ds'.\end{aligned}$$

Since C_0 is a universal constant, we may take N_1 large enough so that $N_1 - 2(C_0+1) > 0$. Moreover, as $\mathcal{R}_L(e^{\frac{s'}{\epsilon^2}},\cdot)$ is supported for $s' \leq \frac{3T_*}{4}$, $(T_* - s')^{-1}$ stays bounded on the support of that function. Using (4.2.17), we get finally

$$\|L_+w_+(e^{\frac{s}{\epsilon^2}},\cdot)\|_{L^2} \leq \epsilon^{2-\theta}(K(N,A_0) + \epsilon^{\theta''}K(N,B))(T_* - s)^{N_1}$$

for some constants depending on N, A_0, B and a new $\theta'' > 0$. By (4.2.11) and the second a priori inequality in (4.2.1), we get

$$\|L_+ v_+(t,\cdot)\|_{L^2} \leq \epsilon^{2-\theta}(K(N,A_0) + \epsilon^{\theta''}K(N,B))(T_* - \epsilon^2 \log t)^{N_1}$$

for new constants $K(N,A_0), K(N,B)$, using again that $N_0 \geq N_1 + 1 + 2\delta$. We take A_1 large enough so that $K(N,A_0) \leq \frac{A_1}{4}$ and $\epsilon < \epsilon_0$ small enough so that $\epsilon^{\theta''}K(N,B) \leq \frac{A_1}{4}$ in order to obtain (4.2.2). □

4.3 Proof of the main theorem

We shall deduce from the preceding subsections the proof of Theorem 1.2.1. Let us recall how the constants are chosen:
• One fixes first $\theta > 0$ small and δ, δ', γ satisfying (3.1.12), with δ, δ' small. One fixes also $\rho_0 \in \mathbb{N}$ large enough so that the estimates in Proposition A.2.2 hold true (for a fixed large enough ν) and ρ_0 larger than $\tilde{\rho}_0$ in Proposition A.3.1. This ρ_0 is universal and does not depend on any of the constants that we shall introduce in the forthcoming points. It determines the constant C_0 in Lemma 4.1.2.
• On chooses next $s_0 \in \mathbb{N}$ large enough, such that Proposition A.3.1 holds true and large enough in function of ρ_0 so that Proposition A.4.1 holds true.
• One takes N_1 large enough as in Proposition 4.2.1. Once N_1 has been chosen, we take N_0 so that Proposition 4.2.1 and Proposition 4.1.1 hold true. Once N_1 and N_0 have been fixed, the order N at which one has to construct the approximate solution so that $u_+^{\rm app}$ in Proposition 3.1.3 satisfies (3.1.30)-(3.1.32) is also determined.
• Once N_0 is determined, the constant A_0 is taken large enough in Proposition 4.1.1.
• Once A_0 is fixed, the constant A_1 is determined by Proposition 4.2.1. Next we choose B large enough relatively to A_0, A_1 as in (4.3.4) below.
• Finally, ϵ is taken in $]0, \epsilon_0]$ for some ϵ_0 small enough in function of all preceding constants.

Proof of Theorem 1.2.1: To construct the solution u of the theorem, one considers the solution $(u_+, u_- = -\bar{u}_+)$ of the equivalent system (3.1.7), (3.1.8): one looks for u_+ under the form $u_+ = u_+^{\rm app} + v_+$ where $u_+^{\rm app}$ is defined in (3.1.16) and v_+ satisfies equation (3.1.42). One then wants to solve this equation for v_+ backwards from $t = T(\epsilon)$, with zero initial data at $t = T(\epsilon)$, and prove that the solution exists down to time $t = 1$. By local existence theory, there is $T_0 < T(\epsilon)$ such that the solution exists on $[T_0, T(\epsilon)]$ and we denote by $T \geq 1$ the infimum of the $\tilde{T} \geq 1$ such that the solution exists on $[\tilde{T}, T(\epsilon)]$ and satisfies for all $t \in [\tilde{T}, T(\epsilon)]$ a priori estimates

$$\begin{aligned} \|v_+(t,\cdot)\|_{H^{s_0}} &\leq A_0 \epsilon^{2-\theta}(T_* - \epsilon^2 \log t)^{N_0} \\ \|L_+ v_+(t,\cdot)\|_{L^2} &\leq A_1 \epsilon^{2-\theta}(T_* - \epsilon^2 \log t)^{N_1} \\ \|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} &\leq B\frac{\epsilon^{2-\theta}}{\sqrt{t}} \end{aligned} \tag{4.3.1}$$

where the parameters $s_0, \rho_0, N_0, N_1, A_0, A_1, B$ are chosen as explained at the beginning of this subsection. If we apply Proposition 4.1.1, we get that it implies that for t in the same interval

$$\|v_+(t,\cdot)\|_{H^{s_0}} \leq \frac{A_0}{2}\epsilon^{2-\theta}(T_* - \epsilon^2 \log t)^{N_0} \tag{4.3.2}$$

if $\epsilon < \epsilon_0$ small enough. Then, applying Proposition 4.2.1, we get for $\epsilon < \epsilon_0$,

$$\|L_+ v_+(t,\cdot)\|_{L^2} \leq \frac{A_1}{2}\epsilon^{2-\theta}(T_* - \epsilon^2 \log t)^{N_1}. \tag{4.3.3}$$

By (A.4.7), we deduce from the first two inequalities (4.3.1)

$$\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \leq C\frac{\epsilon^{2-\theta}}{\sqrt{t}}(A_1+\sqrt{A_0}\sqrt{A_0+A_1}) \leq \frac{B}{2\sqrt{t}}\epsilon^{2-\theta} \tag{4.3.4}$$

if B is chosen large enough relatively to A_0, A_1.
By the bootstrap (4.3.2), (4.3.3), (4.3.4), we get that the solution v_+ exists on $[1, T(\epsilon)]$ and satisfies (4.3.1) at any t in that interval. Writing these estimates at $t=1$, we get from (4.3.1)

$$\begin{aligned}\|u_+(1,\cdot)-u_+^{\rm app}(1,\cdot)\|_{H^{s_0}} &= O(\epsilon^{2-\theta})\\ \|x(u_+(1,\cdot)-u_+^{\rm app}(1,\cdot))\|_{L^2} &= O(\epsilon^{2-\theta}).\end{aligned} \tag{4.3.5}$$

By (3.1.16) and recalling that χ_1 vanishes close to zero, we get for small enough ϵ,

$$\begin{aligned}\|u_+(1,\cdot)-\tilde{u}_+^{\rm app}(1,\cdot)\|_{H^{s_0}} &= O(\epsilon^{2-\theta})\\ \|x(u_+(1,\cdot)-\tilde{u}_+^{\rm app}(1,\cdot))\|_{L^2} &= O(\epsilon^{2-\theta}).\end{aligned} \tag{4.3.6}$$

The definition (3.1.4) (resp. (3.1.14)) of u_+ (resp. $\tilde{u}_+^{\rm app}$) from u (resp. $u_{\rm app}$) and the fact that u, $u_{\rm app}$ are real valued functions imply

$$\begin{aligned}\|u(1,\cdot)-u_{\rm app}(1,\cdot)\|_{H^{s_0+1}}+\|D_tu(1,\cdot)-D_tu_{\rm app}(1,\cdot)\|_{H^{s_0}} &= O(\epsilon^{2-\theta})\\ \|x(u(1,\cdot)-u_{\rm app}(1,\cdot))\|_{H^1}+\|x(D_tu(1,\cdot)-D_tu_{\rm app}(1,\cdot))\|_{L^2} &= O(\epsilon^{2-\theta}).\end{aligned} \tag{4.3.7}$$

By (2.2.44), (2.1.33) and (2.1.3), $(u_{\rm app}(1,\cdot),\partial_tu_{\rm app}(1,\cdot))$ are the initial conditions $(\epsilon f_0,\epsilon g_0)$ chosen in the statement of the theorem so that (1.2.3) holds. Thus (4.3.7) shows that the initial conditions of our solution u have structure (1.2.7), with the perturbation $(f(x,\epsilon),g(x,\epsilon))$ satisfying (1.2.6).
It remains to prove (1.2.8). At time $t=T(\epsilon)$, the value of u (resp. ∂_tu) is given by $u_{\rm app}(T(\epsilon),\cdot)$ (resp. $\partial_tu_{\rm app}(T(\epsilon),\cdot)$). By (2.2.44), these quantities are equal to $u^2_{\rm app}(T(\epsilon),\cdot)$ (resp. $\partial_tu^2_{\rm app}(T(\epsilon),\cdot)$) with $u^2_{\rm app}$ given by (2.2.2). All contributions corresponding to $\ell\geq 3$ in (2.2.2), as well as their derivatives, have modulus bounded from above by

$$\epsilon^{2-\ell}T(\epsilon)^{-\frac12}(T_*-\epsilon^2\log T(\epsilon)+|y-y_0|^{2\kappa_0})^{-\frac12}\big(T(\epsilon)^{-\frac12}(T_*-\epsilon^2\log T(\epsilon))^{-\frac12-\delta}\big)^{\ell-1}. \tag{4.3.8}$$

Since $\ell\geq 3$, (3.1.10) implies a bound in

$$T(\epsilon)^{-\frac12}(T_*-\epsilon^2\log T(\epsilon))^{-\frac12}\Big(\epsilon^{\frac{\gamma}{2}-\delta'}(T_*-\epsilon^2\log T(\epsilon))^{\delta'-\delta}\Big)^{\ell-1}\epsilon^{2-\ell}. \tag{4.3.9}$$

By (1.2.5), (1.2.4), $T_*-\epsilon^2\log T(\epsilon)=\epsilon^2u(\epsilon')$ is exponentially small in $e^{-\frac{c}{\epsilon^2}}$, so that since $\delta'>\delta$ (4.3.9), and thus all terms with $\ell\geq 3$ in (2.2.2) computed at $t=T(\epsilon)$, are negligible relatively to $\epsilon T(\epsilon)^{-\frac12}(T_*-\epsilon^2\log T(\epsilon))^{-\frac12}$. On the other hand, by (2.1.12) and (1.2.3), (1.2.2), the coefficient of $\frac{\epsilon}{\sqrt{t}}e^{it\varphi(x/t)}$ in (2.2.2), computed at $t=T(\epsilon)$, $\frac{x}{t}=y_0$ satisfies

$$|a_{1,1}(\epsilon^2\log T(\epsilon),y_0)|=|a_1^0(y_0)|\Big(1-\frac{\epsilon^2\log T(\epsilon)}{T_*}\Big)^{-\frac12}$$

since $\Gamma(y_0)\phi(y_0)=T_*^{-1}$ by (1.2.2), (1.2.3). Moreover, since $|a_1^0(y_0)|=(1-y_0^2)^{-\frac14}\Gamma(y_0)^{\frac12}$ by (2.1.8), (1.2.1), with $\Gamma(y_0)\neq 0$ by (1.2.2), we get that all terms with $\ell\geq 3$ in (2.2.2) at time $T(\epsilon)$ are $o(\epsilon T(\epsilon)^{-\frac12}|a_{1,1}(\epsilon^2\log T(\epsilon),y_0)|)$. We conclude that the main contribution to (2.2.2) at time $t=T(\epsilon)$ and $x=y_0T(\epsilon)$ is

$$2{\rm Re}\left[\epsilon T(\epsilon)^{-\frac12}e^{iT(\epsilon)\varphi(y_0)}a_{1,1}(\epsilon^2\log T(\epsilon),y_0)\right]$$

and its time derivative is

$$2\mathrm{Re}\left[\epsilon T(\epsilon)^{-\frac{1}{2}} i\omega(y_0) e^{iT(\epsilon)\varphi(y_0)} a_{1,1}(\epsilon^2 \log T(\epsilon), y_0)\right].$$

Thus

$$\begin{aligned} |u^2_{\mathrm{app}}(T(\epsilon), y_0 T(\epsilon))| + |\partial_t u^2_{\mathrm{app}}(T(\epsilon), y_0 T(\epsilon))| &\sim \epsilon T(\epsilon)^{-\frac{1}{2}} |a_{1,1}(\epsilon^2 \log T(\epsilon), y_0)| \\ &\sim \epsilon T(\epsilon)^{-\frac{1}{2}} (T_* - \epsilon^2 \log T(\epsilon))^{-\frac{1}{2}} \\ &\sim T(\epsilon)^{-\frac{1}{2}} u(\epsilon')^{-\frac{1}{2}} \sim T(\epsilon)^{-\frac{1}{2}} \epsilon'^{-\frac{1}{2}} \end{aligned} \tag{4.3.10}$$

by (1.2.5). If $c > 0$ is given and if δ' in (1.2.4) is taken small enough with respect to c, one has $\epsilon' \leq e^{-\frac{T_*}{\epsilon^2}(1-2c)} \sim T(\epsilon)^{-1+2c}$. Thus (4.3.10) is bounded from below by $T(\epsilon)^{-c}$ which gives the first equality (1.2.8).
To get the second one, we proceed in the same way, except that we have to estimate from below

$$\frac{\epsilon}{\sqrt{T(\epsilon)}} \big\| a_{1,1}(\epsilon^2 \log T(\epsilon), \frac{x}{T(\epsilon)}) \big\|_{L^2(dx)} = \epsilon \| a_{1,1}(\epsilon^2 \log T(\epsilon), y) \|_{L^2(dy)}.$$

By (1.2.3), the example following Definition 2.2.1 and the expression (2.1.12) of $a_{1,1}$, one has

$$|a_{1,1}(s, y)| \sim (T_* - s + |y - y_0|^{2\kappa_0})^{-\frac{1}{2}}$$

so that if $\kappa_0 > 0$,

$$\| a_{1,1}(\epsilon^2 \log T(\epsilon), y) \|_{L^2(dy)} \sim (T_* - \epsilon^2 \log T(\epsilon))^{-\frac{1}{2} + \frac{1}{4\kappa_0}}.$$

We have seen above that $(T_* - \epsilon^2 \log T(\epsilon))^{-1} \geq T(\epsilon)^{1-2c} \epsilon^{-2}$. Then the second inequality (1.2.8) for the $a_{1,1}$-term in (2.2.2) with the lower bound (1.2.9) follows from that, up to changing the definition of c. Since the contributions to (2.2.2) indexed by $\ell \geq 3$ are bounded point-wise by (4.3.8) and thus by $T(\epsilon)^{-\frac{1}{2}} |a_{1,1}(\epsilon^2 \log T(\epsilon), \frac{x}{T(\epsilon)})| e^{-\frac{c}{\epsilon^2}}$ for some $c > 0$, as seen after (4.3.9), they are negligible perturbations, so that (1.2.8), (1.2.9) hold for $u^2_{\mathrm{app}}(T(\epsilon), \cdot)$. This concludes the proof. □

A Appendix

A.1 Pseudo-differential operators

In this subsection, we prove several results on pseudo-differential operators used in the bulk of the proof.

Definition A.1.1 *Let $p(x, \xi)$ be a smooth function on $\mathbb{R} \times \mathbb{R}$, satisfying for some $\mu \in \mathbb{R}$ and all α, β in $\mathbb{N}$*

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{\mu - |\beta|}. \tag{A.1.1}$$

Then if $u \in \mathcal{S}(\mathbb{R})$, we set

$$p(x, D_x) u = \frac{1}{2\pi} \int e^{ix\xi} p(x, \xi) \hat{u}(\xi)\, d\xi \tag{A.1.2}$$

and if $h \in]0,1]$ is a semi-classical parameter, we set

$$\begin{aligned} p(x,hD_x)v &= \frac{1}{2\pi}\int e^{ix\xi}p(x,h\xi)\hat{v}(\xi)\,d\xi \\ &= \frac{1}{2\pi h}\int e^{i\frac{(x-y)\xi}{h}}p(x,\xi)v(y)\,dyd\xi \end{aligned} \tag{A.1.3}$$

where the last integral is an oscillatory one. This is related to (A.1.2) by the conjugation formula

$$\Theta_h^{-1}\circ p(x,hD_x)\circ\Theta_h = p(hx,D_x) \tag{A.1.4}$$

if we define

$$(\Theta_h u)(x) = \frac{1}{\sqrt{h}}u(\frac{x}{h}). \tag{A.1.5}$$

We want to study the action of $p(D_x)$ on oscillating expressions of the form used to construct an approximate solution in section 2. We define first

Definition A.1.2 *Let $x_0 \in]-1,1[$, $\kappa_0 \in \mathbb{N}$. We denote for $m \in \mathbb{R}$ by $\tilde{\Sigma}^m$ the space of continuous functions*

$$\begin{gathered}(x,\lambda,h,\epsilon) \to \sigma(x,\lambda,h,\epsilon) \\ \mathbb{R}\times[1,+\infty[\times]0,1]\times[0,1] \to \mathbb{C}\end{gathered}$$

smooth in (x,λ,h), supported for $|x|\leq 1$, that satisfy for any α,β,ζ,N in $\mathbb{N}$, any (x,λ,h,ϵ) in $[-1,1]\times[1,+\infty[\times]0,1]\times[0,1]$

$$|\partial_x^\alpha\partial_\lambda^\beta(h\partial_h)^\zeta\sigma(x,\lambda,h,\epsilon)| \leq C_{\alpha,\beta,\zeta}\lambda^{-m+\alpha-\beta}(1+\lambda|x-x_0|)^{m-\alpha-2\kappa_0\beta}(1-|x|)^N. \tag{A.1.6}$$

Let $\psi :]-1,1[\to\mathbb{R}$ be a smooth function such that for some $A\in\mathbb{R}_+$ and any $\alpha\in\mathbb{N}$

$$|\partial_x^\alpha\psi(x)| \leq C_\alpha(1-|x|)^{-A-|\alpha|},\ \forall x\in]-1,1[. \tag{A.1.7}$$

Finally, let $\xi\to p(\xi)$ be a symbol independent of x, satisfying (A.1.1).

Proposition A.1.3 *Let σ be in $\tilde{\Sigma}^m$. Then for any (x,λ,h,ϵ) satisfying $\lambda h\leq 1$, we have*

$$\begin{aligned} p(hD_x)[e^{\frac{i}{h}\psi(x)}\sigma(x,\lambda,h,\epsilon)] &= p(d\psi(x))\sigma(x,\lambda,h,\epsilon)e^{\frac{i}{h}\psi(x)} \\ &\quad + h\sigma_1(x,\lambda,h,\epsilon)e^{\frac{i}{h}\psi(x)} + \underline{r}(x,\lambda,h,\epsilon) \end{aligned} \tag{A.1.8}$$

where $\sigma_1\in\tilde{\Sigma}^{m-1}$ and where $\underline{r}$ is a continuous function on $\mathbb{R}\times[1,+\infty[\times]0,1]\times[0,1]$ smooth in (x,λ,h), satisfying for any $\alpha,\beta,\zeta,N\in\mathbb{N}$

$$|\partial_x^\alpha\partial_\lambda^\beta(h\partial_h)^\zeta\underline{r}(x,\lambda,h,\epsilon)| \leq Ch^N(1+|x|)^{-N}. \tag{A.1.9}$$

Proof: The left hand side of (A.1.8) is

$$\frac{1}{2\pi h}\int e^{\frac{i}{h}[(x-y)\xi+\psi(y)]}p(\xi)\sigma(y,\lambda,h,\epsilon)\,dyd\xi. \tag{A.1.10}$$

Let $(x,y)\to\theta(x,y)$ be a smooth function on $\mathbb{R}\times]-1,1[$, supported for $|x-y|\ll 1-|y|$, such that for any α,β

$$|\partial_x^\alpha\partial_y^\beta\theta(x,y)| \leq C(1-|y|)^{-\alpha-\beta}. \tag{A.1.11}$$

Assume also $\theta(x,y)=1$ if $|x-y|\leq c(1-|y|)$ for some small $c>0$. If we insert the cut-off $1-\theta$ under the integral, and make N' integration by parts in ξ, we get an integrand bounded by

$$Ch^{N'}\lambda^{-m}(1+\lambda|y-x_0|)^m(1-|y|)^{N-N'}\langle\xi\rangle^{\mu-N'}(1+|x-y|)^{-N'} \tag{A.1.12}$$

by (A.1.1), (A.1.6). If we make act on (A.1.10) $h\partial_h$, we get also a similar bound, with a different N', using also (A.1.7). We thus see that (A.1.10) with the cut-off $1-\theta$ under the integral brings a contribution to $\underline{r}$ in (A.1.8), using that by assumption $\lambda=O(1/h)$ in order to control any positive power of λ like those coming from ∂_x-derivatives. We are thus reduced to

$$\frac{1}{2\pi h}\int e^{\frac{i}{h}[(x-y)\xi+\psi(y)]}\theta(x,y)p(\xi)\sigma(y,\lambda,h,\epsilon)\,dyd\xi. \tag{A.1.13}$$

Define

$$\psi_1(x,y)=\int_0^1\psi'(\tau y+(1-\tau)x)\,d\tau. \tag{A.1.14}$$

As on the support of θ, $1-|x|\sim 1-|y|$, we see using (A.1.7) that for $\theta(x,y)\neq 0$

$$|\partial_x^\alpha\partial_y^\beta\psi_1(x,y)|\leq C_{\alpha,\beta}(1-|y|)^{-A-1-\alpha-\beta} \tag{A.1.15}$$

and $\psi(y)=\psi(x)-\psi_1(x,y)(x-y)$, so that (A.1.13) may be written

$$\frac{1}{2\pi h}e^{i\frac{\psi(x)}{h}}\int e^{\frac{i}{h}(x-y)\eta}\theta(x,y)p(\eta+\psi_1(x,y))\sigma(y,\lambda,h,\epsilon)\,dyd\eta. \tag{A.1.16}$$

Inside this integral, we decompose

$$p(\eta+\psi_1(x,y))=p(\psi_1(x,y))+\eta q(x,y,\eta) \tag{A.1.17}$$

where $q(x,y,\eta)=\int_0^1 p'(\psi_1(x,y)+\tau\eta)\,d\tau$ satisfies according to (A.1.1), (A.1.15) and for (x,y) staying in the support of θ, bounds of the form

$$|\partial_x^\alpha\partial_y^\beta\partial_\eta^\gamma(\eta\partial_\eta)^\zeta q(x,y,\eta)|\leq C\langle\eta\rangle^{\max(\mu-1,0)}(1-|y|)^{-K_{\alpha,\beta,\gamma,\zeta}} \tag{A.1.18}$$

for some positive exponents $K_{\alpha,\beta,\gamma,\zeta}$. We substitute (A.1.17) inside (A.1.16). The first term in the right hand side of (A.1.17) gives the first term in the right hand side of (A.1.8). Consider next the term in (A.1.16) coming from the last term in (A.1.17) i.e. the product of $e^{i\frac{\psi(x)}{h}}$ by

$$-\frac{i}{2\pi}\int e^{\frac{i}{h}(x-y)\eta}\tilde{q}(x,y,\eta,\lambda,h,\epsilon)\,dyd\eta, \tag{A.1.19}$$

with

$$\tilde{q}(x,y,\eta,\lambda,h,\epsilon)=\partial_y[\theta(x,y)q(x,y,\eta)\sigma(y,\lambda,h,\epsilon)].$$

It follows from (A.1.6), (A.1.11), (A.1.18) that

$$\begin{aligned}|\partial_y^{\alpha'}\partial_\eta^\gamma(\eta\partial_\eta)^{\gamma'}(h\partial_h)^\zeta\tilde{q}(x,y,\eta,\lambda,h,\epsilon)|\\ \leq C\lambda^{-m+1+\alpha'}(1+\lambda|y-x_0|)^{m-1}(1-|y|)^N\langle\eta\rangle^{\max(\mu-1,0)}\end{aligned} \tag{A.1.20}$$

for any $\alpha',\gamma,\gamma',\zeta,N$. We perform inside integral (A.1.19) integrations by parts using the operator $\langle\frac{\eta}{\lambda h}\rangle^{-2}\big(1-\frac{\eta}{\lambda^2 h}D_y\big)$. We shall obtain a new expression

$$\int e^{\frac{i}{h}(x-y)\eta}\tilde{q}_1(x,y,\eta,\lambda,h,\epsilon)\,dyd\eta \tag{A.1.21}$$

where $\tilde{q}_1$ satisfies since $\lambda h \leq 1$ and since $1-|x| \sim 1-|y|$ on the support of $\theta(x,y)$,

$$(A.1.22)\quad |\partial_y^{\alpha'}\partial_\eta^{\gamma}(\eta\partial_\eta)^{\gamma'}(h\partial_h)^{\zeta}\tilde{q}_1(x,y,\eta,\lambda,h,\epsilon)| \\ \leq C\lambda^{-m+1+\alpha'}(\lambda h)^{-\gamma}\Big\langle\frac{\eta}{\lambda h}\Big\rangle^{-N_0}(1+\lambda|y-x_0|)^{m-1}(1-|x|)^N\langle\eta\rangle^{\max(\mu-1,0)}$$

for an arbitrary large N_0. We perform next integrations by parts in (A.1.21) using

$$\langle\lambda(x-y)\rangle^{-2}(1+\lambda^2 h(x-y)\cdot D_\eta).$$

It follows from (A.1.22) that the modulus of (A.1.21) is bounded by

$$(A.1.23)\quad C\lambda^{-m+1}\int\Big\langle\frac{\eta}{\lambda h}\Big\rangle^{-N_0}\langle\lambda(x-y)\rangle^{-N_0}(1+\lambda|y-x_0|)^{m-1}\langle\eta\rangle^{\max(\mu-1,0)}\,dyd\eta(1-|x|)^N.$$

Since $\lambda h\leq 1$, the modulus of (A.1.23) is $O(h\lambda^{-m+1}(1+\lambda|x-x_0|)^{m-1}(1-|x|)^N)$.
If we make act a ∂_x-derivative on the integral in (A.1.16), one ∂_y-integration by parts together with (A.1.15) and estimates (A.1.6), (A.1.18), shows that we get the same estimates as in (A.1.18)-(A.1.23), with m replaced by $m-1$. In the same way, a ∂_λ-derivative acting on the integral gives rise to an extra factor $\lambda^{-1}(1+\lambda|y-x_0|)^{-2\kappa_0}$, that induces in the estimates of (A.1.23) a corresponding factor $\lambda^{-1}(1+\lambda|x-x_0|)^{-2\kappa_0}$.
Finally, a $h\partial_h$-derivative acting on the exponential in (A.1.16) may be trade off against a $\eta\partial_\eta$-derivative, so that by integration by parts, the final expression (A.1.23) has still the same estimates. We thus see that (A.1.16) with $p(\eta+\psi_1(x,y))$ replaced by $\eta q(x,y,\eta)$ may be written as the second term in the right hand side of (A.1.8). This concludes the proof. □

We shall translate Proposition A.1.3 on the class of symbols Σ^m introduced in Definition 2.2.1. Notice that if a belongs to Σ^m and if for $s\in[0,T_*[$, we set

$$(A.1.24)\qquad \lambda=(T_*-s)^{-\frac{1}{2\kappa_0}},\ s=T_*-\lambda^{-2\kappa_0},$$

then for $a(s,y,h,\epsilon)$ in Σ^m, the function

$$(A.1.25)\qquad \sigma(y,\lambda,h,\epsilon)=a(T_*-\lambda^{-2\kappa_0},y,h,\epsilon)$$

satisfies by (2.2.1)

$$|\sigma(y,\lambda,h,\epsilon)|\leq C\lambda^{-2\kappa_0 m}(1+\lambda|y-x_0|)^{2\kappa_0 m}(1-|y|)^N$$

for any N i.e. bound (A.1.6) with $\alpha=\beta=\zeta=0$ and m replaced by $\tilde{m}=2\kappa_0 m$. If we take a ∂_y-derivative of σ, we get in the same way estimate (A.1.6) with $\tilde{m}=2\kappa_0 m$ and $\alpha=1$. One checks similarly that $\partial_\lambda, h\partial_h$ derivatives acting on (A.1.25) give rise to the similar bounds for (A.1.6). In other words, with definition (A.1.25) of σ in terms of a, we have the equivalence

$$(A.1.26)\qquad a\in\Sigma^m\Leftrightarrow\sigma\in\tilde{\Sigma}^{\tilde{m}}\text{ with }\tilde{m}=2\kappa_0 m.$$

Corollary A.1.4 *Let $p(\xi)$ be a function independent of x and satisfying (A.1.1). Let $(s,y,h,\epsilon)\to a(s,y,h,\epsilon)$ be an element of Σ^m defined as in Definition 2.2.1 for some m in $\mathbb{R}$. Let ψ be a real phase function defined on $]-1,1[$ satisfying (A.1.7). Then if $t(T_*-s)^{\frac{1}{2\kappa_0}}\geq c>0$, we have*

$$(A.1.27)\quad p(D_x)\Big[e^{it\psi(\frac{x}{t})}a\big(s,\frac{x}{t},\frac{1}{t},\epsilon\big)\Big]=e^{it\psi(\frac{x}{t})}p\Big(\psi'\big(\frac{x}{t}\big)\Big)a\big(s,\frac{x}{t},\frac{1}{t},\epsilon\big) \\ +\frac{1}{t}e^{it\psi(\frac{x}{t})}a_1\big(s,\frac{x}{t},\frac{1}{t},\epsilon\big)+r\big(s,\frac{x}{t},\frac{1}{t},\epsilon\big)$$

where $a_1 \in \Sigma^{m-\frac{1}{2\kappa_0}} \subset \Sigma^{m-1}$ *and* r *satisfies*

$$|\partial_s^\alpha \partial_y^\beta (h\partial_h)^\zeta r(s,y,h,\epsilon)| \le C_N h^N (1+|y|)^{-N} \tag{A.1.28}$$

for all α, β, ζ, N.

Proof: If we set $h = \frac{1}{t}$, we have according to (A.1.5)

$$a(s, \frac{x}{t}, h, \epsilon) = \sqrt{t}(\Theta_h^{-1} a)(s,x,h,\epsilon) \tag{A.1.29}$$

so that the left hand side of (A.1.27) may be written according to (A.1.4) as

$$\frac{1}{\sqrt{h}} \Theta_h^{-1} \big[p(hD_x)[e^{i\frac{\psi(x)}{h}} a(s,x,h,\epsilon)] \big]. \tag{A.1.30}$$

We notice that if λ is defined by (A.1.24), the assumption $\lambda h \le c$ of Proposition A.1.3 is equivalent to the condition $t(T_* - s)^{\frac{1}{2\kappa_0}} \ge c^{-1}$ that we impose in the corollary. If we apply Proposition A.1.3 to the symbol σ defined from a by (A.1.25), we deduce that (A.1.30) is equal to

$$\frac{1}{\sqrt{h}} \Theta_h^{-1} \big[p(d\psi(x)) e^{i\frac{\psi(x)}{h}} a(s,x,h,\epsilon) + h e^{i\frac{\psi(x)}{h}} \sigma_1(x,\lambda,h,\epsilon) + \underline{r}(x,\lambda,h,\epsilon) \big] \tag{A.1.31}$$

for some element $\sigma_1 \in \tilde{\Sigma}^{\tilde{m}-1}$ with $\tilde{m} = 2\kappa_0 m$ by (A.1.26). We denote by $a_1 \in \Sigma^{m-\frac{1}{2\kappa_0}}$ the symbol associated to σ_1 by (A.1.26), so that by (A.1.29), we obtain (A.1.27) with $r(s,x,h,\epsilon) = \underline{r}(x,\lambda,h,\epsilon)$ that satisfies (A.1.28) by (A.1.9), (A.1.24) and the fact that $\lambda(s)h \le 1$. This concludes the proof. □

Corollary A.1.5 *Denote for* $|x| < 1$, $\varphi(x) = \sqrt{1-x^2}$ *and set* $p(\xi) = \sqrt{1+\xi^2}$. *Let* m *be an element of* Σ^m *and* q *be in* $\mathbb{Z}$. *We have*

(A.1.32)

$$\begin{aligned}(x + tp'(D_x))[e^{itq\varphi(x/t)} a(s, \frac{x}{t}, \frac{1}{t}, \epsilon)] = x\Big(1 - \frac{q}{\sqrt{1+(q^2-1)(x/t)^2}}\Big) e^{itq\varphi(x/t)} a(s, \frac{x}{t}, \frac{1}{t}, \epsilon) \\ + e^{itq\varphi(x/t)} a_1(s, \frac{x}{t}, \frac{1}{t}, \epsilon) + r(s, \frac{x}{t}, \frac{1}{t}, \epsilon)\end{aligned}$$

for some $a_1 \in \Sigma^{m-\frac{1}{2\kappa_0}} \subset \Sigma^{m-1}$ *and* r *satisfying (A.1.28) (with* $h = \frac{1}{t}$*).*

Proof: We just apply (A.1.27) noticing that $y + p'(q\varphi'(y)) = y\Big(1 - \frac{q}{\sqrt{1+(q^2-1)y^2}}\Big)$ by the definition of p, φ. □

A.2 Properties of multilinear operators

We gather here some properties of multilinear operators that we use in the bulk of the proof. Some of them follow from the appendices in [14].

Lemma A.2.1 *Let $m_1 \in S_{1,0}(M^{\nu_1}, p)$, $m_2 \in S_{1,0}(M^{\nu_2}, q)$ for some $p, q \in \mathbb{N}^*$, some $\nu_1, \nu_2 \in \mathbb{N}$, with the notation introduced in Definition 3.1.1. Assume moreover that m_1, m_2 have constant coefficients. Then there is m in $S_{1,0}(M_0^{\nu_1+\nu_2}, p+q-1)$ such that*

$$\mathrm{Op}(m_1)(u_1, \dots, u_{p-1}, \mathrm{Op}(m_2)(u_p, \dots, u_{p+q-1})) = \mathrm{Op}(m)(u_1, \dots, u_{p+q-1}) \tag{A.2.1}$$

for any functions $u_1, \dots, u_{p+q-1}$.

Proof: Equality (A.2.1) follows from (3.1.3) setting

$$m(\xi_1, \dots, \xi_{p+q-1}) = m_1(\xi_1, \dots, \xi_{p-1}, \xi_p + \cdots + \xi_{p+q-1}) m_2(\xi_p, \dots, \xi_{p+q-1}).$$

The conclusion follows from

$$M_0(\xi_1, \dots, \xi_{p-1}, \xi_p + \cdots + \xi_{p+q-1})^{\nu_1} M_0(\xi_p, \dots, \xi_{p+q-1})^{\nu_2} \leq C M_0(\xi_1, \dots, \xi_{p+q-1})^{\nu_1+\nu_2}$$

since $M_0(\xi_1, \dots, \xi_n)$ is equivalent to the second largest among $|\xi_1|+1, \dots, |\xi_n|+1$. □

We recall some results abound boundedness properties of operators associated to symbols in the class $S_{\kappa,0}(M_0^{\nu}, p)$ from [14]. Recall that we defined

$$\|u\|_{W^{\rho,\infty}} = \|\langle D_x \rangle^{\rho} w\|_{L^\infty}. \tag{A.2.2}$$

Then, by Proposition D.1.1 of [14] (applied with $h = 1$ and to symbols independent of x, y with the notation of that reference), we have

Proposition A.2.2 *Let $n \in \mathbb{N}^*, \kappa \in \mathbb{N}, \nu \geq 0$. There is $\rho_0 \in \mathbb{N}$ such that for any $m \in S_{\kappa,0}(M_0^{\nu}, n)$, independent of x, the following estimates hold for any $s \in \mathbb{N}$, any $v_1, \dots, v_n$*

$$\|\mathrm{Op}(m)(v_1, \dots, v_n)\|_{H^s} \leq C_s \sum_{j=1}^{n} \Big(\prod_{\ell \neq j} \|v_\ell\|_{W^{\rho_0,\infty}} \Big) \|v_j\|_{H^s} \tag{A.2.3}$$

and moreover, for any fixed j in $\{1, \dots, n\}$

$$\|\mathrm{Op}(m)(v_1, \dots, v_n)\|_{L^2} \leq C \Big(\prod_{\ell \neq j} \|v_\ell\|_{W^{\rho_0,\infty}} \Big) \|v_j\|_{L^2} \tag{A.2.4}$$

$$\|\mathrm{Op}(m)(v_1, \dots, v_n)\|_{H^s} \leq C_s \Big(\prod_{\ell \neq j} \|v_\ell\|_{W^{\rho_0+s,\infty}} \Big) \|v_j\|_{H^s} \tag{A.2.5}$$

If one assumes in addition that m is supported for $|\xi_1| + \cdots + |\xi_{n-1}| \leq C(1 + |\xi_n|)$ for some constant C, one gets instead of (A.2.3)

$$\|\mathrm{Op}(m)(v_1, \dots, v_n)\|_{H^s} \leq C_s \Big(\prod_{\ell=1}^{n-1} \|v_\ell\|_{W^{\rho_0,\infty}} \Big) \|v_n\|_{H^s} \tag{A.2.6}$$

and for any $j < n$,

$$\|\mathrm{Op}(m)(v_1, \dots, v_n)\|_{H^s} \leq C_s \|v_j\|_{L^2} \Big(\prod_{\substack{\ell \neq j \\ \ell \neq n}} \|v_\ell\|_{W^{\rho_0,\infty}} \Big) \|v_n\|_{W^{\rho_0+s,\infty}}. \tag{A.2.7}$$

Without the support condition on m, we get instead for any $1 \leq \ell \leq n-1$

$$\|\mathrm{Op}(m)(v_1,\dots,v_n)\|_{H^s} \leq C_s\Big[\sum_{j=1}^{\ell}\Big(\prod_{\ell'\neq j}\|v_{\ell'}\|_{W^{\rho_0,\infty}}\Big)\|v_j\|_{H^s} + \sum_{j=\ell+1}^{n}\sum_{j'=1}^{\ell}\|v_j\|_{W^{\rho_0+s,\infty}}\|v_{j'}\|_{L^2}\prod_{\substack{\ell'\neq j,j'\\ 1\leq\ell'\leq n}}\|v_{\ell'}\|_{W^{\rho_0,\infty}}\Big]. \tag{A.2.8}$$

Finally, inequality (A.2.4) holds also for x-dependent symbols in $S_{\kappa,\beta}(M_0^\nu,n)$ for any $\kappa\geq 0,\beta\geq 0$.

Proof: Estimates (A.2.3) and (A.2.4) are inequalities (D.6) and (D.7) of Proposition D.1.1 of [14]. Inequality (A.2.5) follows from (A.2.4) if we make act s ∂_x-derivatives on $\mathrm{Op}(m)(v_1,\dots,v_n)$ and use Leibniz rule. In addition, (A.2.4) holds for general symbols in $S_{\kappa,\beta}(M_0^\nu,n)$ by (iii) of Proposition D.1.1 of [14]. Estimate (A.2.6) is just inequality (D.5) in [14]. Let us prove (A.2.7) when $j=1$ for instance. Using the support property of m, we may write for any $\alpha\in\mathbb{N}$, $\alpha\leq s$,

$$\partial_x^\alpha \mathrm{Op}(m)(v_1,\cdots,v_n) = \mathrm{Op}(\tilde m)(v_1,\dots,v_{n-1},\langle D_x\rangle^s v_n)$$

for another symbol $\tilde m$ in $S_{\kappa,0}(M_0^\nu,n)$. Applying (A.2.4) we get (A.2.7).
To prove (A.2.8), we decompose

$$m(\xi_1,\dots,\xi_n)=\sum_{j=1}^{n} m_j(\xi_1,\dots,\xi_n)$$

where m_j is in $S_{\kappa,0}(M_0^\nu,n)$ and is supported for $|\xi_1|+\cdots+\widehat{|\xi_j|}+\cdots+|\xi_n|\leq C(1+|\xi_j|)$. For $1\leq j\leq \ell$, we apply (A.2.6) with n replaced by j to bound $\|\mathrm{Op}(m_j)(v_1,\cdots,v_n)\|_{H^s}$ by the first sum in the right hand side of (A.2.8). For $\ell+1\leq j\leq n$ we bound $\|\mathrm{Op}(m_j)(v_1,\cdots,v_n)\|_{H^s}$ using (A.2.7) with (j,n) replaced by (j',j). This concludes the proof. □

A.3 Action of L_+ on characteristic cubic expressions

Consider m an element of $S_{1,0}(M_0,3)$ with constant coefficients, with the notation introduced in Definition 3.1.1. Let $I=(i_1,i_2,i_3)$ be a characteristic index i.e. an element of $\{-1,1\}^3$ with $i_1+i_2+i_3=1$. The goal of this subsection is to obtain L^2 estimates for the action of L_+ on a characteristic cubic term.

Proposition A.3.1 *There are integers $\tilde\rho_0,\tilde s_0$ in $\mathbb{N}$ such that for any functions w_1,w_2,w_3 the following estimate holds true*

$$\|L_+\mathrm{Op}(m)(w_1,w_2,w_3)\|_{L^2}\leq C\sum_{\ell=1}^{3}\big(\|L_{i_\ell}w_\ell\|_{L^2}+\|w_\ell\|_{H^{\tilde s_0}}\big)\prod_{\substack{1\leq j\leq 3\\ j\neq\ell}}\|w_j\|_{W^{\tilde\rho_0,\infty}}. \tag{A.3.1}$$

In addition, one has also the bounds

$$\begin{aligned}\|L_+\mathrm{Op}(m)(w_1,w_2,w_3)\|_{L^2}\leq{}& C\big(\|L_{i_1}w_1\|_{L^2}+\|w_1\|_{H^{\tilde s_0}}\big)\|w_2\|_{W^{\tilde\rho_0,\infty}}\|w_3\|_{W^{\tilde\rho_0,\infty}}\\ &+C\|w_1\|_{W^{\tilde\rho_0,\infty}}\big(\|L_{i_2}w_2\|_{L^2}+\|w_2\|_{H^{\tilde s_0}}\big)\|w_3\|_{W^{\tilde\rho_0,\infty}}\\ &+C\|w_1\|_{L^2}\|w_2\|_{W^{\tilde\rho_0,\infty}}\big(\|L_{i_3}w_3\|_{W^{\tilde\rho_0,\infty}}+\|w_3\|_{W^{\tilde\rho_0,\infty}}\big)\end{aligned} \tag{A.3.2}$$

and

$$\begin{aligned}\|L_+\mathrm{Op}(m)(w_1,w_2,w_3)\|_{L^2} &\le C\big(\|L_{i_1}w_1\|_{L^2}+\|w_1\|_{H^{\tilde s_0}}\big)\|w_2\|_{W^{\tilde\rho_0,\infty}}\|w_3\|_{W^{\tilde\rho_0,\infty}}\\ &+C\|w_1\|_{L^2}\|w_2\|_{W^{\tilde\rho_0,\infty}}\big(\|L_{i_3}w_3\|_{W^{\tilde\rho_0,\infty}}+\|w_3\|_{W^{\tilde\rho_0,\infty}}\big)\\ &+C\|w_1\|_{L^2}\big(\|L_{i_2}w_2\|_{W^{\tilde\rho_0,\infty}}+\|w_2\|_{W^{\tilde\rho_0,\infty}}\big)\|w_3\|_{W^{\tilde\rho_0,\infty}}.\end{aligned}\tag{A.3.3}$$

Moreover estimates similar to (A.3.2), (A.3.3) hold if one makes any permutation of $(1,2,3)$ *in the right hand side.*

To prove the proposition, we shall apply some results of [14]. In order to do so, we reduce ourselves to the framework of the appendices of that reference, using the rescaling (A.1.5). Set $h=\frac{1}{t}$ and

$$v_j=(\Theta_h w_j)(x)=\frac{1}{\sqrt{h}}w_j\big(\frac{x}{h}\big).\tag{A.3.4}$$

Then if we set

$$\|v\|_{H^s_h}=\|\langle hD_x\rangle^s v\|_{L^2},\ \|v\|_{W^{\rho,\infty}_h}=\|\langle hD_x\rangle^\rho\|_{L^\infty},\tag{A.3.5}$$

one has

$$\|v_j\|_{H^s_h}=\|w_j\|_{H^s},\ \|v_j\|_{W^{\rho,\infty}_h}=h^{-\frac12}\|w_j\|_{W^{\rho,\infty}}.\tag{A.3.6}$$

Define

$$\mathrm{Op}_h(m)(v_1,v_2,v_3)=\frac{1}{(2\pi)^3}\int e^{ix(\xi_1+\xi_2+\xi_3)}m(h\xi_1,h\xi_2,h\xi_3)\prod_{j=1}^3\hat v_j(\xi_j)\,d\xi_1d\xi_2d\xi_3.\tag{A.3.7}$$

Then

$$\Theta_h^{-1}\mathrm{Op}_h(m)(\Theta_h w_1,\Theta_h w_2,\Theta_h w_3)=h^{-1}\mathrm{Op}(m)(w_1,w_2,w_3).\tag{A.3.8}$$

Moreover, by (A.1.4), if we set

$$\mathcal{L}_\pm=\frac1h\mathrm{Op}_h(x\pm p'(\xi))=\frac1h(x\pm p'(hD_x))\tag{A.3.9}$$

we get

$$\Theta_h^{-1}\circ\mathcal{L}_\pm\circ\Theta_h w=L_\pm w.\tag{A.3.10}$$

It follows from (A.3.4), (A.3.6)-(A.3.10), that inequality (A.3.1) is equivalent to

$$\|\mathcal{L}_+\mathrm{Op}_h(m)(v_1,v_2,v_3)\|_{L^2}\le C\sum_{\ell=1}^3\big(\|\mathcal{L}_{i_\ell}v_\ell\|_{L^2}+\|v_\ell\|_{H^{\tilde s_0}_h}\big)\prod_{\substack{1\le j\le3\\ j\neq\ell}}\|v_\ell\|_{W^{\tilde\rho_0,\infty}_h}.\tag{A.3.11}$$

In the same way, (A.3.2) is equivalent to

$$\begin{aligned}\|\mathcal{L}_+\mathrm{Op}_h(m)(v_1,v_2,v_3)\|_{L^2}\le C\big[&\big(\|\mathcal{L}_{i_1}v_1\|_{L^2}+\|v_1\|_{H^{\tilde s_0}_h}\big)\|v_2\|_{W^{\tilde\rho_0,\infty}_h}\|v_3\|_{W^{\tilde\rho_0,\infty}_h}\\ &+\|v_1\|_{W^{\tilde\rho_0,\infty}_h}\big(\|\mathcal{L}_{i_2}v_2\|_{L^2}+\|v_2\|_{H^{\tilde s_0}_h}\big)\|v_3\|_{W^{\tilde\rho_0,\infty}_h}\\ &+\|v_1\|_{L^2}\|v_2\|_{W^{\tilde\rho_0,\infty}_h}\big(\|\mathcal{L}_{i_3}v_3\|_{W^{\tilde\rho_0,\infty}_h}+\|v_3\|_{W^{\tilde\rho_0,\infty}_h}\big)\big].\end{aligned}\tag{A.3.12}$$

and (A.3.3) is equivalent to

$$\begin{aligned}\|\mathcal{L}_+\mathrm{Op}_h(m)(v_1,v_2,v_3)\|_{L^2} \le C\big[&\big(\|\mathcal{L}_{i_1}v_1\|_{L^2}+\|v_1\|_{H_h^{\tilde{s}_0}}\big)\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\\ &+\|v_1\|_{L^2}\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}\big(\|\mathcal{L}_{i_3}v_3\|_{W_h^{\tilde{\rho}_0,\infty}}+\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\big)\\ &+\|v_1\|_{L^2}\big(\|\mathcal{L}_{i_2}v_2\|_{W_h^{\tilde{\rho}_0,\infty}}+\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}\big)\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\big].\end{aligned} \tag{A.3.13}$$

Moreover, estimates of Proposition A.2.2 hold (uniformly in $h\in]0,1]$) if we replace everywhere $\mathrm{Op}(m)$ by $\mathrm{Op}_h(m)$, $\|\cdot\|_{H^s}$ by $\|\cdot\|_{H_h^s}$, $\|\cdot\|_{W^{\rho,\infty}}$ by $\|\cdot\|_{W_h^{\rho,\infty}}$.

Proof of Proposition A.3.1: Let us decompose

$$\begin{aligned}m(\xi_1,\xi_2,\xi_3)&=m^L(\xi_1,\xi_2,\xi_3)+m^H(\xi_1,\xi_2,\xi_3)\\ m^H(\xi_1,\xi_2,\xi_3)&=\sum_{j=1}^3 m_j^H(\xi_1,\xi_2,\xi_3)\end{aligned} \tag{A.3.14}$$

where for some $\beta>0$ small, m^L is supported for $|\xi_1|+|\xi_2|+|\xi_3|\le Ch^{-\beta}$, while m_j^H is supported for $|\xi_\ell|\le C|\xi_j|$, $\ell\neq j$ and $|\xi_j|\ge ch^{-\beta}$, each of these symbols being in $S_{1,0}(M_0,3)$.

• **Contribution of m^H to (A.3.11)-(A.3.13)**

Write

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m_1^H)(v_1,v_2,v_3)&=\mathrm{Op}_h(\tilde{m}_1^H)(v_1,v_2,v_3)+\mathrm{Op}_h(m_1^H)(\mathcal{L}_{i_1}v_1,v_2,v_3)\\ &\quad+h^{-1}p'(hD_x)\mathrm{Op}_h(m_1^H)(v_1,v_2,v_3)-i_1h^{-1}\mathrm{Op}_h(m_1^H)(p'(hD_x)v_1,v_2,v_3)\end{aligned} \tag{A.3.15}$$

where $\tilde{m}_1^H=i\frac{\partial m_1^H}{\partial\xi_1}\in S_{1,0}(M_0^2,3)$. In the arguments of each term in the right hand side, we may replace v_1 by $\mathrm{Op}_h((1-\chi_0)(h^\beta\xi_1))v_1$ for $\chi_0\in C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support, by the support property of m_1^H. We estimate then the L^2-norm of (A.3.15) using the version of (A.2.4) for $\mathrm{Op}_h(m)$. We obtain

$$\begin{aligned}\|\mathcal{L}_+\mathrm{Op}_h(m_1^H)(v_1,v_2,v_3)\|_{L^2}\le C\big(&h^{-1}\|\mathrm{Op}_h((1-\chi_0)(h^\beta\xi))v_1\|_{L^2}+\|\mathcal{L}_{i_1}v_1\|_{L^2}\big)\\ &\times\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\end{aligned} \tag{A.3.16}$$

if $\tilde{\rho}_0$ is taken large enough. Moreover, in the first factor in the right hand side, we may bound

$$h^{-1}\|\mathrm{Op}_h((1-\chi_0)(h^\beta\xi))v_1\|_{L^2}\le Ch^{-1+\beta\tilde{s}_0}\|v_1\|_{H_h^{\tilde{s}_0}}\le C\|v_1\|_{H_h^{\tilde{s}_0}} \tag{A.3.17}$$

if $\tilde{s}_0$ is chosen large enough so that $\tilde{s}_0\beta\ge 1$. Thus the left hand side of (A.3.16) is bounded from above by the first term in the right hand side of (A.3.11). By symmetry, we thus get that (A.3.11) for m replaced by m^H holds.

Let us prove (A.3.12) for m^H. By (A.3.15) to (A.3.17), the contribution of m_1^H to the left hand side of (A.3.12) is estimated by the first term in the right hand side of this inequality. In the same way, the contribution of m_2^H is bounded by the second term in the right hand side. For m_3^H, write instead of (A.3.15)

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m_3^H)(v_1,v_2,v_3)&=\mathrm{Op}_h(\tilde{m}_3^H)(v_1,v_2,v_3)+\mathrm{Op}_h(m_3^H)(v_1,v_2,\mathcal{L}_{i_3}v_3)\\ &\quad+h^{-1}p'(hD_x)\mathrm{Op}_h(m_3^H)(v_1,v_2,v_3)-i_3h^{-1}\mathrm{Op}_h(m_3^H)(v_1,v_2,p'(hD_x)v_3).\end{aligned} \tag{A.3.18}$$

We use next (A.2.4) with $j=1$. The L^2-norm of (A.3.18) is bounded from above by

$$C\|v_1\|_{L^2}\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}\big(\|\mathcal{L}_{i_3}v_3\|_{W_h^{\tilde{\rho}_1,\infty}}+h^{-1}\|\mathrm{Op}_h(1-\chi(h^\beta\xi))v_3\|_{W_h^{\tilde{\rho}_1,\infty}}\big) \tag{A.3.19}$$

for some large enough $\tilde{\rho}_1$. If $\tilde{\rho}_0$ is such that $(\tilde{\rho}_0 - \tilde{\rho}_1)\beta > 1$, we may bound the last term by $\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}$, using that operators of negative order are bounded on L^∞-spaces. This gives an estimate of (A.3.19) by the last term in the right hand side of (A.3.12).
Finally, let us prove (A.3.13) for m^H. The contribution of m_1^H, m_3^H are treated as in the study of (A.3.11) and (A.3.12) above. For m_2^H, we write (A.3.18) for m_2^H instead of m_3^H with indices 2 and 3 interchanged in the right hand side. This gives an estimate for the m_2^H-contribution to the left hand side of (A.3.13) by the third term in the right hand side.

• Contribution of m^L to (A.3.11)-(A.3.13)

Since m^L is supported for $|\xi_1| + |\xi_2| + |\xi_3| \leq Ch^{-\beta}$ by construction, m_L satisfies estimate (3.1.2) with $\beta > 0$, $\nu = 1, \kappa = 1$, i.e. belongs to the class $S_{1,\beta}(M_0, 3)$. This allows us to apply Proposition F.2.1 of [14] that asserts that a Leibniz rule holds, in that sense that if (i_1, i_2, i_3) is characteristic,

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m^L)(v_1,v_2,v_3) = \mathrm{Op}_h(m_1^L)(\mathcal{L}_{i_1}v_1, v_2, v_3)\\ +\mathrm{Op}_h(m_2^L)(v_1, \mathcal{L}_{i_2}v_2, v_3)\\ +\mathrm{Op}_h(m_3^L)(v_1, v_2, \mathcal{L}_{i_2}v_3)\\ +\mathrm{Op}_h(r)(v_1, v_2, v_3)\end{aligned} \tag{A.3.20}$$

where m_j^L, $j = 1, 2, 3$ and r are elements of the class $S_{1,\beta}(M_0^\nu, 3)$ for some $\nu \in \mathbb{N}$. Actually, in [14], there is also a weight $\prod_{j=1}^3 \langle \xi_j \rangle^{-1}$ in the right hand side of the inequalities (3.1.2) that define the symbols, but that does not play any role in the proofs. There is also in Proposition F.2.1 of [14] an extra term in the right hand side of (A.3.20), of the form $h^{-1}\mathrm{Op}_h(r')(v_1, v_2, v_3)$ for some r'. Such a term does not appear here because our symbols are constant coefficients and in particular do not depend on the y-variable in Proposition F.2.1 of [14]: see the last three lines in Proposition B.2.1 of [14].
To obtain (A.3.11) for m^L, we now just have to use estimate (A.2.4) for each term in the right hand side of (A.3.20), putting the L^2 norm on the factor in $\mathcal{L}_{i_j}v_j$ for the first three terms in the right hand side.
One obtains (A.3.12) for m^L in the same way, except that we treat the $\mathrm{Op}_h(m_3^L)$-term in the right hand side of (A.3.20) putting the L^2 norms on the factor v_1 in estimate (A.2.4). Finally, to get (A.3.13) for m^L, we argue in the same way, controlling the L^2 norms of the $\mathrm{Op}_h(m_2^L)$ and $\mathrm{Op}_h(m_3^L)$ terms using (A.2.4) where we put the L^2 norm on the v_1 term in the right hand side. This concludes the proof. □

A.4 Klainerman-Sobolev estimates

We prove in this subsection a Klainerman-Sobolev estimate for the one dimensional Klein-Gordon equation. This estimate is not new and may be found implicitly on a weaker form in [13, 31] for instance. We first introduce some notation.
If $\delta \in [0, 1]$, let us introduce $\tilde{S}_\delta(1)$ the space of smooth functions $(x, \xi) \to a(x, \xi, h)$ from $\mathbb{R}^2$ to $\mathbb{C}$, depending also on a parameter $h \in]0, 1]$, such that for any α, β in $\mathbb{N}$

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_\alpha h^{-\delta(\alpha+\beta)}. \tag{A.4.1}$$

For $u \in \mathcal{S}(\mathbb{R})$, define the semi-classical Weyl quantization of a acting on u by

$$\mathrm{Op}_h^{\mathrm{W}}(a)u = \frac{1}{2\pi h}\int e^{i(x-y)\frac{\xi}{h}} a\Big(\frac{x+y}{2}, \xi, h\Big)u(y)\, dyd\xi. \tag{A.4.2}$$

If $b(x, \xi, y, \eta)$ is a smooth function, define

$$\sigma(D_x, D_\xi, D_y, D_\eta)b(x, \xi, y, \eta) = (D_\xi D_y - D_x D_\eta)b(x, \xi, y, \eta) \tag{A.4.3}$$

and recall that if $\delta \in [0, \frac{1}{2}[$, if a_1, a_2 are in $\tilde{S}_\delta(1)$, there is a symbol $a_1 \#_h a_2$ in $\tilde{S}_\delta(1)$, such that for any N

$$a_1 \#_h a_2 - \sum_{k=0}^{N} \frac{1}{k!} \Big(\frac{ih}{2} \sigma(D_x, D_\xi, D_y, D_\eta) \Big)^k (a_1(x,\xi) a_2(y,\eta))|_{x=y, \xi=\eta} \tag{A.4.4}$$

is in $h^{(N+1)(1-2\delta)} \tilde{S}_\delta(1)$ and

$$\mathrm{Op}_h^{\mathrm{W}}(a_1) \circ \mathrm{Op}_h^{\mathrm{W}}(a_2) = \mathrm{Op}_h^{\mathrm{W}}(a_1 \#_h a_2). \tag{A.4.5}$$

In particular, if a_1 and a_2 have disjoint supports, $\mathrm{Op}_h^{\mathrm{W}}(a_1) \circ \mathrm{Op}_h^{\mathrm{W}}(a_2)$ may be written for any N in $\mathbb{N}$ as $h^N \mathrm{Op}_h^{\mathrm{W}}(r)$ with r in $\tilde{S}_\delta(1)$, since $\delta < \frac{1}{2}$.
Recall also that if $a \in \tilde{S}_\delta(1)$, $\mathrm{Op}_h^{\mathrm{W}}(a)$ is bounded on H_h^s with uniform estimates

$$\|\mathrm{Op}_h^{\mathrm{W}}(a) u\|_{H_h^s} \leq C \|u\|_{H_h^s}. \tag{A.4.6}$$

All the results above may be found for instance in Chapter 7 of the book of Dimassi-Sjöstrand [15].
Our goal is to prove:

Proposition A.4.1 *Let $\rho_0 \in \mathbb{N}$. There is $s_0 \in \mathbb{N}$ such that for any function w, one has the bound*

$$\|w\|_{W^{\rho_0,\infty}} \leq \frac{C}{\sqrt{t}} \Big(\big(\|L_+ w\|_{L^2} + \|w\|_{H^{s_0}} \big)^{\frac{1}{2}} \|w\|_{H^{s_0}}^{\frac{1}{2}} + \|L_+ w\|_{L^2} \Big), \tag{A.4.7}$$

where $L_+ = x + t p'(D_x)$ with $p(\xi) = \sqrt{1+\xi^2}$.

We define from w a function v by (A.3.4) and using notation (A.3.9) and (A.3.6), we see that (A.4.7) is equivalent to

$$\|v\|_{W_h^{\rho_0,\infty}} \leq C \Big(\big(\|\mathcal{L}_+ v\|_{L^2} + \|v\|_{H_h^{s_0}} \big)^{\frac{1}{2}} \|v\|_{H_h^{s_0}}^{\frac{1}{2}} + \|\mathcal{L}_+ v\|_{L^2} \Big). \tag{A.4.8}$$

Let us notice that one may further reduce to proving that there is some large enough $\tilde{s}_0$ such that the following estimate holds:

$$\|\langle hD_x \rangle^{-3} v\|_{L^\infty} \leq C \Big(\big(\|\mathcal{L}_+ v\|_{L^2} + \|v\|_{H_h^{\tilde{s}_0}} \big)^{\frac{1}{2}} \|v\|_{H_h^{\tilde{s}_0}}^{\frac{1}{2}} + \|\mathcal{L}_+ v\|_{H_h^{-\tilde{s}_0}} \Big). \tag{A.4.9}$$

Actually, if (A.4.9) is proved, we may apply it to $v_k = \mathrm{Op}_h^{\mathrm{W}}(\chi(2^{-k}\xi)) v$ for some $\chi \in C_0^\infty(\mathbb{R}^*)$ and $k \in \mathbb{N}$. We have then

$$\begin{aligned}
\|v_k\|_{W_h^{\rho_0,\infty}} &\leq C 2^{k(\rho_0+3)} \|\langle hD_x \rangle^{-3} v_k\|_{L^\infty} \\
&\leq C 2^{k(\rho_0+3)} \Big(\big(\|\mathcal{L}_+ v_k\|_{L^2} + \|v_k\|_{H_h^{\tilde{s}_0}} \big)^{\frac{1}{2}} \|v_k\|_{H_h^{\tilde{s}_0}}^{\frac{1}{2}} + \|\mathcal{L}_+ v_k\|_{H_h^{-\tilde{s}_0}} \Big) \\
&\leq C 2^{k(\rho_0+3+\frac{\tilde{s}_0}{2}-\frac{s_0}{2})} \big(\|\mathcal{L}_+ v\|_{L^2} + \|v\|_{H_h^{s_0}} \big)^{\frac{1}{2}} \|v\|_{H_h^{s_0}}^{\frac{1}{2}} + C 2^{k(\rho_0+3-\tilde{s}_0)} (\|\mathcal{L}_+ v\|_{L^2} + \|v\|_{L^2}).
\end{aligned}$$

from which (A.4.8) follows by summation of a Littlewood-Paley decomposition if $s_0 > \tilde{s}_0 + 2(\rho_0 + 3)$ and $\tilde{s}_0 > \rho_0 + 3$.
In the rest of this subsection, we shall prove (A.4.9). Before starting the proof, we make some reductions.

Lemma A.4.2 *Let $\gamma, \chi \in C_0^\infty(\mathbb{R})$, equal to one close to zero, with small enough support. Let $M \in \mathbb{N}$. There is $\beta > 0$ and a family of smooth functions $x \to \theta_h(x)$, depending on a parameter $h \in]0,1]$, with for any $\alpha \in \mathbb{N}$, $\partial_x^\alpha \theta_h(x) = O(h^{-2\beta\alpha})$, θ_h being supported in $[-1+ch^{2\beta}, 1-ch^{2\beta}]$ for some $c > 0$, such that for any function v*

$$\begin{aligned} \|\langle hD_x\rangle^{-2} v - \mathrm{Op}_h^{\mathrm{W}}\big(\gamma((x+p'(\xi))\langle\xi\rangle^2)\chi(h^\beta\xi)\theta_h(x)\langle\xi\rangle^{-2}\big)v\|_{L^\infty} \\ \leq C\big(\|v\|_{H_h^{-1+\frac{1}{2\beta}}} + \|\mathcal{L}_+ v\|_{H_h^{-2M+1}}\big). \end{aligned} \tag{A.4.10}$$

Proof: By semi-classical Sobolev embedding, one has for any $\epsilon > 0$

$$\|\langle hD_x\rangle^{-2}(v - \mathrm{Op}_h^{\mathrm{W}}(\chi(h^\beta\xi))v)\|_{L^\infty} \leq Ch^{-\frac{1}{2}+\beta(s+\frac{3}{2}-\epsilon)}\|v\|_{H_h^s} \tag{A.4.11}$$

if $s > -\frac{3}{2}+\epsilon$, so that we have an upper bound by the right hand side of (A.4.10). We shall study next the L^∞ norm of $\mathrm{Op}_h^{\mathrm{W}}(a(x,\xi))v$ if

$$a(x,\xi) = \chi(h^\beta\xi)\langle\xi\rangle^{-2}(1-\gamma)((x+p'(\xi))\langle\xi\rangle^2) = a_1(x,\xi)(x+p'(\xi)) \tag{A.4.12}$$

where $a_1 = \chi(h^\beta\xi)\gamma_1((x+p'(\xi))\langle\xi\rangle^2)$ with $\gamma_1(z) = \frac{1-\gamma(z)}{z}$. Then a and a_1 belong to $\tilde{S}_\delta(1)$ with $\delta = 2\beta < \frac{1}{2}$ for small enough $\beta > 0$. We use (A.4.5), (A.4.4) to write with some r in $\tilde{S}_\delta(1)$

$$\begin{aligned} \mathrm{Op}_h^{\mathrm{W}}(a)v &= \mathrm{Op}_h^{\mathrm{W}}(a_1) \circ \mathrm{Op}_h^{\mathrm{W}}(x+p'(\xi))v + h^{1-2\delta}\mathrm{Op}_h^{\mathrm{W}}(r) \\ &= h\mathrm{Op}_h^{\mathrm{W}}(a_1)\mathcal{L}_+ v + h^{1-2\delta}\mathrm{Op}_h^{\mathrm{W}}(r). \end{aligned} \tag{A.4.13}$$

In the right hand side write

$$\mathrm{Op}_h^{\mathrm{W}}(a_1)\mathcal{L}_+ v = \mathrm{Op}_h^{\mathrm{W}}(a_1)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{2M})(\langle hD_x\rangle^{-2M}\mathcal{L}_+ v) \tag{A.4.14}$$

and use that since M is an integer, we have an exact composition formula (A.4.5)

$$\mathrm{Op}_h^{\mathrm{W}}(a_1) \circ \mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{2M}) = \sum_{k=0}^{2M} \frac{1}{k!}\Big(i\frac{h}{2}\sigma(D_x, D_\xi, D_y, D_\eta)\Big)^k (a_1(x,\xi)\langle\eta\rangle^{2M})|_{x=y,\xi=\eta}. \tag{A.4.15}$$

Since $|\xi| = O(h^{-\beta})$ on the support of a_1, we get that (A.4.15) is of the form $h^{-2M\delta}\mathrm{Op}_h^{\mathrm{W}}(a_2)$ with some $a_2 \in \tilde{S}_\delta(1)$. Applying again the semi-classical Sobolev inequality, we deduce from (A.4.14), (A.4.15)

$$\|\mathrm{Op}_h^{\mathrm{W}}(a_1)\mathcal{L}_+ v\|_{L^\infty} \leq Ch^{-\frac{1}{2}-2M\delta}\|\mathcal{L}_+ v\|_{H^{-2M+\frac{1}{2}+\epsilon}}.$$

Plugging this in (A.4.13), we get

$$\|\mathrm{Op}_h^{\mathrm{W}}(a)v\|_{L^\infty} \leq Ch^{\frac{1}{2}-2M\delta}\|\mathcal{L}_+ v\|_{H^{-2M+\frac{1}{2}+\epsilon}} + Ch^{\frac{1}{2}-2\delta}\|\mathrm{Op}_h^{\mathrm{W}}(r)v\|_{H^{\frac{1}{2}+\epsilon}}. \tag{A.4.16}$$

If $\delta = 2\beta$ is small enough relatively to $1/M$, this implies that (A.4.16) is bounded by the right hand side of (A.4.10). Taking into account (A.4.11), we thus see that it remains to consider

$$\mathrm{Op}_h^{\mathrm{W}}\big(\chi(h^\beta\xi)\gamma((x+p'(\xi))\langle\xi\rangle^2)(1-\theta_h)(x)\langle\xi\rangle^{-2}\big)v. \tag{A.4.17}$$

We shall be done if we prove that, if $\mathrm{Supp}\,\gamma$ has been taken small enough, we may choose θ_h such that it is equal to one on the support of $\chi(h^\beta\xi)\gamma((x+p'(\xi))\langle\xi\rangle^2)$ so that (A.4.17) vanishes identically. This follows from the fact that, if $\mathrm{Supp}\,\gamma$ is small enough and $\gamma((x+p'(\xi))\langle\xi\rangle^2) \neq 0$, then when $\xi \to +\infty$ (resp. $\xi \to -\infty$), $x+1$ (resp. $x-1$) stays in an interval $[c_1/\xi^2, c_2/\xi^2]$ (resp. $[-c_2/\xi^2, -c_1/\xi^2]$) for some $0 < c_1 < c_2$. If in addition, $|\xi| = O(h^{-\beta})$, this implies that x belongs

to the interval $[-1+ch^{2\beta}, 1-ch^{2\beta}]$ for some $c>0$, which allows to construct the wanted θ_h. This concludes the proof. □

Proof of Proposition A.4.1: We set with the notation of Lemma A.4.2

$$\tilde{v} = \mathrm{Op}_h^{\mathrm{W}}(\gamma((x+p'(\xi))\langle\xi\rangle^2)\theta_h(x)\chi(h^\beta\xi)\langle\xi\rangle^{-3})v \tag{A.4.18}$$

so that, by that lemma, inequality (A.4.9) that implies Proposition A.4.1, will follow if we prove

$$\|\tilde{v}\|_{L^\infty} \leq C(\|v\|_{H_h^{\tilde{s}_0}} + \|\mathcal{L}_+ v\|_{L^2})^{\frac{1}{2}}\|v\|_{H_h^{\tilde{s}_0}}^{\frac{1}{2}}. \tag{A.4.19}$$

Take $\tilde{\theta}_h \in C_0^\infty(]-1,1[)$, equal to one on the support of θ_h, satisfying $\partial_x^\alpha\tilde{\theta}_h = O(h^{-\delta\alpha})$ (with $\delta = 2\beta$) for any α. Since the symbol of the operator defining $\tilde{v}$ in (A.4.18) is in $\tilde{S}_\delta(1)$, it follows from (A.4.5) and the remark following it, that $(1-\tilde{\theta}_h)\tilde{v} = h^N\mathrm{Op}_h^{\mathrm{W}}(r)v$ for some symbol r in $\tilde{S}_\delta(1)$ and any N. Then, using again semi-classical Sobolev estimate and (A.4.6), we see that $\|(1-\tilde{\theta}_h)\tilde{v}\|_{L^\infty}$ is estimated by the right hand side of (A.4.19). We are thus left with studying $\tilde{\theta}_h\tilde{v}$. If $\varphi(x) = \sqrt{1-x^2}$ for $x\in]-1,1[$, write

$$\begin{aligned}\|\tilde{\theta}_h\tilde{v}\|_{L^\infty} = \|e^{-i\frac{\varphi}{h}}\tilde{\theta}_h\tilde{v}\|_{L^\infty} &\leq Ch^{-\frac{1}{2}}\|hD_x(e^{-i\frac{\varphi}{h}}\tilde{\theta}_h\tilde{v})\|_{L^2}^{\frac{1}{2}}\|e^{-i\frac{\varphi}{h}}\tilde{\theta}_h\tilde{v}\|_{L^2}^{\frac{1}{2}}\\ &\leq Ch^{-\frac{1}{2}}\big(\|\tilde{\theta}_h(x)(hD_x - d\varphi(x))\tilde{v}\|_{L^2}^{\frac{1}{2}} + \|(hD_x\tilde{\theta}_h)\tilde{v}\|_{L^2}^{\frac{1}{2}}\big)\|\tilde{v}\|_{L^2}^{\frac{1}{2}}.\end{aligned} \tag{A.4.20}$$

Note that $(hD_x\tilde{\theta}_h)\tilde{v} = -ih^{1-\delta}\tilde{\theta}_h^1(x)\tilde{v}$ for a function $\tilde{\theta}_h^1$ satisfying again $\partial_x^\alpha\tilde{\theta}_h^1 = O(h^{-\delta\alpha})$, whose support does not intersect the support of the symbol defining $\tilde{v}$ in (A.4.18). Using again (A.4.5), we conclude that $\|(hD_x\tilde{\theta}_h)\tilde{v}\|_{L^2} \leq c_N h^N\|v\|_{L^2}$, so that to show that (A.4.20) is bounded by the right hand side of (A.4.19), it is enough to prove that

$$h^{-\frac{1}{2}}\|\tilde{\theta}_h(x)(hD_x - d\varphi(x))\tilde{v}\|_{L^2}^{\frac{1}{2}}\|\tilde{v}\|_{L^2}^{\frac{1}{2}} \leq C(\|v\|_{H_h^{\tilde{s}_0}} + \|\mathcal{L}_+ v\|_{L^2})^{\frac{1}{2}}\|v\|_{H_h^{\tilde{s}_0}}^{\frac{1}{2}}. \tag{A.4.21}$$

Notice that in (A.4.21), $h^\beta\tilde{\theta}_h(x)d\varphi(x)$ is an element of $\tilde{S}_\delta(1)$ and that

$$\tilde{\theta}_h(x)hD_x = \mathrm{Op}_h^{\mathrm{W}}(\tilde{\theta}_h\xi) + i\frac{h}{2}\tilde{\theta}_h'(x). \tag{A.4.22}$$

Again, the last contribution in (A.4.22) will bring a trivial term to estimate in (A.4.21), so that we are reduced to the study of

$$h^{-\frac{1}{2}}\|\mathrm{Op}_h^{\mathrm{W}}(\tilde{\theta}_h(x)(\xi - d\varphi(x)))\tilde{v}\|_{L^2}^{\frac{1}{2}}\|\tilde{v}\|_{L^2}^{\frac{1}{2}}. \tag{A.4.23}$$

If we express $\tilde{v}$ from (A.4.18) and use (A.4.5), (A.4.4) at order $N=1$, we obtain

$$\mathrm{Op}_h^{\mathrm{W}}(\tilde{\theta}_h(x)(\xi - d\varphi(x)))\tilde{v} = \mathrm{Op}_h^{\mathrm{W}}(a_0(x,\xi) + ha_1(x,\xi) + h^{2(1-2\delta)-\beta}r)v, \tag{A.4.24}$$

where $r\in\tilde{S}_\delta(1)$ is the remainder in (A.4.4) (the extra power $h^{-\beta}$ coming from the fact that $\tilde{\theta}_h d\varphi$ is not in $\tilde{S}_\delta(1)$, but only in $h^{-\beta}\tilde{S}_\delta(1)$), and where a_0, a_1 are the first two terms in expansion (A.4.4) and are given explicitly by

$$\begin{aligned}a_0 &= \theta_h(x)(\xi - d\varphi(x))\gamma((x+p'(\xi))\langle\xi\rangle^2)\chi(h^\beta\xi)\langle\xi\rangle^{-3}\\ a_1 &= -\frac{i}{2}\{(\xi - d\varphi(x)), \theta_h(x)\gamma((x+p'(\xi))\langle\xi\rangle^2)\chi(h^\beta\xi)\langle\xi\rangle^{-3}\}.\end{aligned} \tag{A.4.25}$$

If δ, β are small enough, the r term in (A.4.24) brings to (A.4.23) a contribution bounded by the right hand side of (A.4.21). We thus have to study a_0, a_1. We use Lemma 1.8 of [13] to rewrite a_0, a_1. According to (1.28), (1.29) with $\kappa = 2$ in that reference, we may write

$$a_0(x,\xi) = (x+p'(\xi))b_0(x,\xi) \tag{A.4.26}$$

where b_0 is supported for $|\xi| \leq h^{-\beta}$, $1 - x^2 \geq c\langle\xi\rangle^{-2}$ and satisfies estimates of the form $|\partial_x^\alpha \partial_\xi^{\alpha'} b_0(x,\xi)| \leq C\langle\xi\rangle^{2\alpha-\alpha'}$. Actually, as already seen in the proof of Lemma A.4.2, $\theta_h \equiv 1$ on the support of $\gamma((x+p'(\xi))\langle\xi\rangle^2)\chi(h^\beta\xi)$, so that this factor θ_h may be omitted in the definition (A.4.25) of a_0. It follows then that b_0 is in $\tilde{S}_\delta(1)$ with $\delta = 2\beta$.
We may thus apply (A.4.5), (A.4.4) to write

(A.4.27)
$$\begin{aligned}\mathrm{Op}_h^{\mathrm{W}}(a_0)v &= \mathrm{Op}_h^{\mathrm{W}}(b_0(x,\xi)(x+p'(\xi))v \\ &= \mathrm{Op}_h^{\mathrm{W}}(b_0)\mathrm{Op}_h^{\mathrm{W}}(x+p'(\xi))v - \frac{h}{2i}\mathrm{Op}_h^{\mathrm{W}}\big(\frac{\partial b_0}{\partial\xi} - p''(\xi)\frac{\partial b_0}{\partial x}\big)v + h^{2-4\delta}\mathrm{Op}_h^{\mathrm{W}}(r)v\end{aligned}$$

for some r in $\tilde{S}_\delta(1)$. In the above expression, $\frac{\partial b_0}{\partial\xi} - p''(\xi)\frac{\partial b_0}{\partial x}$ is in $\tilde{S}_\delta(1)$ since $p''(\xi) = O(\langle\xi\rangle^{-3})$. Applying (A.4.6) to the three terms in the right hand side of (A.4.27), we get that for $\beta > 0$ small enough

$$\begin{aligned}\|\mathrm{Op}_h^{\mathrm{W}}(a_0)v\|_{L^2} &\leq C(\|\mathrm{Op}_h^{\mathrm{W}}(x+p'(\xi))v\|_{L^2} + h\|v\|_{L^2}) \\ &\leq Ch(\|\mathcal{L}_+ v\|_{L^2} + \|v\|_{L^2}).\end{aligned} \tag{A.4.28}$$

Consider next a_1 given by (A.4.25) (where $\theta_h(x)$ may be removed). As on the support of a_1, $\partial_x^\alpha(d\varphi(x)) = O(\langle\xi\rangle^{1+2\alpha})$, it follows that a_1 is in $\tilde{S}_\delta(1)$, so that the second term in the right hand side of (A.4.24) satisfies

$$h\|\mathrm{Op}_h^{\mathrm{W}}(a_1)v\|_{L^2} \leq Ch\|v\|_{L^2}. \tag{A.4.29}$$

Plugging (A.4.24), (A.4.28) and (A.4.29) inside (A.4.23), we get that this expression is bounded from above by the right hand side of (A.4.21). This concludes the proof. □

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