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On factorization of rank-one auto-correlation matrix polynomials

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Abstract

This article characterizes the rank-one factorization of auto-correlation matrix polynomials. We establish a sufficient and necessary uniqueness condition for uniqueness of the factorization based on the greatest common divisor (GCD) of multiple polynomials. In the unique case, we show that the factorization can be carried out explicitly using GCDs. In the non-unique case, the number of non-trivially different factorizations is given and all solutions are enumerated.

Keywords: matrix polynomial, auto-correlation, rank-one factorization, uniqueness, greatest common divisors

2000 MSC: 15A21,15A23,42A85,47A68,47N70

1. Introduction

Let $\mathbf{\Gamma}(z) = [\Gamma_{ij}(z)]_{i,j=1}^{K,K}$ be an $K \times K$ matrix polynomial of degree at most $2(N-1)$, with complex coefficients. The goal of this paper is to characterize the matrix polynomials that admit the following rank-one factorization:

$$\mathbf{\Gamma}(z) = \begin{bmatrix} \Gamma_{11}(z) & \cdots & \Gamma_{1K}(z) \\ \vdots & & \vdots \\ \Gamma_{K1}(z) & \cdots & \Gamma_{KK}(z) \end{bmatrix} = \begin{bmatrix} X_1(z) \\ \vdots \\ X_K(z) \end{bmatrix} \begin{bmatrix} \tilde{X}_1(z) & \cdots & \tilde{X}_K(z) \end{bmatrix}, \quad (1)$$

where $X_k(z), k \in \{1, \dots, K\}$ are polynomials with degree at most $N-1$; in (1), the notation $\tilde{Y}(z)$ denotes the complex conjugate reversal

$$\tilde{Y}(z) := z^{N-1} \overline{Y(\bar{z}^{-1})} = \sum_{n=0}^{N-1} \overline{y[N-1-n]} z^n \quad (2)$$

of a polynomial $Y(z) = \sum_{n=0}^{N-1} \overline{y[n]} z^n$ of degree at most $N-1$. In this paper we address the following questions:

- when is the factorization (1) unique?
- if it is not unique, how to find all possible factorizations (1)?

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The polynomial factorization problem arises in several applications in signal processing, such as phase retrieval problems [1] and blind system identification [2]. In such applications, one is interested to reconstruct a number of signals (vectors) $\mathbf{x}_k = [x_k[0] \ \cdots \ x_k[N-1]]^\top \in \mathbb{C}^N, k \in \{1, \dots, K\}$, given a set of their correlations sequences $(\gamma_{ij}[n])_{i,j=1, n=-N+1}^{K,K,N-1}$ defined as follows:

$$\gamma_{ij}[n] := \sum_{m=0}^{N-1-n} x_i[m+n] \overline{x_j[m]}. \quad (3)$$

Such correlation functions encode the $K \times K$ auto-correlation matrix sequence of the K -dimensional vector signal $([x_1[n] \ \dots \ x_K[n]]^\top)_{n=0}^{N-1}$. Moreover it can be shown (see Appendix A) that the elements in (3) corresponds exactly to the coefficients of the polynomial $\Gamma_{ij}(z)$ factorized as (1). Thus the problem of recovery of the vectors \mathbf{x}_k from correlations sequences is equivalent to the problem of factorizing¹ a given matrix polynomial as (1).

In this paper we provide a complete characterization of all possible factorizations rank-one matrix polynomial (1); in fact, these factorizations are entirely characterized by the greatest common divisor of all the matrix elements $\Gamma_{ij}(z)$, denoted as $\gcd\{\Gamma_{ij}\}_{i,j=1}^{K,K}$. In particular, we prove the following theorem.

Theorem 1. *The factorization (1) is unique up to global scaling if and only if the greatest common divisor $H(z) = \gcd\{\Gamma_{ij}\}_{i,j=1}^{K,K}$ is 1 or its roots lie only on the complex unit circle $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$.*

By uniqueness up to global scaling in (1) we mean that any alternative factorization $\Gamma_{ij}(z) = Y_i(z) \widetilde{Y}_j(z)$ satisfies $Y_i(z) = cX_i(z)$ with some $c \in \mathbb{T}$. Moreover, in the non-unique case, we provide all possible factorizations modulo the global scaling, which again depend on roots of the polynomial $H(z)$ and their multiplicities.

Related work. Factorizations of matrix polynomials and matrix functions are a classic topic in linear algebra and operator theory [3]. In fact, it can be shown that the matrix polynomials factorized as (1) have the so-called $*$ -palindromic structure [4]. Several previous works have addressed the spectral properties or the Smith normal form of palindromic matrix polynomials, but, up to the authors knowledge, none of them discussed in detail uniqueness and characterization of solutions, which is a very important question in applications mentioned above. The factorization (1) also resembles the problem of spectral factorization of matrix functions [5], however, unlike the latter problem, in the factorization (1) there is no restriction on location of roots of the polynomials inside the unit disk. Very few papers on low-rank factorization treat the low-rank case [6]. It is shown that [6, Theorem 1] that any rank-one matrix polynomial that is positive-semidefinite on the complex unit circle \mathbb{T} admits a factorization (1) that is unique if one imposes additional constraints (minimum phase requirement). The question of characterizing the set of non-minimal-phase spectral factorizations was only analyzed [7] for the full rank case; also, only the generic case (determinant with only simple roots) is treated.

The uniqueness of the factorization (1) is also closely related to uniqueness of the solution in (algebraic) phase retrieval problems (see [1]). For the latter problem, the case $K = 1$ (corresponding to 1×1 matrices) was fully characterized in [8]. In the case $K > 1$, only partial results are available in [2], while our paper provides a complete characterization in the language of matrix polynomial factorizations.

Caveat. In this paper, we employ the formalism of univariate polynomials with roots at infinity, that is crucial for the algebraic theory of Hankel matrices [9, §I.0] and matrix methods for approximate greatest common divisor computations [10]. This formalism simplifies the proofs and allows for a transparent and complete characterization for the polynomial factorization problem. However, the notion of greatest common divisor is slightly different from the standard one, as it takes into account the possible roots at infinity.

¹This is the reason why in [1] we refer to (1) as the polynomial auto-correlation factorization (PAF) problem.

We will also remark how the usual inner product on \mathbb{C}^N can be computed using multiplications of polynomials.

Lemma 3. *Let $X, Y \in \mathbb{C}_{\leq N-1}[z]$. The coefficient of the polynomial $X(z)\tilde{Y}(z)$ at the monomial z^{N-1} is equal to the inner product between the vectors of coefficients, that is:*

$$(X(z)\tilde{Y}(z))\Big|_{z^{N-1}} = \langle X, Y \rangle = \sum_{n=0}^{N-1} x[n]\overline{y[n]} = \langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{C}^N}.$$

In particular, $(X(z)\tilde{X}(z))\Big|_{z^{N-1}} = \langle X, X \rangle = \|X\|_2^2 = \|\mathbf{x}\|_2^2$.

Proof. It follows from straightforward calculation (see also Appendix A, where the relation between (1) and correlation sequences is detailed). \square

2.2. Divisors and greatest common divisors

In this paper, we work with the multiplication defined in Equation (5): as a result, we always take care of the space where polynomials belong to². Therefore, we need a special definition of divisibility.

Definition 4 (Divisors of polynomials). *We say that the polynomial $C \in \mathbb{C}_{\leq N}[z]$ has a divisor $B \in \mathbb{C}_{\leq D}[z] \setminus \{0\}$ if there is a polynomial $A \in \mathbb{C}_{\leq N-D}[z]$ such that $C(z) = A(z)B(z)$ in the sense of Equation (5).*

Example 5. *Consider the following polynomial from $\mathbb{C}_{\leq 5}[z]$:*

$$A(z) = 0 \cdot z^5 + 0 \cdot z^4 + \frac{1}{2}z^3 + \frac{1}{2}z^2 - z \in \mathbb{C}_{\leq 5}[z], \quad (8)$$

which has two zero leading coefficients. Then the polynomial $F(z) = 0 \cdot z^3 + 0 \cdot z^2 + z + 2 \in \mathbb{C}_{\leq 3}[z]$ is a divisor of the polynomial $A(z)$ in (8), because $A(z) = F(z) \cdot (\frac{1}{2}(z-1)z)$ according to definition Equation (5). However, the polynomial $G(z) = 0 \cdot z^4 + 0 \cdot z^3 + 0 \cdot z^2 + z + 2 \in \mathbb{C}_{\leq 4}[z]$ is not a divisor of $A(z)$, because there is no $H(z) \in \mathbb{C}_{\leq 1}[z]$ such that $A(z) = G(z)H(z)$. Note that $F(z)$ and $G(z)$ represent the same polynomial $z + 2$ in $\mathbb{C}[z]$ (if we forget the space in which the polynomial is living).

Remark 6 (Alternative definition of the divisibility). *Example 5 suggests the following equivalent definition of divisibility: the polynomial $C \in \mathbb{C}_{\leq N}[z]$ has a divisor $B \in \mathbb{C}_{\leq D}[z] \setminus \{0\}$ if B is a divisor of C in the usual sense, and, in addition B has at most the same number of zero leading coefficients as C .*

For instance, in Example 5 the polynomial $z + 2$ is a divisor of $A(z) = \frac{1}{2}z^3 + \frac{1}{2}z^2 - z \in \mathbb{C}_{\leq 5}[z]$, but $A(z)$ has two zero leading coefficients. The polynomials $F(z)$ and $G(z)$, in their turn, have 2 and 3 zero leading coefficients, respectively.

Now we are ready to introduce the notion of the greatest common divisor of polynomials.

Definition 7. *For a set of polynomials, $X_1 \in \mathbb{C}_{\leq N_1}[z], \dots, X_K \in \mathbb{C}_{\leq N_K}[z]$, the greatest common divisor H is defined as the polynomial $H \in \mathbb{C}_{\leq D}[z]$ with highest possible D , such that $H(z)$ is a divisor of all polynomials X_1, \dots, X_K in the sense of Definition 4. We set $\gcd\{A, 0\} = A$, and, if all polynomials are zero, then we formally set $\gcd\{X_1, \dots, X_K\} = 0 \in \mathbb{C}_{\leq 0}[z] = \mathbb{C}$.*

²Again, we could work instead homogeneous bivariate polynomials, but we stick to the notation with univariate polynomials used in this paper.

The GCD of polynomials in Definition 7 enjoys the usual properties of the greatest common divisor (such as associativity). In particular, the GCD exists and is unique up to a multiplication by a nonzero scalar. Therefore the notation

$$H = \gcd\{X_1, \dots, X_K\} \quad (9)$$

means that H is a GCD; the equality (9) is meant modulo multiplication by a nonzero scalar (a typical abuse of notation in the literature). We say that the polynomials are coprime if $\gcd\{X_1, \dots, X_K\} = 1$.

Remark 8. Similarly to Remark 6, the GCD of polynomials $X_k \in \mathbb{C}_{\leq N_k}[z]$ coincides with the GCD in the usual sense with the zero leading coefficients equal to the minimal number of zero leading coefficients among X_k .

Finally, we recall a matrix-based criterion of coprimeness of two polynomials.

Theorem 9. Two polynomials $X_1, X_2 \in \mathbb{C}_{\leq N-1}[z]$ are coprime in the sense of Definition 7 if and only if its $2N \times 2N$ Sylvester matrix

$$S(X_1, X_2) = \begin{bmatrix} \mathbf{M}_{N-1}(\mathbf{x}_1) & \mathbf{M}_{N-1}(\mathbf{x}_2) \end{bmatrix}$$

is nonsingular.

We refer the reader to [10, Section 2] for more details and properties of the GCD, as well as the generalizations of Theorem 9 and the matrix-based algorithms to compute the GCD.

2.3. Roots at infinity

An important tool used in this paper is that we operate with ∞ roots. We will say that the polynomial $A \in \mathbb{C}_{\leq D}[z]$ in (4) has a root at ∞ (with multiplicity μ_k) if its leading μ_k coefficients vanishes (i.e., if $a[D] = \dots = a[D - \mu_k + 1] = 0$). We will formally write $(z - \infty)^d B(z)$ to denote that d zero leading coefficients are appended to the polynomial.

Example 10. Consider the polynomial from Example 5. The polynomial has roots $\{\infty, -2, 1, 0\}$, where the root ∞ has multiplicity 2. Hence it has the following factorization

$$A(z) = \frac{1}{2}(z - \infty)^2(z - 1)(z + 2)z. \quad (10)$$

Remark 11. The root at infinity can be formally defined as:

$$(z - \infty) := 0 \cdot z + 1 \in \mathbb{C}_{\leq 1}[z].$$

Then the multiplication by such polynomial in the sense of the definition in (6) corresponds exactly to adding a zero leading coefficient. In particular,

$$(z - \infty)^d := 0 \cdot z^d + 0 \cdot z^{d-1} + \dots + 0 \cdot z + 1 \in \mathbb{C}_{\leq d}[z].$$

With such a convention, the following extended version of the fundamental theorem of algebra holds true: any nonzero polynomial $A \in \mathbb{C}_{\leq D}[z] \setminus \{\mathbf{0}\}$ can be uniquely factorized (up to permutation of roots) as

$$A(z) = \lambda \prod_{i=1}^m (z - \alpha_i)^{\mu_i}, \quad (11)$$

where $\lambda \in \mathbb{C}$, $\alpha_i \in \mathbb{C} \cup \{\infty\}$ are distinct roots and μ_i are the multiplicities of α_i , so that their sum is

$$\mu_1 + \dots + \mu_m = D.$$

Finally, we remark that the conjugate reflection (2) leads to reflection of roots.

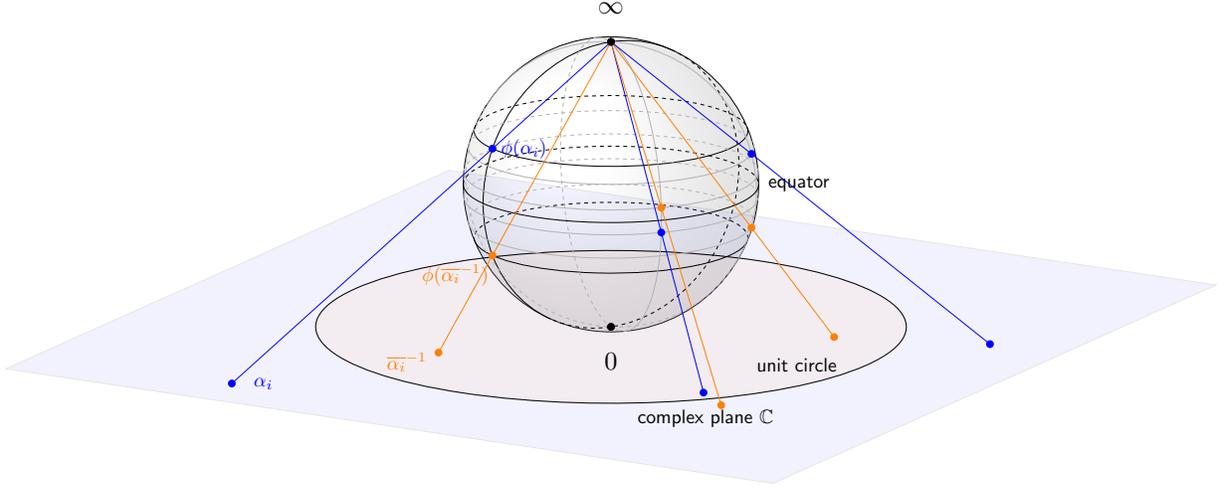


Figure 1: Complex plane and the Riemann sphere (the preimage under the stereographic projection). The conjugate inversion corresponds to reflection with respect to the equator on the Riemann sphere. Here $\phi : \mathbb{C} \rightarrow \mathcal{S}^2$ denote the inverse stereographic mapping onto the sphere \mathcal{S}^2 .

Lemma 12. *The conjugate reflection of the polynomial (11) admits a factorization*

$$\tilde{A}(z) = \tilde{\lambda} \prod_{i=1}^m \left(z - \overline{\alpha_i^{-1}} \right)^{\mu_i}, \quad \text{where } \tilde{\lambda} := \bar{\lambda} \prod_{\substack{i=1 \\ \alpha_i \neq \infty}}^m (-\overline{\alpha_i})^{\mu_i},$$

i.e., the roots α_i are mapped to $\overline{\alpha_i^{-1}}$, where 0 is formally assumed to be the inverse of ∞ and vice versa.

Proof. Follows from straightforward calculation. \square

Example 13. *For Example 10, the conjugate reflection $\tilde{A}(z) \in \mathbb{C}_{\leq 5}[z]$, as well as its factorization becomes:*

$$\tilde{A}(z) = 0 \cdot z^5 - z^4 + \frac{1}{2}z^3 + \frac{1}{2}z^2 = (-1)(z - \infty)(z + \frac{1}{2})(z - 1)z^2.$$

which has roots $\{\infty, 1, -\frac{1}{2}, 0\}$, where the root 0 has multiplicity 2.

Graphically, the conjugate reflection of the roots has a nice interpretation in terms of the Riemann sphere: the mapping of the root under conjugate reflection becomes simply a reflection with respect to the plane passing through the equator, see Fig. 1.

Remark 14. *When dealing with homogeneous polynomials, the roots in fact belong to the projective space \mathbb{P}^1 which corresponds exactly to $\mathbb{C} \cup \{\infty\}$.*

Remark 15. *Let (11) be the factorization of a polynomial $A \in \mathbb{C}_{\leq D}[z]$ with roots α_i of respective multiplicities μ_i . Then $B \in \mathbb{C}_{\leq D'}[z] \setminus \{0\}$ is a divisor of A if and only if it can be factorized as*

$$B(z) = \lambda' \prod_{i=1}^m (z - \alpha_i)^{\nu_i}, \tag{12}$$

where $0 \leq \nu_i \leq \mu_i$ and $\nu_1 + \dots + \nu_m = D'$.

Example 16. *Continuing Example 5, the polynomial $0 \cdot z^4 + 0 \cdot z^3 + 0 \cdot z^2 + z + 2 = (z - \infty)^3(z + 2) \in \mathbb{C}_{\leq 4}[z]$ is not the divisor of $A(z)$ from (8), because there are not enough infinite roots in the expansion of $A(z)$.*

3. Uniqueness of factorizations

The main goal of this section is to provide a proof of Theorem 1, thus giving a characterization of the uniqueness properties of the polynomial factorization problem (1). In fact, we will prove a generalized version of Theorem 1.

3.1. Key lemma and the coprime case

The following key lemma links the GCD of polynomials $X_k(z)$ with the GCD of the elements of the matrix polynomial $\mathbf{\Gamma}(z)$.

Lemma 17. *Let $Q(z) := \gcd\{X_1, X_2, \dots, X_K\}$ where $Q \in \mathbb{C}_{\leq D}[z]$, and define*

$$H(z) = Q(z)\tilde{Q}(z). \quad (13)$$

Then the GCD of the elements of $\mathbf{\Gamma}(z)$ must be equal to $H(z)$:

$$\gcd\{\Gamma_{ij}\}_{i,j=1}^{K,K} = H(z).$$

Proof. Let $R_1, R_2, \dots, R_K \in \mathbb{C}_{\leq N-D-1}[z]$ be the corresponding quotients, i.e., $X_k(z) = Q(z)R_k(z)$ for $k = 1, \dots, K$ with $\gcd\{R_1, R_2, \dots, R_K\} = 1$. Direct calculations show that, for $i, j = 1, \dots, K$,

$$\Gamma_{ij}(z) = X_i(z)\tilde{X}_j(z) = R_i(z)Q(z)\tilde{R}_j(z)\tilde{Q}(z) = R_i(z)\tilde{R}_j(z)H(z).$$

Then the GCD of polynomials $\Gamma_{ij}(z)$ can be explicitly computed as

$$\begin{aligned} & \gcd\{\Gamma_{ij}\}_{i,j=1}^{K,K} \\ &= \gcd\left\{\gcd\{\Gamma_{1j}\}_{j=1}^K, \gcd\{\Gamma_{2j}\}_{j=1}^K, \dots, \gcd\{\Gamma_{Kj}\}_{j=1}^K\right\} \\ &= \gcd\{R_1H, R_2H, \dots, R_KH\} && \text{since } \gcd\{\tilde{R}_1, \tilde{R}_2, \dots, \tilde{R}_K\} = 1 \\ &= H(z) && \text{since } \gcd\{R_1, R_2, \dots, R_K\} = 1, \end{aligned}$$

which concludes the proof. \square

Remark 18. *If the matrix polynomial $\mathbf{\Gamma}(z)$ can be factorized as (1), then any simultaneous rescaling of the polynomials $X_1(z), \dots, X_K(z)$ by $\beta \in \mathbb{T}$ provides an alternative factorization since*

$$\begin{bmatrix} \beta X_1(z) \\ \vdots \\ \beta X_K(z) \end{bmatrix} \begin{bmatrix} \bar{\beta}\tilde{X}_1(z) & \cdots & \bar{\beta}\tilde{X}_K(z) \end{bmatrix} = \begin{bmatrix} X_1(z) \\ \vdots \\ X_K(z) \end{bmatrix} \begin{bmatrix} \tilde{X}_1(z) & \cdots & \tilde{X}_K(z) \end{bmatrix},$$

i.e., polynomials $Y_k(z) = \beta X_k(z)$ provide an equivalent factorization since $\beta\bar{\beta} = 1$.

In what follows, we will say that the solution (1) is essentially unique if it is uniqueness up to a global scaling by $\beta \in \mathbb{T}$, i.e., for any alternative factorization

$$\mathbf{\Gamma}(z) = \begin{bmatrix} Y_1(z) \\ \vdots \\ Y_K(z) \end{bmatrix} \begin{bmatrix} \tilde{Y}_1(z) & \cdots & \tilde{Y}_K(z) \end{bmatrix}, \quad (14)$$

there exists $\beta \in \mathbb{T}$ such that $Y_k(z) = \beta X_k(z)$ for all k .

Armed with Lemma 17, we can already give the characterization of the coprime case.

Proposition 19. Let $\{\Gamma_{ij}\}_{i,j=1}^{K,K}$ be coprime (equivalently, let $X_1(z), \dots, X_K(z)$ be coprime). Then

1. the rank-one factorization (1) is essentially unique (in the sense of Remark 18);
2. in particular, let

$$A_j(z) := \gcd(\Gamma_{j1}, \dots, \Gamma_{jK}) \quad (15)$$

and fix an index j such that $A_j(z) \not\equiv 0$. Then a factorization (14) can be obtained by

$$Y_k(z) = c_j \cdot \frac{\Gamma_{kj}(z)}{A_j(z)}, \quad (16)$$

where c_j is the normalization constant $c_j = \|A_j\| / \sqrt{|\Gamma_{jj}(z)|_{z^{N-1}}}$.

Proof. We first prove 2). We first show that the polynomial $Y_k(z)$ defined in (16) provides an alternative factorization. Indeed, for A_j defined in (15) we have

$$A_j(z) = \gcd(X_j \tilde{X}_1, \dots, X_j \tilde{X}_K) = d_j X_j(z), \quad d_j \neq 0. \quad (17)$$

since $\tilde{X}_1(z), \dots, \tilde{X}_K(z)$ are coprime. Then for the polynomial $Y_k(z)$ defined in (16) we get

$$Y_k(z) = \frac{X_k(z) \tilde{X}_j(z)}{d_j \tilde{X}_j(z)} \cdot \frac{\|d_j X_j(z)\|}{\|X_j(z)\|} = \frac{d_j}{|d_j|} X_k(z), \quad (18)$$

thanks to Lemma 3. By defining $\beta = \frac{d_j}{|d_j|}$, we observe that $\beta \in \mathbb{T}$, and therefore $\Gamma_{ij}(z) = Y_i(z) \tilde{Y}_j(z)$.

Proof of 1) Assume that apart from (14), there exists an alternative factorization $\Gamma_{ij}(z) = U_i(z) \tilde{U}_j(z)$. Note that by Lemma 17, we have $\gcd\{U_1, \dots, U_K\} = 1$. Therefore $\Gamma_{ij}(z) = X_i(z) \tilde{X}_j(z)$ and $\Gamma_{ij}(z) = U_i(z) \tilde{U}_j(z)$ are two valid factorizations that satisfy the conditions of 2). Now assume that Y_k is computed as in (18). Then, by 2) there are two constants $\beta, \delta \in \mathbb{T}$ such that

$$Y_k = \beta X_k \quad \text{and} \quad Y_k = \delta U_k, \quad \text{for all } k.$$

Therefore, we have that $U_k = \delta^{-1} \beta X_k$, for all k , which completes the proof. \square

Proposition 19 already tells us how to deal with the coprime case, and provides us with a constructive way to find the factorization (1) from the matrix polynomial $\Gamma(z)$ (i.e., by computing Y_k as in (18)). Moreover, this algorithm can be further simplified as shown by the following remark.

Remark 20. In the conditions of (19), the polynomials X_k can be retrieved, up to individual constants from (15) (i.e. $A_k = \beta_k X_k$). Therefore, if one is interested, for example, only in the roots of X_k , those can be obtained just by computing A_k as in (15).

We also note that the Proposition 19 covers the generic case, as shown by the following corollary.

Corollary 21 (Almost everywhere uniqueness of (1)). *In the generic case, the solution of (1) is essentially unique: there exists a set of (Lebesgue) measure zero in $A \subset (\mathbb{C}_{\leq N-1}[z])^K$, such that for all $(X_1, \dots, X_K) \in (\mathbb{C}_{\leq N-1}[z])^K \setminus A$ the solution of (1) is essentially unique.*

Proof. By Theorem 9 the two polynomials $X_1, X_2 \in \mathbb{C}_{\leq N-1}[z]$ are coprime if and only if $S(X_1, X_2)$ is invertible. The equation $\det(S_1(X_1, X_2)) = 0$ defines an algebraic variety V of dimension $2N - 1 \leq \dim((\mathbb{C}_{\leq N-1}[z])^2)$, thus it is a set of measure zero. Taking $A = V \times \dim(\mathbb{C}_{\leq N-1}[z])^{K-2}$ concludes the proof. \square

Finally, we remark that the condition of coprimeness in Proposition 19 is only a sufficient condition (and not a necessary condition as mistakenly claimed in [12, Theorem 1]). We establish a necessary and sufficient condition in the next subsection.

3.2. The main uniqueness result

Before proving the main uniqueness result, we establish another important lemma that shows what happens in the non-coprime case.

Lemma 22. *Let $X_1(z), \dots, X_K(z)$ be a tuple of polynomials not vanishing simultaneously. Let $Q = \gcd\{X_1, \dots, X_K\} \in \mathbb{C}_{\leq D}[z]$ and $R_1, R_2, \dots, R_K \in \mathbb{C}_{\leq N-D-1}[z]$ be the corresponding quotients such that $X_k(z) = Q(z)R_k(z)$ for any k . Then $Y_k(z)$ provides a valid alternative factorization (14) if and only if the polynomials have the form*

$$Y_k(z) = S(z)R_k(z) \quad (19)$$

where $S(z)$ satisfies $S(z)\tilde{S}(z) = Q(z)\tilde{Q}(z)$.

Proof. The “if” part is obvious, because for any polynomial of the form (19)

$$Y_i(z)\tilde{Y}_j(z) = S(z)\tilde{S}(z)R_i(z)\tilde{R}_j(z) = Q(z)\tilde{Q}(z)R_i(z)\tilde{R}_j(z) = X_i(z)\tilde{X}_j(z).$$

Now we will prove the “only if” part.

Let $H(z) = Q(z)\tilde{Q}(z)$ (so that $H(z) = \gcd\{\Gamma_{ij}\}_{i,j=1}^{K,K}$ by Lemma 17), and fix an alternative factorization $Y_k(z)$. Then, from Lemma 17, a GCD $S = \gcd\{Y_1, \dots, Y_K\}$ must satisfy $S(z)\tilde{S}(z) = cH(z)$ with $c \in \mathbb{C}$, where we can normalize c to be 1 (thanks to the freedom of choosing the normalization for $S(z)$).

Now denote by $T_k(z)$ the corresponding quotients of $Y_k(z)$ such that $Y_k(z) = S(z)T_k(z)$. Then the matrix polynomial

$$\mathbf{G}(z) := \frac{\mathbf{\Gamma}(z)}{H(z)}$$

must have the factorization of its entries as $G_{ij}(z) = R_i(z)\tilde{R}_j(z) = T_i(z)\tilde{T}_j(z)$. However, since $H = \gcd\{\Gamma_{ij}\}_{i,j=1}^{K,K}$, we have $\gcd\{G_{ij}\}_{i,j=1}^{K,K} = 1$, and therefore, by Proposition 19 it has an essentially unique factorization. Thus there exists a constant $\beta \in \mathbb{T}$ such that $T_k = \beta R_k$ for all k . In turn, we can define $S'(z) := \beta S(z)$, and one gets that $Y_k(z)$ can be expressed as

$$Y_k(z) = S'(z)R_k(z),$$

where $S'(z)\tilde{S}'(z) = Q(z)\tilde{Q}(z)$, which completes the proof. \square

Remark 23. *Lemma 22 shows that the study of the uniqueness properties of (1) is directly related to uniqueness of the univariate polynomial factorization $H(z) = Q(z)\tilde{Q}(z)$, as the quotients can be obtained thanks to the constructive procedure described in Proposition 19 applied to $\frac{\mathbf{\Gamma}(z)}{H(z)}$.*

In other words, all possible factorizations can be obtained from the Smith normal form of the rank-one matrix polynomial $\mathbf{\Gamma}(z)$ [4].

Before giving the sufficient and necessary uniqueness condition, we make a remark about the roots of the product $Q(z)\tilde{Q}(z)$ which are key to understanding uniqueness.

Lemma 24. *Let $Q(z) = \lambda \prod_{i=1}^D (z - \alpha_i)$ (with possibly repeating α_i). Then $H(z) = Q(z)\tilde{Q}(z)$ has the following factorization*

$$H(z) = \lambda \tilde{\lambda} \prod_{i:\alpha_i \neq \infty} (-\bar{\alpha}_i) \prod_{i=1}^D (z - \alpha_i)(z - \bar{\alpha}_i^{-1}). \quad (20)$$

Furthermore, if $\alpha \in \mathbb{T}$, then $\alpha = \bar{\alpha}^{-1}$. Therefore, a unit-modulus α is a root of $Q(z)$ of multiplicity μ if and only if it is a root of $H(z)$ of multiplicity 2μ .

Proof. Follows from Lemma 12. \square

Theorem 25. Let $X_1(z), \dots, X_K(z)$ be a tuple of polynomials not vanishing simultaneously. Then the following equivalences are true:

1. The problem (1) admits a unique solution (in the sense of Remark 18);
2. $X_1(z), \dots, X_K(z)$ have no common roots in $(\mathbb{C} \cup \{\infty\}) \setminus \mathbb{T}$ (common roots may be only on the unit circle);
3. $H(z) = \gcd\{\Gamma_{ij}\}_{i,j=1}^{K,K}$ has no roots in $\mathbb{C} \setminus \mathbb{T}$.

Proof. The proof is organized in several parts.

- $\boxed{2 \Leftrightarrow 3}$ By Lemma 17, $H(z) = cQ(z)\tilde{Q}(z)$, where c is a constant and $Q(z) = \gcd\{X_i(z)\}_{i=1}^K$. Therefore, by Lemma 24, $H(z)$ does not have roots outside the unit circle \mathbb{T} if and only if $Q(z)$ does not. Note that by Lemma 24, the roots of $H(z)$ appear in pairs, and therefore $H(z)$ has an ∞ root if and only if 0 is also a root.
- $\boxed{1 \Rightarrow 2}$ Suppose that the solution of (1) is essentially unique, but the polynomial $Q(z)$ has a root α outside the unit circle. Then easy calculations show that polynomial $S(z) = \frac{Q(z)(z-\alpha^{-1})}{(z-\alpha)}$ satisfies

$$S(z)\tilde{S}(z) = Q(z)\tilde{Q}(z).$$

Note that $S(z)$ is not proportional to $Q(z)$, because

$$\frac{(z - \alpha^{-1})}{(z - \alpha)} \neq \text{const.}$$

Therefore the vector polynomial

$$(Y_1(z), \dots, Y_K(z)) := (S(z)R_1(z), \dots, S(z)R_K(z)),$$

is not proportional to the vector $(X_1(z), \dots, X_K(z))$, but gives an alternative factorization $\Gamma_{ij}(z) = Y_i(z)\tilde{Y}_j(z)$ (a contradiction).

- $\boxed{1 \Leftarrow 2}$ Let $Q(z)$ be the GCD of $\{X_k\}_{k=1}^K$ and $R_k(z)$ be the corresponding quotients. Since $Q(z)$ has only unit-modulus roots, by Lemma 24, the polynomial $H(z) = Q(z)\tilde{Q}(z)$ has the roots with doubled multiplicities. Therefore, there is a unique (up to a constant $\beta \in \mathbb{T}$) way to factorize $H(z)$ (for any alternative factorization, $H(z) = S(z)\tilde{S}(z)$, there exists a constant $\beta \in \mathbb{T}$ such that $S(z) = \beta Q(z)$). Therefore, by Lemma 22, any other valid alternative factorization $Y_k(z)$ has necessarily the form

$$Y_k(z) = \beta Q(z)R_k(z),$$

which completes the proof. □

4. Enumerating the factorizations

In this section, we refine Theorem 25 by providing the number of solutions and by describing the set of solutions of (1) in the non-unique case. As mentioned in Remark 23, this description depends mainly on uniqueness properties of the factorization (13) (i.e., how to find all $Q(z)$ such that $H(z) = Q(z)\tilde{Q}(z)$). The univariate factorization problem, in its turn, is known to be closely related to enumerating the solutions of the so-called univariate phase retrieval problem [8, 13]. In this section we enumerate in Theorem 29 all

possible factorizations based on the technique used for the phase retrieval problem (see [8, Theorem 3.1, Corollary 3.3] and [14, Proposition 6.1]) which goes back to the result of Fejér [15, p.61].

The number of ways we can factorize $H(z)$ depends on multiplicities of its roots, and the following remark is very useful.

Remark 26 (Root pairs of $H(z)$). *From Lemma 24, we know that the roots of $H(z)$ come in conjugate-reflected pairs, i.e., if a root $\delta \notin \mathbb{T}$ is a root of $H(z)$ with multiplicity μ then $\bar{\delta}^{-1}$ is also a root with the same multiplicity. In such a case we will say that the pair $(\delta, \bar{\delta}^{-1})$ has multiplicity μ . In the case where $\delta \in \mathbb{T}$ is a root of $H(z)$, then it must have even multiplicity.*

Example 27. *Consider $Q(z) = A(z)$ that is the polynomial from Example 10 having double ∞ root and simple roots $\{-2, 1, 0\}$. Then the polynomial $H(z) = A(z)\tilde{A}(z)$ is given by*

$$H(z) = -\frac{1}{2}(z - \infty)^3 \left(z + \frac{1}{2}\right) (z + 2)(z - 1)^2 z^3.$$

The multiplicity of the root pair $(0, \infty)$ is 3, the root pair $(-2, -\frac{1}{2})$ has multiplicity 1, and the root pair 1 has multiplicity 2.

Remark 28. *As shown in Example 27, different roots of $Q(z)$ can merge to form root pairs (as it is the case for the roots -2 and $-\frac{1}{2}$ in Example 27). In general, if δ and $\bar{\delta}^{-1}$ are roots of $Q(z)$ with multiplicities ν_1 and ν_2 , then the multiplicity of the pair $(\delta, \bar{\delta}^{-1})$ of $H(z)$ is equal to $\nu_1 + \nu_2$.*

Theorem 29 (Enumerating solutions of (1)). *Let Q, R_1, \dots, R_K be as in Lemma 22, where the root structure of the polynomial $H(z) = Q(z)\tilde{Q}(z)$ is as follows:*

- $H(z)$ has P root pairs $(\delta_1, \bar{\delta}_1^{-1}), \dots, (\delta_P, \bar{\delta}_P^{-1})$ outside the unit circle with multiplicities μ_1, \dots, μ_P and
- roots $\varepsilon_1, \dots, \varepsilon_T \in \mathbb{T}$ with (even) multiplicities ν_1, \dots, ν_T .

Then all the possible alternative factorizations $Y(z)$ such that $\Gamma(z) = Y(z)Y(z)$ may be expressed as

$$\begin{aligned} Y_1(z) &= \beta S(z)R_1(z) \\ &\vdots \\ Y_K(z) &= \beta S(z)R_K(z) \end{aligned}$$

where $\beta \in \mathbb{T}$ is arbitrary, and where

$$S(z) = \lambda \prod_{i=1}^P (z - \delta_i)^{\ell_i} (z - \bar{\delta}_i^{-1})^{\mu_i - \ell_i} \prod_{j=1}^T (z - \varepsilon_j)^{\frac{\nu_j}{2}}, \quad (21)$$

$0 \leq \ell_i \leq \mu_i$ are nonnegative integers, and $\lambda > 0$ is a constant that depends on a particular collection of ℓ_i (see Lemma 24).

Remark 30. *Theorem 29 gives a way to obtain all the polynomials from $\Gamma(z)$. Indeed, the factorization relies on $H(z) = \gcd\{\Gamma(z)\}_{i,j=1}^K$ (see Lemma 17) and the quotient polynomials $R_1(z), \dots, R_K(z)$ can be also obtained from $\Gamma(z)$ thanks to Proposition 19.*

Corollary 31 (Number of factorizations of (1)). *Under the assumptions of Theorem 29, the problem (1) admits exactly*

$$\prod_{i=1}^P (\mu_i + 1) \quad (22)$$

different solutions. In particular, when roots of $H(z)$ are all simple and outside the unit circle, there is exactly 2^D different solutions.

Proof of Theorem 29. Lemma 22 shows that the number of solutions of (1) is exactly the number of different (up to multiplication by a scalar) polynomials $S(z)$ such that $H(z) = S(z)\tilde{S}(z)$. This *spectral factorization* problem is equivalent to selecting the roots of $Q(z)$ amongst the root pairs $(\delta_i, \bar{\delta}_i^{-1})$ of $H(z)$ outside \mathbb{T} .

Consider a non-unit-modulus root pair $(\delta_i, \bar{\delta}_i^{-1})$ with multiplicity μ_i ; then the number of different combinations is equal to the number of outcomes of a random draw of μ_i items with replacement in a set of 2 elements, i.e., $\mu_i + 1$. Repeating the same process for each root pair gives all possible factorizations (21) by selecting the integers $0 \leq \ell_i \leq \mu_i$ (the multiplicity of δ_i in $S(z)$). \square

Example 32. *Continuing Example 27, we see that there are two root pairs not on the unit circle (with multiplicities 3 and 1, respectively). This yields a total of $4 \cdot 2 = 8$ solutions, where the other factorizations are given by permuting 0 and ∞ roots or/and replacing root -2 with $-\frac{1}{2}$. For example, some of possible alternative factorisations are given by $Q(z) = \frac{1}{2}(z+2)(z-1)z^3$ or $Q(z) = (z+\frac{1}{2})(z-1)z^3$.*

4.1. Case of two polynomials

We conclude this paper by providing an explicit expression of solutions of (1) in the simplified case of $K = 2$ and where there are no 0 or ∞ roots in common, meaning that $\mathbf{x}[0] \neq \mathbf{0}$ and $\mathbf{x}[N-1] \neq \mathbf{0}$. This setting is relevant to the context of polarimetric phase retrieval [1].

Proposition 33. *Let $K = 2$, and $H(z)$ be with roots in $\mathbb{C} \setminus \{0, \infty\}$, and fix a factorization of $H(z)$:*

$$H(z) = c \sum_{i=1}^D (z - \beta_i)(z - \bar{\beta}_i^{-1}).$$

Let $\{\alpha_{ji}\}_{i=1}^{N-D-1}$ be the roots (with repetitions) of the quotient polynomials R_j , $j \in \{1, 2\}$. Then the corresponding rank-one factorization has the form

$$X_1(z) = e^{j\theta} \lambda_1 \prod_{i=1}^D (z - \beta_i) \prod_{i=1}^{N-D-1} (z - \alpha_{1i}), \quad (23)$$

$$X_2(z) = e^{j\theta} \lambda_2 \prod_{i=1}^D (z - \beta_i) \prod_{i=1}^{N-D-1} (z - \alpha_{2i}), \quad (24)$$

where the constants $\lambda_1, \lambda_2 \in \mathbb{C}$ are given by

$$\lambda_1 = \sqrt{|\gamma_{11}[N-1]| \prod_{i=1}^D |\beta_i|^{-1} \prod_{i=1}^{N-D-1} |\alpha_{1i}|^{-1}}, \quad (25)$$

$$\lambda_2 = e^{j\Delta} \sqrt{|\gamma_{22}[N-1]| \prod_{i=1}^D |\beta_i|^{-1} \prod_{i=1}^{N-D-1} |\alpha_{2i}|^{-1}}, \quad (26)$$

where Δ reads

$$\Delta = \pi(N-1) + \arg \gamma_{12}[N-1] + \sum_{i=1}^D \arg \beta_i + \sum_{i=1}^{N-D-1} \arg \alpha_{2i}. \quad (27)$$

Proof. To determine λ_1 and λ_2 , one writes the expression of the elements of matrix polynomials in terms of $X_1(z)$ and $X_2(z)$ above. For instance:

$$\begin{aligned} \Gamma_{11}(z) &= X_1(z) z^{N-1} \overline{X_1(\bar{z}^{-1})} \\ &= |\lambda_1|^2 \prod_{i=1}^D (z - \beta_i) \prod_{i=1}^{N-D-1} (z - \alpha_{1i}) \prod_{i=1}^D (1 - \bar{\beta}_i z) \prod_{i=1}^{N-D-1} (1 - \bar{\alpha}_{1i} z) \end{aligned} \quad (28)$$

Using that $\Gamma_{11}(z) := \sum_{n=0}^{2N-2} \gamma_{11}[n-N+1] z^n$, identifying leading order coefficients yields

$$\gamma_{11}[N-1] = |\lambda_1|^2 (-1)^{N-1} \prod_{i=1}^D \bar{\beta}_i \prod_{i=1}^{N-D-1} \bar{\alpha}_{1i} \quad (29)$$

Similarly, one gets

$$\gamma_{22}[N-1] = |\lambda_2|^2 (-1)^{N-1} \prod_{i=1}^D \bar{\beta}_i \prod_{i=1}^{N-D-1} \bar{\alpha}_{2i} \quad (30)$$

$$\gamma_{12}[N-1] = \lambda_1 \bar{\lambda}_2 (-1)^{N-1} \prod_{i=1}^D \bar{\beta}_i \prod_{i=1}^{N-D-1} \bar{\alpha}_{2i} \quad (31)$$

These relations determine uniquely the moduli of λ_1, λ_2 as well as the difference of argument between λ_1 and λ_2 . Thus λ_1, λ_2 are unique up to a global phase factor $\exp(j\theta)$, $\theta \in [-\pi, \pi)$. One obtains eventually the following expressions

$$\lambda_1 = e^{j\theta} \left(|\gamma_{11}[N-1]| \prod_{i=1}^D |\beta_i|^{-1} \prod_{i=1}^{N-D-1} |\alpha_{1i}|^{-1} \right)^{1/2} \quad (32)$$

$$\lambda_2 = e^{j(\theta-\Delta)} \left(|\gamma_{22}[N-1]| \prod_{i=1}^D |\beta_i|^{-1} \prod_{i=1}^{N-D-1} |\alpha_{2i}|^{-1} \right)^{1/2} \quad (33)$$

with

$$\begin{aligned} \Delta &= \arg(\lambda_1 \bar{\lambda}_2) \\ &= \pi(N-1) + \arg \gamma_{12}[N-1] + \sum_{i=1}^D \arg \beta_i + \sum_{i=1}^{N-D-1} \arg \alpha_{2i}. \quad \square \end{aligned} \quad (34)$$

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A. Link between matrix polynomial factorization and autocorrelation

The matrix polynomial rank-one factorization problem (1) arises in multivariate instances of Fourier phase retrieval [1] and blind multichannel system identification [2]. In such applications, one is interested in recovering a deterministic discrete vector signal $\mathbf{x} : \llbracket 0, N-1 \rrbracket \rightarrow \mathbb{C}^R$ from the different cross-correlations functions between the R signal channels. Now, define the polynomial representation of the i -th channel of \mathbf{x} as $X_i(z) = \sum_{n=0}^{N-1} x_i[n]z^n$. Similarly, define the correlation polynomial $\Gamma_{ij}(z) := \sum_{n=0}^{2(N-1)} \gamma_{ij}[n-N+1]z^n$. Then, a key result is that

$$\Gamma_{ij}(z) = X_i(z)\tilde{X}_j(z) \quad (35)$$

since

$$X_i(z)\tilde{X}_j(z) = \left(\sum_{n=0}^{N-1} x_i[n]z^n \right) \left(\sum_{m=0}^{N-1} \overline{x_j[N-1-m]}z^m \right) \quad (36)$$

$$= \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} x_i[n] \overline{x_j[N-1-m]} z^{n+m} \quad (37)$$

$$= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_i[n] \overline{x_j[m]} z^{n+N-1-m} \quad (38)$$

$$= \sum_{n'=0}^{2(N-1)} \gamma_{ij}[n'-N+1]z^{n'} := \Gamma_{ij}(z). \quad (39)$$

Therefore, defining the matrix polynomial $\mathbf{\Gamma}(z)$ such that

$$\mathbf{\Gamma}(z) = \begin{bmatrix} \Gamma_{11}(z) & \cdots & \Gamma_{1R}(z) \\ \vdots & & \vdots \\ \Gamma_{R1}(z) & \cdots & \Gamma_{RR}(z) \end{bmatrix} = \sum_{n=0}^{2N} \begin{bmatrix} \gamma_{11}[n-N+1] & \cdots & \gamma_{1R}[n-N+1] \\ \vdots & & \vdots \\ \gamma_{R1}[n-N+1] & \cdots & \gamma_{RR}[n-N+1] \end{bmatrix} z^n := \sum_{n=0}^{2(N-1)} \mathbf{\Gamma}[n]z^n \quad (40)$$

where $\{\mathbf{\Gamma}[n] \in \mathbb{C}^{R \times R}\}_{n=-N+1}^{N-1}$ is the auto-correlation matrix sequence of the D -dimensional vector signal $\{\mathbf{x}[n] \in \mathbb{C}^R\}_{n=0}^{N-1}$. Plugging (35) into (40) yields the rank-one autocorrelation matrix factorization problem (1).

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