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AN ANALYTIC PROOF OF THE RIEMANN HYPOTHESIS

HUBERT SCHAETZEL

ABSTRACT. The purpose of this article is to give an analytic proof of the Riemann hypothesis which suggests that all non-trivial zeros of the Zeta function are on the critical line.

CONTENTS

1. The tools	1
2. The absence of double zeros of the Zeta function	3
3. The graphical context	6
3.1. The standard context	6
3.2. The simplified context	9
4. The analytic proof	10
Literature and sources	15

1. THE TOOLS

The Riemann Zeta function is defined over the complex plane $Re(s) > 1$, with the standard notation in the complex plane $s = \sigma + i.t$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This function has an analytic continuation over the whole complex plane except at its unique pole $s = 1 + 0.i$. The resulting function is therefore a meromorphic function according to the following definitions.

Definition 1. *A function f is holomorphic on an open set D in the complex plane if for any $z_0 \in D$ one can write*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

in which the coefficients a_n are complex numbers and the series is convergent to $f(z)$ for z in a neighbourhood of z_0 .

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Alternatively, a complex analytic function is an infinitely differentiable function such that the Taylor series at any point z_0 in its domain converges to $f(z)$ for z in a neighbourhood of z_0 pointwise. (See reference [9]).

Definition 2. A meromorphic function on an open subset D of the complex plane is holomorphic on all of D except for a set of isolated points, which are the poles of the function. (See reference [9]).

These reminders being made, the following theorems will also be useful later on.

Theorem 1. *Isolated zeros principle.* The zeros of a non-constant analytic function are isolated.

Proof. See reference [9]. □

Theorem 2. *If U is an open subset of the complex plane \mathbb{C} , then a function $f : U \rightarrow \mathbb{C}$ is conformal if and only if it is holomorphic and its derivative is everywhere non-zero on U . The Riemann mapping theorem states that any non-empty open simply connected proper subset of \mathbb{C} admits a bijective conformal map to the open unit disk in \mathbb{C} .*

Proof. See reference [8]. Note that the theorem applies to a function f of one and only one variable s . □

Theorem 3. *According to the Abel summation formula, for $Re(s) > 0$, the zeta function can be expressed as*

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{u\}}{u^{1+s}} du$$

where $\{u\}$ is the mantissa of u .

Proof. See reference [7]. □

The Riemann's hypothesis, formulated in 1859 [1] [6], is that the non-trivial zeros of the function are such that $Re(s) = \frac{1}{2}$, the zeros quoted as trivial being $s = -2n$, $n \in N^*$.

A well-established result is that all the non-trivial zeros are located within the critical band $0 < Re(s) < 1$. In search of zeros, one can reduce the review to the domain $0 < \sigma \leq 1/2$ (or $1/2 \leq \sigma < 1$) thanks to the following fact:

Theorem 4. *Within the critical band, the non-trivial ζ -function zeros are symmetrical to the axis $s = 1/2$.*

Proof. It is an immediate result of the functional equation (see references [2] and [7])

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s),$$

in which the expression $2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)$ never cancels if $0 < \sigma < 1$. □

The Pari gp application is chosen here whenever we wish to provide some numerical data and the illustrations are given after injecting the data on an Excel spreadsheet.

Given some coordinate $s = \sigma + i.t$ value, we will call, here and there, σ the abscissa and t the ordinate.

2. THE ABSENCE OF DOUBLE ZEROS OF THE ZETA FUNCTION

Theorem 5. *The Riemann Zeta function and its first derivative never cancel simultaneously, that is there is no solution s such that $\zeta(s) = \zeta'(s) = 0$.*

Proof. Let us suppose $\Re(s) > 0$. Recalling theorem 3, let us take the derivatives on each side of the corresponding expression. We get using the product rule for derivatives

$$\begin{aligned}\zeta'(s) &= -\frac{s}{(s-1)^2} + \left(\frac{1}{s-1} - \int_1^\infty \frac{\{u\}}{u^{1+s}} du\right) + s \int_1^\infty \frac{(\ln(u))\{u\}}{u^{1+s}} du \\ &= -\frac{s}{(s-1)^2} + \frac{\zeta(s)}{s} + s \int_1^\infty \frac{(\ln(u))\{u\}}{u^{1+s}} du\end{aligned}$$

Then

$$\int_1^\infty \frac{\{u\}}{u^{1+s}} du = \frac{1}{s-1} - \frac{\zeta(s)}{s}$$

and

$$\int_1^\infty \frac{(\ln(u))\{u\}}{u^{1+s}} du = \frac{1}{(s-1)^2} - \frac{\zeta(s)}{s^2} + \frac{\zeta'(s)}{s}.$$

Thus

$$\begin{aligned}\frac{\int_1^\infty \frac{(\ln(u))\{u\}}{u^{1+s}} du}{\left(\int_1^\infty \frac{\{u\}}{u^{1+s}} du\right)^2} &= \frac{\frac{1}{(s-1)^2} - \frac{\zeta(s)}{s^2} + \frac{\zeta'(s)}{s}}{\left(\frac{1}{s-1} - \frac{\zeta(s)}{s}\right)^2} \\ &= \frac{\frac{1}{(s-1)^2} - 2\frac{\zeta(s)}{s(s-1)} + \left(\frac{\zeta(s)}{s}\right)^2 - \frac{\zeta(s)}{s^2} + \frac{\zeta'(s)}{s} + 2\frac{\zeta(s)}{s(s-1)} - \left(\frac{\zeta(s)}{s}\right)^2}{\frac{1}{(s-1)^2} - 2\frac{\zeta(s)}{s(s-1)} + \left(\frac{\zeta(s)}{s}\right)^2} \\ &= 1 + \frac{-\frac{\zeta(s)}{s^2} + \frac{\zeta'(s)}{s} + 2\frac{\zeta(s)}{s(s-1)} - \left(\frac{\zeta(s)}{s}\right)^2}{\left(\frac{1}{s-1} - \frac{\zeta(s)}{s}\right)^2} \\ &= 1 + \frac{\frac{1}{s}(\zeta'(s) + \zeta(s)\left(\frac{s+1}{s(s-1)} - \frac{\zeta(s)}{s}\right))}{\left(\frac{1}{s-1} - \frac{\zeta(s)}{s}\right)^2}\end{aligned}$$

Let us have s_0 a double root (at least) of the zeta function, that is $\zeta(s_0) = \zeta'(s_0) = 0$.

Then

$$\int_1^\infty \frac{\{u\}}{u^{1+s_0}} du = \frac{1}{s_0 - 1}$$

and

$$\int_1^\infty \frac{(\ln(u))\{u\}}{u^{1+s_0}} du = \frac{1}{(s_0 - 1)^2}$$

and therefore

$$\frac{\int_1^\infty \frac{(\ln(u))\{u\}}{u^{1+s_0}} du}{\left(\int_1^\infty \frac{\{u\}}{u^{1+s_0}} du\right)^2} = 1$$

The Riemann Zeta function is continuous. Therefore, when $s \rightarrow s_0$, we get

$$\frac{\frac{1}{s}(\zeta'(s) + \zeta(s)(\frac{s+1}{s(s-1)} - \frac{\zeta(s)}{s}))}{(\frac{1}{s-1} - \frac{\zeta(s)}{s})^2} \rightarrow 0$$

while the following expressions tend towards fixed non-null values

$$\frac{1}{s} \rightarrow \frac{1}{s_0} \text{ and } (\frac{1}{s-1} - \frac{\zeta(s)}{s})^2 \rightarrow (\frac{1}{s_0-1} - \frac{\zeta(s_0)}{s_0})^2 \rightarrow \frac{1}{(s_0-1)^2}$$

Therefore, as $\zeta(s) \rightarrow \zeta(s_0) = 0$,

$$\zeta'(s) + \zeta(s)(\frac{s+1}{s(s-1)} - \frac{\zeta(s)}{s}) \rightarrow \zeta'(s) + \frac{s+1}{s(s-1)}\zeta(s) \rightarrow 0$$

Thus, the zeros being isolated, and therefore $\zeta(s) \neq 0$,

$$\frac{\zeta'(s)}{\zeta(s)} \rightarrow -\frac{s+1}{s(s-1)}$$

Let us choose some complex coordinate s , not a zero but close to a non-trivial zero s_0 of ζ , thus $s = s_0 + \Delta s_0$, with n the multiplicity of the zero s_0 . With this choice, the complex value s has an inverse. There is no zero near the pole of the studied function and therefore we can also make the choice of s such that $1/(s-1)$ and $1/(s_0-1)$ have always finite values in our arguments. From definitions 2 and 1, the terms of higher degrees being negligible in the Taylor series, we deduce

$$\zeta(s) \rightarrow \alpha(s - s_0)^n \text{ as } s \rightarrow s_0$$

where α is some finite non-zero value. The number n being the multiplicity of the zero s_0 , we are interested in the cases $n \geq 2$. Let us start by evaluating the first derivative near that zero

$$\zeta'(s) \rightarrow n\alpha(s - s_0)^{n-1} \text{ as } s \rightarrow s_0$$

Therefore

$$\frac{\zeta'(s)}{\zeta(s)} \rightarrow \frac{n}{s - s_0} \text{ as } s \rightarrow s_0$$

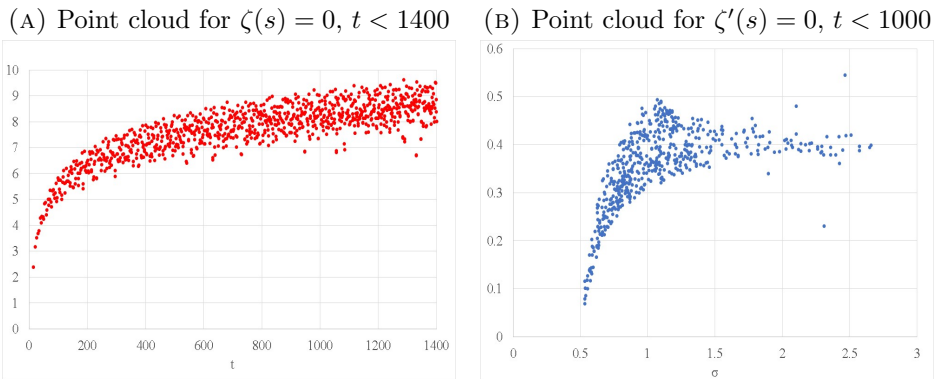
Thus we have necessarily

$$\frac{n}{s - s_0} \rightarrow -\frac{s+1}{s(s-1)}$$

As s tends towards s_0 , the first term is diverging so that $s \rightarrow 0$ or $s \rightarrow 1$ (if $s \rightarrow \infty$, the second term is tending towards 0 and cannot be a solution). But there are definitively no zeros of the Riemann Zeta function near the coordinates $s = 0$ or $s = 1$, therefore the hypothesis $\zeta(s_0) = \zeta'(s_0) = 0$ is impossible. The result being true within the critical band, it is true in the whole complex plane. \square

Note. One may find interesting to evaluate in some way the "distance" that exists really to the simultaneous events $\zeta(s) = 0$ and $\zeta'(s) = 0$. That can be done, for example, by evaluating the ratio $\frac{\int_1^\infty \frac{(\ln(u))\{u\}}{u^{1+s}} du}{(\int_1^\infty \frac{\{u\}}{u^{1+s}} du)^2}$ and comparing it to 1. The strategy could be to research the closest value to 1 within the critical band as t is taking increasing values. This is however somewhat cumbersome to implement. We choose here to get the point clouds of the ratio when only one of the events occurs. It won't give the optimum data but the approach still gives some notable information. The figures 1A and 1B represents the point clouds obtained. These figures provide the Napierian logarithm and therefore the two events' coincidence would to achieved for $Ln(1) = 0$, hence at the ordinate 0. Figure 1A shows that the more likely solutions, if they existed, would be near $t = 0$, an expected consequence of the former proof. As there is no expected case where $\zeta'(s) = 0$ for $\sigma < 1/2$ (see reference [3]), there can be no tendency towards $\sigma = 0$ in the second point cloud, the way it is calculate. In figure 1B, we see instead a tendency towards the average value of the expected $s = 0$ or $s = 1$ (thus $\sigma = (0 + 1)/2$). We see also that the upper side of the point cloud culminates mainly around $\sigma = 1$ and the point cloud get its broadest size around that abscissa. To finish with, let us note also that the four points that are "escaping" the point cloud are part of the seven first solutions of $\zeta'(s) = 0$, where $0 < t < 70$, an interval where the $\frac{s}{s-1}$ contribution in the Abel summation of $\zeta(s)$ is still of some highly differentiating importance.

FIGURE 1. Point clouds of $f = Ln(\frac{\int_1^\infty \frac{(\ln(u))\{u\}}{u^{1+s}} du}{(\int_1^\infty \frac{\{u\}}{u^{1+s}} du)^2})$
 $\zeta(s) = \zeta'(s) = 0$ if $f = 0$.



3. THE GRAPHICAL CONTEXT

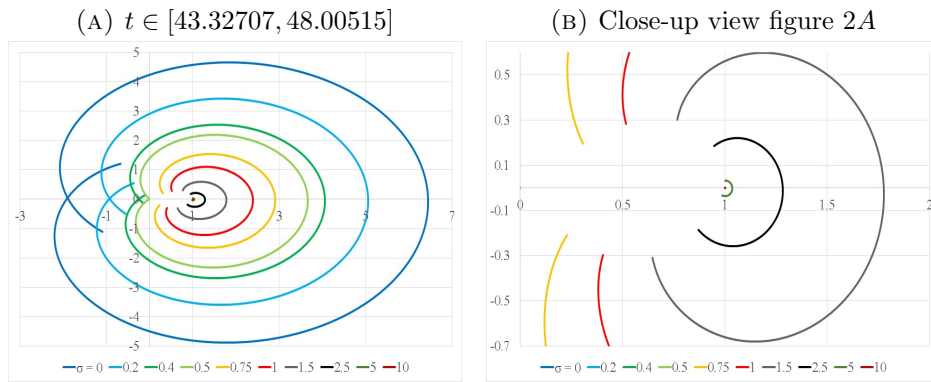
3.1. The standard context. Let us consider two oriented parallel vertical segments in the complex plane s and $s + \epsilon$ where ϵ is a real valued infinitesimal, $s = \sigma + i.t$ and $t \in [t_1, t_2]$, $0 < t_1 < t_2$. The rectangular section between the two segments does not contains the pole of the zeta function (because $t > 0$). Then the oriented images by the zeta function of these lines stay on the same side one in regard to the other. For a line on the right side of its neighbour, its image is on the right side of its neighbour's image also.

This results from the fact that the ζ -function is analytic (except at the pole). Therefore, locally according to theorem 2, for a small enough region of the domain, it provides a bijective conformal map from the domain to the codomain except if the domain contains a zero of the first derivative (or the pole). The zeros of the first derivative (function which is analytic also) are isolated zeros by theorem 1 and therefore the preceding argument holds in any sufficiently infinitesimal domain near but not containing the zero. Considering two parallel oriented lines in that region, it provides two oriented lines in the codomain which do not cross. Of course, by extending the domain, the bijection at some step will eventually fail.

The respective side in the image for some given respective side in the domain being necessarily always the same according to the first part of this proof, a numerical example enables to conclude immediately that right side provides right side (and left provides left).

Let us then consider the complex plane domain $t \in [t_1, t_2]$ where t_1, t_2 are the imaginary parts of two successive non-trivial positive zeros of the zeta function, choosing here expressly zeros with real part value $1/2$. The image of the continuous network of vertical segments of that domain by the Riemann Zeta function provides a continuous network of curves that divides a priori into two types of configurations.

FIGURE 2. Trajectories $\zeta(\sigma + i.t)$ for constant σ



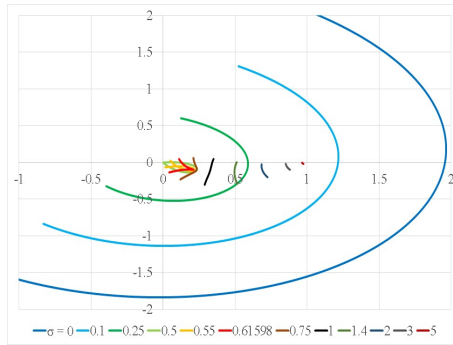
The first one derives from the absence of a zero of the first derivative of the Zeta function in the chosen domain. The second one derives from the

presence of one and only one zero of the first derivative of the Zeta function at some abscissa σ greater than $1/2$.

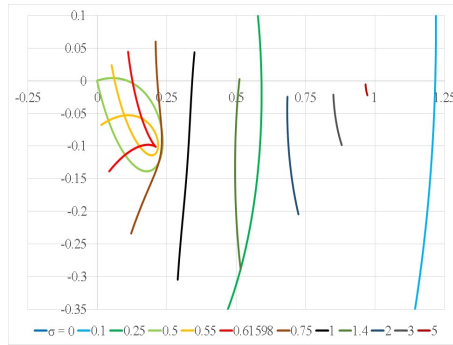
To gather these two kind of drawings, it suffices to consider a rectangular domain $\sigma \in [\sigma_1, \sigma_2]$ and $t \in [t_1, t_2]$ where for example $\sigma_1 = 0$ or some finite sufficient lesser value and $\sigma_2 = 5$ or some finite sufficient greater value.

FIGURE 3. Trajectories $\zeta(\sigma + i.t)$ for constant σ

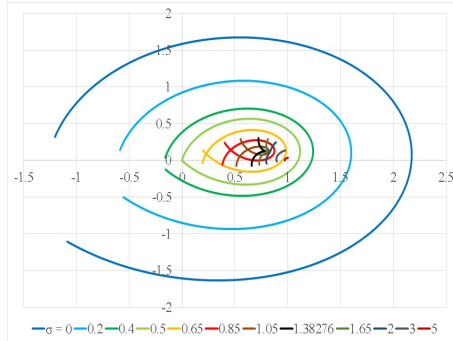
(A) $t \in [185.59878, 184.87447]$



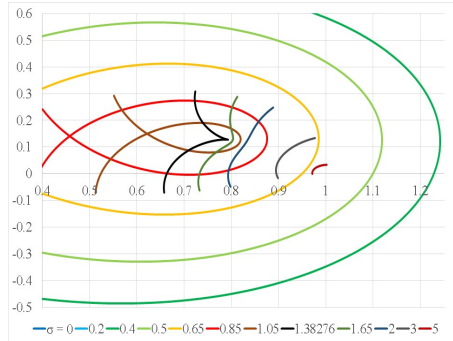
(B) Close-up view figure 3A
 $\zeta'(0.61598 + i.185.21481) \simeq 0$



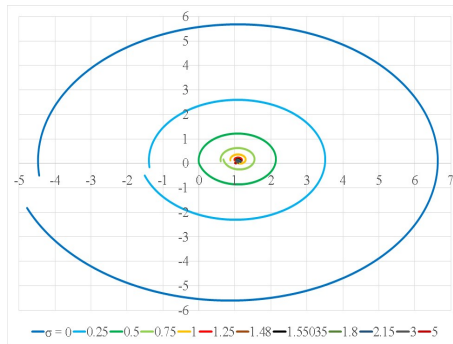
(C) $t \in [40.91872, 43.32707]$



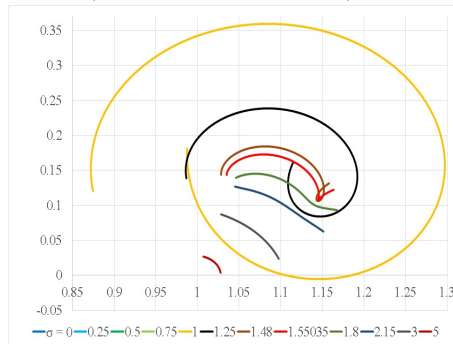
(D) Close-up view figure 3C
 $\zeta'(1.38276 + i.42.29096) \simeq 0$



(E) $t \in [161.18896, 163.030710]$



(F) Close-up view figure 3E
 $\zeta'(1.55035 + i.162.66556) \simeq 0$



Before distinguishing the two configurations, let us start by noting the common feature. There is a progressive evolution from a large scale image for "large" negative values of σ to minute forms for "large" positive values of σ , the later forms converging to the coordinate $(1,0)$. The exponential evolution results from the $\frac{1}{n^\sigma}$ contribution of each of the Riemann zeta function terms. It leads, and that is seen clearly on any sample, to a rapid reducing scaling effect. Figures 2 and 3 are such samples as σ is given higher values while keeping the same sized vertical segments in the domain.

The image of two infinitesimal close segments shift progressively on the same side in the absence of a zero of the derivative within the domain and therefore the image will head asymptotically to the coordinate $(1,0)$ as illustrated by figures 2A and 2B, this example being the first and simplest type of configuration, the case where, let us say abruptly, nothing special happens.

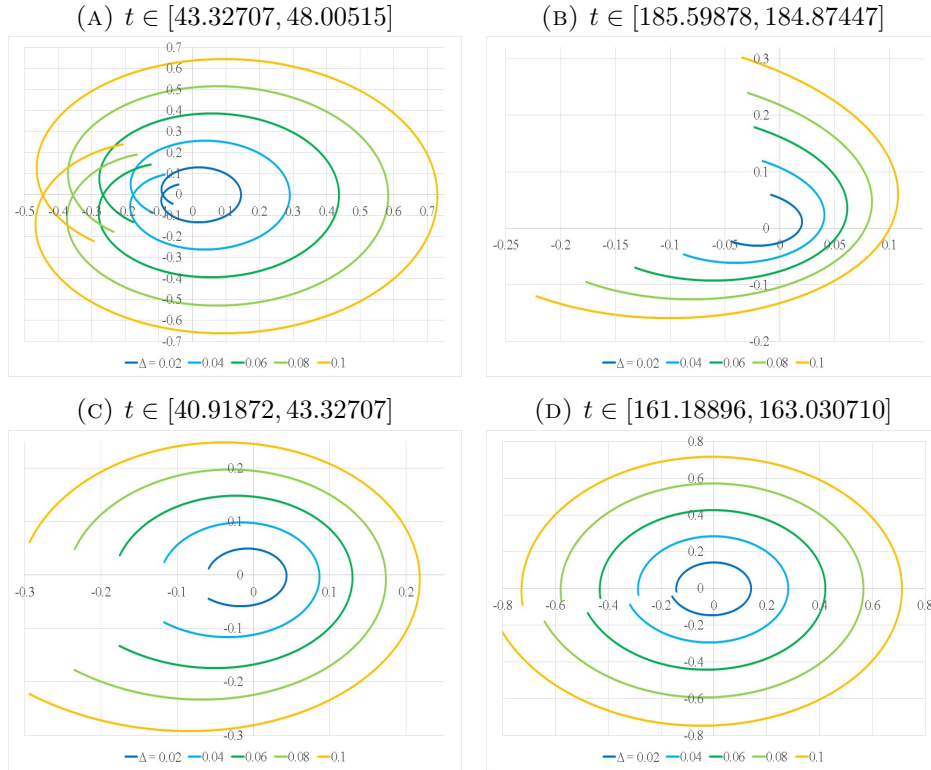
The second type is produced when $\zeta'(s) = 0$ for some coordinate s within the chosen wide enough rectangular domain. Supposing any such event, we get the examples of the figures 3A to 3F. Here the conformal map argument holds for the image of the left part of the rectangular domain up to the abscissa of the zero of the derivative on one hand and again separately for the image of the right part of rectangular domain starting at the abscissa of the said zero of the derivative on the other hand. This time, the exponential reducing trend may be false locally. Indeed, by definition, $\zeta(s + \epsilon) - \zeta(s) \rightarrow \zeta'(s) \cdot \epsilon$ as $\epsilon \rightarrow 0$ and therefore the distances $\zeta(s + \epsilon) - \zeta(s)$ are locally shorter near the image's locus where $\zeta'(s)$ is small, including thus the case $\zeta'(s) = 0$. Therefore, the same way that there is logic for a reduction of size of the images before the event $\zeta'(s) = 0$, there is no surprise for eventually an increase of the lengths of the vertical lines' images after that event is crossed (see for example figure 3B where that occurs between $\sigma \approx 0.616$ and $\sigma \approx 1$). The effect is much more visible if one reduces the domain to a smaller height band which still includes the zero of the derivative.

Note. For the first type of configuration, in the absence of "disturbance" by a zero of the derivative, all curves shifting on the same side to each other, there exists only the double intersection solution, among all image's segments on the right side of the critical line, this line included, with the $(0,0)$ coordinate which results from the initial choice of the two successive zeros on the said critical line. No intersection for the strict right side means, as we know, no intersection also on the strict left side neither, thus the Riemann hypothesis confirmation for this first case.

For the second type of configuration, the remarks are quite analogous but the arguments in favour of the Riemann hypothesis are trickier to explicit and somewhat opposable in the case of the existence of a zero of the derivative with abscissa lesser than $1/2$. Here, one can recall the Speicher's theorem which provides the Riemann conjecture's proof but only in the case $\sigma > 1/2$ for all Zeta function derivative's zeros (see reference [3]).

3.2. The simplified context. The "complexity" of the graphics' designs are due entirely to the zeros of the derivatives of the Zeta function. It is possible however to simplify greatly the drawings by using another function.

FIGURE 4. Trajectories
 $\zeta(\sigma - \Delta + i.t) - \zeta(\sigma + \Delta + i.t)$, $\sigma = 1/2$



Indeed, trying to solve $\zeta(1/2 - \Delta + i.t) = \zeta(1/2 + \Delta + i.t) = 0$, we may rather start by solving $\chi(\Delta, t) = \zeta(1/2 - \Delta + i.t) - \zeta(1/2 + \Delta + i.t) = 0$ (and then $\zeta(1/2 + \Delta + i.t) = 0$ for example), thus to study $\chi(\Delta, t)$ in a general way. The figures 4A to 4D are the "cousins" of the figures 2 and 3 as we step from the ζ -function to the χ -function. These figures keep similitudes as long as the zeros of the derivative are not reached (in the initial figures), in particular for the extent of the rotations around their respective "centres". But no first derivative cancellations to be considered any more at least apparently. Each of the graphs seems to be almost homothetic in regard to the coordinate (0,0) and therefore the only solution to $\chi(\Delta, t) = 0$ would be $\Delta = 0$. The questions are then "can we trust the obvious?" and "where are now the zeros of the derivative hiding?".

4. THE ANALYTIC PROOF

Without divulging prematurely the answers to the two previous questions, let us say that we need to deviate from our so far choice $\sigma = 1/2$. Let us have a more general approach by writing $\chi(\sigma, \Delta, t) = \zeta(\sigma - \Delta + i.t) - \zeta(\sigma + \Delta + i.t)$ and let us choose $0 \leq \Delta \leq 1/2$ and $t > 0$.

Definition 3. *Let us consider a closed circular neighbourhood of $s = \sigma + i.t$, not reduced to a point, in the complex plane and $[\sigma_{min}, \sigma_{max}]$ the range of values of the abscissas over that neighbourhood, $\sigma_{min} \leq \sigma \leq \sigma_{max}$. The function $\chi(\sigma, \Delta, t)$ is said to be locally well-ordered over the previous neighbourhood of s if its trajectories are strictly on the same side to each other as Δ strictly increases over the whole range $[\sigma_{min} - \sigma, \sigma_{max} - \sigma]$, σ and t is being kept constant. One can also say that the trajectories are well-ordered (instead of the function).*

Lemma 1. *The function $\chi(\sigma, \Delta, t)$, $t > 0$, is locally well-ordered except at the zeros of the derivative of the Zeta function. Moreover, it is a local approximate homothety in regard to Δ .*

Proof. Note first that the condition $t \neq 0$ secludes the Zeta function pole. We can write then the Taylor series:

$$\chi(\sigma, \Delta, t) = -2 \sum_{n=0}^{n=+\infty} \frac{\Delta^{2n+1}}{(2n+1)!} \zeta^{(2n+1)}(\sigma + i.t).$$

Now, for some parameter r and using $s = \sigma + i.t$, we get

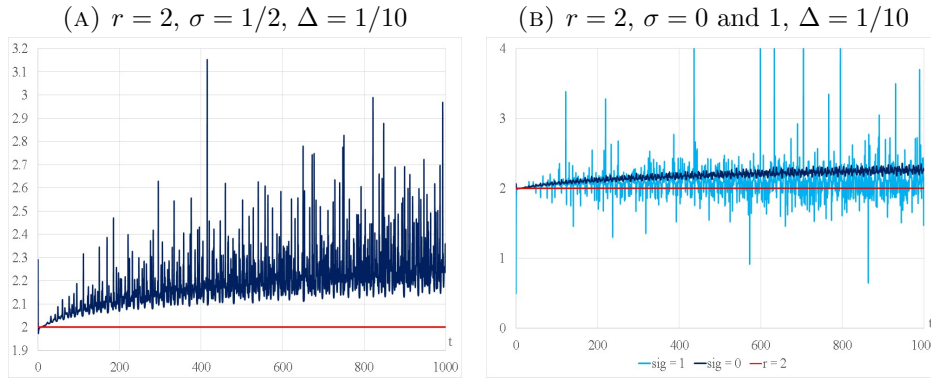
$$\rho(r, \sigma, \Delta, t) = \frac{\chi(\sigma, r \cdot \Delta, t)}{\chi(\sigma, \Delta, t)} = r \frac{\zeta'(s) + r^2 \frac{\Delta^2}{6} \zeta^{(3)}(s) + r^4 \frac{\Delta^4}{120} \zeta^{(5)}(s) + \dots}{\zeta'(s) + \frac{\Delta^2}{6} \zeta^{(3)}(s) + \frac{\Delta^4}{120} \zeta^{(5)}(s) + \dots}.$$

Whatever the finite given values of σ and Δ , $\chi(\sigma, r \cdot \Delta, t)$ is finite (being off the pole of the Zeta function). For small enough values of Δ and supposing $\zeta'(s) \neq 0$, $\rho(r, \sigma, \Delta, t) \approx r$, hence the local approximate homothety. \square

As Δ is given larger values, and the first terms in the numerator and denominator are no more the leading ones, the most likely to become so are then the second terms of the fraction, thus $\rho(r, \sigma, \Delta, t)$ will tend towards r^3 . One therefore would expect, taking this time a fixed moderate value of Δ and varying instead t , to get for $\rho(r, \sigma, \Delta, t)$ values between r and r^3 , mostly near but superior to r and experiencing in general peaks smaller than r^3 . Figure 5A gives an illustration of such evolution of $\rho(r, \sigma, \Delta, t)$ choosing $\sigma = 1/2$ and $\Delta = 1/10$. Except close to the origin, we see the systematic values over the chosen homothety threshold $r = 2$ and the narrow peaks. One can of course envision to have the first and second terms negligible providing even higher ratios. Such occurrences are obviously expected to be exceptional. Now, checking the evolution of $\rho(r, \sigma, \Delta, t)$ in regard of σ , let us signal figure 5B in addition to figure 5A. For moderate Δ , on the left hand side of the critical line, the expression does not vary much. It oscillates mildly above

the ratio r . Heading towards the right hand side instead, the ratio will start to show counterexamples to the interval $[r, r^3]$, notably with value smaller than r . Indeed near a zero of the derivative, with an arbitrary choice of Δ , one cannot guess in advance which of the numerator or denominator will be larger in $\frac{\rho(r, \sigma, \Delta, t)}{r}$. However, the reader can always choose a smaller value of Δ to re-establish the local homothety.

FIGURE 5. Evolution of $\|\rho(r, \sigma, \Delta, t)\|_2$



Lemma 2. *A solution s to $\zeta(s + \Delta) = 0$ and $\chi(\sigma, \Delta, t) = 0$ is necessarily a double zero of the Zeta function.*

Proof. If $\zeta'(s) = 0$,

$$\chi(\sigma, \Delta, t) = -2 \sum_{n=1}^{+\infty} \frac{\Delta^{2n+1}}{(2n+1)!} \zeta^{(2n+1)}(s)$$

and thus

$$\rho(r, \sigma, \Delta, t) = r^3 \frac{\frac{\Delta^2}{6} \zeta^{(3)}(s) + r^2 \frac{\Delta^4}{120} \zeta^{(5)}(s) + \dots}{\frac{\Delta^2}{6} \zeta^{(3)}(s) + \frac{\Delta^4}{120} \zeta^{(5)}(s) + \dots} \rightarrow r^3.$$

We may have simplified the fraction by $\frac{\Delta^2}{6}$ but we left it here on purpose to show that, as Δ tends towards 0, numerator and denominator tend also towards 0, that is $\chi(\sigma, \Delta, t) \rightarrow 0$. The figures 6A to 6C illustrate in a concrete way what happens in the case where we choose a small enough fixed value Δ and some interval $[t_1, t_2]$ and we shift progressively the parameter σ from an abscissa on the left side of a zero of the derivative of the Zeta function up to the abscissa of the said zero and then beyond that same zero. Here we chose the starting abscissa $\sigma = 0.5$ as we picked for our illustration the 168th zero of the derivative characterized by its relative proximity to the critical line. The following arguments would be the same if we had to face a zero of the derivative with abscissa lower than 1/2 as the knowledge of its effective value is not needed to proceed. The first picture, figure 6A, shows the almost homothetic pattern over the whole range $[t_1, t_2]$. In the

absence of a particular event, the crossing of the center $(0, 0)$ cannot start except if $\Delta = 0$. The second picture, figure 6B, which is brought with a closer view, shows, because of the evolution towards the degree 3 of r , an accelerated narrowing of the curves near the position corresponding to the zero of the derivative. Still being an almost local homothety (but with larger local ratio near the image coordinate $(0,0)$), no crossover of the curves can occur on this stage. That crossover can and will only occur as σ takes values beyond the derivative's zero abscissa (see figure 6C). Note before going further that this is in no way contradictory to the same sides' pattern cited for the Zeta function context. Effectively, the function $\chi(\sigma, \Delta, t)$ is analytic (away from the pole of the Zeta function). But the conformal map property does not apply here because we cannot write the function as an expression of only one variable s , but we need instead two of them s and Δ in $\chi(s, \Delta) = \zeta(s - \Delta) - \zeta(s + \Delta)$. Back to the main subject, we get the start of the crossing of the coordinate $(0,0)$ and the failure of the well-ordered trajectories by $\chi(\sigma, \Delta, t)$ at the precise place where simultaneously $\zeta'(s) = 0$ and Δ is taking its limit value 0. This is equivalent to say that $\zeta'(s + \Delta) = 0$ at that peculiar event. Now, seeking to solve the additional event $\zeta(s + \Delta) = 0$, this means that the latest zero is necessarily a double zero. If $\chi(\sigma, \Delta, t) = 0$, the events $\zeta(s + \Delta) = 0$, $\zeta'(s + \Delta) = 0$ are concomitant events, hence the lemma. \square

Note. If some event doesn't exist at all, one can only simulate what would happen if the said event would occur. Obviously it is impossible to display it, hence $\Delta \rightarrow 0$ necessarily in our presentation.

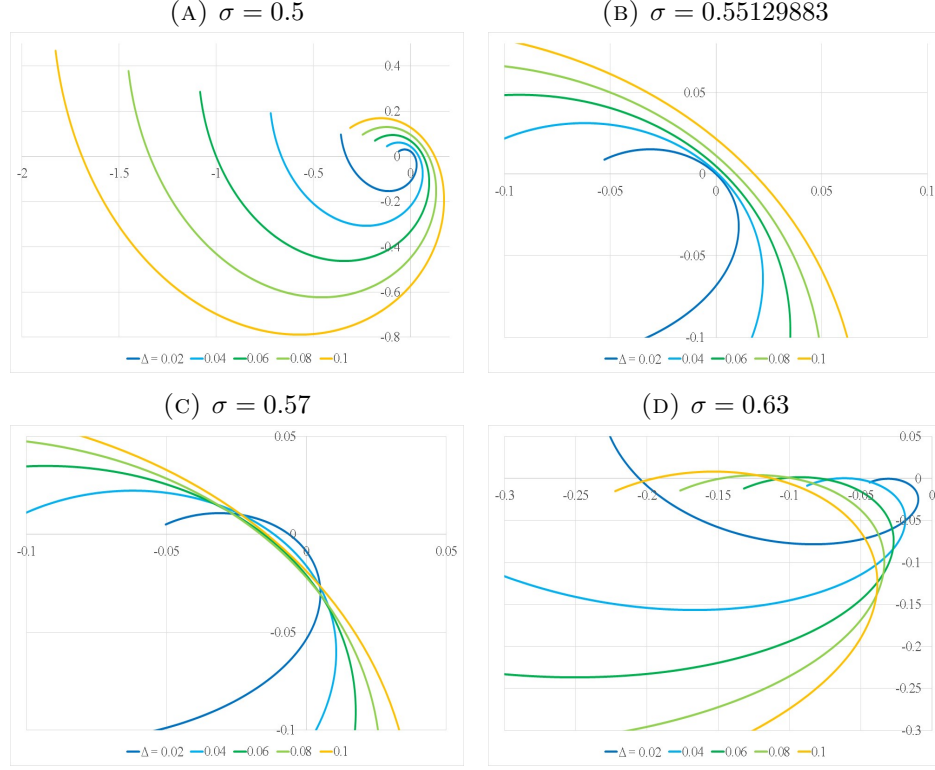
Note. In the previous argument, we mentioned the evolution of figures 6A to 6C with increasing σ . However, we didn't prove that this evolution is necessarily in that direction. We may have instead that order of drawings' patterns with decreasing σ . Indeed, in the vicinity s where $\zeta'(s) \rightarrow 0$, and with $r \approx 1$, we recall to have

$$\rho(r, \sigma, \Delta, t) = r \frac{\zeta'(s) + r^2 \frac{\Delta^2}{6} \zeta^{(3)}(s) + \dots}{\zeta'(s) + \frac{\Delta^2}{6} \zeta^{(3)}(s) + \dots}.$$

The local homothety ratio's progressive evolution between r and r^3 is only true on one side of the σ_0 value such that $\zeta'(s_0) = 0$, this side being either on the left or on the right of σ_0 according to the local sign of the ratio $\zeta^{(2k+1)}(s)/\zeta'(s)$ where k is the smallest natural number such that $\zeta^{(2k+1)}(s_0) \neq 0$. Indeed this is the result of the following calculation, where we use $(1 + \epsilon_1)/(1 + \epsilon_2) \approx 1 + \epsilon_1 - \epsilon_2$ to evaluate the former ratio (at $s \neq s_0$ and small enough Δ),

$$\begin{aligned} \rho(r, \sigma, \Delta, t) &\approx r \cdot (1 + (r^2 - 1) \frac{\Delta^{2k}}{2k+1} \frac{\zeta^{(2k+1)}(s)}{\zeta'(s)}) \\ &\approx r \cdot (1 + (r^2 - 1) \frac{\Delta^{2k}}{2k+1} (\Re(\frac{\zeta^{(2k+1)}(s)}{\zeta'(s)}) + i \Im(\frac{\zeta^{(2k+1)}(s)}{\zeta'(s)}))) \end{aligned}$$

FIGURE 6. Trajectories $\chi(\sigma, \Delta, t)$,
 $t \in [415.018810, 415.455215]$,
 $\zeta'(0.55129883 + 415.247512i) \approx 0$



Hence, using $\Delta^{2k} \gg \Delta^{4k}$ in the second equality underneath and $(1 + 2\epsilon)^{1/2} \approx 1 + \epsilon$ in the third one,

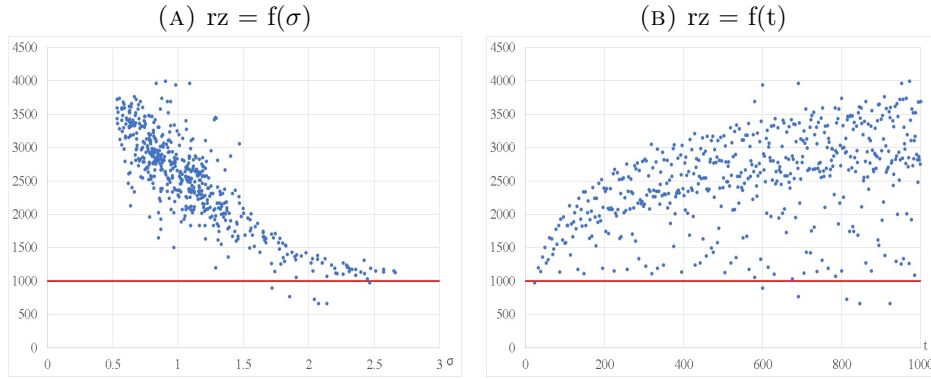
$$\begin{aligned}
 \|\rho(r, \sigma, \Delta, t)\|_2 &\approx r \cdot \left\{ (1 + (r^2 - 1) \frac{\Delta^{2k}}{2k+1} \Re(\frac{\zeta^{(2k+1)}(s)}{\zeta'(s)}))^2 \right. \\
 &\quad \left. + ((r^2 - 1) \frac{\Delta^{2k}}{2k+1} \Im(\frac{\zeta^{(2k+1)}(s)}{\zeta'(s)}))^2 \right\}^{\frac{1}{2}} \\
 &\approx r \cdot \sqrt{1 + 2(r^2 - 1) \frac{\Delta^{2k}}{2k+1} \Re(\frac{\zeta^{(2k+1)}(s)}{\zeta'(s)})} \\
 &\approx r \cdot (1 + (r^2 - 1) \frac{\Delta^{2k}}{2k+1} \Re(\frac{\zeta^{(2k+1)}(s)}{\zeta'(s)}))
 \end{aligned}$$

It happens that the left case (that is here $\Delta > 0$) seems to be always the well-ordered side. It is at least so for the sample up to $0 < t \leq 1000$ for which we chose to draw the figures 7A and 7B. The ratio $\frac{\zeta^{(3)}(s)}{\zeta'(s)}$ is not only always positive but usually greater than $1/\Delta$. In the studied interval, and with the choice $\Delta = 1/1000$, only 1 in 100 are lower than the resulting "1000" threshold. Note also that $\zeta^{(3)}(s_0)$ is very likely different from 0, and therefore the usually adapted value of k is 1 (hence we can rely on the ratio $\frac{\zeta^{(3)}(s)}{\zeta'(s)}$ to conduct the calculation of a sample). In order to prove the systematic

left hand side conjecture, one would need for example a precise Weierstrass factorisation of $\zeta'(s)$, which would allow to get the exact expression of the ratio $\frac{\zeta^{(2)}(s)}{\zeta'(s)}$ and then of $\frac{\zeta^{(3)}(s)}{\zeta'(s)} = \left(\frac{\zeta^{(2)}(s)}{\zeta'(s)}\right)' + \left(\frac{\zeta^{(2)}(s)}{\zeta'(s)}\right)^2$.

Of course, as σ increase further on the right side of the abscissa of the zero of the derivative, where the homothety fails, there will be nevertheless usually a progressive return to the near homothety of ratio r as shown in figure 6D.

FIGURE 7. Evaluation $rz = \Re\left(\frac{\zeta^{(3)}(s)}{\zeta'(s)}\right)$



Theorem 6. *The Riemann hypothesis is true.*

Proof. Theorem 5 and lemma 2 are incompatible. Therefore it is impossible to have $\zeta(s + \Delta) = 0$ and $\chi(\sigma, \Delta, t) = 0$ simultaneously if $\Delta \neq 0$, therefore $\zeta(s + \Delta) = 0$ and $\zeta(s - \Delta) = \chi(\sigma, \Delta, t) + \zeta(s + \Delta) = 0$ in the same time. Theorem 4 allows us then to conclude. \square

Note. To be very precise, following the proof in lemma 2 and the note that follows it, our full argumentation is not immediately equivalent to say that $\zeta(s + \Delta) = 0$ and $\zeta(s - \Delta) = 0$ is impossible if $\Delta \neq 0$, but it proves that $\zeta(s + \Delta) = 0$ and $\zeta(s - \Delta) = 0$ if and only if $\Delta = 0$ which is reputedly the same.

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