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A short ODE proof of the Fundamental Theorem of Algebra

Ramona Anton[1], Nicolae Mihalache[2] and François Vigneron[3]

Abstract

We propose a short proof of the Fundamental Theorem of Algebra based on the ODE that describes the Newton flow and the fact that the value $|P(z)|$ is a Lyapunov function. It clarifies an idea that goes back to Cauchy.

Given a polynomial $P \in \mathbb{C}[z]$ of degree $d \geq 1$, we denote by $\mathcal{Z}(P) = P^{-1}(0)$ the set of its zeros and by $\mathrm{Crit}(P) = \mathcal{Z}(P')$ the set of its critical points, a finite set (by Euclidean division). Let also $\mathcal{V}(P) = P(\mathrm{Crit}(P))$ the set of critical values of P.

The *Newton flow* of P is defined by the ODE

$$\varphi_z(0) = z \quad \text{and} \quad \varphi_z'(t) = -\frac{P(\varphi_z(t))}{P'(\varphi_z(t))}. \tag{1}$$

The set $\mathcal{Z}(P)$ is composed of stationary solutions of (1). Thanks to the Cauchy-Lipschitz theorem, the Newton flow is locally well defined starting from any point $z \notin \mathrm{Crit}(P)$ and can be extended locally as long as $\varphi_z(t) \notin \mathrm{Crit}(P)$. For $z \notin \mathrm{Crit}(P)$, let us denote by $T(z) \in (0, +\infty]$ the maximal forward time of existence of φ_z. One has

$$\forall z \notin \mathrm{Crit}(P), \quad \forall t \in [0, T(z)) \qquad P(\varphi_z(t)) = e^{-t} P(z) \tag{2}$$

because Leibniz's rule and (1) imply

$$\frac{d}{dt}\left[e^t P(\varphi_z(t))\right] = e^t P(\varphi_z(t)) + e^t P'(\varphi_z(t))\varphi_z'(t) = 0,$$

thus the Fundamental Theorem of Calculus gives $e^t P(\varphi_z(t)) = e^t P(\varphi_z(t))\big|_{t=0} = P(z)$. In particular, Newton's flow is *iso-angle i.e.* $\arg P(\varphi_z(t)) = \arg P(z)$ if $P(z) \neq 0$. Moreover, for $t \in (0, T(z))$, $\varphi_z(t) \in \mathcal{B}(z) = \{y \in \mathbb{C}\,;\, |P(y)| \leq |P(z)|\}$, which is a compact set because $\lim\limits_{|z|\to\infty} |P(z)| = +\infty$.

Theorem 1. *Every non-constant polynomial $P \in \mathbb{C}[z]$ has at least one root in $\mathbb{C}$.*

Proof. Our strategy to find a root of P is to choose a starting point z_0 with $T(z_0) = +\infty$ and then pass to the limit in (2). We will see that it is enough that $\arg(P(z_0))$ avoids the finite set $\arg(\mathcal{V}(P))$. Observe that $0 \notin \mathcal{V}(P)$, otherwise P has a root in $\mathrm{Crit}(P)$ and we stop here.

If $T(z) < \infty$ for some $z \in \mathbb{C} \setminus \mathrm{Crit}(P)$, then $P(z) \neq 0$. As $t \to T(z)$, the orbit $\varphi_z(t)$ leaves any compact subset of $\mathbb{C} \setminus \mathrm{Crit}(P)$. As $\varphi_z(t) \in \mathcal{B}(z)$, a compact set, there

is a sequence $t_n \to T(z)$ such that $\varphi_z(t_n)$ converges to some point $c \in \mathrm{Crit}(P)$. The continuity of P at c implies that $\arg P(z) = \arg P(\varphi_z(t_n)) = \arg(P(c)) \in \arg(\mathcal{V}(P))$.

Let us find z_0 with $\arg P(z_0) \notin \arg(\mathcal{V}(P))$. As $\lim_{|z|\to\infty} z^{-d}P(z) = \alpha \in \mathbb{C}^*$, taking $z_0 = re^{i\vartheta}$ with $d\vartheta + \arg\alpha \notin \arg(\mathcal{V}(P)) \mod 2\pi$ and $r > 0$ large enough ensures that $\arg(P(z_0)) \notin \arg(\mathcal{V}(P))$. Therefore $T(z_0) = +\infty$ and the bounded sequence $(\varphi_{z_0}(n))_{n\in\mathbb{N}} \subset \mathcal{B}(z_0)$ admits an accumulation point z_*. The continuity of P at z_* and (2) imply that $P(z_*) = \lim_{n\to\infty} e^{-n}P(z_0) = 0$. □

For a historical review on the Fundamental Theorem of Algebra see *e.g.* [6], [2, Chap. II] or [5] and, for a general survey of Newton's method, see [1]. The idea of the proof presented in this note can also be found, at a higher level of generality, in Hirsch-Smale [3].

However, the proof above is also of great practical importance. For example, (2) implies that, away from roots and critical points, each step of Newton's method divides approximately the value of the polynomial by e. In [4] we use this idea to split a non-trivial polynomial of record degree 10^{12}. We believe it is important to highlight it.

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References

[1] A. Galantai, *The theory of Newton's method.* J. of Comp. App. Math. 124 (2000), 25-44.

[2] J. Guérindon and J. Dieudonné, *L'Algèbre et la géométrie jusqu'en 1840* in *Abrégé d'histoire des Mathématiques.* Hermann, 1986.

[3] M.-W. Hirsch, S. Smale, *On algorithms for solving $f(x) = 0$.* Communications in Pure and Applied Mathematics 32, 3 (1979), 281-312.

[4] N. Mihalache, F. Vigneron. *How to split a tera-polynomial.* In preparation.

[5] R. Remmert, *The Fundamental Theorem of Algebra* in Graduate Texts in Mathematics 123, *Numbers.* Springer, 1991.

[6] S. Smale, *The fundamental Theorem of Algebra and complexity theory.* Bulletin of the American Mathematical Society 4-1 (1981), 1-36.

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