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# Null controllability of quasilinear parabolic equations with gradient dependent coefficients

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## Abstract

The aim of this paper is to study the null controllability of a large class of quasilinear parabolic equations. In a first step we prove that the associated linear parabolic equations with non-constant diffusion coefficients are approximately null controllable by the means of regular controls and that these controls depend continuously to the diffusion coefficient. A fixed-point strategy is employed in order to prove the null approximate controllability for the considered quasilinear parabolic equations. We also show the exact null controllability in arbitrary small time for a class of parabolic equations including the parabolic  $p$ -Laplacian with  $\frac{3}{2} < p < 2$ . The theoretical results are numerically illustrated combining a fixed point algorithm and a reformulation of the controllability problem for linear parabolic equation as a mixed-formulation which is numerically solved using a finite elements method.

## 1 Introduction

This paper consider the approximate null controllability of quasilinear equations of the following form:

$$\begin{cases} \partial_t y - \operatorname{div}(F(|\nabla y|)\nabla y) = \chi_\omega \varphi & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y(0) = u^0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^N$  is an open bounded domain with a smooth boundary  $\partial\Omega$ ,  $Q_T = (0, T) \times \Omega$ ,  $\Sigma_T = (0, T) \times \partial\Omega$ , the initial data  $u^0$  belongs to  $L^2(\Omega)$ , and the function  $F : (0, +\infty) \rightarrow (0, +\infty)$  verifies the following assumptions:

(A1) for every  $a > 0$ ,  $F$  is a globally Lipschitz function over  $[a, +\infty)$ ;

(A2)  $F$  belongs to  $L^2_{\text{loc}}([0, +\infty))$

(A3) the operator  $A : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  defined for every  $\phi \in H_0^1(\Omega)$  by

$$A\phi = -\operatorname{div}(F(|\nabla\phi|)\nabla\phi) \quad (1.2)$$

is monotone (see [21, Chapitre 2, Définition 1.2.]).

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The control  $\varphi$  acts in the open and non empty set  $\omega \subset \Omega$ . More precisely, we denote by  $\chi_\omega \in C^\infty(\overline{\Omega})$  a regular function such that

$$\chi_\omega(x) = \begin{cases} 1 & \text{for } x \in \omega_\delta \\ 0 & \text{for } x \in \overline{\Omega \setminus \omega}, \end{cases}$$

where  $\omega_\delta = \{x \in \omega \text{ such that } \text{dist}(x, \partial\omega) > \delta\}$  for a given  $\delta > 0$  small enough. We also denote  $q_T = (0, T) \times \omega$ .

We point out that the existence of a unique weak solution to equation (1.1) is a direct consequence of [21, Chapitre 2, Section 1, Théorème 1.2.- bis] applied to the nonlinear monotone operator  $A$  given by (1.2), when  $F$  is chosen such that it leads to the monotonicity of the operator  $A$ . From a historic point of view, the analysis of quasilinear parabolic equations and the properties of their solutions took off in the 1960's, with the pioneering works [18, 21, 1]. Quasilinear equations as (1.1) are to be compared to the parabolic  $p$ -Laplacian from which it derived their study. Such equations have been the source of a large number of publications in the last decades. As in the case of linear parabolic equations, the question of the boundedness of the solution or its gradient arises naturally. Such results are now well known and have been established for large classes of quasilinear systems, we can refer to [28, 12, 1, 6, 5, 11, 2, 29] in the case of a bounded domain with a regular boundary.

The controllability of quasilinear equations has been recently studied, as in [7, 9, 8], in the framework of optimal control, or [26, 25, 15] in the framework of exact controllability. In these last papers, the results for exact controllability hold for systems where the nonlinear term depends on the solution of the system, but not on its gradient. At our knowledge, the controllability of quasilinear systems with a gradient dependent term remains open and is the main purpose of the present work.

With the objective of applying a fixed point method in order to control the quasilinear equation, we first investigate the existence of smooth distributed controls for the following linear heat equation in divergence form with a space and time dependent diffusion coefficient:

$$\begin{cases} \partial_t u - \text{div}(a(t, x)\nabla u) = \chi_\omega \varphi & \text{in } Q_T \\ u = 0 & \text{on } \Sigma_T \\ u(0) = u^0 & \text{in } \Omega \end{cases} \quad (1.3)$$

in which we consider diffusion coefficients  $a \in C^\infty(\overline{Q_T})$  satisfying:

$$0 < \rho_\star \leq a(t, x) \quad ((t, x) \in \overline{Q_T}), \quad (1.4)$$

where  $\rho_\star > 0$  is a constant.

The distributed controllability of equation (1.3) is a well studied subject (see for example [17] or the more recent review paper [16]). The existence of an optimal distributed control, in the sense of minimal  $L^2$ -norm, can be obtained by applying the Hilbert Uniqueness Method (HUM) introduced in [22]. This classical approach allows to show the existence of an exact distributed control whose regularity is a priori only  $L^2(q_T)$ . The main idea of the method is to consider the dual final boundary value problem of (1.3) given by:

$$\begin{cases} \partial_t \varphi + \text{div}(a(t, x)\nabla \varphi) = 0 & \text{in } Q_T \\ \varphi = 0 & \text{on } \Sigma_T \\ \varphi(T) = \varphi^0 & \text{in } \Omega \end{cases} \quad (1.5)$$

for some  $\varphi^0 \in L^2(\Omega)$  and then to deduce the controllability of (1.3), as long as  $\varphi$  is regular enough, by the existence of a suitable observability inequality. More precisely, the following observability inequality holds.

**Proposition 1.1** (Observability inequality [16, Theorem 1.5.]). *There exists a constant  $C_0 > 0$ , depending of  $\Omega$ ,  $T$ ,  $\omega$ , and  $\|a\|_{L^\infty(Q_T)}$ , such that the following inequality holds:*

$$\|\varphi(0)\|^2 \leq C_0 \iint_{q_T} \chi_\omega |\varphi|^2 dx dt \quad (\varphi^0 \in L^2(\Omega)), \quad (1.6)$$

where  $\varphi$  is the solution to (1.5).

Here and henceforth we denote by  $(\cdot, \cdot)$  the inner product in  $L^2(\Omega)$  and by  $\|\cdot\|$  the associated norm. Moreover, by  $(\cdot, \cdot)_k$  we denote the inner product in  $H_0^k(\Omega)$  and by  $\|\cdot\|_k$  its corresponding norm.

Let us recall the relation between the observability inequality and the exact distributed controllability. Consider the following continuous and coercive convex functional  $G_n : L^2(\Omega) \rightarrow \mathbb{R}$  defined by

$$G_n(\varphi^0) = \frac{1}{2} \iint_{qT} \chi_\omega |\varphi|^2 dx dt + \frac{1}{n} \|\varphi^0\| + (\varphi(0), u^0) \quad (\varphi^0 \in L^2(\Omega)),$$

where  $n \in \mathbb{N}^*$  is a penalization parameter and  $\varphi$  is the unique solution to the dual problem (1.5). It is then a matter of minimizing the above functional and extracting a subsequence of minimizers to obtain a unique control. The uniqueness of the control is given by the coercivity of the family of penalized functionals  $(G_n)_n$ . Let us quickly give some more details about this method. First,  $G_n$  admits a unique minimum denoted  $\varphi_n^0$ . This minimum is a critical point of  $G_n$ , namely:

$$\iint_{qT} \chi_\omega \varphi_n \varphi dx dt + \frac{1}{n} \|\varphi_n^0\|^{-1} (\varphi_n^0, \varphi^0) + (\varphi(0), u^0) = 0 \quad (\varphi^0 \in L^2(\Omega)), \quad (1.7)$$

where  $\varphi_n$  is the solution of (1.5) associated to the final data  $\varphi_n^0$  and  $\varphi$  is the solution of the same problem associated to the final data  $\varphi^0$ . We also recall the definition of weak solutions of (1.3).

**Definition 1.1** (Weak solution of (1.3)). *Let  $u^0 \in L^2(\Omega)$  and  $\varphi \in L^2(Q_T)$ . We say that a function  $u \in L^2((0, T), H_0^1(\Omega)) \cap L^\infty((0, T), L^2(\Omega))$  such that  $u' \in L^2((0, T), H^{-1}(\Omega))$  is a weak solution of (1.3) if and only if  $u(0) = u^0$ , and for every  $\phi \in L^2((0, T), H_0^1(\Omega))$  and for almost all  $t \in (0, T)$ :*

$$\int_0^t \langle u', \phi \rangle_{-1,1} ds + \iint_{(0,t) \times \Omega} a(s, x) \nabla u \cdot \nabla \phi dx ds = \iint_{(0,t) \times \Omega} \chi_\omega \varphi \phi dx ds. \quad (1.8)$$

The existence and the uniqueness of weak solutions in the sense of Definition 1.1 is classical and it is not discussed here (see e.g. [14] for a proof). Let us denote by  $u_n$  and  $\psi_n$  the solutions of the following equations:

$$\begin{cases} \partial_t u_n - \operatorname{div}(a(t, x) \nabla u_n) = \chi_\omega \varphi_n & \text{in } Q_T \\ u_n = 0 & \text{on } \Sigma_T \\ u_n(0) = u^0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \partial_t \psi_n + \operatorname{div}(a(t, x) \nabla \psi_n) = 0 & \text{in } Q_T \\ \psi_n = 0 & \text{on } \Sigma_T \\ \psi_n(T) = u_n(T) & \text{in } \Omega, \end{cases}$$

where  $\varphi_n$  is the solution of (1.5) with final data  $\varphi_n^0$ . Up to make the change of variables in time  $t \mapsto t - T$ ,  $\psi_n$  is a weak solution of (1.3) in the sense of Definition 1.1 in the homogeneous setting. Then, we can write:

$$\iint_{Q_T} \partial_t u_n \psi_n dx dt + \iint_{Q_T} a(t, x) \nabla u_n \cdot \nabla \psi_n dx dt = \iint_{qT} \chi_\omega \varphi_n u_n dx dt \quad (1.9)$$

$$\iint_{Q_T} u_n \partial_t \psi_n dx dt - \iint_{Q_T} a(t, x) \nabla u_n \cdot \nabla \psi_n dx dt = 0. \quad (1.10)$$

Summing relations (1.9)-(1.10), we get:

$$\|u_n(T)\|^2 = \iint_{qT} \chi_\omega \varphi_n u_n dx dt + (\psi_n(0), u^0). \quad (1.11)$$

Combining (1.7), Cauchy-Schwarz's inequality and a standard energy estimate, we obtain:

$$\|u_n(T)\|^2 \leq \frac{1}{n} \|u_n(T)\| \leq \frac{C\|u^0\|}{n}. \quad (1.12)$$

If we can pass to the limit for  $n \rightarrow +\infty$  in (1.12), we would have that:

$$u_n \xrightarrow{n \rightarrow +\infty} u \quad \text{in } L^2(Q_T), \quad (1.13)$$

with  $u(T) = 0$ . This is made possible by using the observability inequality (1.6), since, testing against  $\varphi_n^0$  into (1.7), we get:

$$\|\chi_\omega \varphi_n\|_{L^2(Q_T)} \leq \|\varphi_n\|_{L^2(Q_T)} \leq \sqrt{C_0} \|u^0\|. \quad (1.14)$$

We point out that the control  $\varphi$  obtained as the weak limit of  $\varphi_n$  is a priori just belonging to  $L^2(Q_T)$ . Nevertheless, as mentioned in [13], it is interesting to improve the regularity of the obtained control, since if one wants to achieve exact controllability of some nonlinear system, for example by a fixed point result, it is in general necessary to have a more regular control. We point out that we can obtain a smooth approximate control of (1.3) by mollifying the classical exact control. Nevertheless, this non-constructive method does not lend itself well to numerical simulations and is expensive in terms of computing time. The first aim of the present work is to modify the standard HUM method as presented in [16] by introducing a nonlinear double-penalization term, which leads to the existence of a regular control providing the approximate controllability of the system (1.3).

More precisely, we consider the functional  $J_{\alpha,n} : H_0^k(\Omega) \rightarrow \mathbb{R}$  defined by

$$J_{\alpha,n}(\varphi^0) = \frac{1}{2} \iint_{Q_T} \chi_\omega |\varphi|^2 dx dt + \frac{1}{ne^{\alpha n}} \|\varphi^0\|_k^n + (\varphi(0), u^0), \quad (1.15)$$

where  $\varphi$  is the solution to (1.5) associated to the final data  $\varphi^0$  and  $\alpha > 0$ ,  $n \geq 2$ . Since there are two penalization parameters  $\alpha$  and  $n$  in the expression of  $J_{\alpha,n}$ , we call this functional a double-penalized HUM functional. As usual in HUM-like methods, we are interested to use the minimizer of  $J_{\alpha,n}$ . More precisely, it is clear that for every  $\alpha > 0$   $(J_{\alpha,n})_{n \geq 2}$  is a family of convex, continuous, and coercive functionals over  $H_0^k(\Omega)$  with  $k > 0$ . Then, there exists a sequence of associated minimizers  $(\varphi_{\alpha,n}^0)_{n \geq 2}$  in  $H_0^k(\Omega)$  satisfying the Euler-Lagrange equality:

$$\iint_{Q_T} \chi_\omega \varphi_{\alpha,n} \psi dx + e^{-\alpha n} \|\varphi_{\alpha,n}^0\|_k^{n-2} (\varphi_{\alpha,n}^0, \psi^0)_k + (\psi(0), u^0) = 0 \quad (\psi^0 \in H_0^k(\Omega)). \quad (1.16)$$

As usual, we denote by  $\psi$  the solution of (1.5) associated to the final data  $\psi^0$  and by  $\varphi_{\alpha,n}$  the solution of (1.5) associated to the final data  $\varphi_{\alpha,n}^0$ . Also, we are able to prove that the controls thus obtained depend  $L^2$ -continuously on the diffusion coefficient. This is summarized in the first result of the paper, given thereafter.

**Theorem 1.1.** *Let  $u^0 \in H_0^k(\Omega)$ ,  $a \in C^\infty(\overline{Q_T})$  satisfying (1.4) and  $T > 0$ . Then, for every  $\varepsilon > 0$  there exists a regular control  $\varphi \in L^2((0, T), H_0^k(\Omega)) \cap C([0, T], H_0^{k-1}(\Omega)) \cap H^{\frac{k}{2}}([0, T], L^2(\Omega))$  such that for every  $\mu \in (0, T)$ , we have  $\varphi \in C^\infty([\mu, T] \times \overline{\Omega})$  and the corresponding solution  $u$  of (1.3) verifies*

$$\|u(T)\| < \varepsilon. \quad (1.17)$$

Moreover, if  $k > \frac{N+4}{2}$  the control  $\varphi$  depends Lipschitz continuously to the diffusion coefficient  $a$  for the norm  $\|\cdot\|_{L^2(Q_T)}$ .

The main result of the paper is a consequence of Theorem 1.1 and provides the approximate null controllability of quasilinear equation (1.1).

**Theorem 1.2.** *Assume that  $F$  satisfies assumption (A1)-(A3) and  $u^0$  belongs to  $L^2(\Omega)$  are chosen such that there exists a unique solution of (1.1). Then, there exists a distributed control  $\varphi$ , whose regularity is given by Theorem 1.1, such that (1.1) is approximately null controllable in any time  $T > 0$ , i.e., for every  $u^0 \in L^2(\Omega)$  and every  $\varepsilon > 0$  there exists a control  $\varphi \in H^k(q_T)$  such that the solution  $u$  of (1.1) satisfies*

$$\|u(T)\| \leq \varepsilon.$$

In particular, Theorem 1.2 implies the approximate null controllability of the so-called parabolic  $p$ -Laplacian:

$$\begin{cases} \partial_t \mathbf{v} - \operatorname{div} (|\nabla \mathbf{v}|^{p-2} \nabla \mathbf{v}) = \chi_\omega \varphi & \text{in } Q_T \\ \mathbf{v} = 0 & \text{on } \Sigma_T \\ \mathbf{v}(0) = u^0 & \text{in } \Omega. \end{cases} \quad (1.18)$$

with  $\frac{3}{2} < p < \infty$ .

More exactly, we prove the following corollary.

**Corollary 1.1.** *Let  $u^0 \in L^2(\Omega)$  and  $\frac{3}{2} < p < \infty$ . Then (1.18) is approximate null controllable in any time  $T > 0$ , i.e., for every  $\varepsilon > 0$  there exists a control  $\varphi \in H^k(q_T)$  such that the solution  $\mathbf{v}$  of (1.18) verifies*

$$\|\mathbf{v}(T)\| \leq \varepsilon.$$

In fact, in the case where the solution stops in finite time, and where this stopping time is well controlled by the norm of the initial data, we can show the global exact controllability of (1.1). More precisely, the following result holds.

**Theorem 1.3.** *Assume that  $F$  satisfies assumptions (A1)-(A3), and that  $y$  is the solution of (1.1) associated to an initial data  $u^0 \in L^2(\Omega)$ . Moreover, let us consider that  $y$  stops in finite time, which is that, if  $\varphi = 0$  then, there exists  $T_s \in (0, T)$ ,  $\gamma > 0$  and  $\mu > 0$  such that:*

$$\|y(T_s)\| = 0 \quad \text{and} \quad T_s \leq \mu \|u^0\|^\gamma. \quad (1.19)$$

Then, one can choose the force term  $\varphi$  such that  $y$  is exactly null controllable in any time  $T^* \in (0, T)$ .

Applying the results in [10, Proposition 2.1.], the following corollary is a direct consequence of Theorem 1.3.

**Corollary 1.2.** *Let  $u^0 \in L^\infty(\Omega)$ ,  $u \geq 0$  and  $\frac{3}{2} < p < 2$ . Then (1.18) is null controllable in any time  $T > 0$ , i.e., there exists a control  $\varphi \in H^k(q_T)$  such that the solution  $\mathbf{v}$  of (1.18) verifies*

$$\mathbf{v}(T) = 0.$$

For the sake of clarity, we will omit throughout the article the dependence of the constants and will generically denote positive constants by  $C$ .

The remaining part of this paper is structured as follows. Section 2 is dedicated to the existence of approximate null control in  $H^k(q_T)$  for the linear equation (1.3). These controls are obtained as the minima of a well chosen functional. In Section 3 the minimization of the functional is replaced by a mixed formulation which will be particularly useful for the numerical approximation of the controls. In order to prove the Theorem 1.2 we employ a fixed point strategy described in Section 4. Finally, Section 5 numerically illustrate the computation of controls in both the linear and non-linear frameworks.

## 2 Approximate controllability of the linear equation

This section is devoted to the study of the approximate controllability of the equation (1.3) by the double penalization method stated previously. First, we have the following result.

**Proposition 2.1.** For every  $\alpha > 0$  there exists  $\varphi_\alpha^0 \in H_0^k(\Omega)$  such that the minimizers  $\varphi_{\alpha,n}^0$  of the functional  $J_{\alpha,n}$  satisfy:

$$\varphi_{\alpha,n}^0 \xrightarrow{n \rightarrow +\infty} \varphi_\alpha^0 \text{ in } H_0^k(\Omega). \quad (2.1)$$

Moreover,  $\chi_\omega \varphi_{\alpha,n} \xrightarrow{n \rightarrow +\infty} \chi_\omega \varphi_\alpha$  in  $L^2(q_T)$ , where  $\varphi_\alpha$  is the solution of (1.5) associated to the final data  $\varphi_\alpha^0$  and  $\varphi_{\alpha,n}$  is the solution of (1.5) associated to the final data  $\varphi_{\alpha,n}^0$ .

*Proof.* Setting  $\psi^0 = \varphi_{\alpha,n}^0$  in (1.16), we get:

$$\iint_{q_T} \chi_\omega |\varphi_{\alpha,n}|^2 dx dt + e^{-\alpha n} \|\varphi_{\alpha,n}^0\|_k^n + (\varphi_{\alpha,n}(0), u^0) = 0. \quad (2.2)$$

From (2.2) we deduce that

$$\iint_{q_T} \chi_\omega |\varphi_{\alpha,n}|^2 dx dt + (\varphi_{\alpha,n}(0), u^0) \leq 0. \quad (2.3)$$

Then, from (2.3) and (1.6), we get, following the same argument as for the classical HUM (see, for instance, [16] or Section 1):

$$\|\chi_\omega \varphi_{\alpha,n}\|_{L^2(q_T)}^2 \leq \sqrt{C_0} \|u^0\| \|\chi_\omega \varphi_{\alpha,n}\|_{L^2(q_T)} \quad (2.4)$$

which gives

$$\|\chi_\omega \varphi_{\alpha,n}\|_{L^2(q_T)} \leq \sqrt{C_0} \|u^0\|. \quad (2.5)$$

Using the reflexivity of  $L^2(q_T)$ , since the sequence  $(\chi_\omega \varphi_{\alpha,n})_{n \geq 2}$  is uniformly bounded in  $L^2(q_T)$ , we get that this sequence converges in  $L^2(q_T)$ .

Moreover, (2.2) combined to (1.6) and (2.5) leads to:

$$\|\varphi_{\alpha,n}^0\|_k^n \leq \frac{C_0 \|u^0\|^2}{2} e^{\alpha n} \quad (2.6)$$

and, therefore,

$$\|\varphi_{\alpha,n}^0\|_k \leq \left( \frac{C_0 \|u^0\|^2}{2} \right)^{\frac{1}{n}} e^\alpha \leq \max\{C_0 \|u^0\|^2, 1\} e^\alpha \quad (n \geq 2). \quad (2.7)$$

Then the sequence  $(\varphi_{\alpha,n}^0)_{n \geq 2}$  is uniformly bounded with respect to  $n$  in  $H_0^k(\Omega)$ . Hence, using Eberlein-Smülyan's theorem and the reflexivity of  $H_0^k(\Omega)$  there exists  $\varphi_\alpha^0 \in H_0^k(\Omega)$  such that (2.1) holds.

Now, from (2.5), we can use once again Eberlein-Smülyan's theorem to get that  $\varphi_{\alpha,n} \rightharpoonup v$  in  $L^2(q_T)$ . Since  $\varphi_{\alpha,n}$  is a solution of (1.5) with initial data  $\varphi_{\alpha,n}^0$ , we get, after passing to the weak limit, that for every test function  $\phi$ :

$$-\int_\omega \chi_\omega v(t) \phi(t) dx + \int_\omega \chi_\omega \varphi_\alpha^0 \phi(0) dx + \iint_{Q_T} a \nabla v \cdot \nabla \phi dx dt = 0.$$

Hence,  $v$  is the weak solution of (1.5) associated to the initial data  $\varphi_\alpha^0 \in H_0^k(\Omega)$ . We deduce that  $v = \varphi_\alpha$ . Then, it follows from Rellich-Kondrachov's theorem and [23, Theorem 4.1.] that  $\varphi_\alpha$  belongs to  $C([0, T], H_0^{k-1}(\Omega))$ . Then, from [24, Theorem 6.2.], we get that  $\varphi_\alpha \in L^2((0, T), H_0^k(\Omega)) \cap H^{\frac{k}{2}}([0, T], L^2(\Omega))$ . Finally, [4, Théorème X.10.] implies that  $\varphi_\alpha$  belongs to  $C^\infty([\mu, T] \times \bar{\Omega})$  for every  $\mu > 0$ .  $\square$

*Proof of Theorem 1.1.* The proof of the theorem will be divided in two steps. First, we are going to show the continuity of minimizers  $\varphi_{\alpha,n}^0$  of functionals  $J_{\alpha,n}$  with respect to the diffusion coefficient  $a$ . Then, we will prove that these minimizers are providing approximate null controls for equation (1.3).

First step: *Lipschitz continuity of the minimizers*  $\varphi_{\alpha,n}^0$

First, we define  $\varphi_{\alpha,n}$  as being the solution of (1.5) with the final data  $\varphi_{\alpha,n}^0$  and the solution  $\tilde{\varphi}_{\alpha,n}$  of (1.5) for the coefficient of diffusion  $(a+h)$ :

$$\begin{cases} \partial_t \tilde{\varphi}_{\alpha,n} + \operatorname{div}((a+h)\nabla \tilde{\varphi}_{\alpha,n}) = 0 & \text{in } Q_T \\ \tilde{\varphi}_{\alpha,n} = 0 & \text{on } \Sigma_T \\ \tilde{\varphi}_{\alpha,n}(T) = \tilde{\varphi}_{\alpha,n}^0 & \text{in } \Omega, \end{cases} \quad (2.8)$$

where  $\varphi_{\alpha,n}^0$  is the minimizer of  $J_{\alpha,n}$  and  $\tilde{\varphi}_{\alpha,n}^0$  is the minimizer of a similar functional  $\tilde{J}_{\alpha,n}$  but with  $\tilde{\varphi}_{\alpha,n}$  solving (2.8). One can check that:

$$\begin{aligned} J'_{\alpha,n}(\varphi_{\alpha,n}^0)\psi^0 - \tilde{J}'_{\alpha,n}(\tilde{\varphi}_{\alpha,n}^0)\psi^0 &= \iint_{qT} \chi_\omega \left( \varphi_{\alpha,n}\psi - \tilde{\varphi}_{\alpha,n}\tilde{\psi} \right) dx dt \\ &\quad + ne^{-\alpha n} \left( \|\varphi_{\alpha,n}^0\|_k^{n-2} (\varphi_{\alpha,n}^0, \psi^0)_k - \|\tilde{\varphi}_{\alpha,n}^0\|_k^{n-2} (\tilde{\varphi}_{\alpha,n}^0, \psi^0)_k \right) \\ &\quad - (\psi(0) - \tilde{\psi}(0), u^0) \\ &= \iint_{qT} \chi_\omega \left( (\varphi_{\alpha,n} - \tilde{\varphi}_{\alpha,n})\psi + \tilde{\varphi}_{\alpha,n}(\psi - \tilde{\psi}) \right) dx dt \\ &\quad + ne^{-\alpha n} \left( \|\varphi_{\alpha,n}^0\|_k^{n-2} (\varphi_{\alpha,n}^0, \psi^0)_k - \|\tilde{\varphi}_{\alpha,n}^0\|_k^{n-2} (\tilde{\varphi}_{\alpha,n}^0, \psi^0)_k \right) \\ &\quad - (\psi(0) - \tilde{\psi}(0), u^0) \\ &= \iint_{qT} \chi_\omega |\varphi_{\alpha,n} - \tilde{\varphi}_{\alpha,n}|^2 dx dt \\ &\quad + \iint_{qT} \chi_\omega (\varphi_{\alpha,n} - \tilde{\varphi}_{\alpha,n})(\psi - \varphi_{\alpha,n} + \tilde{\varphi}_{\alpha,n}) dx dt \\ &\quad + \iint_{qT} \chi_\omega \tilde{\varphi}_{\alpha,n}(\psi - \tilde{\psi}) dx dt \\ &\quad + ne^{-\alpha n} \left( \|\varphi_{\alpha,n}^0\|_k^{n-2} (\varphi_{\alpha,n}^0, \psi^0)_k - \|\tilde{\varphi}_{\alpha,n}^0\|_k^{n-2} (\tilde{\varphi}_{\alpha,n}^0, \psi^0)_k \right) \\ &\quad - (\psi(0) - \tilde{\psi}(0), u^0). \end{aligned} \quad (2.9)$$

We recall that  $\psi$  is the solution of (1.5) with final data  $\psi^0$  and  $\tilde{\psi}$  is the solution of (2.8) with final data  $\tilde{\psi}^0$ . Then, since  $\varphi_{\alpha,n}^0$  and  $\tilde{\varphi}_{\alpha,n}^0$  are respectively the minima of  $J_{\alpha,n}$  and  $\tilde{J}_{\alpha,n}$ , the left-hand side of the above equality (2.9) is null. Also, (2.9) leads to, Hölder's and parametrized Young's inequalities, for  $0 < s < 1$

$$\begin{aligned} (1-s) \iint_{qT} \chi_\omega |\varphi_{\alpha,n} - \tilde{\varphi}_{\alpha,n}|^2 dx dt &\leq \frac{1}{4s} \|\psi - \varphi_{\alpha,n} + \tilde{\varphi}_{\alpha,n}\|_{L^2(qT)}^2 + \|\tilde{\varphi}_{\alpha,n}\|_{L^2(qT)} \|\psi - \tilde{\psi}\|_{L^2(qT)} \\ &\quad + ne^{-\alpha n} \left( \|\varphi_{\alpha,n}^0\|_k^{n-2} (\varphi_{\alpha,n}^0, \psi^0)_k - \|\tilde{\varphi}_{\alpha,n}^0\|_k^{n-2} (\tilde{\varphi}_{\alpha,n}^0, \psi^0)_k \right) \\ &\quad + \|\psi(0) - \tilde{\psi}(0)\|_{L^2(Q_T)} \|u^0\|. \end{aligned} \quad (2.10)$$

From now on, we set  $\psi^0 := \varphi_{\alpha,n}^0 - \tilde{\varphi}_{\alpha,n}^0$ . We recall that  $\psi$ ,  $\varphi_{\alpha,n}$ , and  $\tilde{\varphi}_{\alpha,n}$  are all solutions to (1.5) corresponding to final data  $\psi^0$ ,  $\varphi_{\alpha,n}^0$  and  $\tilde{\varphi}_{\alpha,n}^0$  respectively. We point out that, in view of Proposition 2.1 and some standard energy estimates, all the terms in the right-hand side of (2.10) go to 0 or are bounded (up to a multiplicative constant) by  $\|h\|_{L^2(Q_T)}^2$  as  $n \rightarrow +\infty$ , except eventually  $\|\psi - \varphi_{\alpha,n} + \tilde{\varphi}_{\alpha,n}\|_{L^2(qT)}^2$ . Denoting  $W := \psi - \varphi_{\alpha,n} + \tilde{\varphi}_{\alpha,n}$ , we get that  $W$  solves:

$$\begin{cases} \partial_t W + \operatorname{div}(a\nabla W) = -\operatorname{div}(h\nabla \tilde{\varphi}_{\alpha,n}) & \text{in } Q_T \\ W = 0 & \text{on } \Sigma_T \\ W(T) = 0 & \text{in } \Omega. \end{cases} \quad (2.11)$$



A classical energy estimate over (2.11) leads, if  $s > 0$  is small enough, to:

$$\frac{1}{2}\|W\|_{L^\infty((0,T),L^2(\Omega))}^2 + (\rho_\star - s)\|W\|_{L^2((0,T),H_0^1(\Omega))}^2 \leq \frac{1}{s}\|\nabla\tilde{\varphi}_{\alpha,n}\|_{L^\infty(Q_T)}^2\|h\|_{L^2(Q_T)}^2, \quad (2.12)$$

where  $\rho_\star$  is as in (1.4). As already mentioned, the same kind of estimate holds for  $\|\psi - \tilde{\psi}\|_{L^2(Q_T)}$ . Also, we get from a classical energy estimate:

$$\|\psi(0) - \tilde{\psi}(0)\|_{L^2(Q_T)} \leq 2\|\tilde{\psi}\|_{W_0^{1,\infty}(\Omega)}\|\psi - \tilde{\psi}\|_{L^2((0,T),H_0^1(\Omega))}\|h\|_{L^2(Q_T)}. \quad (2.13)$$

Since  $\tilde{\varphi}_{\alpha,n}$  belongs to  $C([0,T],H_0^{k-1}(\Omega))$ , we can combine now the Sobolev embedding  $H_0^{k-1}(\Omega) \hookrightarrow W_0^{1,\infty}(\Omega)$  (since  $k > \frac{N+4}{2}$ , see [14, Theorem 6.(ii), section 5.6.3.]), and using [20, Theorem 5.7.], we get the estimate:

$$\|\nabla\tilde{\varphi}_{\alpha,n}\|_{L^\infty(Q_T)} \leq C\|\tilde{\varphi}_{\alpha,n}\|_{H^k(Q_T)} \leq C\|\tilde{\varphi}_{\alpha,n}^0\|_k.$$

From the above estimate and (2.12) it follows:

$$\frac{1}{2}\|W\|_{L^\infty((0,T),L^2(\Omega))}^2 + (\rho_\star - s)\|W\|_{L^2((0,T),H_0^1(\Omega))}^2 \leq \frac{C}{s}\|\tilde{\varphi}_{\alpha,n}^0\|_k^2\|h\|_{L^2(Q_T)}^2. \quad (2.14)$$

Combining now (2.14) together with (2.10) and (2.13) leads to:

$$(1-s)\iint_{q_T}\chi_\omega|\varphi_{\alpha,n} - \tilde{\varphi}_{\alpha,n}|^2 dx dt \leq \frac{C}{4s^2(\rho_\star - s)}\|\tilde{\varphi}_{\alpha,n}^0\|_k^2\|h\|_{L^2(Q_T)}^2 + ne^{-\alpha n}(\|\varphi_{\alpha,n}^0\|_k^{n-2}(\varphi_{\alpha,n}^0, \psi^0)_k - \|\tilde{\varphi}_{\alpha,n}^0\|_k^{n-2}(\tilde{\varphi}_{\alpha,n}^0, \psi^0)_k). \quad (2.15)$$

From (2.15), we get:

$$\|\chi_\omega(\varphi_{\alpha,n} - \tilde{\varphi}_{\alpha,n})\|_{L^2(q_T)}^2 \leq \frac{C}{4s^2(\rho_\star - s)^2}\|\tilde{\varphi}_{\alpha,n}^0\|_k^2\|h\|_{L^2(Q_T)}^2 + \frac{ne^{-\alpha n}}{\rho_\star - s}(\|\varphi_{\alpha,n}^0\|_k^{n-2}(\varphi_{\alpha,n}^0, \psi^0)_k - \|\tilde{\varphi}_{\alpha,n}^0\|_k^{n-2}(\tilde{\varphi}_{\alpha,n}^0, \psi^0)_k). \quad (2.16)$$

From Proposition 2.1 and since  $\chi_\omega(\varphi_{\alpha,n} - \tilde{\varphi}_{\alpha,n}) \xrightarrow{n \rightarrow +\infty} \chi_\omega(\varphi - \tilde{\varphi})$  in  $L^2(q_T)$ , we get making  $n \rightarrow +\infty$  and using (2.16), that:

$$\|\chi_\omega(\varphi - \tilde{\varphi})\|_{L^2(q_T)} \leq C\|h\|_{L^2(Q_T)}, \quad (2.17)$$

which is the desired result. Moreover, we point out that  $\varphi$  is a weak solution of (1.3) with final data in  $H_0^k(\Omega)$ . Hence, it belongs to  $C^\infty([\varepsilon, T] \times \Omega)$  for all  $\varepsilon > 0$  (see, e.g. [4, Théorème X.10.]).

Second step: Proof of the existence of an approximate control  $\varphi$

We can follow here the usual method presented in Section 1 or in [16, Section 1.1.]. First, let us consider the solution  $\phi_n$  of (1.5) with final data  $\phi_n(T) := u_{\alpha,n}(T)$ . Then, one can set  $\psi^0 := u_{\alpha,n}(T)$  into (1.16) to get:

$$\iint_{q_T}\chi_\omega\varphi_{\alpha,n}\phi_{\alpha,n} dx dt + e^{-\alpha n}\|\varphi_{\alpha,n}^0\|_k^{n-2}(\varphi_{\alpha,n}^0, \phi_{\alpha,n}^0)_k + (\phi_{\alpha,n}(0), u^0) = 0, \quad (2.18)$$

and, testing against  $\phi_{\alpha,n}$  and  $u_{\alpha,n}$  (which is the solution of (1.3)) respectively into the weak formulations of (1.3) and (1.5), we get:

$$\begin{cases} \iint_{Q_T} \partial_t u_{\alpha,n} \phi_{\alpha,n} \, dx \, dt + \iint_{Q_T} a \nabla u_{\alpha,n} \cdot \nabla \phi_{\alpha,n} \, dx \, dt = \iint_{q_T} \chi_\omega \varphi_{\alpha,n} \phi_{\alpha,n} \, dx \, dt \\ \iint_{Q_T} \partial_t \phi_{\alpha,n} u_{\alpha,n} \, dx \, dt - \iint_{Q_T} a \nabla \phi_{\alpha,n} \cdot \nabla u_{\alpha,n} \, dx \, dt = 0. \end{cases} \quad (2.19)$$

Summing the two equations in (2.19), we get:

$$\|u_{\alpha,n}(T)\|^2 = \iint_{q_T} \chi_\omega \varphi_{\alpha,n} \phi_{\alpha,n} \, dx \, dt + (\phi_{\alpha,n}(0), u^0) \quad (2.20)$$

which, from (2.18), leads to:

$$\begin{aligned} \|u_{\alpha,n}(T)\|^2 &\leq e^{-\alpha n} \|\varphi_{\alpha,n}^0\|_k^{n-2} \left| (\varphi_{\alpha,n}^0, \phi_{\alpha,n}^0)_k \right| \\ &\leq e^{-\alpha n} \|\varphi_{\alpha,n}^0\|_k^{n-2} \|u_{\alpha,n}(T)\|_k \\ &\leq e^{-\alpha n} \|\varphi_{\alpha,n}^0\|_k^{n-2} \|u_{\alpha,n}\|_{L^\infty((0,T), H_0^k(\Omega))}. \end{aligned} \quad (2.21)$$

From (2.6) we deduce that

$$\|\varphi_{\alpha,n}^0\|_k^{n-2} \leq \left( \frac{C_0 \|u^0\|^2}{2} \right)^{\frac{n-2}{n}} e^{\alpha(n-2)} \leq \max\{C_0 \|u^0\|^2, 1\} e^{\alpha(n-2)}.$$

Therefore,

$$e^{-\alpha n} \|\varphi_{\alpha,n}^0\|_k^{n-2} \leq \max\{C_0 \|u^0\|^2, 1\} e^{-2\alpha} \quad (n \geq 2).$$

Combining (2.21) and the previous estimate, we get

$$\|u_{\alpha,n}(T)\|^2 \leq \max\{C_0 \|u^0\|^2, 1\} e^{-2\alpha} \|u_{\alpha,n}\|_{L^\infty((0,T), H_0^k(\Omega))}. \quad (2.22)$$

Using a standard estimate for parabolic equations with smooth coefficients (see e.g. [20, Theorem 5.7.]), there exists a constant  $C > 0$  independent of  $n$  such that

$$\|u_{\alpha,n}\|_{L^\infty((0,T), H_0^k(\Omega))} \leq C (\|u^0\|_k + \|\varphi_{\alpha,n}^0\|_k).$$

Putting together this estimate and (2.7) we obtain:

$$\|u_{\alpha,n}\|_{L^\infty((0,T), H_0^k(\Omega))} \leq C (\|u^0\|_k + \max\{C_0 \|u^0\|^2, 1\} e^\alpha). \quad (2.23)$$

Finally, from (2.22) and (2.23) there exists a constant  $C > 0$  independent of  $\alpha$  and  $N$  such that

$$\|u_{\alpha,n}(T)\| \leq C e^{-\alpha} \quad (n \geq 2).$$

Now, setting  $\alpha = \log(C) - \log(\varepsilon)$  and letting  $n \rightarrow +\infty$  in the weak limit, leads to, up to extracting a subsequence:

$$\|u(T)\| \leq \varepsilon$$

and the conclusion of the theorem follows.  $\square$

### 3 Controllability of the linear equation as a mixed formulation

We consider  $W$  the space of functions defined over  $Q_T$  by

$$W = \left\{ \varphi \in C^2(\overline{Q_T}), \varphi(T) \in H_0^k(\Omega), \varphi = 0 \text{ on } \Sigma_T \right\}$$

and we define  $\Phi$  as the completion of  $W$  with respect to the norm  $\|\cdot\|$  defined by

$$\|\varphi\|^2 = \iint_{q_T} \chi_\omega |\varphi|^2 dx dt + \varepsilon \|\varphi(T)\|_k^2 + \eta \|\partial_t \varphi + \operatorname{div}(a \nabla \varphi)\|_{L^2(Q_T)}^2,$$

for  $\varepsilon \geq 0$  and  $\eta > 0$ . Indeed, using the unique continuation property for the heat equation and the observability inequality (1.6), one can check that the seminorm  $\|\cdot\|$  is a norm for every  $\varepsilon \geq 0$  and  $\eta > 0$ . Moreover,  $\Phi$  is a Hilbert space with respect to the inner product

$$\langle \varphi, \bar{\varphi} \rangle = \iint_{q_T} \chi_\omega \varphi \bar{\varphi} dx dt + \varepsilon (\varphi(T), \bar{\varphi}(T))_k + \eta \iint_{Q_T} (\partial_t \varphi + \operatorname{div}(a \nabla \varphi)) (\partial_t \bar{\varphi} + \operatorname{div}(a \nabla \bar{\varphi})) dx dt.$$

We also denote  $\Phi_0 = \{\varphi \in \Phi \text{ such that } \varphi \text{ verifies the first equation in (1.5)}\}$  and we consider the following minimization problems:

$$\min_{\varphi \in \Phi_0} \mathcal{J}_n(\varphi) := \frac{1}{2} \iint_{q_T} \chi_\omega |\varphi|^2 dx dt + \frac{\varepsilon}{2} \|\varphi(T)\|_k^2 + (\varphi(0), u^0), \quad (3.1)$$

$n \geq 2$  being an integer. Remark that the minimization problem (3.1) is equivalent to the minimization of  $J_{\alpha, n}$  in the case where  $\varepsilon = 2ne^{-\alpha n}$ . We consider the following mixed formulation: find  $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$  solution to

$$\begin{cases} \mathfrak{a}(\varphi, \bar{\varphi}) + \mathfrak{b}(\bar{\varphi}, \lambda) = \mathfrak{l}(\bar{\varphi}) & (\bar{\varphi} \in \Phi) \\ \mathfrak{b}(\varphi, \bar{\lambda}) = 0 & (\bar{\lambda} \in L^2(Q_T)), \end{cases} \quad (3.2)$$

where

$$\mathfrak{a} : \Phi \times \Phi \rightarrow \mathbb{R}, \quad \mathfrak{a}(\varphi, \bar{\varphi}) = \iint_{q_T} \chi_\omega \varphi \bar{\varphi} dx dt + \varepsilon (\varphi(T), \bar{\varphi}(T))_k \quad (3.3)$$

$$\mathfrak{b} : \Phi \times L^2(Q_T) \rightarrow \mathbb{R}, \quad \mathfrak{b}(\varphi, \lambda) = - \iint_{Q_T} (\partial_t \varphi + \operatorname{div}(a \nabla \varphi)) \lambda dx dt \quad (3.4)$$

$$\mathfrak{l} : \Phi \rightarrow \mathbb{R}, \quad \mathfrak{l}(\varphi) = -(\varphi(0), u^0). \quad (3.5)$$

We also set:

$$\bar{\mathfrak{b}} : \Phi \rightarrow L^2(Q_T), \quad \bar{\mathfrak{b}}(\varphi) = \mathfrak{b}(\varphi, \cdot) \quad (3.6)$$

First of all, let us recall the following result.

**Theorem 3.1** ([3, Theorem 4.2.3. and Remark 4.2.1.]). *Let  $\mathfrak{a} : \Phi \times \Phi \rightarrow \mathbb{R}$  be a symmetric, bilinear, and continuous map over  $\Phi \times \Phi$ , coercive over  $\operatorname{Ker}(\bar{\mathfrak{b}})$ , where  $\mathfrak{b} : \Phi \times L^2(Q_T) \rightarrow \mathbb{R}$  is a bilinear and continuous map over  $\Phi \times L^2(Q_T)$  satisfying the so-called Ladyzhenskaya-Babuška-Brezzi condition (or again called, the inf-sup condition):*

$$\exists \delta > 0 \quad \text{such that} \quad \inf_{\lambda \in L^2(Q_T)} \sup_{\varphi \in \Phi} \frac{\mathfrak{b}(\varphi, \lambda)}{\|\varphi\| \|\lambda\|} \geq \delta, \quad (3.7)$$

$\mathfrak{l} : \Phi \rightarrow \mathbb{R}$  is a linear and continuous map over  $\Phi$  and  $\bar{\mathfrak{b}}$  is defined by (3.6). Then, there exists a unique solution to (3.2) being the (unique) saddle point of the Lagrangian:

$$\mathcal{L}(\varphi, \lambda) = \frac{1}{2} \mathfrak{a}(\varphi, \varphi) + \mathfrak{b}(\varphi, \lambda) - \mathfrak{l}(\varphi). \quad (3.8)$$

Thereafter, we prove the next result, following the arguments in [27, Theorem 1].

**Theorem 3.2.** *Let  $\mathfrak{a}$ ,  $\mathfrak{b}$  and  $\mathfrak{l}$  be as in (3.3)-(3.5). The mixed formulation (3.2) is well-posed and its unique solution  $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$  is the unique saddle-point of the Lagrangian  $\mathcal{L}$  given by (3.8). Moreover  $\varphi$  is the minimizer of the functional  $\mathcal{J}_2$  defined by (3.1) and the multiplier  $\lambda$  is the corresponding controlled solution of (1.3) in the sense of transposition.*

*Proof.* In order to apply Theorem 3.1, we first prove the continuity of the considered mappings. Using the definition of  $\Phi$  and its norm, one can see that:

$$|\mathfrak{a}(\varphi, \bar{\varphi})| = |\langle \varphi, \bar{\varphi} \rangle - \eta \iint_{Q_T} (\partial_t \varphi + \operatorname{div}(a \nabla \varphi)) (\partial_t \bar{\varphi} + \operatorname{div}(a \nabla \bar{\varphi})) \, dx \, dt|. \quad (3.9)$$

Then, we check that  $(\varphi, \bar{\varphi}) \mapsto \eta \iint_{Q_T} (\partial_t \varphi + \operatorname{div}(a \nabla \varphi)) (\partial_t \bar{\varphi} + \operatorname{div}(a \nabla \bar{\varphi})) \, dx \, dt$  is bilinear, symmetric and continuous over  $\Phi \times \Phi$ . More precisely, we can write, arguing by positivity:

$$\begin{aligned} \left| \eta \iint_{Q_T} (\partial_t \varphi + \operatorname{div}(a \nabla \varphi)) (\partial_t \bar{\varphi} + \operatorname{div}(a \nabla \bar{\varphi})) \, dx \, dt \right| &\leq \eta \|\partial_t \varphi + \operatorname{div}(a \nabla \varphi)\|_{L^2(Q_T)} \|\partial_t \bar{\varphi} + \operatorname{div}(a \nabla \bar{\varphi})\|_{L^2(Q_T)} \\ &\leq \frac{1}{\eta} \|\varphi\| \|\bar{\varphi}\|. \end{aligned} \quad (3.10)$$

Combining (3.9) and (3.10), we get:

$$|\mathfrak{a}(\varphi, \bar{\varphi})| \leq \left(1 + \frac{1}{\eta}\right) \|\varphi\| \|\bar{\varphi}\|$$

which proves the continuity of the positive and symmetric bilinear map  $\mathfrak{a}$ . Then,  $\mathfrak{a}$  is clearly coercive over  $\operatorname{Ker}(\mathfrak{b})$  since we get for every  $\varphi \in \operatorname{Ker}(\mathfrak{b})$ :

$$\mathfrak{a}(\varphi, \varphi) = \|\varphi\|^2.$$

It is clear that  $\mathfrak{b}$  is bilinear and continuous over  $\Phi \times L^2(Q_T)$ . Indeed,

$$|\mathfrak{b}(\varphi, \lambda)| = \left| \iint_{Q_T} (\partial_t \varphi + \operatorname{div}(a \nabla \varphi)) \lambda \, dx \, dt \right| \leq \eta^{-1} \|\varphi\| \|\lambda\|_{L^2(Q_T)}.$$

Then, the continuity of  $\mathfrak{l}$  over  $\Phi$  can be obtained as follows. First, by Cauchy-Schwarz's inequality, we get:

$$|\mathfrak{l}(\varphi)| = |(u_0, \varphi(0))| \leq \|u_0\| \|\varphi(0)\|. \quad (3.11)$$

Let us now denote  $f = \partial_t \varphi + \operatorname{div}(a \nabla \varphi) \in L^2(Q_T)$ . Then, we can write  $\varphi = \varphi_1 + \varphi_2$  where  $\varphi_1$  and  $\varphi_2$  solve the following boundary final value problems:

$$\begin{cases} \partial_t \varphi_1 + \operatorname{div}(a \nabla \varphi_1) = 0, & \text{in } Q_T \\ \varphi_1 = 0, & \text{on } \Sigma_T \\ \varphi_1(T) = \varphi(T), & \text{in } \Omega, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \varphi_2 + \operatorname{div}(a \nabla \varphi_2) = f, & \text{in } Q_T \\ \varphi_2 = 0, & \text{on } \Sigma_T \\ \varphi_2(T) = 0, & \text{in } \Omega. \end{cases}$$

Since  $\varphi \in \Phi$ , applying the observability inequality (1.6) for estimating  $\|\varphi_1(0)\|$  and the standard energy estimate for  $\|\varphi_2(0)\|$ , there exists  $C_1 > 0$  such that:

$$\begin{aligned} \|\varphi(0)\|^2 &\leq 2\|\varphi_1(0)\|^2 + 2\|\varphi_2(0)\|^2 \leq 2C_0 \iint_{q_T} \chi_\omega |\varphi_1|^2 \, dx \, dt + C_1 \|\partial_t \varphi + \operatorname{div}(a \nabla \varphi)\|_{L^2(Q_T)}^2 \\ &\leq 2C_0 \iint_{q_T} \chi_\omega |\varphi|^2 \, dx \, dt + 2C_0 \iint_{q_T} \chi_\omega |\varphi_2|^2 \, dx \, dt + C_1 \|\partial_t \varphi + \operatorname{div}(a \nabla \varphi)\|_{L^2(Q_T)}^2. \end{aligned}$$

Applying the standard energy estimate for  $\varphi_2$ , there exists a positive  $C > 0$  such that the above estimate reads:

$$\|\varphi(0)\|^2 \leq C \iint_{Q_T} \chi_\omega |\varphi|^2 dx dt + C \|\partial_t \varphi + \operatorname{div}(a \nabla \varphi)\|_{L^2(Q_T)}^2 \leq C \max(1, \eta^{-1}) \|\varphi\|^2.$$

Finally, combining (3.11) and the above inequality, we get the continuity of  $\mathbb{l}$ . For applying Theorem 3.1, it remains to check that  $\mathbb{b}$  satisfies the Ladyzhenskaya-Babuška-Brezzi condition (3.7) over  $\Phi \times L^2(Q_T)$ . In this aim, we consider for  $\lambda_0 \in L^2(Q_T)$  the solution  $\varphi_0$  to the equation:

$$\begin{cases} \partial_t \varphi_0 + \operatorname{div}(a \nabla \varphi_0) = -\lambda_0 & \text{in } Q_T \\ \varphi_0 = 0 & \text{on } \Sigma_T \\ \varphi_0(T) = 0 & \text{in } \Omega. \end{cases}$$

From a classical energy estimate for the last system, and using its associated weak formulation, we get that there exists a positive constant  $C$  depending of  $\Omega$  and  $\rho$  such that:

$$\|\varphi_0\|_{L^2(Q_T)}^2 \leq C \|\lambda_0\|_{L^2(Q_T)}^2 \quad \text{and} \quad \mathbb{b}(\varphi_0, \lambda_0) = \|\lambda_0\|_{L^2(Q_T)}^2. \quad (3.12)$$

Therefore, using (3.12), we can write:

$$\sup_{\varphi \in \Phi} \frac{\mathbb{b}(\varphi, \lambda_0)}{\|\varphi\| \|\lambda_0\|_{L^2(Q_T)}} \geq \frac{\mathbb{b}(\varphi_0, \lambda_0)}{\|\varphi_0\| \|\lambda_0\|_{L^2(Q_T)}} \geq \frac{\|\lambda_0\|_{L^2(Q_T)}^2}{(\|\varphi_0\|_{L^2(Q_T)}^2 + \eta \|\lambda_0\|_{L^2(Q_T)}^2)^{\frac{1}{2}} \|\lambda_0\|_{L^2(Q_T)}} \geq \frac{1}{\sqrt{C + \eta}}.$$

Then the existence and the uniqueness of solutions to (3.2) follows from Theorem 3.1. Let us now prove the second point of the present theorem. First, let us point out that if  $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$  solves the mixed problem (3.2), then  $\mathbb{b}(\varphi, \bar{\lambda}) = 0$  for every  $\bar{\lambda} \in L^2(Q_T)$ , that is  $\partial_t \varphi + \operatorname{div}(a \nabla \varphi) = 0$  in the transposition's sense in  $L^2(Q_T)$ . Also, the solution  $(\varphi, \lambda) \in \Phi \times L^2(Q_T)$  of (3.2) satisfies  $\varphi \in \operatorname{Ker}(\bar{\mathbb{b}})$  and  $\mathcal{L}(\varphi, \lambda) = \mathcal{J}_2(\varphi)$ . Moreover, from (3.2) we have that for every  $\bar{\varphi} \in \Phi$ :

$$\iint_{Q_T} \chi_\omega \varphi \bar{\varphi} dx dt + \varepsilon(\varphi(T), \bar{\varphi}(T))_k - \iint_{Q_T} (\partial_t \bar{\varphi} + \operatorname{div}(a \nabla \bar{\varphi})) \lambda dx dt = -(\bar{\varphi}(0), u^0) \quad (3.13)$$

Using the uniqueness of solution to (1.3), we deduce that  $\lambda$  is the unique solution of (1.3) and satisfies  $\lambda(\cdot, T) = -\varepsilon \varphi(\cdot, T)$ , and the result follows.  $\square$

**Remark 1.** From the point of view of numerical analysis, it is convenient to augment the Lagrangian in the following way, defining (for some  $r > 0$ )  $\mathcal{L}_r$  as:

$$\mathcal{L}_r(\varphi, \lambda) = \frac{1}{2} \left( \mathfrak{a}(\varphi, \varphi) + r \iint_{Q_T} |\partial_t \varphi + \operatorname{div}(a \nabla \varphi)|^2 dx dt \right) + \mathbb{b}(\varphi, \lambda) - \mathbb{l}(\varphi)$$

This new formulation makes sense because the saddle point of  $\mathcal{L}_r$  is identical to that of  $\mathcal{L}$ , taking into account that for every  $\bar{\varphi} \in \operatorname{Ker}(\bar{\mathbb{b}})$ , we have:

$$\frac{1}{2} \left( \mathfrak{a}(\bar{\varphi}, \bar{\varphi}) + r \iint_{Q_T} |\partial_t \bar{\varphi} + \operatorname{div}(a \nabla \bar{\varphi})|^2 dx dt \right) = \frac{1}{2} \mathfrak{a}(\bar{\varphi}, \bar{\varphi})$$

and that the saddle point of  $\mathcal{L}$  belongs to  $\operatorname{Ker}(\bar{\mathbb{b}})$ .

**Remark 2.** In order to approach the minimum of the non-quadratic functional  $\mathcal{J}_n$  for  $n > 2$  we propose an iterative method. More exactly, for every integer  $m > 2$  we define

$$\tilde{\mathcal{J}}_m(\varphi) = \iint_{q_T} \chi_\omega |\varphi|^2 dx dt + \frac{\varepsilon}{2} \|\varphi(T)\|_k^2 \|\varphi_{m-1}^*(T)\|_k^{n-2} + (\varphi(0), u^0), \quad (3.14)$$

where

$$\varphi_m^* = \begin{cases} \arg \min_{\varphi \in \Phi} \mathcal{J}_2(\varphi), & \text{if } m = 2 \\ \arg \min_{\varphi \in \Phi} \tilde{\mathcal{J}}_m(\varphi), & \text{if } m > 2. \end{cases} \quad (3.15)$$

Remark that, for every value of  $m$ ,  $\tilde{\mathcal{J}}_m$  is nothing else than  $\mathcal{J}_2$  with an adapted value of  $\varepsilon$ . Then, the results of Theorem 3.2 can be applied for this functional and therefore, the definition of the sequence of minimizers  $(\varphi_m^*)_{m \geq 2}$  is well-posed and its terms can be computed as solutions of the corresponding mixed formulation (3.2). Moreover, if the sequence  $(\varphi_m^*)_{m \geq 2}$  converges its limit is the minimum of the functional  $\mathcal{J}_n$ .

## 4 Controllability of the quasilinear equation

In order to extend the controllability properties of the linear equation (1.3) to the quasilinear equation (1.1), we aim to apply a fixed point theorem. In this purpose, we first consider the double penalized HUM strategy proposed in Section 2, which allow us to obtain regular approximate controls for a regularized version of equation (1.3). The existence of approximate regular controls for the linear equation (1.3) allows to define an application associating to the diffusion coefficient  $a$  the quantity  $F_\varepsilon(|\nabla y_\varepsilon|)$  from a bounded closed convex set with values in itself,  $y_\varepsilon$  being the controlled solution of the regularized version of (1.3) and  $F_\varepsilon$  being a regularization of the function  $F$  still verifying hypotheses (A1)-(A3). The objective is then to show the continuity of such applications on some weakly sequentially compact sets, in order to apply a suitable fixed point theorem. Remark that, since we aim to prove only a null approximate controllability result, we do not need to take the limit with respect to the regularization parameter  $\varepsilon$ , which would make us lose boundedness in the most interesting cases.

First, let us define a regularisation process  $R_\varepsilon$  such that for every  $\varepsilon > 0$ ,  $R_\varepsilon : L^2(Q_T) \rightarrow C^\infty(\overline{Q_T})$ , and considering a mollifier  $(\nu_\varepsilon)_\varepsilon$ , we set:

$$R_\varepsilon(g) = \nu_\varepsilon * (\chi_\varepsilon g) + \varepsilon, \quad (4.1)$$

where we denote by  $\chi_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a smooth cut-off function with  $\text{supp}(\chi_\varepsilon) = Q_T$  and by “\*” we denote the convolution product. We get that  $R_\varepsilon(g) \xrightarrow{\varepsilon \rightarrow 0} g$  in  $L^2(Q_T)$ . Moreover, let us observe that  $R_\varepsilon$  is continuous over  $L^2(Q_T)$ . Taking  $(a, h) \in K_\varepsilon \times L^2(Q_T)$  such that  $(a + h) \in K_\varepsilon$ , we can write:

$$R_\varepsilon(a + h) - R_\varepsilon(a) = \nu_\varepsilon * (\chi_\varepsilon h). \quad (4.2)$$

Hence, the result follows using Hölder’s inequality. We define a similar regularisation process over the real line that we denote  $r_\varepsilon$ , that is considering a mollifier  $(\zeta_\delta)_\delta$  over the real line,  $r_\varepsilon$  is given by:

$$r_\varepsilon(F) := \zeta_\varepsilon * (\sigma_\varepsilon F) + \varepsilon, \quad (4.3)$$

where we denote by  $\sigma_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  the smooth cutoff function satisfying:

$$\sigma_\varepsilon = \begin{cases} 1 & \text{in } [\varepsilon, \frac{1}{\varepsilon}] \\ 0 & \text{in } (-\infty, 0] \cup [\frac{1}{\varepsilon} + \varepsilon, +\infty). \end{cases} \quad (4.4)$$

We consider the following bounded convex closed subset  $K_\varepsilon$  of the set  $L_+^2(Q_T)$  composed by the functions of  $L^2(Q_T)$  which are non negative almost everywhere, such that

$$\overline{B}_{L^2(Q_T)} \left( 0, \sqrt{2(C_{L,\varepsilon} + 1) \left( \frac{1}{\varepsilon} + \frac{C_0}{2\varepsilon^2} \right) \|u^0\|^2 + \varepsilon |\Omega| T} \right) \cap L_+^2(Q_T) \subset K_\varepsilon,$$

$C_{L,\epsilon}$  being the Lipschitz constant associated to the function  $r_\epsilon(F)$  defined in (4.3).

Let us now define the function  $G_\epsilon : K_\epsilon \rightarrow K_\epsilon$  by

$$G_\epsilon(a) = r_\epsilon(F)(|\nabla v_a|), \quad (4.5)$$

where  $v_a$  is the weak solution to:

$$\begin{cases} \partial_t v_a - \operatorname{div}(R_\epsilon(a)(t, x)\nabla v_a) = \chi_\omega \varphi_a & \text{in } Q_T \\ v_a = 0 & \text{on } \Sigma_T \\ v_a(0) = u_\epsilon^0 & \text{in } \Omega, \end{cases} \quad (4.6)$$

with  $\varphi_a \in H^k(Q_T)$  being the approximate control provided by Theorem 1.1 applied to (4.6) which is nothing else than an alternative version of (1.3) with a regularised operator  $L_\epsilon : u \mapsto -\operatorname{div}(R_\epsilon(a)\nabla u)$ . Notice that in (4.6) the function  $u_\epsilon^0 \in C^\infty(\overline{\Omega})$  is chosen such that  $\|u_\epsilon^0 - u^0\|_{L^2(\Omega)} \rightarrow 0$  when  $\epsilon \rightarrow 0$ .

Let us recall the following fixed point theorem (see [19] for a proof).

**Theorem 4.1** ([19, Theorem 2.1]). *Let  $K$  be a closed convex subset of a Banach space  $Y$ . Let us consider  $G : K \rightarrow K$  such that for all sequence  $(a_n)_n \subset K$  which converges weakly toward  $a$ , then  $(G(a_n))_n$  admits a subsequence which converges strongly toward  $G(a)$ . Hence, if  $G$  is continuous and  $G(K)$  is weakly compact,  $G$  admits a fixed point.*

Before proving the Theorem 1.2, let us prove the following lemma.

**Lemma 4.1.** *For every  $\epsilon > 0$ , the function  $G_\epsilon : K_\epsilon \rightarrow K_\epsilon$  defined by (4.5) is continuous.*

*Proof.* First, let us put forward the fact that  $G_\epsilon$  is well defined. Since  $K_\epsilon$  contains  $0_{L^2(Q_T)}$ , one can write, from assumption (A1) and since  $r_\epsilon(F) \in C^\infty(\overline{Q_T})$ :

$$\begin{aligned} \|G_\epsilon(a) - G_\epsilon(0)\|_{L^2(Q_T)}^2 &= \|r_\epsilon(F)(|\nabla v_a|) - r_\epsilon(F)(|\nabla v_0|)\|_{L^2(Q_T)}^2 \\ &\leq C_{L,\epsilon} \|v_a - v_0\|_{L^2((0,T),H_0^1(\Omega))}^2. \end{aligned} \quad (4.7)$$

Now, we point out that for every  $a \in K_\epsilon$ ,  $\|v_a\|_{L^2((0,T),H_0^1(\Omega))}$  is bounded by  $\|u_\epsilon^0\| \sqrt{\frac{1}{\epsilon} + \frac{C_0}{2\epsilon^2}}$ , this bound following from the following energy estimate:

$$\frac{1}{2} \|v_a\|_{L^2(Q_T)}^2 + \frac{\epsilon}{2} \|v_a\|_{L^2((0,T),H_0^1(\Omega))}^2 \leq \frac{1}{2} \|u_\epsilon^0\|^2 + \frac{1}{4\epsilon} \|\chi_\omega \varphi_a\|_{L^2(Q_T)}^2, \quad (4.8)$$

which implies, using (1.14)

$$\|v_a\|_{L^2((0,T),H_0^1(\Omega))}^2 \leq \frac{1}{\epsilon} \|u_\epsilon^0\|^2 + \frac{1}{2\epsilon^2} \|\chi_\omega \varphi_a\|_{L^2(Q_T)}^2 \leq \left( \frac{1}{\epsilon} + \frac{C_0}{2\epsilon^2} \right) \|u_\epsilon^0\|^2. \quad (4.9)$$

Now, from (4.7), we can write:

$$\|G_\epsilon(a)\|_{L^2(Q_T)}^2 \leq C_{L,\epsilon} \left( \|v_a\|_{L^2((0,T),H_0^1(\Omega))}^2 + \|v_0\|_{L^2((0,T),H_0^1(\Omega))}^2 \right) + \|G_\epsilon(0)\|_{L^2(Q_T)}^2. \quad (4.10)$$

On the other hand, we have:

$$\|G_\epsilon(0)\|_{L^2(Q_T)}^2 \leq \|v_0\|_{L^2((0,T),H_0^1(\Omega))}^2 + \epsilon |\Omega| T. \quad (4.11)$$

Hence,  $G_\epsilon$  is well defined, combining (4.11) together with (4.10) and (4.9).

Let us now show that  $G_\epsilon$  is continuous. Since  $r_\epsilon(F)$  is globally Lipschitz, one have:

$$\|G_\epsilon(a+h) - G_\epsilon(a)\|_{L^2(Q_T)}^2 = \|r_\epsilon(F)(|\nabla v_{a+h}|) - r_\epsilon(F)(|\nabla v_a|)\|_{L^2(Q_T)}^2 \leq C_{L,\epsilon} \|\nabla v_{a+h} - \nabla v_a\|_{L^2(Q_T)}^2. \quad (4.12)$$

Then, we have that  $w_{a,h} := v_{a+h} - v_a$  solves:

$$\begin{cases} \partial_t w_{a,h} - \operatorname{div}(R_\epsilon(a+h)\nabla w_{a,h}) = \chi_\omega(\varphi_{a+h} - \varphi_a) - \operatorname{div}((R_\epsilon(a+h) - R_\epsilon(a))\nabla v_{a+h}) & \text{in } Q_T \\ w_{a,h} = 0 & \text{on } \Sigma_T \\ w_{a,h}(0) = 0 & \text{in } \Omega. \end{cases} \quad (4.13)$$

Hence, an energy estimate over (4.13) leads to, using parametrized Young's inequality and Poincaré's inequality:

$$\begin{aligned} \frac{1}{2}\|w_{a,h}\|_{L^\infty((0,T),L^2(\Omega))}^2 + \frac{\epsilon}{2}\|w_{a,h}\|_{L^2((0,T),H_0^1(\Omega))}^2 &\leq \frac{1}{\epsilon}\|\varphi_{a+h} - \varphi_a\|_{L^2(Q_T)}^2 \\ &+ \frac{1}{\epsilon}\|\nabla v_{a+h}\|_{L^\infty(Q_T)}^2 \|R_\epsilon(a+h) - R_\epsilon(a)\|_{L^2(Q_T)}^2. \end{aligned}$$

We then get, using Theorem 1.1 and (4.12) that:

$$\|G_\epsilon(a+h) - G_\epsilon(a)\|_{L^2(Q_T)}^2 \leq \frac{C(\epsilon)}{\epsilon^2} \left( \|h\|_{L^2(Q_T)}^2 + \|u_\epsilon^0\|_k^2 \|R_\epsilon(a+h) - R_\epsilon(a)\|_{L^2(Q_T)}^2 \right), \quad (4.14)$$

which proves the lemma.  $\square$

**Remark 3.** We point out that is essential here to consider solutions to (4.6), since the space of functions which are essentially positively lower and upper bounded do not give rise to regular enough solutions of (1.3) (namely, at least Hölder continuous) since we can construct discontinuous solutions with respect to the space variable of (1.3) for some diffusion coefficient in this space, given by Serrin's example (see [30]).

We are now able to prove the result.

*Proof of Theorem 1.2.* In order to apply Theorem 4.1, we first show that if  $(a_n)_n$  is a sequence which converges weakly toward  $a$ , then  $G_\epsilon(a_n)$  converges weakly, up to a subsequence, toward  $G_\epsilon(a)$ . First, let us observe that the weak convergence of  $(a_n)_n$  implies that  $R_\epsilon(a_n) \xrightarrow[n \rightarrow +\infty]{} R_\epsilon(a)$  strongly in  $K_\epsilon$ , by definition of  $R_\epsilon$ . Also, arguing by continuity, one can see that the associated controls provided by Theorem 1.1 in (4.6) leads to

$$\|\chi_\omega \varphi_{R_\epsilon(a_n)} - \chi_\omega \varphi_{R_\epsilon(a)}\|_{L^2(Q_T)} \xrightarrow[n \rightarrow +\infty]{} 0.$$

From this, an energy estimate leads to, setting  $w_n := v_a - v_{a_n}$ , where  $v_a$  and  $v_{a_n}$  are respectively solutions to (4.6) associated to the diffusion coefficient  $R_\epsilon(a)$  and  $R_\epsilon(a_n)$  and to the controls  $\varphi_{R_\epsilon(a)}$  and  $\varphi_{R_\epsilon(a_n)}$ , respectively:

$$\begin{aligned} \frac{1}{2}\|w_n\|_{L^\infty((0,T),L^2(\Omega))}^2 + \frac{\epsilon}{2}\|w_n\|_{L^2((0,T),H_0^1(\Omega))}^2 &\leq \frac{1}{\epsilon}\|\chi_\omega(\varphi_{R_\epsilon(a)} - \varphi_{R_\epsilon(a_n)})\|_{L^2(Q_T)}^2 \\ &+ \frac{C\|u^0\|_k}{\epsilon}\|R_\epsilon(a_n) - R_\epsilon(a)\|_{L^2(Q_T)}^2. \end{aligned} \quad (4.15)$$

Then, we can write, from (4.12):

$$\|G_\epsilon(a_n) - G_\epsilon(a)\|_{L^2(Q_T)}^2 = \|r_\epsilon(F)(|\nabla v_{a_n}|) - r_\epsilon(F)(|\nabla v_a|)\|_{L^2(Q_T)}^2 \leq C_{L,\epsilon} \|\nabla v_{a_n} - \nabla v_a\|_{L^2(Q_T)}^2. \quad (4.16)$$



Then, (4.15) combined with (4.16) leads to the fact that  $\|G_\epsilon(a_n) - G_\epsilon(a)\|_{L^2(Q_T)} \xrightarrow{n \rightarrow +\infty} 0$ . Applying now Lemma 4.1 to (4.6), we get from Theorem 4.1 that  $G_\epsilon$  admits a unique fixed point in  $K_\epsilon$ . Namely, we get that in the solution to the equation

$$\begin{cases} \partial_t \mathbf{v} - \operatorname{div}(R_\epsilon(r_\epsilon(F))(|\nabla \mathbf{v}|)) \nabla \mathbf{v} = \chi_\omega \varphi & \text{in } Q_T \\ \mathbf{v} = 0 & \text{on } \Sigma_T \\ \mathbf{v}(0) = u_\epsilon^0 & \text{in } \Omega. \end{cases} \quad (4.17)$$

$\varphi$  can be chosen as an approximate control of (4.17), from Theorem 1.1. For the sake of simplicity, we denote  $R_\epsilon(r_\epsilon(F)(\cdot))$  as  $F_\epsilon$ . Next, we consider the solution to (1.1), namely the solution of:

$$\begin{cases} \partial_t y - \operatorname{div}(F(|\nabla y|)) \nabla y = \chi_\omega \varphi & \text{in } Q_T \\ y = 0 & \text{on } \Sigma_T \\ y(0) = u^0 & \text{in } \Omega. \end{cases}$$

Setting  $w := y - \mathbf{v}$ , an energy estimate leads to:

$$\frac{1}{2} \|w(T)\|^2 + \iint_{Q_T} (F_\epsilon(|\nabla y|) - F_\epsilon(|\nabla \mathbf{v}|) + F_\epsilon(|\nabla \mathbf{v}|) - F(|\nabla \mathbf{v}|)) \cdot \nabla w \, dx \, dt = \frac{1}{2} \|u_\epsilon^0 - u^0\|^2. \quad (4.18)$$

Then, (4.18) leads, using the monotonicity of the operator  $A_\epsilon : v \mapsto -\operatorname{div}(F_\epsilon(|\nabla v|)) \nabla v$  (see [21, Chapitre 2, section 1.3.] and [31, section 25.3]) obtained from that of  $A$  and the linearity of the integral to:

$$\begin{aligned} \frac{1}{2} \|w(T)\|^2 &\leq \frac{1}{2} \|w(T)\|^2 + \iint_{Q_T} (F_\epsilon(|\nabla y|) - F_\epsilon(|\nabla \mathbf{v}|)) \cdot \nabla w \, dx \, dt \\ &\leq \|F_\epsilon(|\nabla \mathbf{v}|) - F(|\nabla \mathbf{v}|)\|_{L^2(Q_T)} \|w\|_1 + \frac{1}{2} \|u_\epsilon^0 - u^0\|^2. \end{aligned}$$

Then, the result follows using assumption (A2), the fact that  $\mathbf{v}$  belongs to  $W^{1,\infty}(Q_T)$ , and from the boundedness of  $\|w\|_1$  up to take  $\epsilon$  small enough.  $\square$

The case of the parabolic  $p$ -Laplacian is directly taken into account directly by Theorem 1.2, and thus we immediately get the Corollary 1.1.

Let us now consider Theorem 1.3 and Corollary 1.2. In fact, when the solution stops in finite time, it is enough to bring its energy to be almost null so that it becomes null in an arbitrarily short time. In other words, the global approximate controllability implies the global exact controllability, as soon as the stopping time is controlled by the energy of the initial data via a relation as in (1.19).

*Proof of Theorem 1.3.* Let  $T^* \in (0, T)$  and  $\varepsilon = \left(\frac{T^*}{2\mu}\right)^{\frac{1}{\gamma}}$ . Applying Theorem 1.2 for an approximate control in time  $\frac{T^*}{2}$  there exists a control  $\varphi_c \in L^2((0, \frac{T^*}{2}) \times \Omega)$  such that the solution  $y$  of (1.1) with the control given by

$$\varphi(t) = \begin{cases} \varphi_c(t) & \text{for } t \in (0, \frac{T^*}{2}) \\ 0 & \text{for } t \geq \frac{T^*}{2} \end{cases}$$

verifies

$$\left\| y\left(\frac{T^*}{2}\right) \right\| \leq \varepsilon.$$

Combining the above inequality to the estimate (1.19) we obtain that  $y(T^*) = 0$ , which is the desired result.  $\square$

As previously mentioned, Corollary 1.2 is an immediate consequence of Theorem 1.3 applied to [10, Proposition 2.1.].

## 5 Numerical simulations

The aim of this section is to propose a numerical strategy for the computation of an approximate null control for quasilinear equations (1.1). In a first step we approach an approximate control  $\varphi \in H^1(Q_T)$  for the linear equation (1.3) by a method consisting in solving several mixed formulations as explained in Section 3. Once the method for the controllability of the linear heat equation is validated, we propose a fixed point algorithm employed to numerically compute an approximate control for the quasilinear equation (1.1). This approach is illustrated by several examples in dimension one of the space. For the remaining part of this section we consider  $\Omega = (0, 1)$ ,  $\omega = (0.1, 0.5)$  and  $T = 0.5$ .

From a practical point of view, the proposed strategy needs to efficiently compute the solutions of mixed formulations of the form (3.2). In order to numerically approach the solutions of such mixed formulations, we consider structured triangulations  $\mathcal{T}_h$  of the domain  $Q_T$  with  $h > 0$  being the diameter of triangles forming  $\mathcal{T}_h$ . Then we define the finite dimensional sub-spaces  $\Phi_h \subset \Phi$  and  $\Lambda_h \subset L^2(Q_T)$  as follows:

$$\Phi_h = \{ \phi_h \in C^1(\overline{Q_T}) : \phi_h|_K \in \mathbb{P}(K) \forall K \in \mathcal{T}_h, \phi_h = 0 \text{ on } \Sigma_T \}, \quad (5.1)$$

where  $\mathbb{P}(T)$  denotes the reduced Hsieh-Clough-Tocher (HCTr for short)  $C^1$  finite element space, and

$$\Lambda_h = \{ \lambda_h \in C(\overline{Q_T}) : \lambda_h|_K \in \mathbb{P}_1(K) \forall K \in \mathcal{T}_h \}, \quad (5.2)$$

with  $\mathbb{P}_1(T)$  being the space of affine functions with respect to both  $x$  and  $t$ . We then approach the mixed formulation (3.2) by its following discrete version: find  $(\varphi_h, \lambda_h) \in \Phi_h \times \Lambda_h$  solution to

$$\begin{cases} \mathfrak{a}(\varphi_h, \overline{\varphi}_h) + \mathfrak{b}(\overline{\varphi}_h, \lambda_h) = \mathfrak{l}(\overline{\varphi}_h) & (\overline{\varphi}_h \in \Phi_h) \\ \mathfrak{b}(\varphi_h, \overline{\lambda}_h) = 0 & (\overline{\lambda}_h \in \Lambda_h). \end{cases} \quad (5.3)$$

Remark that for every  $h > 0$  the mixed-formulation (5.3) is well posed. Nevertheless, in order to have a convergence of the solutions  $(\varphi_h, \lambda_h)$  to the solution  $(\varphi, \lambda)$  a inf-sup condition similar to (3.7) should be verified for the discrete mixed-formulation (5.3) with a inf-sup constant uniform with respect to  $h$ . Proving such a uniform inf-sup condition is generally a difficult question. An alternative avoiding the necessity of this condition is to stabilize the mixed formulation (5.3) by an appropriate term.

We denote by  $N_x$  the number of right triangles in the triangulation  $\mathcal{T}_h$  having one side on the boundary  $\Omega \times \{0\}$  and by  $N_y$  the number of right triangles having one side on the boundary  $\{0\} \times (0, T)$ . We take  $N_y$  such that the vertical side  $h_y$  of every triangle in  $\mathcal{T}_h$  is much smaller than  $h_x$  where  $h_x$  is the length of the horizontal side of the triangle. Then  $h_x = 1/N_x$  and  $N_y = N_x \gamma^{-1} T$  with  $\gamma \in (0, 1]$  being such that  $N_y$  is an integer. Two such triangulations are represented in Figure 1.

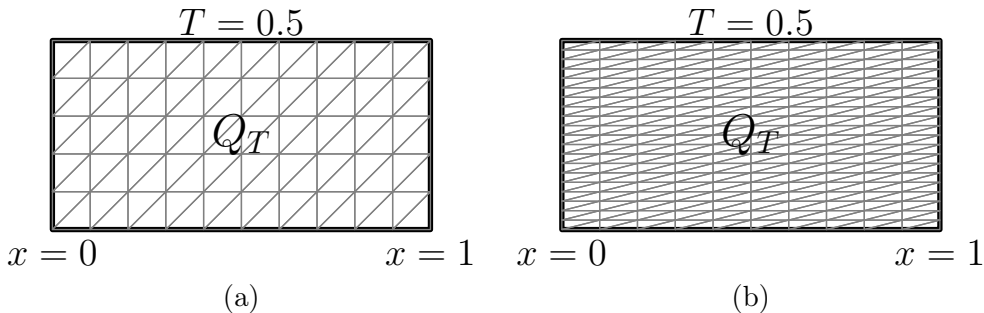


Figure 1: Two structured triangulations of  $Q_T$  with  $N_x = 10$ . (a)  $\gamma = 1$ . (b)  $\gamma = 0.25$ .

Since the controls of minimal  $L^2$  norm for the heat equation obtained by HUM are highly oscillating in time near the control time  $T$ , for all the simulations discussed in this work we consider meshes that are finer in time than in space. More exactly, we take  $N_y = 320$  and  $N_x \in \{20, 40, 80, 160\}$ .

## 5.1 Approximation of $H^1(q_T)$ controls for linear parabolic equations

In this section we consider a non-homogeneous diffusion coefficient given by

$$a(t, x) = \frac{1}{10} (1 + x^2 + t). \quad (5.4)$$

In order to compute an approximate control in  $H^1(q_T)$  for the equation (1.3) we numerically approach the minima of the functional  $\mathcal{J}_2$  given by (3.1) by solving the mixed formulation (5.3). We adapt the value of the penalization parameter  $\varepsilon$  by a fixed point algorithm such that we are able to approach the minima of the functional  $\mathcal{J}_n$  for every integer  $n \geq 2$ .

In what follows, we consider two examples of regular initial data to control.

### 5.1.1 Example 1: linear equation with $u^0(x) = \sin(\pi x)$

As a first example we consider the approximate control of the linear equation (1.3) with initial data given by

$$u^0(x) = \sin(\pi x). \quad (5.5)$$

In Table 1 we gather the  $L^2$  norm of the approximate control  $\chi_\omega \varphi$  obtained for different meshes and three different values of  $n$ . We observe that the norm of the control converges with respect to the size of the mesh for each value of  $n \in \{2, 4, 8\}$ . The control is robust with respect to  $n$ : when the mesh is fine enough we observe that the  $L^2$  norm of the control slightly decrease with from  $n = 4$  to  $n = 8$ . Similar results are obtained for the  $H^1$  norm of the control and are gathered in Table 2. The control  $\chi_\omega \varphi$  and its associated controlled solution  $\lambda$  computed for  $N_x = 160$  and  $n = 8$  are displayed in Figure 2.

	$n = 2$	$n = 4$	$n = 8$
$N_x = 20$	2.068	2.068	2.068
$N_x = 40$	2.265	2.265	2.264
$N_x = 80$	2.358	2.358	2.338
$N_x = 160$	2.387	2.387	2.351

Table 1: Example 1:  $L^2(q_T)$  norm of the control of the linear equation (1.3) with a diffusion coefficient  $a$  given by (5.4) and initial data (5.5) as a function of  $n$  and  $N_x$ .

	$n = 2$	$n = 4$	$n = 8$
$N_x = 20$	40.380	40.380	40.380
$N_x = 40$	47.697	47.697	47.604
$N_x = 80$	59.623	59.623	57.233
$N_x = 160$	63.714	63.714	59.248

Table 2: Example 1:  $H^1(q_T)$  norm of the control of the linear equation (1.3) with a diffusion coefficient  $a$  given by (5.4) and initial data (5.5) as a function of  $n$  and  $N_x$ .

### 5.1.2 Example 2: linear equation with $u^0(x) = \chi_{(0.6, 0.9)}(x)$

As a second example we consider a localized but still regular initial data to control:

$$u^0(x) = \chi_{(0.6, 0.9)}(x) = \begin{cases} 1, & \text{if } x \in [0.6 + \delta, 0.9 - \delta] \\ 0, & \text{if } x \in (0, 1) \setminus (0.6, 0.9) \\ e^{\alpha \left( \frac{1}{\delta^2} - \frac{1}{(x-0.6)(0.6+2\delta-x)} \right)}, & \text{if } x \in (0.6, 0.6 + \delta) \\ e^{\alpha \left( \frac{1}{\delta^2} - \frac{1}{(x-0.9+2\delta)(0.9-x)} \right)}, & \text{if } x \in (0.9 - \delta, 0.9), \end{cases} \quad (5.6)$$

with  $\delta = 0.1$  and  $\alpha = 0.02$ .

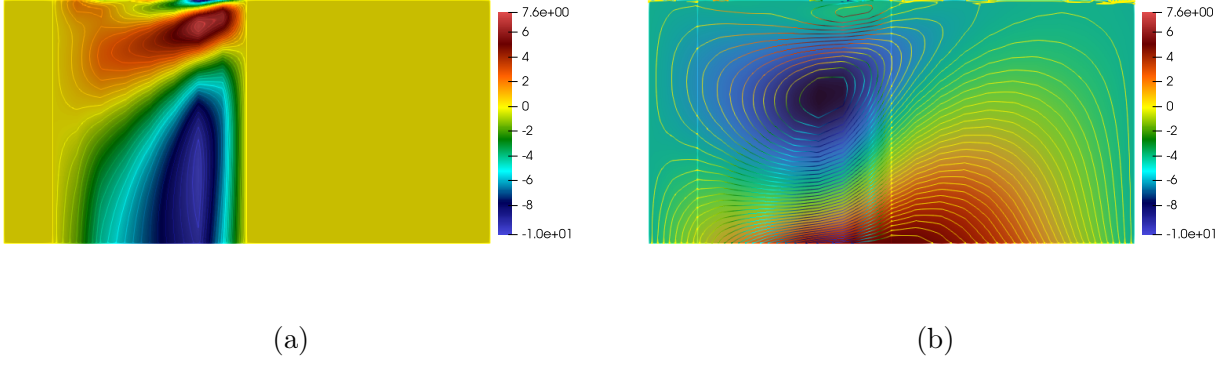


Figure 2: Example 1: (a) Control  $\chi_\omega \varphi$  for the linear equation (1.3) with  $a$  given by (5.4) and initial data (5.5) computed for  $N_x = 160$  and  $n = 8$ . (b) The corresponding controlled solution  $\lambda$ .

We obtain results similar to the ones in the Example 1 described in Section 5.1.1. The  $L^2$  and  $H^1$  norm of the obtained control are listed in Table 3 and Table 4 respectively. We also depict the control and corresponding controlled solution computed on the mesh with  $N_x = 160$  and  $n = 8$  in Figure 3.

	$n = 2$	$n = 4$	$n = 8$
$N_x = 20$	0.672	0.672	0.672
$N_x = 40$	0.746	0.746	0.746
$N_x = 80$	0.787	0.787	0.787
$N_x = 160$	0.800	0.800	0.800

Table 3: Example 2:  $L^2(q_T)$  norm of the control of the linear equation (1.3) with a diffusion coefficient  $a$  given by (5.4) and initial data (5.6) as a function of  $n$  and  $N_x$ .

	$n = 2$	$n = 4$	$n = 8$
$N_x = 20$	14.847	14.847	14.847
$N_x = 40$	17.708	17.708	17.708
$N_x = 80$	22.888	22.888	22.882
$N_x = 160$	24.625	24.625	24.608

Table 4: Example 2:  $H^1(q_T)$  norm of the control of the linear equation (1.3) with a diffusion coefficient  $a$  given by (5.4) and initial data (5.6) as a function of  $n$  and  $N_x$ .

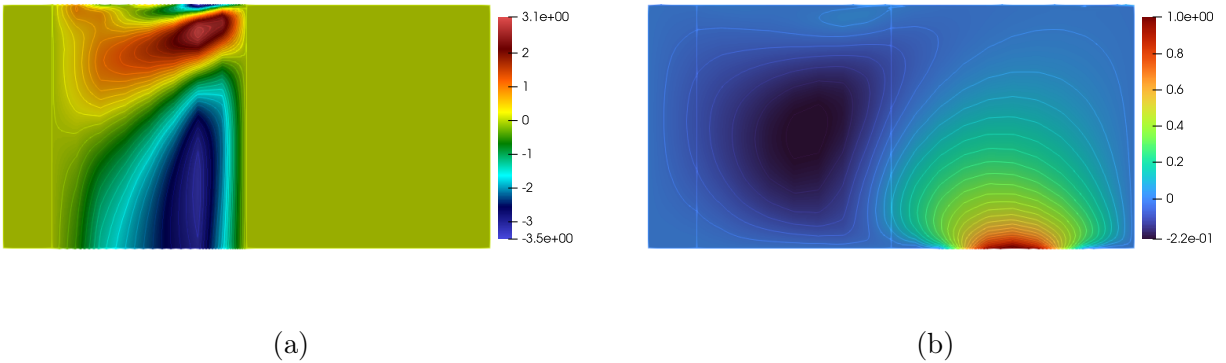


Figure 3: Example 2: (a) Control  $\chi_\omega \varphi$  for the linear equation (1.3) with  $a$  given by (5.4) and initial data (5.6) computed for  $N_x = 160$  and  $n = 8$ . (b) The corresponding controlled solution  $\lambda$ .

## 5.2 Approximation of $H^1(q_T)$ controls for quasilinear equations

For the remaining part of this section we consider the following non-linearity:

$$F(X) = \frac{1}{10} \left( 1 + (1 + X^2)^{-\frac{1}{2}} \right). \quad (5.7)$$

Remark that this nonlinear function  $F$  verifies the hypotheses (A1)-(A3).

In order to numerically approach the control and the corresponding controlled solution we employ a fixed-point algorithm combined to the strategy proposed in Section 5.1 for the approximation of  $H^1(q_T)$  controls for linear parabolic equations. More exactly, the following algorithm is employed for the computation of an approximate null control for the quasilinear equation (1.1):

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**Algorithm 1** Fixed point algorithm for the approximation of the control and the controlled solution for the quasilinear problem

---

**Require:**  $F, u^0, T, \varepsilon, i_{\max}$   $\triangleright i_{\max}$  is the maximal number of iterations

$a \leftarrow 1$

$i \leftarrow 0$

$\text{err} \leftarrow +\infty$

Compute the control and controlled solution  $(\varphi_0, \lambda_0)$  for the linear problem.

**while**  $i \leq i_{\max}$  and  $\text{err} > \text{tol}$  **do**  $\triangleright$  The tolerance  $\text{tol}$  is taken equal to  $h$

$a \leftarrow F(|\nabla \lambda_i|)$

$i \leftarrow i + 1$

Compute the control and controlled solution  $(\varphi_i, \lambda_i)$  for the linear problem.

$\text{err} \leftarrow \|\chi_\omega(\varphi_i - \varphi_{i-1})\|_{L^2(q_T)}$

**end while**

**if**  $\text{err} \leq \text{tol}$  **then**

The algorithm converged.

The control and solution of the quasilinear problem are  $(\varphi, \lambda) \leftarrow (\varphi_i, \lambda_i)$ .

**end if**

---

In what follows we consider the same initial data as in Section 5.1 for the control of the quasilinear equation (1.1) corresponding to this choice of  $F$ . Since, in the previous section, we observe that the control is very robust with respect to the value of  $n$ , for the remaining part of this section we only consider the case  $n = 2$ . We then consider different levels of meshes and several values of the penalization parameter  $\varepsilon$ . For each mesh of the domain  $Q_T$  and every value of  $\varepsilon$  we compute the  $L^2$  norm of the control provided by Algorithm 1 and we report the number of iterations needed for its convergence.

### 5.2.1 Example 3: quasilinear equation with $u^0(x) = \sin(\pi x)$

In this section we consider again the control of initial data (5.5) in the case of the quasilinear equation (1.1) with  $F$  given by (5.7). The first question we would want to investigate is related to the convergence of Algorithm 1. In this purpose we list in Table 5 the number of iterations needed for the convergence of the fixed point algorithm for four levels of meshes and for four different values of the penalization parameter  $\varepsilon$ . We observe that, for every  $\varepsilon \in \{10^{-6}, 10^{-8}, 10^{-10}, 10^{-12}\}$  the number of iterations needed for the convergence slightly increases with  $N_x$ . This is due to the fact that tolerance parameter in the algorithm is smaller for larger values of  $N_x$ . The second observation is that the fixed point algorithm does not converge for small values of  $\varepsilon$  and fine enough meshes.

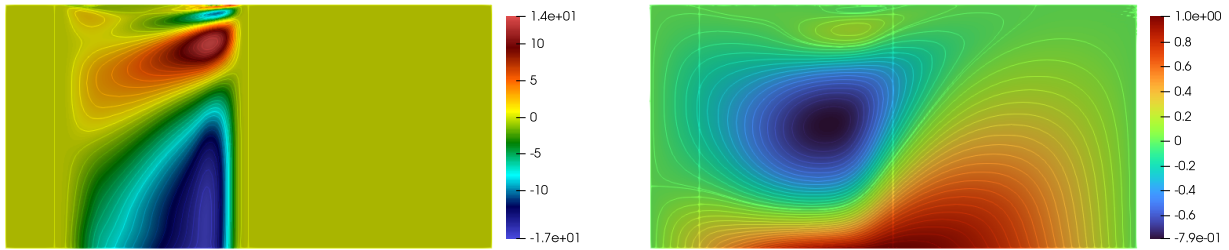
In Table 6 we gather the norm of the approximate control computed for different of values of  $N_x$  and  $\varepsilon$ . As for the control of the linear equation we observe a convergence of the norm of the control with respect to  $h$ . The control obtained for  $N_x = 160$  and  $\varepsilon = 10^{-8}$  and its associated controlled solution are illustrated in Figure 4.

	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$	$\varepsilon = 10^{-12}$
$N_x = 20$	4	4	5	5
$N_x = 40$	5	5	5	5
$N_x = 80$	5	6	6	-
$N_x = 160$	6	6	-	-

Table 5: Example 3: The number of iterations needed for the convergence of Algorithm 1 as a function of  $\varepsilon$  and  $N_x$  for the control of quasilinear equation (1.1) with  $F$  given by (5.7) and initial data (5.5).

	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$	$\varepsilon = 10^{-12}$
$N_x = 20$	2.362	3.004	3.355	3.388
$N_x = 40$	2.300	2.929	3.315	3.400
$N_x = 80$	2.315	2.956	3.359	-
$N_x = 160$	2.319	2.963	-	-

Table 6: Example 3:  $L^2(q_T)$  norm of the control of the quasilinear equation (1.1) with  $F$  given by (5.7) and initial data (5.5) as a function of  $\varepsilon$  and  $N_x$ .



(a)

(b)

Figure 4: Example 3: (a) Control  $\chi_\omega \varphi$  of the quasilinear equation (1.1) with  $F$  given by (5.7), initial data given by (5.5) and for  $N_x = 160$ ,  $\varepsilon = 10^{-10}$ . (b) The corresponding controlled solution  $\lambda$ .

### 5.2.2 Example 4: quasilinear equation with $u^0(x) = \chi_{(0.6,0.9)}(x)$

This last example consider the numerical approximation of the approximate null control for equation (1.1) with  $F$  given by (5.7) and initial data (5.6). For this choice of initial data we conduct the same experiments as for Example 3. We obtain similar results with the difference that Algorithm 1 has a better convergence for this initial data. As reported in Table 7 the fixed point algorithm converge for  $\varepsilon = 10^{-12}$  and for all the values of the discretization parameter  $N_x$ . Nevertheless, the number of iterations augment for  $N_x = 160$  and the convergence will probably deteriorate for smaller values of  $\varepsilon$ .

	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$	$\varepsilon = 10^{-12}$
$N_x = 20$	3	4	4	4
$N_x = 40$	4	4	5	5
$N_x = 80$	4	5	5	5
$N_x = 160$	5	5	6	16

Table 7: Example 4: the number of iterations needed for the convergence of Algorithm 1 as a function of  $\varepsilon$  and  $N_x$  for the control of quasilinear equation (1.1) with  $F$  given by (5.7) and initial data (5.6).

The values of the  $L^2$  norm of the computed controls, reported in Table 8, indicate that controls converge with respect to  $N_x$  for fixed values of  $\varepsilon$ . This convergence seems faster for larger value of the penalization parameter  $\varepsilon$ . The control and the corresponding controlled solution associated to the initial data (5.6) are displayed in Figure 5.

	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$	$\varepsilon = 10^{-12}$
$N_x = 20$	0.736	1.009	1.160	1.175
$N_x = 40$	0.718	0.972	1.146	1.184
$N_x = 80$	0.721	0.982	1.159	1.211
$N_x = 160$	0.723	0.982	1.163	1.005

Table 8: Example 4:  $L^2(q_T)$  norm of the control of the quasilinear equation (1.1) with  $F$  given by (5.7) and initial data (5.6) as a function of  $\varepsilon$  and  $N_x$ .

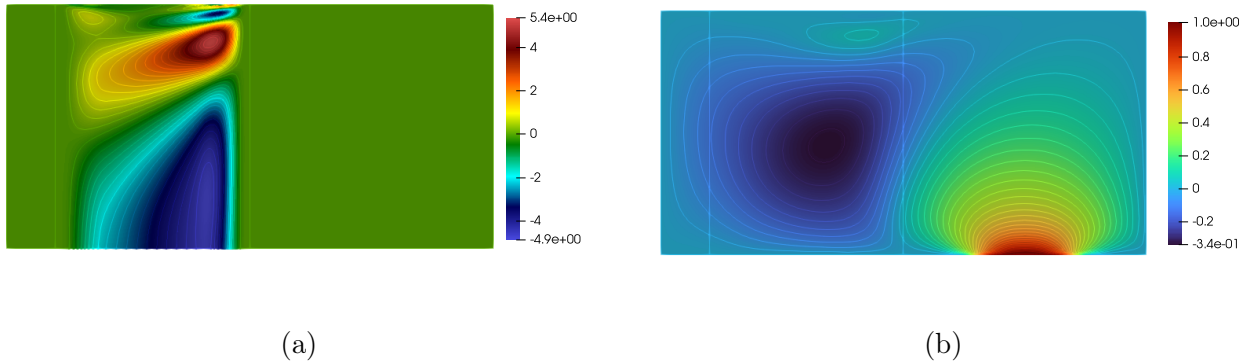


Figure 5: Example 4. (a) Control  $\chi_\omega \varphi$  of the quasilinear equation (1.1) with  $F$  given by (5.7), initial data given by (5.6) and for  $N_x = 160$ ,  $\varepsilon = 10^{-8}$ . (b) The corresponding controlled solution  $\lambda$ .

## 6 Conclusion and research perspectives

In this paper, we proved the approximate null controllability in arbitrarily small time  $T$  of quasilinear equations of the parabolic  $p$ -Laplacian type, when  $\frac{3}{2} < p$ , and the global exact one in the particular case when  $\frac{3}{2} < p < 2$ . Numerical simulations illustrate the proposed control strategy.

A first open question is that, under the hypothesis of showing the Lipschitz continuity of the control associated to the linear problem in the  $L^q$  frame for some  $q > 2$ , it is possible to extend our controllability result for the  $L^q$  controllability of the parabolic  $p$ -Laplacian. We could then obtain the exact controllability of the parabolic  $p$ -Laplacian for some  $p_* \leq p < 2$ , still applying [10, Proposition 2.1.].

Another interesting question is that the results presented in this paper could be extended to the controllability of non-Newtonian fluid flows, e.g. of power law or Carreau-Yasuda type. More precisely, the issue is that for a system with solutions being divergence free in the weak  $L^2$  sense, the addition of the nonlinear quadratic term will probably cause some difficulties.

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