HAL open science

Poincaré and Arithmetic Revisited

Catherine Goldstein

To cite this version:

Catherine Goldstein. Poincaré and Arithmetic Revisited. 2023. hal-04059991

HAL Id: hal-04059991

https://hal.science/hal-04059991v1

Preprint submitted on 5 Apr 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Poincaré and Arithmetic Revisited

Catherine Goldstein

Abstract: *Henri Poincaré's forays into number theory have often been reduced to his pioneering use of automorphic forms or his contribution to the arithmetic of elliptic curves. We examine here all of his arithmetical papers, in particular the earliest, those devoted to the study of forms. From this apparently marginal standpoint, we will be able to grasp several characteristic features of Poincaré's work at large. We show in particular the coherence of Poincaré's point of view and his mastery of the disciplinary issues of his day. We also come back to his knowledge of the contemporary mathematical literature and to his links with the program of Charles Hermite, in particular for a unity of mathematics built less on the reduction to concepts than on a circulation of methods, tools and inspiration in all fields of mathematics and for a theory of numbers revealed by the domains of the continuous.*

Henri Poincaré is not especially renowned as an arithmetician. According to André Weil who devoted an article to the issue,

> Poincaré's writings that touch upon arithmetic occupy an entire volume (volume V) of his Complete Works. One could not deny that they are of unequal value. Some of them have hardly any other interest than to show us how carefully Poincaré studied all of Hermite's work and how he assimilated his methods and results.[. . .] What is particularly striking in this volume is that in it Poincaré shows very little knowledge of German-language works.[1]

Nonetheless Weil singled out two major contributions, which may attest to Poincaré's universal and long-lasting genius: Poincaré's use of non-Euclidean ge-

Catherine Goldstein
Institut de mathématiques de Jussieu-Paris-Gauche, UMR 7586, CNRS, Sorbonne Université, Université Paris Cité, Case 247 4 place Jussieu, F-75252 Paris Cedex 05, France, e-mail: `catherine.goldstein@imj-prg.fr`

[1] [Weil 1955, p. 207]: " Les écrits de Poincaré qui touchent à l'Arithmétique occupent un volume entier (tome V des *Œuvres*). On ne saurait nier qu'ils sont de valeur inégale. Certains n'ont guère d'autre intérêt que de nous faire voir combien attentivement Poincaré à ses débuts a étudié toute l'œuvre d'Hermite et comme il s'en est assimilé les méthodes et les résultats. [. . .] Ce qui frappe dans le volume de ses *Œuvres* dont il s'agit, c'est surtout qu'il s'y montre fort peu instruit des travaux en langue allemande."

ometry in the theory of ternary quadratic forms [Poincaré 1882b, Poincaré 1886b, Poincaré 1887], which Weil prefers presenting as the first example of an arithmetically-defined discontinuous group, and Poincaré's celebrated paper on points with rational coordinates on elliptic curves defined over the field of rational numbers (the origin of Weil's own 1928 thesis) [Poincaré 1901]. To these two gems, which still survive in the living memory of mathematics, Nicolas Bergeron has more recently added an article on invariants, published by Poincaré on the occasion of the centenary of Peter Gustav Lejeune Dirichlet, [Poincaré 1905a], viewed from the perspective of the modern theory of automorphic forms [Bergeron 2018].

Seen from such points of view which correspond to the disciplinary context of the second half of the twentieth century and beyond, Poincaré's results appear as isolated and scattered, of little significance in illuminating the global characteristics of his work. Here, on the contrary, I would like to restore the synchronic configuration in which these results are inscribed, that of the last third of the nineteenth century. My first and main objective is to show the coherence of Poincaré's interventions in number theory, a coherence that testifies, in a domain that was a priori marginal for him, to several characteristics of his work, in particular his great mastery of the disciplinary issues of his time and his ability to reformulate them in an approach that was specific to him. Such a mastery is rooted in a knowledge of the state of the art in mathematics and thus of previous mathematical literature in one form or another. As Poincaré's relation to his predecessors has often puzzled his commentators, two secondary objectives of the present paper are to assess more clearly Poincaré's awareness of the international literature, in particular of the work of German mathematicians, and to discuss in more detail the components of his interactions with Charles Hermite and with Hermite's mathematics.[2]

1 Poincaré's corpus on arithmetic

To delineate precisely the corpus to be constructed for our purpose is not as obvious as it may seem. As is well-known, a complete numbered bibliography was provided by Poincaré himself in the classical exercice of his *Notice sur les travaux scientifiques*, preparatory to a candidacy for the French Academy of Sciences.[3] This bibliography was extended by him in 1901, at the request of Gösta Mittag-Leffler, then reproduced and completed after his death in a special issue of *Acta mathematica*[4] and of course in Volume V of Poincaré's *Œuvres* in 1950.

[2] In this respect, I am only completing for arithmetic, and confirming, the conclusions of the fine article that Frédéric Brechenmacher devoted to Poincaré's algebraic practices [Brechenmacher 2011].

[3] See in particular [Poincaré 1886c]. Poincaré was elected in 1887, after having been ranked increasingly higher on the lists of several previous elections, as was the tradition at the time [Gray 2013, 162].

[4] [Poincaré 1921]. The number (38) of the volume appears to date it to the year 1915, but, according to Mittag-Leffler, it was finally printed only in 1921 because of the war.

The bibliography is organized by journal and not by theme. But in the structured presentation of his work that accompanies it, Poincaré gathered 16 articles under the heading "Arithmetic", to which Albert Châtelet, the editor of the Volume V of Poincaré's *Œuvres* in 1950, added several more, most of them published after 1900.[5] However, the Volume V of Poincaré's *Œuvres* is explicitly devoted to both algebra and arithmetic. In the bibliography at the beginning of the volume, Châtelet thus indiscriminately gathered together articles that Poincaré himself had put under the headings "Algèbre" or "Algèbre de l'infini" (Algebra of the infinite) in 1901, as well as later texts belonging (according to Châtelet) to one of these headings ("Arithmétique", "Algèbre", "Algèbre de l'infini"). Then, the whole volume is organized in sixteen sections, the first five being on algebraic, the last eleven on arithmetical themes—this thematic distribution being itself only loosely based on Poincaré's presentation for the earlier texts, without respecting a chronological order. It provides a list of 20 articles devoted to arithmetic (20 among the 491 items of the bibliography published in 1921).

However, the classification of this small set of texts as "arithmetic" was not shared by all mathematicians in Poincaré's era. In the *Jahrbuch über die Fortschritte der Mathematik*, 3 of these 20 articles were not reviewed at all (probably because their journal of publication was not included at the time among the titles taken into account in the *Jahrbuch*). Châtelet had included in his list the part devoted to arithmetic and algebra of a survey on the future of mathematics, presented *in absentia* at the fourth International Congress of Mathematicians, held in Rome, and reproduced several times in a variety of journals [Poincaré 1908a]; it was classified as "Philosophy" by the *Jahrbuch*. Moreover, 4 articles among the 20 were classified as algebra; for one of them, this classification is understandable, as the article is the second half of a two-part investigation, one on the algebraic, the other on the arithmetic, theory of forms, and the two parts were simply reviewed together. But, more surprisingly, and despite the word "arithmetic" in their titles, articles on the application of Fuchsian functions to arithmetic and on arithmetical invariants were also classified as algebra or function theory. As for the celebrated paper on the arithmetic of curves [Poincaré 1901], it was classified as analytic geometry![6] Another classification is that of the *Répertoire bibliographique des sciences mathématiques* (launched by a committee headed by Poincaré himself!): only 2 articles—on the distribution of prime numbers, published in 1891—were indexed in the section on number theory (section I). The others were either in the section B (which includes linear substitutions, invariants and the algebraic theory of forms) or in the section D (general theory of functions). As for the *Catalogue of Scientific Papers 1800-1900*, issued by a committee of the Royal Society

[5] There are small discrepancies in the lists. For instance, the two earliest items, two notes to the *Comptes rendus de l'Académie des sciences* in 1879, are counted as either one or two, depending on the list. Note that there is also a misprint in Châtelet's list, the article 122 should be instead 127 cf. [Poincaré 1921, p. 9] and [Poincaré 1950, p. 16].

[6] Some of these papers have been reclassified as number theory in the recently-created online database integrating the *Jahrbuch* and zbMATH Open, using a modern classification of all articles. However, the 1901 paper escaped totally the attention of the reviewers, probably because of its original classification and did not receive any modern MSC-headings (September 2022).

of London in 1908, it also classified most of the relevant part of Poincaré's papers published before 1900 not in Number Theory (rubrics 2800–2920), but under the headings of "Non-Euclidean Geometry" (6410), "Automorphic Functions" (4440) or "General Theory of Quantics" (the English terminology for algebraic forms, 2040).[7]

Such variations are indicative of the uncertain status of number theory circa 1900. They may also hint at the role played by Poincaré in the extension and restructuration of the domain during the twentieth century. How then to select the texts on which to focus for our study? Confronted by the same problem for algebra, Frédéric Brechenmacher took a unique text as his point of departure, from which he unfolded an intricate web of conceptual and disciplinary settings [Brechenmacher 2011]. Here, the chronology provides us with a clue; the distribution of our 20 potential candidates for the study of Poincaré's arithmetic (see Fig.1) clearly displays an initial concentrated period, while Poincaré's best known, later papers are rather chronologically isolated.

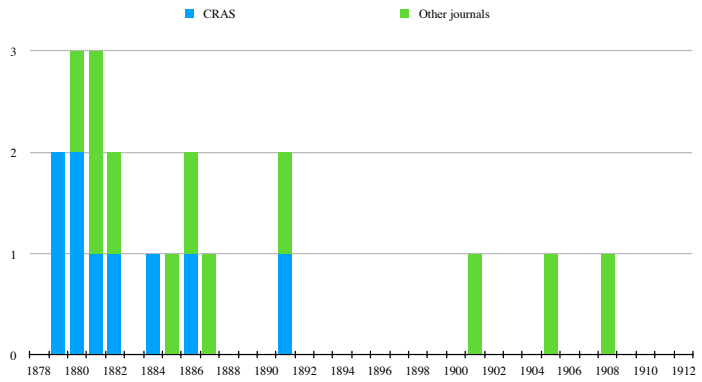

Fig. 1: The distribution of Poincaré's articles on arithmetic by year.

In his 1886 presentation, Poincaré explains that his research on arithmetic concerns exclusively the theory of forms.[8]. As explained above, my purpose is not to study Poincaré's influence on the development of number theory, but the coherence of his arithmetical work and its sources. I shall thus first examine in some detail the

[7] On the history of mathematical reviewing and the use of these classifications, see [Siegmund-Schulze 1993, Nabonnand & Rollet 2002, Goldstein 1999, Goldstein & Schappacher 2007b].

[8] [Poincaré 1886c, p. 61]: "Mes recherches arithmétiques ont exclusivement porté sur la théorie des formes.

15 articles published before 1890, then briefly, for reasons of space, discuss some of their relations to the later articles.[9]

2 The arithmetic theory of forms in the nineteenth century

Although the work of German number theorists, such as Ernst-Eduard Kummer, Richard Dedekind or David Hilbert, is better known, hundreds of papers on number theory (even according, say, to the classification of the *Jahrbuch*) have been published by French authors and in French journals in the last decades of the nineteenth century, in particular in the *Comptes rendus* of the French Academy. They can be gathered roughly in three main clusters, which can be defined and distinguished by their references, their sources of inspiration, their methods and some of their practices of publication. One of these clusters blossomed in particular thanks to the French Association for the Avancement of Science (launched in 1872), which coordinated teachers, engineers, military and amateurs with a strong interest in mathematics. The two others were mostly restricted to academia and strongly relied on analytic methods as well as complex functions and numbers; the difference between them was mostly thematic and testified to their sources, Hermite and Kronecker's works for one, Dirichlet's and Riemann's on the distribution of prime numbers for the other.[10] Poincaré figured prominently in the second cluster, besides authors like Camille Jordan, Émile Picard, Léon Charve, Georges Humbert, or, later, the editor of Volume V of Poincaré's *Œuvres* himself, Albert Châtelet. Their main topic was the arithmetical study of algebraic forms, that is, of homogeneous polynomials in n variables x_i, with coefficients a_{ij} in various sets of numbers (ordinary integers, real numbers, and sometimes algebraic integers, in particular).

Two important aspects should be emphasized. First of all, this cluster was international; we may mention for instance the work of Henry Smith in England, of Eduard Selling or Paul Bachmann in Germany or of Luigi Bianchi in Italy. Then, it largely benefited from the legacy of a research field which blossomed in mid-century and that Norbert Schappacher and myself have called "arithmetic algebraic analysis" [Goldstein & Schappacher 2007a, 24–55]; however, after 1870, several parts of this research field became proper disciplines—in the sense of an "object-oriented system of scientific activities" [Guntau & Laitko 1987]—with its own subject matter, its key concepts, its main problems and soon its textbooks and theses. This is particularly the case for the (arithmetical) theory of forms, see for example [Smith 1861–1865, Charve 1880, Bachmann 1898].

If the study of sums of squares, in particular, began much earlier, a common source for the arithmetic theory of forms in the nineteenth century was the fifth section of Carl Friedrich Gauss's *Disquisitiones Arithmeticae*, published in 1801.

[9] This part of Poincaré's work is presented in the ninth chapter of Jeremy Gray's biography of Poincaré [Gray 2013, 466-488], to which my present text can be considered as a footnote.

[10] For the constitution of these clusters and details on each of them, see [Goldstein 1994, Goldstein 1999, Goldstein & Schappacher 2007b].

In this section, Gauss studies binary quadratic forms with integer coefficients, that is expressions $f(x, y) = axx + 2bxy + cyy$, with $a, b, c \in \mathbb{Z}$ and two variables x, y (and launched the study of ternary quadratic forms, with three variables x, y, z). Two such forms f and g are said to be equivalent (or in the same class) if they are the same up to an invertible linear change of variables with integer coefficients: $g(x, y) = f(\alpha x + \beta y, \gamma x + \delta y)$, where $\alpha, \beta, \gamma, \delta$ are integers and $\alpha\delta - \beta\gamma = \pm 1$.[11] Gauss singled out the determinant $D = b^2 - ac$ of the form f as a key quantity; any two equivalent forms have the same determinant, that is, D is invariant under the linear transformations considered above. Reciprocally, for a given determinant D, Gauss proved that there are only finitely many different classes of equivalent forms. He also defined what would be the two main problems of the theory of forms for the whole of the nineteenth century:

> I. Given any two forms having the same determinant, we want to know whether or not they are equivalent [. . .] Finally we want to find all the transformations of one form into the other [. . .].
>
> II. Given a form, we want to find whether a given number can be represented by it and to determine all the representations [Gauss 1801/1966, §158, p. 113].

While the second problem, for particular forms such as sums of squares, had been in the spotlight previously, the first would take center stage during the remainder of the century. This is indeed the problem that Poincaré (who nonetheless devoted two articles to the second problem) emphasizes at the beginning of his first paper on quadratic forms:

> The main problems relating to quadratic forms can be reduced as one knows to a single one: Recognizing whether two given forms are equivalent, and by what means one can pass from one to the other.[12]

This shift in interest went hand in hand with a view of classification as a central object of research in mathematics as well as in the natural sciences.[13] "Science", Poincaré would write in 1905, "is above all a classification, a manner of bringing together facts which appearances separate, although they are bound together by some natural and hidden kinship. Science, in other words, is a system of relations." [14]

Relations between forms, from the point of view described above, manifest themselves through linear transformations. Their study, including the study of those transforming a form into itself, became an important topic in the nineteenth century, as well as the search for quantities, such as the determinant, that are invariant under such transformations and can be used as characteristics in the classification. Gauss

---

[11] Gauss calls the two forms f and g properly equivalent if $\alpha\delta - \beta\gamma = 1$. I shall not comment here on these different types of equivalences, see [Goldstein & Schappacher 2007a, 8–13].

[12] [Poincaré 1879a, 344]: Les principaux problèmes relatifs aux formes quadratiques se ramènent comme on le sait à un seul : Reconnaître si deux formes données sont équivalentes, et par quel moyen on peut passer de l'une à l'autre.

[13] On this issue, see [Knight 1981, Tort 1989, Rey 1994, Lê & Paumier 2016].

[14] [Poincaré 1905b, 172]: "[Q]u'est-ce que la science ? [. . .] c'est avant tout une classification, une façon de rapprocher des faits que les apparences séparaient, bien qu'ils fussent liés par quelque parenté naturelle et cachée. La science, en d'autres termes, est un système de relations.

also completed what would be a model for the classifications of forms in the future; to find good, "simple" (in a sense to be explained) representatives of each class of forms (the so-called reduced forms), to study the possible equivalence among the reduced forms and to explain how to transform any form into a reduced form of its class. In the case of the binary quadratic forms with integer coefficients and with a strictly negative determinant (the case of a definite form), for instance, one can choose in each class an essentially unique reduced form such that

$$-a < 2b \leq a < c \qquad \text{or} \qquad 0 \leq 2b \leq a = c. \tag{1}$$

There are a finite number of such reduced forms for a given D and every form is equivalent to a reduced form. It is notable that for a reduced form, the coefficient a is then less than $2\sqrt{\frac{-D}{3}}$ (a bound which only depends on D, but not on the particular class of the form); since $a = f(1,0)$ is obviously a value of the reduced form at integer values of (x, y), it is also a value at integers of all the forms in the same class, obtained by a linear change of variables from the reduced form. This remark gave rise to questions about the smallest non-zero value of the numbers represented by a form. For $D > 0$, there is no longer a unique reduced form in each class, but Gauss organized equivalent reduced forms into finite "periods". In this case, there are also infinitely many transformations of a form into itself, such transformations being associated to the solutions of the Pell-Fermat equation $T^2 - DU^2 = 1$.

Equivalent forms obviously represent the same integers, but the reciprocal is not true. Again, in order to solve the second problem, Gauss refined his solution to the first problem—more precisely, to his classification of forms—with new criteria, leading him to the concepts of order and genus, two new characteristics attached to a form. Last, but not least, he defined a relation called the composition of forms: a form F is said to be a compound of the forms g and h if there exists linear functions X and Y of xu, xv, yu, yv such that $F(X, Y) = g(x, y)h(u, v)$ (with extra technical conditions). This construction is of course useful for the representation of integers by forms, as a form composed of two others of the same determinant can represent the product of two integers which are represented respectively by the two forms. But its importance is more subtle; this relation among forms is not a binary operation, that is, it is not possible to define "the" compound of two forms. However, it behaves nicely with respect to equivalence and turns into such a binary operation on classes of forms — that is, Gauss could define "the" composition of two classes—a remarkable idea, as it displayed an operation on less than familiar objects, *classes* of forms, made of infinitely many algebraic expressions. As such it became a model for operation on sets of mathematical objects. In the *Disquisitiones arithmeticae*, however, composition of forms relied upon complicated and extensive algebraic computations; a conundrum for many readers of Gauss who would then try to simplify or redefine it.

Among the numerous works inspired by Gauss, some were decisive in the construction of a discipline around algebraic forms. At the end of the 1830s, Peter Gustav Lejeune-Dirichlet introduced infinite series, built from inverses of quadratic forms

with integer coefficients, in order to compute the number of classes of forms for a given determinant. Other analytic means, in particular elliptic functions, were also used by various authors to refine or generalize such computations. A decade later, Hermite began a series of articles devoted to forms; considering first quadratic forms with any number of variables and with real coefficients, he established through a close reading of Gauss's Section V bounds for the values of such forms at integers, which depended only on the determinant and the number of variables (and not on the coefficients of the form). This result, closely linked to the theory of reduction as explained above, led him to a variety of applications, from the approximation of real numbers by rationals to the properties of algebraic numbers or even of complex periodic functions [Goldstein 2007]. Hermite also introduced his method of "continuous reduction"; he associated to a given problem a family of positive definite quadratic forms, indexed by *real* parameters, and thus transferred to the initial situation the reduction procedures for this family (continuously, by changing the values of the real parameters, hence the name of the method), in particular through a study of the transformations leading to the reduction.

For instance, if $f(x, y) = ax^2 + 2bxy + cy^2$ is an indefinite quadratic form, with a positive determinant $D = b^2 - ac$, one can write $f(x, y) = (x - \alpha y)(x - \beta y)$ for two real numbers α, β. Hermite thus associated to f the family of definite quadratic forms $f_\Delta(X, Y) = (x - \alpha y)^2 + \Delta(x - \beta y)^2$, with a real positive parameter Δ. For each Δ, there exists a linear transformation such that the transformed form F_Δ of f_Δ is reduced. Applying to f the transformation(s) reducing f_Δ, for each Δ, Hermite showed that there are only a finite number of transformed forms, reproducing themselves periodically; they define the reduced forms associated to f. Remarkably, here, the focus shifted from the forms to the transformations and these transformations became the key elements in the reduction process. The mid-century witnessed a blossoming of the study of such linear transformations, and it was the nature of their coefficients that defined what the nature of the domain of research: it was considered to be arithmetic when the coefficients were integers, algebra when they were real or complex general numbers [Brechenmacher 2016]. The determinant, as explained, appeared as the simplest instance of invariants of forms—functions built from the coefficients of the forms which certain types of transformations leave unchanged (sometimes up to a well-controlled term). In turn, such invariants played a key role in the classification of forms and invariant theory was seen then as the new and fruitful direction for algebra at large [Fisher 1966, Crilly 1986, Parshall 1989, Parshall 2006, Parshall 2023].

When Poincaré entered the scene at the end of the 1870s, a whole discipline attached to the arithmetic of forms had thus been established, one of the first in number theory [Goldstein & Schappacher 2007a, p. 54].[15] Besides its subject matter, it included core concepts such as invariants and reduced objects, theorems about the two main problems (equivalence and the representation of integers by forms), a systematization based on the classification of forms, methods of proof based on the study and use of linear transformations such as that of continuous reduction. It

[15] It is important to keep in mind that so-called "algebraic number theory", that is, in fact, the theory of algebraic number fields, has yet not reached that same stage at the time, despite the publication of Dedekind's first papers, nor had analytic number theory.

constituted a separate subsection of the section on number theory in the recently founded *Jahrbuch über die Fortschritte der Mathematik*. As we will see, Poincaré's memoirs took its place on this map in a quite natural way.

3 Lattices as a framework for forms

Poincaré's first articles on forms were published at the beginning of his career as a mathematician, in 1879.[16] The two first items on his list of works are these notes at the Academy of Sciences in August and November 1879, parts of a memoir for which Hermite, Joseph Bertrand and Victor Puiseux were designated as reviewers. A version of these results was then expanded into a longer article in *Journal de l'Ecole polytechnique* [Poincaré 1880c]. This situation is standard and it is difficult to establish a strict chronology for the subtopics relative to forms that Poincaré handled between 1879 and 1889. Most often, he presented a memoir to the Academy for review, publishing one or two short notes to announce his results, sometimes withdrawing the larger memoir before any referee report and publishing a long version of his results in another journal several years later. Roughly speaking, Poincaré discussed two main situations. First of all, that of quadratic forms, mostly binary and ternary, for which the main results were well-known; for them, Poincaré introduced in particular new geometrical viewpoints, based either on lattices ([Poincaré 1879a, Poincaré 1879b, Poincaré 1880c]) or on non-Euclidean geometry ([Poincaré 1882b, Poincaré 1886b, Poincaré 1887]) and new analytical invariants ([Poincaré 1880c, Poincaré 1882a]). Then, that of cubic forms, mostly ternary and quaternary, as Hermite and others had already thoroughly explored the binary cubic case ([Poincaré 1880a, Poincaré 1880b, Poincaré 1881c, Poincaré 1882c, Poincaré 1886a]); for these forms, Poincaré aimed at classifying them, establishing in particular relations between his classification and some already known classifications of algebraic curves. Moreover, if some aspect of his research lent itself easily to generalization, for instance, to a greater number of variables, he would discuss this general situation ([Poincaré 1881b, Poincaré 1885]). For reasons of space, I will thus not strictly respect chronology, but will discuss each subtopic separately.

The 1879 manuscript and notes on forms were apparently the occasion of a renewal of scientific links with Hermite, who had been his teacher at the Ecole polytechnique a few years earlier.[17] At least until Poincaré's 1887 entry into the Academy of Sciences, which Hermite had supported for several years in a row with

[16] As as student, Poincaré also published a small contribution to the *Nouvelles annales de mathématiques*, in 1874, see [Gray 2013, p. 157].

[17] [Hermite & Mittag-Leffler, vol. 5, p. 110] : "Je crois à ce jeune homme, qui a été mon élève à l'Ecole polytechnique en 1875, un véritable génie" (I believe that this young man, who was my student at the Polytechnique in 1875, has a true genius). Hermite wrote to Poincaré on November 22, 1879 that he had not yet seen the August manuscript, but would be delighted to read at the Academy the new note prepared by Poincaré (it would take place on November 24) [Poincaré 1986, p. 164]. Let us remind our readers that Poincaré also defended his thesis on differential equations and lacunary series, preceded by a short article on this topic, in August 1879; but Hermite was

laudatory reports, and with the promotion of Poincaré among his colleagues, both maintained regular and close relations—and Poincaré would play a decisive role in the organization of Hermite's Jubilee in 1892. Hermite's judgment on Poincaré, entrusted in a letter to Gösta Mittag-Leffler in March 1882, is well-known:

> In confidence, with great fear of being overheard by Madame Hermite, I will tell you that of our three mathematical stars [Paul Appell, Emile Picard and Poincaré], Poincaré seems to me the brightest. And then, he is a charming young man, who, like me, is from Lorraine and who knows my family very well.[18]

This statement is not an isolated one. Despite some opposition, on family as well as on institutional grounds, Hermite insisted that "Poincaré is unquestionably superior to Appell and Picard in terms of both the importance of his discoveries and the number of published works."[19] This on-going support is expressed in several ways: Hermite sent Poincaré's thesis to Mittag-Leffler [Hermite & Mittag-Leffler, I, p. 118], asking him to recommend Poincaré's work to Hugo Gyldén or Karl Weierstrass [Hermite & Mittag-Leffler, I, p. 150]; he provided explanations on Poincaré's work to Georges Halphen [Poincaré 1986, p. 158]. Reciprocally, he fed Poincaré with mathematical literature, commented on his results and, as we shall see, pushed him to rewrite, develop or explore more deeply and more precisely certain topics.

In Poincaré's first papers on forms, however, Hermite is not mentioned.[20] Poincaré acted here as he often would do in the future; he first re-read or re-established in a specific framework results that were already known, at least partially. In our case, the basic results are those of Gauss's Section V, but the framework is more surprising. In Poincaré's terms:

> The link between Bravais' theory of parallelogrammatic lattices and that of quadratic forms was noticed long ago, but was restricted until now to definite forms. The main objective of this Memoir is to show that nothing is easier than applying the same geometrical representation to indefinite forms. First I had to study the properties of these parallelogrammatic lattices

---

not a member of the defence committee, composed of Ossian Bonnet, Claude Bouquet and Gaston Darboux.

[18] [Hermite & Mittag-Leffler] : "Tout bas et en confidence, ayant grande crainte d'être entendu de Madame Hermite, je vous dirai que de nos trois étoiles mathématiques [Appell, Picard et Poincaré], Poincaré me semble la plus brillante. Et puis, c'est un charmant jeune homme, qui est lorrain comme moi et qui connaît parfaitement ma famille." Also quoted in [Gray 2013, p. 161]. Appell was Hermite's nephew by marriage and Picard his son-in-law, and they were thus both supported by Hermite's family members. On the other hand, both had been students at the Ecole normale supérieure, which at that time was beginning to take over the training of scientific elites, under the leadership of Louis Pasteur, against the influence of the Polytechnique.

[19] [Hermite & Mittag-Leffler, vol. 5, p. 214]: "Poincaré est incontestablement supérieur à Appell et à Picard sous le double rapport de l'importance des découvertes et du nombre des travaux publiés".

[20] The link to Poincaré's other early works, on differential equations, with the topic proposed in 1879 for an Academy Prize has been noted by historians [Gray 2006], [Gray 2013, ch. 3]. But I have not been able to find an explicit incentive for his work on forms. The theme of the decomposition of a number as a sum of squares was proposed only in 1881. We note that a topic on crystals was also proposed by the Academy for a mathematical prize in 1879 and, given Poincaré's reference to Auguste Bravais's work on crystallography, this might have played a role in his interest in lattices and forms, but I have not found any source to substantiate this.

> and to sketch out, so to speak, their arithmetic. [. . .]. The lattices enjoy properties which recall several of the properties of numbers.[21]

The representation of positive forms by lattices had been indeed popularized much earlier among mathematicians through Gauss's review of Ludwig August Seeber's theory of reduction for ternary quadratic forms [Seeber 1824, Gauss 1831]. Seeber worked on crystallography and his interest in quadratic forms was primarily linked to the modelization of crystal properties. In his review, Gauss explains that, if on a plane one chooses two coordinate axes making an angle of cosine $\frac{b}{\sqrt{ac}}$, the value at x, y of a positive definite form, $ax^2 + 2bxy + cy^2$, with a, b, c integers, represents the square of the distance to the origin of the point with coordinates $(x\sqrt{a}, y\sqrt{c})$ with respect to these axes. For x, y integers, the form is thus associated to a discrete grid of points, situated at the intersection of two systems of lines which are parallel respectively to each of the two axes and evenly spaced ($\sqrt{a}$ for one system of lines, $\sqrt{c}$ for the other) (see Fig. 2); this double system of lines defines a lattice. The plane is thus cut into equal elementary parallelograms (such that no point of the lattice lies inside such a parallelogram); the area of each such parallelogram is $ac - b^2$ (that is, the absolute value of the determinant of the form). Different lattices can be associated to the same distribution of lattice points, for different choices of the systems of lines joining them; the forms associated with these different lattices are then equivalent. In this framework, reduction theory can also be described in geometrical terms: among the various lattices associated to the same given regular discrete distribution of points, the lattice corresponding to a reduced form in Gauss's sense is the only one for which the fundamental triangle, joining the chosen origin to the nearest points of the lattice, has acute angles; it is also the only one for which the elementary parallelograms have their sides smaller than their diagonals [Gauss 1831, Dirichlet 1850]. This geometrical representation of the theory of forms can be extended in the same way to a three-dimensional space and ternary forms $ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy$, with three variables, the triple system of lines defining a lattice with spacings of $\sqrt{a}, \sqrt{b}, \sqrt{c}$, respectively and the cosine of the angle between two of the three axes being $\frac{a'}{\sqrt{bc}}, \frac{b'}{\sqrt{ac}}, \frac{c'}{\sqrt{ab}}$ respectively.

Poincaré, however, did not refer to Seeber or to Gauss's review in his arithmetical work. Besides Gauss's *Disquisitiones arithmeticae* and, for one specific result, Eisenstein, Poincaré mentions only Auguste Bravais in [Poincaré 1880c]. Bravais, also an engineer from the Polytechnique and a professor at this school (before Poincaré's time) developed his own theory of lattices, first on the plane in a botanical context, then in a 3-dimensional setting for crystallography.[22] Bravais' viewpoint was neither

---

[21] [Poincaré 1880c, pp. 177–178]: Le lien qui existe entre la théorie des réseaux parallélogrammatiques de Bravais et celle des formes quadratiques a été remarqué depuis longtemps, mais on s'est restreint jusqu'ici aux formes définies ; le but principal de ce Mémoire est de faire voir que rien n'est plus facile que d'appliquer la même représentation géométrique aux formes indéfinies. J'ai dû d'abord étudier les propriétés de ces réseaux parallélogrammatiques et en ébaucher pour ainsi dire l'arithmétique [. . .]. Les réseaux jouissent de propriétés qui rappellent quelques-unes des propriétés des nombres.

[22] [Bravais 1850, Bravais 1851] and the posthumous collection [Bravais 1866]. Bravais calls a 3-dimensional lattice an "assemblage". For a survey of Bravais's approach in crystallography from

arithmetical, nor centered on quadratic forms; it relied on the study of symmetries and of the effect on lattices of various transformations, in particular rotations. But it constituted a common reference among Polytechnicians: Henry Résal, the editor of the *Journal de mathématiques pures et appliquées* following Joseph Liouville, referred for instance to Bravais in a footnote added to an article by Eduard Selling on the reduction of quadratic forms which Hermite strongly recommended to all his students [Selling 1877].[23] And Camille Jordan explicitly followed Bravais' study of symmetries when he analysed the groups of space motions at the end of the 1860s [Jordan 1868-1869]. Poincaré used Bravais not only as a mere designation of objects (the so-called "Bravais lattices"), but also for his proofs of several basic properties on lattices; he would also mention Bravais's work elsewhere, in particular in his lectures on the theory of light [Poincaré 1889, p. 195].

In [Poincaré 1880c], Poincaré introduces several notations for plane lattices and their points. First of all, a plane lattice can be defined by four numbers (a, b, c, d), the coordinates of the points of the lattice being given as

$$x = am + bn$$
$$y = cm + dn$$

with integers m, n.[24] Poincaré also designates this lattice (a, b, c, d) by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. He calls the quantity $ad - bc$ the norm of the lattice (this is the area of an elementary parallelogram of the lattice).

The first part of the 1880 memoir [Poincaré 1880c] is then explicitly devoted to the development of an *arithmetic of lattices*, for the case where a, b, c, d are integers. The objective is to mimic standard concepts of the arithmetic of integers, such as multiples, divisors and primes. A multiple of a lattice, for instance, is simply a sublattice—that is, all its points are contained among the points of the original lattice. Two lattices are then said to be equivalent if each of them is a multiple of the other. It is always possible, up to equivalence, to assume that $d = 0$; then, two lattices $(a, b, c, 0)$ and $(a', b', c', 0)$ are equivalent if and only if $c = c', b = b', a \equiv a' \pmod{b}$.[25]

Using Bravais's results, Poincaré asserts that the norm of a lattice is the limit of the ratio of the area of a circle to the number of lattice points inside the circle when the radius increases indefinitely. This allows him to show that the norm of a lattice

a group theoretical point of view, see [Scholz 1989]; for a general presentation of Bravais' work using lattices, see [Boucard, Goldstein & Malécot, 2023].

[23] But it is the German version of Selling's paper, published in 1874, and not its French 1877 translation, that Poincaré would mention in [Poincaré 1882b].

[24] The directions of the two systems of lines defining the lattice are then $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$.

[25] As pointed out by Châtelet in the comments of his edition of Poincaré's works, a unique representative of each class of equivalence is obtained if one requires that a, b, c are positive and $0 < a < b$.

is divisible by the norms of its divisors and that the norms of two equivalent lattices are equal. Poincaré then characterizes the smallest common multiple and the largest common divisor of two lattices. He defines a “prime" lattice as a lattice the norm of which is a prime number, and a “second" lattice as one the norm of which is the power of a prime number.[26] He concludes with the theorem that any lattice is the least common multiple of co-prime second lattices.

Poincaré’s procedure, and his results, are thus very close to those of Richard Dedekind’s theory of ideals, which had been published in French only a few years earlier [Dedekind 1876, Dedekind 1877]. For instance, Dedekind defines a multiple of an ideal in the following way: the ideal $\mathfrak{a}$ is a multiple of the ideal $\mathfrak{b}$ when “all the numbers of the ideal $\mathfrak{a}$ are contained in $\mathfrak{b}$” [Dedekind 1876, p. 287]; he then also develops an arithmetic of ideals, in order to generalize and simplify Ernst Kummer’s preceding theory of ideal numbers. However, while Poincaré, as we shall see, alluded indeed to “ideal numbers” in his memoirs, he mentioned explicitly neither Kummer nor Dedekind, reinterpreting their theories in his lattice framework.

Following Poincaré, let us then come back to quadratic forms. He represents now a (binary quadratic) form $am^2 + 2bmn + cn^2$ (with $a > 0$) by the lattice

$$\begin{bmatrix} \frac{b}{\sqrt{a}} & \sqrt{a} \\ \sqrt{\frac{b^2-ac}{Da}} & 0 \end{bmatrix}$$

or by

$$\begin{bmatrix} b & a \\ \sqrt{\frac{b^2-ac}{D}} & 0 \end{bmatrix}$$

and its multiples.[27] We recall that $D = b^2 - ac$, which makes it obvious that the entries in the last expression are integers when the form has integer coefficients. A key point of this expression is to underline its relation with the usual association between a quadratic form and a lattice that we have outlined above. In the usual representation, valid for definite binary quadratic forms (with $D < 0$), the third term would have been $\sqrt{\frac{ac-b^2}{a}}$. As Poincaré explains, his representation allows a similar treatment for definite and indefinite forms (those with $D > 0$). For definite forms, he introduces it as a projection of the usual plane representation on another plane, which makes an angle with the first that depends on D. It means in particular that the plane hosting the lattices depends on the determinant D (or, from our modern perspective, that the representation is proper to one specific quadratic field $\mathbb{Q}(\sqrt{D})$.[28] The whole theory Poincaré then develops is associated to the forms of a given determinant

[26] In French, the word “premier" is both ‘prime’ and ‘first’, which explains this somewhat strange terminology. We would now prefer “primary" instead of “second".

[27] There is here of course an abrupt change of notation, as a, b, c do not mean the same thing as before! This type of change is frequent in Poincaré’s early papers. Darboux, for instance, complained of precisely this when he reviewed Poincaré’s thesis in 1878-1879.

[28] This apparent disadvantage could be seen in a positive way as changing the implicit Euclidean metric of the Gauss-Seeber representation into a Lorentzian one.

D, up to the square of an integer (which amounts to normalizing the size of the elementary parallelograms). Again, on the plane, the same regular distribution of points may correspond to different lattices, and thus give rise to different quadratic forms, depending on the way the two systems of parallel lines joining these points are chosen.

Let us now explain Poincaré's geometrical definition of the theory of reduction for binary quadratic indefinite forms. Let O, A, B be a fundamental triangle of the lattice associated to an indefinite form, that is, a triangle with three points of the lattice as vertices and such that no other point of the lattice lies inside the triangle. Poincaré then constructs an elementary parallelogram of the lattice $OABC$, OA and OB being sides of this parallelogram.[29] Then OBC and OAC are two other fundamental triangles of the lattice, that Poincaré calls "derived" from the first one OAB. In the same way, OAB is itself derived from two other fundamental triangles OAE and OBD (that Poincaré calls the "primitive" of OAB) (see Fig. 2).[30]

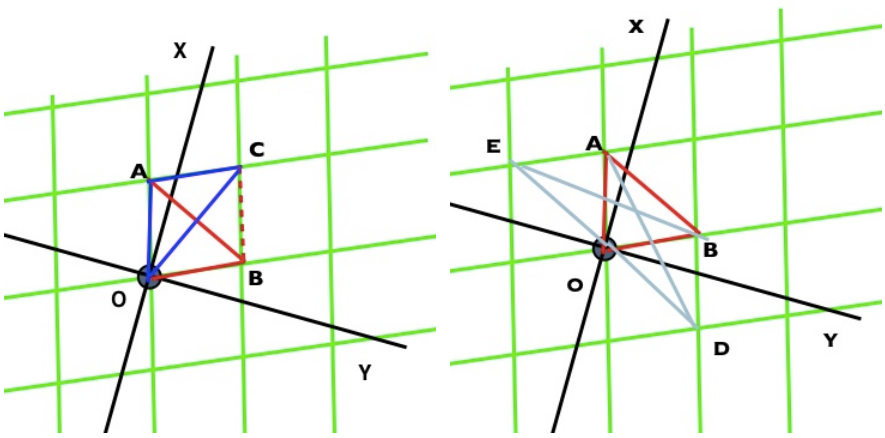

Fig. 2: A fundamental triangle OAB of a lattice, its derivatives OAC and OBC (left) and its primitives OEB and OAD (right).

For instance, let $x^2 - 2y^2$ be our initial form: its determinant is $D = 2$, the associated lattice is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. On the appropriate projection plane, the elementary

[29] In the original text, the drawings do not show the lattices, they only sketch the construction of the derived and primitive triangles. For convenience, our representation Fig. 2 is thus slightly different from the original, as well as its normalization.

[30] Again, Poincaré's terminology is slightly confusing. The words "derived" and "primitive" (in Latin or German, in particular) appear in the theory of forms developed by Gauss and some of his successors. In their work, a "derived form" is also obtained by applying a certain transformation to the initial form. But "primitive" here describes an intrinsic property of the form, for instance that its coefficients a, b, c are co-prime. In this perspective, unlike Poincaré, 'primitive' and 'derived' do not correspond to inverse transformations of each other. Poincaré's terminology seems more akin to that coming from function theory.

parallelograms of the lattice are squares (the area being renormalized as 1 in the projection). The points of the lattice are those of coordinates ($x = n, y = m$), with n and m integers; the corresponding points A and B are respectively $(0, 1)$ and $(1, 0)$, the point C being thus $(1, 1)$. The new elementary parallelogram associated to the derived triangle $0AC$, for example, is thus $0, A, C', C$, with $C' = (1, 2)$ (the other derived triangle $0BC$ gives rise to $0BCC$" with C" $= (2, 1)$). Reciprocally, $0AB$ is one of the two primitive triangles of $0AC$. The new lattice corresponding to the elementary parallelogram $0, A, C', C$ is thus given by $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and the new quadratic form of determinant 2 associated to it is $x^2 + 2xy + 2y^2$.

The lines $X : \sqrt{D}x = y$ and $Y : \sqrt{D}x = -y$ are called the asymptotes. Poincaré calls "ambiguous" a fundamental triangle (such as OAB) such that the first asymptote cuts it and the second does not. Exactly one derivative and one primitive of an ambiguous triangle are also ambiguous (in our example, OAC is ambiguous and OBC is not). The procedure of derivation thus allows Poincaré to construct sequences of ambiguous triangles. Relying on a theorem of Bravais, Poincaré is then able to show that the binary quadratic forms associated with the successive triangles of such a sequence are periodically reproduced. As seen in our example Fig. 2, a triangle has a common side with its derivatives; thus, among the finite sequence of triangles in a period, subsequences of ambiguous triangles and their successive derivatives share a common side, before another side occurs as the common side of another subsequence. Poincaré shows that the last triangle (or the first) of such a finite subsequence is associated to a reduced binary quadratic form.

Poincaré also displays the correlated geometrical interpretation of the development into continued fractions of $\frac{a\sqrt{D}}{1-b\sqrt{D}}$: the successive reduced fractions provide the coordinates of a vertex of a triangle associated to a reduced form. This interpretation will be used by Poincaré in a later note to embody the rational approximation of a real number α, by using the line $y = \alpha x$ and the Bravais lattice of the points with integral coordinates [Poincaré 1884].[31]

The following section of [Poincaré 1880c] reinterprets in the language of lattices the difficult composition of forms introduced by Gauss in the *Disquisitiones arithmeticae*. For this, Poincaré introduces a new multiplication on lattices, which is different from the non-commutative general multiplication on which the arithmetic of lattices discussed above was based. Let $Am + Bn$ and $A_1m_1 + B_1n_1$ be two lattices (with A, B, A_1, B_1 complex quadratic numbers for the same D), the result of this new multiplication is the lattice $AA_1\mu_1 + AB_1\mu_2 + BA_1\mu_3 + BB_1\mu_4$, with $m, n, m_1, n_1, \mu_1, \mu_2, \mu_3, \mu_4$ integers.[32] Poincaré proves that if a form is composed of two others, the corresponding lattice is the (new) product of the lattices associated

[31] In this note, he also generalizes this construction to the rational approximation of two real numbers, by using this time the 3-dimensional version of Bravais lattices. We recalled earlier the link between the theory of reduction and bounds on the values of the form at integers. Hermite had already applied it, for suitable quadratic forms, to the rational approximation of several real numbers, [Hermite 1850].

[32] The infelicitous notation, again, is Poincaré's.

to the two forms, that is, the composition on forms corresponds to a true operation on lattices, from which he can deduce Gauss's results relative to composition.

At the beginning of the last section of this memoir, Poincaré announces:

> The above considerations make it possible to present in a simple and concrete manner the theory of ideal complex numbers that correspond to quadratic forms of determinant D.[33]

This reference to ideals is very rare in the French landscape at the time [Goldstein 1999]. Hermite had displayed an interest in Kummer's work on ideal numbers early on [Hermite 1850], but his research aimed at offering alternative proposals to handle the arithmetic of algebraic numbers rather than promoting the reception of Kummer's (or later, Dedekind's) conceptual enterprise.[34] Poincaré situates himself in a Hermitian vein here, when he proposes (at least for the quadratic case) to substitute plane lattices—themselves embodied in tables of familiar "true" numbers—to an "ideal" family constructed by means of divisibility properties, as Kummer did, or with a set-theoretical perspective, as Dedekind did. [35] To a (quadratic) real complex number $\lambda + \mu\sqrt{D}$, Poincaré associates the lattice $\begin{bmatrix} \lambda & \mu\sqrt{D} \\ \mu & \lambda \end{bmatrix}$. The points of the lattice represent the multiple of this number and the (new) product of lattices, as defined for the composition of forms, corresponds to the product of two associated complex numbers.[36] An ideal complex number is then defined by Poincaré as a lattice with some simple conditions (which are of course verified for the lattices associated to a "true" complex number). The main arithmetical properties of lattices are then transferred to these ideal numbers; a prime ideal number, for instance, is one for which the norm (of the defining lattice) is a prime and Poincaré proves in particular that every ideal number, in his sense, can be decomposed in a unique way into ideal prime factors.[37]

Although Poincaré only addresses well-known cases in his first articles on forms, he clearly had hopes regarding the framework of lattices he has deployed, in particular for a generalization to any number of variables. The equivalence of two forms (under linear, invertible, transformations with integer coefficients) correspond to the equality of the two associated lattices, that is, (as in Bravais's memoirs), the two lattices differ from each other only by a rotation around the fixed origin O, of

---

[33] [Poincaré 1880c]: "Les considérations qui précèdent permettent d'exposer d'une manière simple et concrète la théorie des nombres complexes idéaux qui correspondent aux formes quadratiques de déterminant D".

[34] See [Goldstein 2007] and on Hermite's emphasis on a "clear" and "concrete" approach in mathematics, see [Goldstein 2011], and below.

[35] On Kummer's construction of ideal numbers, see [Edwards 1977]. On the issues at stake with ideal numbers at the time and Dedekind's approach, see [Edwards 1980, Edwards 1992, Haubrich 1992]. Once again, Poincaré seems close to the practical preferences expressed by Dedekind, even if their proposals to solve the problems differ, [Ferreiros 2007, Haffner 2014, Haffner 2019].

[36] See [Edwards 2007] on the link between the composition of forms and a theory of ideal numbers.

[37] As pointed out by Châtelet, Poincaré's viewpoint amounts for complex numbers to the representation by a matrix of the multiplication by this number of the elements of a well-chosen basis of the quadratic field [Poincaré 1950, p. 174, footnote 2]. This matrix point of view on ideal theory was adopted by several authors after Poincaré, including Châtelet himself.

an angle θ. Poincaré computes the transformations of the forms in terms of their coefficients and the angle θ and reciprocally. But an original and, for Poincaré, decisive step is the introduction of new types of invariants. As explained before, for a binary quadratic form $ax^2 + 2bxy + cy^2$, the quantity $b^2 - ac$ is invariant under all linear transformations of the variables of determinant 1, whatever the nature of the coefficients of this transformation; this property had given rise to the search for other invariant algebraic expressions, associated to forms of various degrees and number of variables, during all of the nineteenth century; as the nature of the coefficients does not intervene, it was considered to be the *algebraic* part of the study of forms by most authors, in particular Hermite, Jordan or Poincaré himself [Brechenmacher 2011, Brechenmacher 2016]. For binary quadratic forms, $b^2 - ac$ was the only algebraic invariant. But, as noted by Poincaré, there exist (many) arithmetical invariants, that is, expressions which are unchanged under a linear, invertible, transformation with *integer* coefficients. For instance, the series $\sum_{-\infty}^{\infty} \frac{1}{(am^2+2bmn+cn^2)^k}$ (where the sum is taken over all the integer couples $(m, n) \neq (0, 0)$) are such arithmetical invariants. Poincaré also considers $\sum_{-\infty}^{\infty} \frac{1}{(\sqrt{a}m+\frac{b+\sqrt{ac-b^2}}{\sqrt{a}}n)^{2k}}$, which a linear transformation with integer coefficients changes by a function of the angle of rotation θ.[38]

He then provides an effective (if not efficient) procedure to decide if two definite binary quadratic forms $ax^2+2bxy+cy^2$ and $a'x^2+2b'xy+c'y^2$ with the same determinant $D = ac - b^2 = a'c' - b'^2$ are equivalent [Poincaré 1879b, Poincaré 1882a]. Let us consider the (convergent) series

$$\phi_k(q) = \sum \frac{1}{(qm + n)^{2k}},$$

the sum being taken over all integers m, n, except $(0, 0)$. Assuming that the two forms are equivalent, Poincaré expresses the coefficients $\alpha, \beta, \gamma, \delta$ of a linear transformation between the forms as a function of the coefficients of the forms and the real and imaginary parts of

$$\sqrt{\frac{a\phi_1(\frac{b'+i\sqrt{D}}{a'})}{a'\phi_1(\frac{b+i\sqrt{D}}{a})}}.$$

If one computes the values of ϕ_1 with a sufficient approximation, the values of $\alpha, \beta, \gamma, \delta$ can be known up to less than $1/2$ and one thus gets exact values for these integers. It is then sufficient to check if this transformation indeed sends the form $ax^2 + 2bxy + cy^2$ into the form $a'x^2 + 2b'xy + c'y^2$. Other series are introduced for deciding on the equivalence of indefinite forms [Poincaré 1882a]. In both cases, Poincaré also shows that the series can be represented by definite integrals (using in particular then recent results on elliptic functions by Appell).

[38] Analogous series had appeared in Dirichlet's work on the computation of class numbers of binary quadratic forms (as well as in his work on prime numbers in arithmetic progressions).

4 Representation of numbers

Poincaré's foray into ideal theory did not go unnoticed. For instance, Arthur Cayley wrote to Poincaré on October 12, 1883:

> I have to thank you very much for the valuable series of memoirs which you have kindly sent me. I see that you have in one of them applied the theory of ideal numbers to the case of binary quadratic forms ; it had occurred to me that a very good illustration of the general theory would thus be obtained and I am vey glad to find that the case has been worked out [Poincaré 1986, p. 116].

Nor was this an isolated instance. As explained earlier, a classical question of the theory was the study of the values of forms at integers, and, in particular, the representation of integers by such values.[39] After a short communication to the Academy of Sciences in 1881 [Poincaré 1881b], Poincaré handled this problem for general binary forms in a memoir in the *Bulletin of the French Mathematical Society*, of which he had been elected to membership on April 21, 1882 [Poincaré 1885]. In this article, he puts ideal theory center-stage, but this time he explicitly adopts Dedekind's terminology, mentioning ideals instead of ideal numbers.[40] Poincaré's point of departure, however, is a type of form that Hermite had singled out in several memoirs [Goldstein 202?]:

$$\Psi(x_0, x_1, \cdots, x_{m-1}) = (x_0 + \alpha_1 x_1 + \alpha_1^2 x_2 + \cdots + \alpha_1^{m-1} x_{m-1})$$
$$(x_0+\alpha_2 x_1+\alpha_2^2 x_2+\cdots+\alpha_2^{m-1} x_{m-1}) \cdots (x_0+\alpha_m x_1+\alpha_m^2 x_2+\cdots+\alpha_m^{m-1} x_{m-1}),$$

where $\alpha_1, \cdots, \alpha_m$ are the roots of an algebraic equation. That is, $\Psi(x_0, x_2, \cdots, x_{m-1})$ is the norm of the complex integer $x_0 + \alpha_1 x_1 + \alpha_1^2 x_2 + \cdots + \alpha_1^{m-1} x_{m-1}$ (as well as of its conjugates).

Such decomposable norm-forms were here used by Poincaré (as had been done earlier by Hermite) as a link between ideal theory and the representation of integers by binary forms. More precisely, let F be an arbitrary binary form

$$F(x, y) = B_m x^m + B_{m-1} x^{m-1} y + \cdots + B_1 x y^{m-1} + B_0 y^m$$

with the B_i integers. The question of the representation of an integer N by F is easily reduced to that of the representation of $B_m^{m-1} N$ by the form $(x+\alpha_1 y)(x+\alpha_2 y) \cdots (x+\alpha_m y) = \Psi(x, y, 0, \cdots, 0)$ (where the α_i are now the roots of the algebraic equation obtained by dehomogenizing the form F). Poincaré was thus led to study in general the representation of an integer, say N', by a form $\Psi(x_0, x_1, \cdots, x_{m-1})$. To do this, he proceeds by studying all the ideals of norm N' in what we would call the ring

[39] Poincaré also made use of ideals in 1891 while extending to them some analytical results of Pafnuty Chebyshev on the distribution of prime numbers, but his application was limited to the ring $\mathbb{Z}[i]$ of Gaussian integers, for which all ideals are principal [Poincaré 1891a, Poincaré 1891b].

[40] Besides Dedekind, this article includes references to Eisenstein and Kummer, and, as we shall see, to Hermite.

generated by the α_i. The question is thus to decide if the ideals representing N are principal, that is, if they are composed of the multiples of one complex number, as required for the initial problem.

Poincaré represents the elements of such an ideal (this concept being more or less understood in Dedekind's sense) as a module stable under multiplication by any complex number of the type $x_0 + \alpha_1 x_1 + \alpha_1^2 x_2 + \cdots + \alpha_1^{m-1} x_{m-1}$ [41] as $x^{(1)} m_1 + x^{(2)} m_2 + \cdots + x^{(n)} m_n$, with integers m_i and $(x^{(i)})$ generators of the ideal. The norm of the elements of the ideal then defines a form of the same degree and the same number of variables as Ψ, and it is possible to study its equivalence with the form Ψ by using the Hermitian method of continuous reduction we mentioned earlier [Hermite 1851].

To summarize, Poincaré's procedure is to construct all ideals of norm N, then to examine if they are or are not principal by deciding on the equivalence of two forms. Hermite's technique even provides theoretically the transformation that is needed to express N as the value at integers of the initial binary form.

Most of Poincaré's article is thus devoted to the determination of ideals with a given norm. The generators $(x^{(i)})$ of an ideal are represented as a function of the powers of the α_j by a table of coefficients.[42]The first decisive step is to reduce this table to a triangular form which describes this possible ideal-solution—a step also arising from Hermite's work [Hermite 1851]. Then, Poincaré computes successively the conditions required such that a table represents an ideal (in his sense), then an ideal with a prime number as its norm, then an ideal with a power of a prime number as its norm, which finally allows him to exhibit ideals with a given norm N.

To give a flavor of the computations involved, let us illustrate them by the first step, in the case of a reduced 3 by 3 table $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$.[43] The three generators are thus here $a, b + d\alpha_1, c + e\alpha_1 + f\alpha_1^2$, with a, b, c, d, e, f integers. If a complex integer $x_0 + x_1\alpha_1 + x_2\alpha_1^2$ is in this module, it should be a linear combination with integral coefficients of the generators, say $pa + q(b + d\alpha_1) + r(c + e\alpha_1 + f\alpha_1^2)$ (with p, q, r integers), thus the coefficient of the term in ${\alpha_1}^2$ should be a multiple of f. Moreover, for this module to be an ideal, the multiplication by α_1 of the generators should again be in the module, that is, $a\alpha_1 = qd\alpha_1$ (the term $rf{\alpha_1}^2$ being 0, one should have here $r = 0$), thus d divides a. In the same way, expressing that $b\alpha_1 + d\alpha_1^2$ is in the

[41] By taking only integral coefficients, Poincaré does not obtain in all cases the principal ideal generated by an element in the complete ring of integers, and here, as elsewhere in his writings, several assumptions are missing. Châtelet completed them carefully in his comments to the *OE uvres* and I will not discuss them further.

[42] We would now call this table a matrix, but F. Brechenmacher has convincingly discussed the conceptual nuances of the two terms in [Brechenmacher 2011]. As for Poincaré, he spoke of "notation" and later of "tableaux" (tables, or charts).

[43] In his paper, Poincaré uses a representation with 3, 4 or 5 variables, while asserting the generality of his construction. Such a tension appears again in other papers of the same period and has also been analyzed in [Brechenmacher 2011].

module provides the fact that f divides d. A repetition of the same argument with ${\alpha_1}^2$ provides the final conditions:

$$a \equiv d \equiv 0 \mod f \quad b \equiv e \equiv c \equiv 0 \mod f \quad a \equiv 0 \mod d \quad b \equiv 0 \mod d.$$

5 Cubic ternary forms: another geometrical outlook

As we have seen, the use of lattices could be extended to ternary (quadratic) forms. But Poincaré came back to the basics of the theory of forms when he turned to cubic forms [Poincaré 1881c, Poincaré 1882c]:

> The various problems connected with the theory of binary quadratic forms have long been solved by the notion of reduced forms [. . .]. To generalize such a useful idea, to find forms playing in the general case the same role as reduced forms do in the case of quadratic forms, such is the problem which naturally arises and which M. Hermite has solved in the most elegant way. M. Hermite has confined himself to the study of definite or indefinite quadratic forms and of forms decomposable into linear factors; but his method can be extended without difficulty to the most general case. I believe that this generalization can lead to interesting results, and this is what determined me to undertake this work. [. . .] The simplest of all forms, after the quadratic forms and the forms decomposable into linear factors, are the ternary cubic forms. [. . .] In addition to [their] simplicity, other considerations have influenced my choice. These forms have indeed, from the algebraic point of view, been the object of very interesting and very complete works, and thanks to the close connection between Higher Algebra and Higher Arithmetic, these results have been of great help to me.[44]

The "very interesting and very complete works" mentioned by Poincaré were, according to his references, those of Otto Hesse, Siegfried Aronhold, Jakob Steiner and Alfred Clebsch, which concern invariant theory. Though Poincaré describes them as "algebraic", their relevancy here relies on a geometrical interpretation of the problem; for a ternary form $F(x_1, x_2, x_3)$, the equation $F(x_1, x_2, x_3) = 0$ indeed gives rise to a (projective) plane curve. Following Hermite, Poincaré first studies and classifies the *linear transformations* reproducing the form (that is, leaving the form unchanged when the transformation is applied to its variables), by means of what we now call eigenvalues. Then, he transfers the results to the corresponding plane

[44] [Poincaré 1881c, 190–191]: Les divers problèmes qui se rattachent à la théorie des formes quadratiques binaires ont été résolus depuis longtemps, grâce à la notion de réduite [. . .]. Généraliser une idée aussi utile, trouver des formes jouant dans le cas général, le même rôle que les réduites remplissent dans le cas des formes quadratiques, tel est le problème qui se pose naturellement et que M. Hermite a résolu de la façon la plus élégante [. . .]. M. Hermite s'est borné à l'étude des formes quadratiques définies ou indéfinies et des formes décomposables en facteurs linéaires ; mais sa méthode peut s'étendre sans difficulté au cas le plus général. Je crois que cette généralisation peut conduire à des résultats intéressants ; et c'est ce qui m'a déterminé à entreprendre ce travail. [. . .] Les plus simples de toutes les formes, après les formes quadratiques et les formes décomposables en facteurs linéaires, sont les formes cubiques ternaires. [. . .] Outre [leur] simplicité [. . .] d'autres considérations ont influé sur mon choix. Ces formes ont été en effet, au point de vue algébrique, l'objet de travaux très intéressants et très complets, et grâce au lien étroit qui rapproche l'Algèbre supérieure de l'Arithmétique supérieure, ces résultats m'ont été d'un grand secours.

curves, linking their (algebraic) invariants and their geometrical characteristics to the various categories of transformations. For each associated family of forms, Poincaré also provides a canonical one, whose equation is considered as particularly simple, and he explicitly computes the invariants.[45]

In the second part of his memoir, Poincaré addresses the properly arithmetical problems of the cubic ternary forms: their equivalence and classification, and the description of the transformations (this time with integer coefficients) which reproduce them (we now call them automorphisms and for sake of simplicity, we will use this terminology freely). To do this, Poincaré, following Hermite and other authors, in particular Selling, uses a (real) transformation sending the form into a canonical one (which is thus algebraically equivalent) and then transfers to the original form the reduction and the automorphisms of the canonical form. As for the quadratic case, there exist several possible definitions of the reduction of a form and/or of the canonical forms; the explicit description of the reduced forms depends on these choices, but their general properties, in particular the finiteness or unicity of the reduced forms in each class of algebraically equivalent forms, does not. For instance, the first "family" (in Poincaré's terminology) of ternary cubic forms identified by him is made of forms algebraically equivalent to the form $6\alpha xyz + \beta(x^3 + y^3 + z^3)$, with $\alpha \neq 0$,[46] chosen as the canonical form. Poincaré computes its Hessian $\Delta = 6(\beta^3 + 2\alpha^3)xyz - 6\alpha^2\beta(x^3 + y^3 + z^3)$ and the two Aronhold invariants $S = 4\alpha(\alpha^3 - \beta^3)$ and $T = 8\alpha^6 + 20\alpha^3\beta^3 - \beta^6$. The distribution of the nine inflection points on the associated cubic curve shows that a real transformation reproducing the canonical form can only exchange the three lines $x = 0$, $y = 0$, $z = 0$, i.e., that these transformations (except the identity, which is never mentioned by Poincaré) should be given by one of the following five "tables":

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Poincaré then proves that there is in general a unique reduced form arithmetically equivalent to a given form of this family and provides bounds on the coefficients of the reduced forms in terms of the invariants S and T. In the case of cubic ternary forms, this gives a new proof of a recent result of Camille Jordan, stating there are only finitely many classes of forms with integer coefficients algebraically equivalent to a given form (here the chosen canonical form).[47]

A more subtle case arises when the cubic form can be decomposed into several factors. For instance, in the case where the form represents a conic and a line which are not tangent, the canonical forms can be chosen to be $6\alpha xyz + z^3$ or $3\alpha x^2z + 3\alpha y^2z + z^3$, whether or not.the line and the conic intersect each other. The first canonical form is reproducible by the family of transformations with one

[45] On this algebraic work, and its relation to both Hermite and Jordan's works, see [Brechenmacher 2011]. On the history of the classification of algebraic curves, see [Lê 2023].

[46] This means that the equation is not reducible to a sum of three cubes.

[47] [Jordan 1879]. Jordan excludes the case of determinant 0.

parameter $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix}$. When the double points of the associated curve are imaginary (in particular when the canonical form is $3\alpha x^2 z + 3\alpha y^2 z + z^3$), there are a finite number of reduced forms, thus a finite number of classes. But when the double points are real, several cases occur, whether or not the invariant $4S$ is a fourth-power: there may be a finite number of classes or infinitely many classes, each of them containing a finite number of reduced forms.

The preliminary notes presented to the Academy of Sciences, as well as the longer memoirs on ternary cubic forms, are, as we have seen, quite explicit, giving for each family the concrete equations of canonical forms, their automorphisms, and the possible distribution of the reduced forms in arithmetical classes (and also genus). However, this did not satisfy Hermite completely. Hermite, who presented Poincaré's note [Poincaré 1880a] in June 1880 to the Academy, wrote to him a few days earlier to fix an appointment in order to discuss his memoir. He also suggested some further readings and concluded:

> Your search for the substitutions that reproduce a given form, and the distinction between the cases where these substitutions are entirely determined or depend on one or two variable parameters, seem to me to be entirely new, and I attach great importance to them. You have seen perfectly well that there is no arithmetical question in the search for the equivalence of cubic forms unless there are an infinite number of algebraic substitutions that change them into themselves. But then we leave the field of cubic forms and the question that you had the merit of posing—an entirely new question and one that I consider very beautiful and very fruitful —is that of the simultaneous reduction, that is to say, by the same linear substitutions, and with integer coefficients, of the system of a ternary quadratic form and of a linear function. [. . .] But you must not be satisfied with having thus opened the way, you must, in reality and in fact, give the means of calculating these reduced forms, and produce numerical applications. Many things can be revealed in this way of which neither you nor anyone else has any idea, so hidden are the properties of numbers and so far beyond any prediction. It is with regard to them that observation plays an absolutely necessary role; you need elements of observation, and these elements you will be the first to have obtained and to have given. [48]

Hermite's aphorism about observation is a recurrent one [Goldstein 2011], but Poincaré took the request seriously. "Following M. Hermite's advice", he wrote later,

[48] [Poincaré 1986, pp. 164–165]: Votre recherche des substitutions qui reproduisent une forme donnée, et la distinction des cas où ces substitutions sont entièrement déterminées ou bien dépendent d'un ou deux paramètres variables, me semblent entièrement nouvelles, et j'y attache une grande importance. Vous avez parfaitement vu qu'il n'y a de question arithmétique, dans la recherche de l'équivalence des formes cubiques qu'autant qu'il existe une infinité de substitutions algébriques qui les changent en elles-mêmes. Mais alors on quitte le champ des formes cubiques et la question que vous avez eu le mérite de poser question entièrement neuve et que je juge très belle et très féconde, est celle de la réduction simultanée, c'est-à-dire par la même substitution linéaire, et à coefficients entiers, du système d'une forme quadratique ternaire et d'une fonction linéaire. [. . .] Mais il ne faut point vous contenter d'avoir ainsi ouvert la voie, il faut, en réalité et en fait, donner les moyens de calculer ces réduites, et faire des applications numériques. Bien des choses peuvent se révéler ainsi dont ni vous ni personne n'a eu l'idée, tant les propriétés des nombres sont cachées et en dehors de toute prévision. C'est à leur égard que l'observation joue un rôle absolument nécessaire : il faut donc des éléments d'observation, et ces éléments vous serez le premier à les avoir obtenus et donnés.

he investigated more deeply the simultaneous reduction of a quadratic and a linear form—which corresponds to one of the more complicated case alluded to above, in which the cubic form can be decomposed and which is more delicate to handle from an arithmetical point of view. Using both complex congruences and Pell-Fermat type equations, Poincaré exhibited for instance the reduced forms associated to the system $x+y+z$, $x^2+4y^2-z^2+2xy+2xz+2yz$ or the automorphisms of the system $14x+y+2z$, y^2-6z^2 as the powers of the transformation

$$\begin{bmatrix} 1 & 5981360 & 14651280 \\ 0 & 46099201 & 112919520 \\ 0 & 18819920 & 46099201 \end{bmatrix}.$$ [49]

Hermite continued to encourage him to make new explicit calculations.

> Your result on the transformations of a system composed of a ternary [quadratic] form and a linear form is excellent, but I confess that I would have preferred that, at the cost of greater difficulty, you had been led to a new algorithm of calculation, instead of reducing the solution to the transformation into themselves of the simple binary forms. It is therefore necessary to persevere in the research which concerns only ternary forms.[50]

He had in fact a model for this, as his student Léon Charve defended a thesis on November 1880, a few months after this letter, on the reduction of ternary quadratic forms with completely effective (and extremely laborious) computations [Charve 1880]. Hermite mentioned this thesis several times in his own correspondence in laudatory terms.[51] However, when Poincaré returned to this quadratic case, it was not to have the effect that Hermite wished for.

---

[49] [Poincaré 1880b] and the developed article [Poincaré 1886a, pp. 135–142].

[50] [Poincaré 1986, p. 168]: Votre résultat sur les transformations semblables d'une système composé d'une forme ternaire [quadratique] et d'une forme linéaire est excellent, mais je vous avoue que j'aurais préféré qu'au prix d'une difficulté plus grande vous eussiez été amené à un nouvel algorithme de calcul, au lieu de ramener la solution à la transformation en elles-mêmes des simples formes binaires. Il faut donc persévérer dans la recherche qui concerne les seules formes ternaires.

[51] For instance, to Thomas Stieltjes [Hermite & Stieltjes, II, p. 12]: "La réduction n'est point un procédé facile ni commode et il n'a rien moins fallu que le talent et l'opiniâtreté de M. Charve pour en faire application dans quelques cas particuliers, et cependant il serait si utile et même absolument indispensable de pouvoir faire de nombreuses applications, pour s'éclairer et se diriger, j'ajouterai pour s'inspirer puisqu'il s'agit d'Arithmétique" [Reduction is neither an easy nor a convenient procedure and it took nothing less than the talent and obstinacy of Mr. Charve to apply it in a few specific cases, and yet it would be so useful and even absolutely indispensable to be able to make numerous applications, to enlighten and to direct, and since we are dealing with Arithmetic I will add, to inspire ourselves.].

6 Back to quadratic forms: Fuchsian functions and non-Euclidean geometry in arithmetic

Poincaré's discovery of a link between Fuchsian functions and non-Euclidean geometry is well-known, as the mathematician used it to illustrate the art of invention in mathematics [Poincaré 1908b]. It is the famous story of the omnibus:

> At the moment I set foot on the step, the idea came to me, without anything in my previous thoughts seeming to have prepared me for it, that the transformations I had used to define the Fuchsian functions are identical to those of non-Euclidean geometry.[52]

The scene probably took place in June 1880,[53] at a time when Poincaré was working on the classification of cubic ternary forms, and more specifically on the case when the form is composed of a linear and a quadratic factor, which led him again to ternary quadratic forms. As is well-known, he was involved simultaneously in the writing of a contribution to the 1880 prize in mathematics of the French Academy of Sciences, on differential equations, and in his own creation, description and classification of specific Fuchsian functions and of their associated transformations.[54]

I would like to underline once more the close relations, at several levels, between these works and the way Poincaré transfers methods, intuitions and objets from one topic to another, as we have already seen for the invariants of quadratic forms or the classification of cubic ones. In his very first note on Fuchsian matters, Poincaré defines a Fuchsian function as a uniform function on the plane which is reproduced by a discontinuous subgroup of the homographic transformations on the unit disk. He then remarks that some of these subgroups are isomorphic to groups of linear transformations with integer coefficients that reproduce an indefinite ternary form with integer coefficients, concluding that this "highlights the intimate links between number theory and the analytical question in hand" [Poincaré 1881a, p. 335]. Indeed the omnibus story continues on, displaying influences in both directions:

> On my return to Caen [. . .] I then began to study questions of Arithmetic without much result and without suspecting that this could have the slightest connection with my previous research. Disgusted with my failure, I went to spend a few days at the seaside and thought of other things. One day, while walking on a cliff, the idea came to me, always with the same characteristics of brevity, suddenness and immediate certainty, that the arithmetic transformations of the ternary indefinite quadratic forms are identical to those of non-Euclidean geometry. Back in Caen, I meditated on this result and drew the consequences: the example of quadratic forms showed me that there were Fuchsian groups which are different from those corresponding to the hypergeometric series.[55]

---

[52] [Poincaré 1908b, p. 363]: Au moment où je mettais le pied sur le marche-pied, l'idée me vint, sans que rien de mes pensées antérieures parut m'y avoir préparé, que les transformations dont j'avais fait usage pour définir les fonctions fuchsiennes sont identiques à celles de la Géométrie non-euclidienne.

[53] See [Gray 2013, p. 216–217].

[54] We will not restate the details of this episode, which has been thoroughly studied, in particular by Jeremy Gray [Gray 2000], [Gray 2013, ch. 3].

[55] [Poincaré 1908b, p. 363]: "De retour à Caen [. . .] je me mis alors à étudier des questions d'Arithmétique sans grand résultat apparent et sans soupçonner que cela pût avoir le moindre

Poincaré was here perfectly in line with several of his fellow mathematicians, in particular the contemporary research of Émile Picard on substitutions with 3 variables on the hypersphere or Camille Jordan's study of the groups of motions, which Jordan directly connected to Bravais's works.[56] While the appearance of non-Euclidean geometry in the story is a more spectacular feature, a close reading of all these papers suggests that the explicit writing out of the various transformations used in these different situations was a driving element and a decisive factor in favoring the thematic rapprochements.

As Poincaré explained at the Algiers meeting of the French Association for the Advancement of Science where he presented his new viewpoint in April 1881 [Poincaré 1882b], his point of departure was Hermite's method for ternary quadratic forms [Hermite 1854]. In 1854, Hermite had studied the reduction of indefinite ternary quadratic forms—the forms algebraically equivalent to, say, $X^2 + Y^2 - Z^2$—by a variant of his technique of continuous reduction. To such a form f, Hermite associated a family of *definite* ternary quadratic forms $\phi(x, y, z) = f(x, y, z) + 2(\lambda x + \mu y + \nu z)^2$, λ, μ, ν real numbers with some suitable conditions. Each ϕ could then be reduced by a suitable transformation, by the general theory of definite quadratic forms. Hermite then applied this transformation to the initial f and thus obtained, by varying ϕ, a family of transformed forms, which he considered as the reduced forms of the initial form f. He showed that the coefficients of these reduced forms satisfy certain bounds; in particular, if the coefficients of the initial f are integers, it implies that there are a finite number of such reduced forms. Hermite also studied the automorphisms of the form, proving for instance that what we now call their characteristic equation has solutions of the type $\pm 1, l, \frac{1}{l}$.[57]

Poincaré followed exactly the same path in 1881. He associated to the indefinite form $F(x, y, z) = (ax+by+cz)^2+(a'x+b'y+c'z)^2-(a''x+b''y+c''z)^2 = \xi^2+\eta^2-\zeta^2$ the definite forms $\xi^2 + \eta^2 - \zeta^2 + 2(\xi_1\xi + \eta_1\eta - \zeta_1\zeta)^2$, with ξ_1, η_1, ζ_1 satisfying the condition (analogous to that of Hermite) $\xi_1^2 + \eta_1^2 - \zeta_1^2 = -1$. Again, he used the transformations reducing these definite forms, applying them in turn to F to get what he defined as the reduced (forms) for F.

As he noted, however, since $\xi_1^2 + \eta_1^2 - \zeta_1^2 = -1$, the point with coordinates $\frac{\xi_1}{\zeta_1+1}, \frac{\eta_1}{\zeta_1+1}$ is inside the unit disk. To each definite form of the family is then associated such a point, as well as a reduced form. When the parameters ξ_1, η_1, ζ_1 change, the point moves inside the disk. However, the reduced form remains the same so long as the point lies inside a certain region of the disk, then it changes. The

rapport avec mes recherches antérieures. Dégoûté de mon insuccès, j'allai passer quelques jours au bord de la mer et je pensai à tout autre chose. Un jour, en me promenant sur une falaise, l'idée me vient, toujours avec les mêmes caractères de brièveté, de soudaineté et de certitude immédiate, que les transformations arithmétiques des formes quadratiques ternaires indéfinies sont identiques à celles de la Géométrie non-euclidienne. Étant revenu à Caen, je réfléchis sur ce résultat, et j'en tirai les conséquences ; l'exemple des formes quadratiques me montrait qu'il y a des groupes fuchsiens autres que ceux qui correspondent à la série hypergéométrique.

[56] The analogies and differences with Jordan, in particular with respect to the concept of group, are discussed in [Brechenmacher 2011].

[57] The description of the automorphisms was completed by several authors c. 1870, in particular by Georg Cantor in his Habilitationschrift, by Paul Bachmann, and by Hermite himself.

transformations providing the reduction can be then studied geometrically, by looking at the corresponding regions delimited inside the disk.[58] To do this, Poincaré used non-Euclidean geometry on the disk, more specifically, a non-Euclidean description of the tessellation of the disk in domains delimited by polygons. This approach, as might be expected, was not to the taste of Hermite who asked Poincaré several times to reformulate his results:

> In renewing my request to you to present your results on the classification of functions $\frac{az+b}{cz+d}$ in order to obtain the elements of the formation of the Fuchsian functions, without resort to the use of non-Euclidean geometry, and after having presented them by the method by which you discovered them, I beg you, Sir, to receive the renewed assurance of my highest esteem for your work and of my most devoted sentiments.[59]

The explicit, detailed, connection with the Fuchsian groups was presented only a few years later, Poincaré choosing at that time another expression of the canonical ternary quadratic form [Poincaré 1886b]:

> An indefinite ternary quadratic form may always be written . . . in the following way:
>
> $$F(x, y, z) = Y^2 - XZ,$$
>
> where
>
> $$X = ax + by + cz, \quad Y = a'x + b'y + c'z, \quad Z = a''x + b''y + c''z,$$
>
> a, b, c being arbitrary real numbers.

Let now $\alpha, \beta, \gamma, \delta$ be four real numbers such that $\alpha\delta - \beta\gamma = 1$. Poincaré introduces the transformations:

$$\begin{aligned} X' &= \alpha^2 X && +2\alpha\gamma Y && + \gamma^2 Z \\ Y' &= \alpha\beta X && +(\alpha\delta + \beta\gamma)Y && + \gamma\delta Z \\ Z' &= \beta^2 X && +2\beta\delta Y && + \delta^2 Z \end{aligned}$$

If $X = ax'+by'+cz', \quad Y = a'x'+b'y'+c'z', \quad Z = a''x'+b''y'+c''z'$, it is then easy to check that the transformation changing x, y, z into x', y', z' leaves F invariant. If the coefficients of F and the $\alpha, \beta, \gamma, \delta$ are integers, these transformations form a discontinuous group and the associated substitutions $z \to \frac{\alpha z+\beta}{\gamma z+\delta}$ form a Fuchsian

[58] In other words, as Châtelet explains in a footnote, Poincaré studies the fundamental domain of the automorphisms, seen as homographic transformations.

[59] For instance, [Poincaré 1986, p. 174]: “En vous renouvelant la prière de présenter sans recourir à l’emploi de la géométrie non euclidienne, après les avoir exposés par la méthode qui vous les a fait découvrir, vos résultats sur la classification des fonctions $\frac{az+b}{cz+d}$ afin de posséder les éléments de la formation des fonctions fuchsiennes, je vous prie, Monsieur de recevoir la nouvelle assurance ma plus haute estime pour vos travaux et de mes sentiments bien dévoués". Hermite was not hostile to all geometrical arguments; he did not complain about Poincaré’s lattices or Hermann Minkowski’s geometry of numbers (also based on lattices). But for him, non-Euclidean geometry was not helpful in representing analytical facts, see [Goldstein 2011].

group. In a longer memoir, Poincaré emphasizes the particular properties of these arithmetically-defined Fuchsian groups, in particular the algebraic relations satisfied by the associated Fuchsian functions, analogous to those already known for elliptic and modular functions [Poincaré 1887].

7 Classification again

In the wake of his research on the classification of forms, Poincaré also devoted two short notes to a generalization of the tools Gauss had introduced for his own refined classification of binary quadratic forms, “order" and “genus", [Poincaré 1882d].[60] In the *Disquisitiones arithmeticae*, two classes of binary quadratic forms, represented for example by the forms $ax^2+2bxy+cy^2$ and $a'x^2+2b'xy+c'y^2$, are said to belong to the same order if the g.c.d. of (a, b, c) is equal to the g.c.d. of (a', b', c') and the g.c.d. of $(a, 2b, c)$ is equal to the g.c.d. of $(a', 2b', c')$. Following a proposal of Eisenstein for ternary forms, for forms of higher degree or with more variables, Poincaré imposes equality conditions not only on the coefficients of the forms, but also on those of some of their invariants and covariants. For instance, for the binary cubic form $f = ax^3+3bx^2y+3cxy^2+dy^3$, the Hessian $6(ac-b^2)x^2+6(ad-bc)xy+6(bd-c^2)y^2$ should be taken into account; the order is determined by four quantities, the g.c.d of a, b, c, d, the g. c. d. of $a, 3b, 3c, d$, the g.c.d. of $ac - b^2, ad - bc, bd - c^2$ and the g.c.d. of $2(ac - b^2), ad - bc, 2(bd - c^2)$.

As for the genus, its definition in the *Disquisitiones arithmeticae* relied on the following fact: for a binary quadratic form $ax^2+2bxy+cy^2$, such that g.c.d.(a, b, c)=1, and a prime factor p of its determinant $ac-b^2$, two integers which are represented by the form are both quadratic residues modulo p or both non-quadratic residues. The various cases (“characters") for the various p then define the genus of the form (in fact, of its whole class). Poincaré defined the equivalence of two forms f and f' according to a modulus m when there exists a linear transformation T with integer coefficients and determinant $\equiv 1 \pmod{m}$ such that $f \circ T \equiv f' \pmod{m}$. Two (algebraically equivalent) forms are then in the same genus if they are equivalent according to all moduli. Again, in these notes, Poincaré gives only very general statements, without proofs, and illustrates his definitions with a few numerical examples. He computes in particular the distribution of binary cubic forms according to moduli 2, 3 and 5, but he does not seem to have gone further in the 1880s in this attempt of a classification of higher-degree forms. In particular, he does not appear to have then seen Eisenstein’s suggestion, developed in particular by Smith in the 1860s, and then much later by Minkowski, of defining the genus by means of transformations with *rational* coefficients [Smith 1861–1865, Dickson 1919].

[60] Several authors, in particular Eisenstein and Dirichlet, had then contributed to simplifications, reformulations and partial extensions of these notions, which also appear in other domains of mathematics, see [Lê 2023].

8 Poincaré's arithmetic revisited

We have tried to show that Poincaré's arithmetical work is highly coherent as soon as one restores the collective program in which it is embedded, i.e. the disciplinary configuration of the theory of forms in the last third of the nineteenth century. With its own questions, concepts and resources, it largely guided Poincaré's objectives, in the perspective of the classification of forms: to find well-chosen, preferably effective, invariants; to identify adequate representatives of classes (and of other levels of classification), such as the canonical forms of the algebraic classification and the reduced forms of the arithmetic one; to explain the operations allowing the transformation of each form into its representatives; and to study the automorphisms of a form. Conversely, we have seen that the recourse to ideal numbers did not indicate a change of discipline (for instance, as David Hilbert would define it in his own presentation of the theory of algebraic numbers), but an attempt to integrate (or even disintegrate . . .) these concepts (whose importance Poincaré clearly perceived) in the disciplinary framework of forms.

We have seen this program at work in all his early research, including the famous papers linking quadratic forms, Fuchsian functions and non-Euclidean geometry. What shaped these papers was also reflected in his later, more famous, work. His 1905 article, on the centenary of Dirichlet's birth, is thus in many respects, a microcosm of the larger mathematical world we have just presented. Its results, as their recent commentator Nicolas Bergeron describes them, may seem disparate, the only clear arithmetical application being a new proof of a well-known formula of Dirichlet on the number of classes of forms. But, as before, essential and varied analytical tools (in particular those linked to automorphic functions) were mobilized to search for invariants of linear and quadratic forms [Bergeron 2018].

As for Poincaré's celebrated memoir of 1901, today's readers see it as one of the main origins of so-called Diophantine geometry and focus on the way Poincaré, with the help of the parametrization of cubic curves by elliptic functions, defined more or less adequately the rank of (the group of points with rational coordinates on) an elliptic curve.[61] However, our study provides another context for this article. In 1880, Poincaré had employed a geometric interpretation of ternary forms, considering them as defining equations of plane (algebraic) projective curves. It allowed him to link the classifications of curves (by their invariants, singular points and other geometric characteristics) with those of forms (by linear transformations). In 1901, Poincaré proposed a new classification of algebraic curves directly inspired by the theory of forms, one based on birational transformations between curves. The link to the mode of classification of forms is explicit, Poincaré refers directly to the *Disquisitiones arithmeticae* as providing the principles to classify conics according

[61] See [Weil 1955], [Gray 2013, pp. 486–488]. The parametrization was already widely used, see [Schappacher 1991, Lê 2018]. Norbert Schappacher has discussed the problems raised by Poincaré's definition of rank in [Schappacher 1991]. On this paper and its geometrical viewpoint, see [Schneider 2000].

to his own program [Goldstein & Schappacher 2007b, pp. 95–96].[62] And it is indeed to the study of birational transformations between curves defined by equations of different degrees that the major part of Poincaré's memoir is devoted, as well as to the reduction of any algebraic curve to a curve defined by equations of the lowest possible degree— coherent with the guidelines of the theory of forms.

In this respect, the presentation of arithmetic in Poincaré's lecture on the future of mathematics is illuminating :

> Among the words which have had the happiest influence, I would mention "group" and "invariant". [. . .] Progress in arithmetic has been slower than that in algebra and analysis and it is easy to understand the reasons. The feeling for continuity is a precious guide which the arithmetician lacks [. . .] [He] must therefore take analogies with algebra for his guide [. . .] The theory of forms, and in particular that of quadratic forms, is intimately bound to the theory of ideals. One of the earliest to take form among arithmetic theories, it arose with the successful introduction of unity through the use of linear transformation groups. These transformations have allowed a classification with its consequent introduction of order.[63]

This coherence also manifests itself on a more subterranean level, that of practices. As the word has been widely used recently in the philosophy of mathematics, let me specify that I use it here in a rather informal way, to designate a concrete way to carry on an activity (as opposed to official rules or principles, and to theory). They have to do with "real individuals, their actions and their material conditions of life"[64] In mathematics, practices can thus be attached to the pervasive use of a certain tool or technique, or to a way of reading the articles of other mathematicians or of publishing one's own work or of exchanging mathematical information. They can also be detected through an epistemic privilege attached to specific features, like effectivity or proofs, or a recurrent representational device, be it diagrams or lattices.

Poincaré's way of practising mathematics in his early arithmetical work displays a striking mixture of a particularly vague mode of writing and of an impressive mobilisation of ideas and techniques from several branches of mathematics. The second point has been obvious on several occasions here, with the recourse to several kinds of geometry or to a large variety of analytical tools. On the first point, let us note,

---

[62] We do not know if Minkowski's use of transformations with rational coefficients to define the genus of forms played a role in Poincaré's conception. Nor does Poincaré mention contemporary work on birational geometry, even that connecting it with Diophantine equations.

[63] [Poincaré 1908a, p. 175, pp. 179–180]: Les progrès de l'Arithmétique ont été plus lents que ceux de l'Algèbre et de l'Analyse, et il est aisé de comprendre pourquoi. Le sentiment de la continuité est un guide précieux qui fait défaut à l'arithméticien. [. . .] L'arithméticien doit donc prendre pour guide les analogies avec l'Algèbre. [. . .] La théorie des formes, et en particulier celle des formes quadratiques, est intimement liée à celle des idéaux. Si parmi les théories arithmétiques elle a été l'une des premières à prendre figure, c'est quand ont est parvenu à y introduire l'unité par la considération des groupes de transformations linéaires. Ces transformations ont permis la classification et par conséquent l'introduction de l'ordre.

[64] [Marx & Engels 1846/1969, p. 20]: "Es sind die wirklichen Individuen, ihre Aktion und ihre materiellen Lebensbedingungen". Or, as Michel Foucault writes [Foucault 1982/2001, p. 1039]: "L'on tient plus aux manières de voir, de dire, de faire et de penser qu'à ce qu'on voit, qu'à ce qu'on pense, qu'à ce qu'on dit" ("We care more about the ways of seeing, saying, doing and thinking than about what we see, what we think, what we say"). On this issue, see among many others, [Bourdieu 1994, Lepetit 1995, Chateauraynaud & Cohen 2016].

for example, that necessary hypotheses are often missing—the examples include the irreducibility of an algebraic equation under scrutiny, the non-vanishing of certain expressions like the determinant or the fact that some of his ideals are not defined over the ring of integers, but only over a subring. As seen in the section on quadratic forms and lattices, Poincaré may also use the same symbols to designate different things in the same article[65]. Browsing through his articles gives the impression of flying over a vast textual landscape (in the vernacular) with the occasional example or calculation serving as anchor points. More than the precise statements and detailed proofs we are now used to, computations of examples are the warrants of the solidity of Poincaré's whole construction.[66]

It is also quite tricky to identify Poincaré's sources of inspiration—he often quotes some predecessors in a general way at the beginning of his text, very rarely for a specific result inside the text (we have seen an example with some references to Bravais or Eisenstein. As those close to him sometimes explained after his death, Poincaré was particularly gifted for roughly grasping ideas or problems and then integrating them into his own framework.[67] Moreover, several correspondents pointed out to Poincaré that such and such a result had already appeared in one of the sources that he mentioned. However, he was also one of the rare authors (French or not) to mention Dedekind's theory of ideals in the 1880s and he quoted and relied on numerous German authors; we have mentioned Gauss of course, but also Eisenstein, Dirichlet, Selling, Hesse, Steiner. . . . Weil's assertion in this respect seems a little misleading—or perhaps a little anachronistic, in that he seems to be referring to Poincaré's neglect of what will be considered in the interwar period as "the royal road" to a structuralist point of view.[68]

An obvious source, however, is Hermite, whose influence operates at several levels, in addition to the direct interactions we have already mentioned. Like Poincaré in 1880, Hermite had revisited the classical results of the classification of forms of the *Disquisitiones arithmeticae*, in the light of his procedure of continuous reduction [Hermite 1851]. At another level, the emphasis on (linear) transformations is one of

---

[65] As mentioned earlier, Gaston Darboux wrote to Poincaré in 1878 about his thesis: "I still believe that we will make a good thesis out of it, but it seems essential to me to recast the writing and to correct all the errors of calculation or the changes of notation which make it almost unreadable." ["Je persiste à croire que nous en ferons une bonne thèse, mais il me parait indispensable de fondre la rédaction et de corriger toutes les erreurs de calcul ou les changements de notation qui la rendent presque illisible."] [Poincaré 1986, p. 132].

[66] Poincaré is almost describing his own practice when, advising Mittag-Leffler on the translation of Georg Cantor's memoirs on set theory, he writes: "To make it accessible, it would be necessary to give a few specific examples after each definition and then put the definitions at the beginning instead of at the end", [Hermite & Mittag-Leffler, p. 278]: "Il faudrait pour la rendre accessible donner quelques exemples précis à la suite de chaque définition et puis mettre les définitions au commencement au lieu de les mettre à la fin."

[67] The contrast with Châtelet's painstaking corrections and complements of Poincaré's memoirs is in this respect quite striking.

[68] The rumor that Poincaré did not know or mention German sources spread through Mittag-Leffler, in particular among French students in Germany at the time of the rivalry between Poincaré and Klein around automorphic functions, despite Hermite's protests; see for instance [Hermite & Mittag-Leffler, I, pp. 129, 251].

the characteristics of Hermite's work during his whole career. His use of "tableaux" (our matrices) to work out transformations is pervasive, as it is in Poincaré's work.[69]. The reduction of transformations, in particulier, is carried out on these "tableaux", playing a key role for both mathematicians. Poincaré also took from Hermite the idea that decomposable forms constitute a fruitful entry into the study of algebraic numbers. Some specific constructions were directly borrowed from Hermite's articles: for example, Poincaré followed and generalized the approach Hermite had introduced to factorize into complex factors prime numbers congruent to 1 modulo 5 or 7 [Hermite 1850, Goldstein 2007].

Two other instructive shared features deserve to be highlighted. First, the importance of reduced forms in their scheme of work. Reduced forms are particular representatives of classes (sets of forms connected by suitable linear transformations). The later, structural, viewpoint would privilege classes, which are intrinsic. Poincaré, like Hermite, was perfectly aware that several (rather arbitrary) choices were possible for the reduced forms; indeed, he modified his choice, for example, in the course of his research on ternary quadratic forms. This freedom of choice, however, like that of the "tableaux" (whose writing depends on a choice of generators), favors calculation. Hermite is quite explicit about his predilection [Goldstein 2011] and Poincaré-, who nevertheless built, as we have shown, an arithmetic of lattices, followed him on this point. This can be seen in particular in what Poincaré called a classification: his are not based on classes per se but on the construction of specific, and in principle, calculable characteristics such as invariants.

Another point that brought the two mathematicians together was their vision of a larger research field that would merge arithmetic, algebra and analysis, and exclude the disciplinary purity which was at the time defended by many mathematicians, such as Edouard Lucas or Leopold Kronecker. On the contrary, the use of continuous tools in arithmetic was favored and praised by Hermite as well as by Poincaré. We have emphasized this direction here, but, reciprocally, the search for automorphisms, for instance, was exported into the study of Fuchsian or Abelian functions, as well as differential equations.[70] Poincaré's famous sentence—"The only natural object of mathematical thought is the integer"—might thus lead to a misinterpretation if read in isolation. It is in fact a mere concession to the defenders of a pure number theory, stripped of its analytical tools, a concession immediately corrected into a promotion of a unified field of mathematics.[71].

> The only natural object of mathematical thought is the integer. [. . . We] have devoted almost all our time and energy to the study of the continuous. Who will regret it? Who will believe that this time and these efforts have been wasted? Analysis unfolds for us infinite perspectives

[69] This has already been underlined by F. Brechenmacher on the basis of algebraic works in the same period, in particular Poincaré's 1884 paper on complex numbers [Brechenmacher 2011]

[70] Other examples are given by Frédéric Brechenmacher in [Brechenmacher 2011]. He shows how the theory of forms redefined the multiple domains where Poincaré intervenes.

[71] Hermite repeated on several occasions that the theory of numbers is only an anticipation of the theory of elliptic functions. For the importance he attached to the use of analytical tools, see [Goldstein 2007, Goldstein 2011]. It should be noted that Hermite himself, a priori unaware of advances in geometry, sometimes used elementary geometric representations

> that arithmetic does not suspect, it shows us at a glance a grandiose whole, the order of which is simple and symmetrical; on the contrary, in the theory of numbers, where the unforeseen reigns, the view is, so to speak, blocked at every turn. [. . .] Let us be grateful to the continuum which, if everything comes out of the whole number, was alone capable of bringing out so much. Need I remind you, moreover, that M. Hermite drew a surprising advantage from the introduction of continuous variables into the theory of numbers? Thus, the proper domain of the whole number is itself invaded, and this invasion has restored order where disorder reigned.[72]

In Poincaré, however, the continuous is not restricted to the theory of functions. It extends to geometric representations or even geometric techniques, themselves borrowed from several branches of mathematics, from Bravais's theory of polyhedra and lattices to that of projective curves or to non-Euclidean geometry.[73] This justifies his well-known reputation as one of the last universalist mathematicians. But what is striking when reading his early work on arithmetic is his professionalism (all the more paradoxical for us who are now used to a very different writing style); his mastery of both the disciplinary issues and the tools available, his ability to intervene effectively in order to fill in all the gaps in a program, rarely explained in detail, but whose reconstruction allows us to see that Poincaré had identified its stakes and components perfectly. [74]

The historian Gil Bartholeyns suggests that:

> The evolution of the object of history during the twentieth century can be described as the change from the extraordinary (the particular, the unique) to the ordinary (the collective, the structural, the trivial). In place of the exceptional individuals, the chefs-d'oeuvre, the memorable events, [the historians] have preferred the forgotten, the unpretentious documents, the repetitive and shared dimensions of existence.[75]

---

[72] [Poincaré 1897]: Le seul objet naturel de la pensée mathématique, c'est le nombre entier. [. . . N]ous avons consacré à l'étude du continu presque tout notre temps et toutes nos forces. Qui le regrettera? Qui croira que ce temps et ces forces ont été perdus? L'analyse nous déroule des perspectives infinies que l'arithmétique ne soupçonne pas; elle nous montre d'un coup d'œil un ensemble grandiose, dont l'ordonnance est simple et symétrique; au contraire, dans la théorie des nombres, où règne l'imprévu, la vue est pour ainsi dire arrêtée à chaque pas. [. . . S]oyons reconnaissants au continu qui, si tout sort du nombre entier, était seul capable d'en faire tant sortir. Ai-je besoin, d'ailleurs, de rappeler que M. Hermite a tiré un parti surprenant de l'introduction des variables continues dans la théorie des nombres? Ainsi, le domaine propre du nombre entier est envahi lui-même, et cette invasion a rétabli l'ordre là où régnait le désordre.

[73] Châtelet also underlines Poincaré's hope, again like Hermite, in the specific approach of Hermann Minkowski with his geometry of numbers [Poincaré 1908a]. Minkowski's approach would be regularly integrated into the arithmetical work developed in France before the First World War, in particular by Châtelet himself [Gauthier 2009, Gauthier 2011]. On the importance of geometry in Poincaré's arithmetical works, see [Schneider 2000].

[74] This situation can be compared to that described for Albert Einstein's famous articles of 1905, also often perceived as isolated, but whose coherence can be restored [Rynasiewicz & Renn 2005].

[75] [Bartholeyns 2010]: "L'évolution de l'objet de l'histoire au XXe siècle peut être décrite comme le passage de l'extraordinaire (le particulier, l'unique) à l'ordinaire (le collectif, le structurel, le banal). Aux individus exceptionnels, aux chefs-d'œuvre et aux événements mémorables, on a donné préférence aux oubliés, aux documents sans prétention, aux dimensions répétitives et partagées de l'existence".

As I have tried to show here, the forgotten and the repetitive may also draw a path to a better understanding of the exceptional.

9 Acknowledgment

This text has benefited greatly from careful review and helpful suggestions for which I am deeply grateful. For them I would like to warmly thank Jinze Du, Lizhen Ji, François Lê, Juan Li, Jim Ritter, Norbert Schappacher, Erhard Scholz and Jiwei Zhao.

References

Bachmann 1898. Bachmann, Paul. 1898. *Die Arithmetik der quadratischen Formen*. Leipzig: Teubner.

Bartholeyns 2010. Bartholeyns, Gil. 2010. Le paradoxe de l'ordinaire et l'anthropologie historique. *L'Atelier du Centre de recherches historiques* 6: `http://journals.openedition.org/acrh/1928`.

Bergeron 2018. Bergeron, Nicolas. 2018. Les "invariants arithmétiques" de Poincaré. *Graduate Journal of Mathematics* 3(1):1–14.

Boucard, Goldstein & Malécot, 2023. Boucard, Jenny, Catherine Goldstein and Valéry Malécot. 2023. Dispositions discrètes: les transferts entre botanique et cristallographie chez Auguste Bravais. In *Rencontres autour de la notion d'ordre au XIXe siècle: Pratiques savantes et discours dans les sciences, la philosophie et l'art ornemental*, ed. Jenny Boucard and Christophe Eckes. Paris: Hermann, to appear.

Bourdieu 1994. Bourdieu, Pierre. 1994. *Raisons pratiques: sur la théorie de l'action*. Paris: Seuil.

Bravais 1850. Bravais, Auguste. 1850. Mémoire sur les systèmes formés par des points distribués régulièrement sur un plan ou dans l'espace. *Journal de l'École polytechnique* 19: 1-128.

Bravais 1851. Bravais, Auguste. 1851. Études cristallographiques. *Journal de l'École polytechnique* 20: 101–278.

Bravais 1866. Bravais, Auguste. 1866. *Études cristallographiques*, ed. Léonce Élie de Beaumont. Paris: Gauthier-Villars.

Brechenmacher 2008. Brechenmacher, Frédéric. 2008. La controverse de 1874 entre Camille Jordan et Leopold Kronecker. *Revue d'histoire des mathématiques* 2 (13): 187–257.

Brechenmacher 2011. Brechenmacher, Frédéric. 2011. Autour de pratiques algébriques de Poincaré : héritages de la réduction de Jordan. Preprint, https://hal.archives-ouvertes.fr/hal-00630959.

Brechenmacher 2016. Brechenmacher, Frédéric. 2016. Algebraic generality vs arithmetic generality in the controversy between C. Jordan and L. Kronecker (1874). *The Oxford Handbook of Generality in Mathematics and the Sciences*, ed. Karine Chemla, Renaud Chorlay, David Rabouin, 433–467. Oxford: Oxford University Press.

Charve 1880. Charve Léon. 1880. De la réduction des formes quadratiques ternaires et de leur application aux irrationnelles du 3e degré. *Annales scientifiques de l'École normale supérieure* 2e s. 9: 3–156.

Chateauraynaud & Cohen 2016. Chateauraynaud, Francis and Yves Cohen (dir.). 2016. *Histoires pragmatiques*. Paris: Editions EHESS.

Crilly 1986. Crilly, Tony. The Rise of Cayley's Invariant Theory (1841–1862). *Historia Mathematica* 15: 241–254.

Dedekind 1876. Dedekind, Richard. 1876. Sur la théorie des nombres entiers algébriques. *Bulletin des sciences mathématiques* 11: 278-288, 310.

Dedekind 1877. Dedekind, Richard. 1877. Sur la théorie des nombres entiers algébriques. *Bulletin des sciences mathématiques et astronomiques* 2e s. 1: 17-41, 69–92, 144–164, 207–248.

Dickson 1919. Dickson, Leonard Eugene. 1919. *History of the Theory of numbers,* vol. III*: Quadratic and Higher Forms*. Washington: Carnegie Institute. Repr. New York: Chelsea, 1952.

Dirichlet 1850. Dirichlet, Peter Gustav Lejeune-. 1850. Über die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen. *Journal für die reine und angewandte Mathematik* 40: 209–227.

Edwards 1977. Edwards, Harold. 1977. *Fermat's Last Theorem: A Genetic Introduction to Algebraic Number Theory*. Berlin, New York, etc. : Springer.

Edwards 1980. Edwards, Harold. 1980. The Genesis of Ideal Theory. *Archive for the History of Exact Sciences* 23: 321–378.

Edwards 1992. Edwards, Harold. 1992. Mathematical Ideas, Ideals and Ideology. *The Mathematical Intelligencer* 14: 6–19.

Edwards 2007. Edwards, Harold. 2007. Composition of Binary Quadratic Forms and the Foundations of Mathematics. [Goldstein, Schappacher & Schwermer 2007]: 129–144.

Ferreiros 2007. Ferreiros, Jose. 2007. The Rise of Pure Mathematics as Arithmetic with Gauss. [Goldstein, Schappacher & Schwermer 2007]: 235–268.

Fisher 1966. Fisher, Charles. The Death of a Mathematical Theory. *Archive for the History of Exact Sciences* 3: 137–159.

Foucault 1982/2001. Foucault, Michel. *Dits et Ecrits*. Paris : Gallimard.

Gauss 1801/1966. Carl Friedrich Gauss. 1801. *Disquisitiones Arithmeticae*. Leipzig: Fleischer. English transl. by Arthur Clarke, New Haven: Yale University Press, 1966.

Gauss 1831. Gauss, Carl Friedrich. 1831. Recension der Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seeber. *Göttingische gelehrte Anzeigen* 2: 1065-1077.

Gauthier 2009. Gauthier, Sébastien. 2009. La géométrie dans la géométrie des nombres : histoire de discipline ou histoire de pratiques à partir des exemples de Minkowski, Mordell et Davenport. *Revue d'histoire des mathématiques* 15-2: 183–230.

Gauthier 2011. Gauthier, Sébastien. 2011. Justifier l'utilisation de la géométrie en théorie des nombres : des exemples chez C.F. Gauss et H. Minkowski. *Justifier en mathématiques*, ed. Domminique Flament and Philippe Nabonnand: 103-128. Paris : Éditions de la Maison des sciences de l'homme.

Goldstein 1994. Goldstein, Catherine. 1994. La théorie des nombres dans les *Notes aux Comptes rendus de l'Académie des sciences* (1870–1914): un premier examen. *Rivista di Storia della scienza*, II, 2, 2:137-160.

Goldstein 1999. Goldstein, Catherine. 1999. Sur la question des méthodes quantitatives en histoire des mathématiques: le cas de la théorie des nombres en France (1870–1914). *Acta historiae rerum naturalium nec non technicarum* New series 3-28:187-214.

Goldstein 2007. Goldstein, Catherine. 2007. The Hermitian Form of Reading the *Disquisitiones*. [Goldstein, Schappacher & Schwermer 2007]: 377–410.

Goldstein 2011. Goldstein, Catherine. 2011. Un arithméticien contre l'arithmétisation : les principes de Charles Hermite. *Justifier en mathématiques*, ed. Dominique Flament & Philippe Nabonnand, 129–165. Paris: MSH.

Goldstein 202?. Goldstein, Catherine. 202?. The Unity of Mathematics according to Charles Hermite, in preparation.

Goldstein & Schappacher 2007a. Goldstein, Catherine and Norbert Schappacher. 2007. A Book in Search of A Discipline (1801-1860). [Goldstein, Schappacher & Schwermer 2007]: 3–65.

Goldstein & Schappacher 2007b. Goldstein, Catherine and Norbert Schappacher. 2007. Several Disciplines and a Book (1860-1900). [Goldstein, Schappacher & Schwermer 2007]: 67–103.

Goldstein, Schappacher & Schwermer 2007. Goldstein, Catherine, Norbert Schappacher and Joachim Schwermer, eds. 2007. *The Shaping of Arithmetic after C. F. Gauss's Disquisitiones Arithmeticae*. Heidelberg, Berlin, etc.: Springer.

Gray 2000. Gray, Jeremy. 2000. *Linear differential equations and Group theory from Riemann to Poincaré*. 2nd ed. Boston: Birkhäuser.

Gray 2006. Gray, Jeremy. 2006. A History of Prizes in Mathematics. *The Millennium Prize Problems*, ed. James Carlson, Arthur Jaffe & Andrew Wiles, 3–30. Cambridge, Mass. USA, and Providence, Rhode Island, USA: CMI/AMS.

Gray 2013. Gray, Jeremy. 2013. *Henri Poincaré: A Scientific Biography*. Princeton: Princeton University Press.

Guntau & Laitko 1987. Guntau, Martin & Laitko, Hubert. 1987. *Der Ursprung der modernen Wissenschaften: Studien zur Entstehung wissenschaftlicher Disziplinen*. Berlin: Akademie Verlag.

Haffner 2014. Haffner, Emmylou. 2014. D'un point de vue rigoureux et parfaitement général : pratique des mathématiques rigoureuses chez Richard Dedekind. *Philosophia Scientiae* 18: 131–156.

Haffner 2019. Haffner, Emmylou. 2019. From modules to lattices: Insight into the genesis of Dedekind's Dualgruppen. *British Journal for the History of Mathematics* 34-1: 23–42,

Haubrich 1992. Haubrich, Ralf. 1992. *Zur Entstehung der algebraischen Zahlentheorie Richard Dededkinds*. Dissertation. Göttingen: Georg-August-Universität.

Hermite 1850. Hermite, Charles. 1850. Lettres à M. Jacobi sur différents objets de la théorie des nombres. *Journal für die reine und angewandte Mathematik* 40: 261–315.

Hermite 1851. Hermite, Charles. 1851. Sur l'introduction des variables continues dans la théorie des nombres. *Journal für die reine und angewandte Mathematik* 41: 191–216.

Hermite 1854. Hermite, Charles. 1854. Sur la théorie des formes quadratiques ternaires indéfinies. *Journal für die reine und angewandte Mathematik* 47, 307-312.

Hermite & Mittag-Leffler. Hermite, Charles and Gösta Mittag-Leffler. 1984. Lettres à Gösta Mittag-Leffler, [ed. P. Dugac]. *Cahiers du séminaire d'histoire des mathématiques* (1 part. 1874–1883) 5: 49–285; (2nd part.1884–1891): 6 (1985), 69-217; (3rd part. 1891–1900): 10 (1989), 1–82.

Hermite & Stieltjes. Hermite, Charles and Thomas Stieltjes. 1905. *Correspondance*, ed. Benjamin Baillaud and Henri Bourget. Paris: Gauthier-Villars.

Jordan 1868-1869. Jordan, Camille.1868-1869. Sur les groupes de mouvements. *Annali di matematica* s. 2, 2: 167–215, 322–345.

Jordan 1879. Jordan, Camille. 1879. Sur l'équivalence des formes algébriques. *Comptes rendus hebdomadaires des séances de l'Académie des sciences* 88: 906–908.

Knight 1981. Knight, David. 1981. *Ordering the World: A History of Classifying Man*. London: urnett Books.

Lê 2018. Lê, François. 2018. Le paramétrage elliptique des courbes cubiques par Alfred Clebsch. *Revue d'histoire des mathématiques* 24-1: 1–39.

Lê 2023. Lê, François. 2023. Des taxons et des nombres : quelques remarques sur les ordres, classes et genres des courbes algébriques. *Revue d'histoire des sciences*, à paraître.

Lê & Paumier 2016. Lê, François and Anne-Sandrine Paumier. 2016. De la science comme classification à la classification comme pratique scientifique. *Cahiers François Viète*, s. III, 1: 9–34.

Lepetit 1995. Lepetit, Bernard (dir.). 1995. *Les formes de l'expérience*. Paris: Albin Michel.

Marx & Engels 1846/1969. Marx, Karl and Friedrich Engels. 1846/1969. Die Deutsche Ideologie. *Werke*, vol. 3. Berlin: Dietz Verlag.

Nabonnand & Rollet 2002. Nabonnand, Philippe and Laurent Rollet. 2002. Une bibliographie mathématique idéale? Le Répertoire bibliographique des sciences mathématiques. *Gazette des mathématiciens* 92: 11–26.

Parshall 1989. Parshall, Karen. 1989. Toward a History of Nineteenth-Century Invariant Theory. *The History of Modern Mathematics*, ed. David E. Rowe and John Mc-Cleary: vol. 1: 157–206. Boston: Academic Press

Parshall 2006. Parshall, Karen. 2006. The British development of the theory of invariants (1841–1895). *The Bulletin of the British Society for History of Mathematics* 21: 186–199.

Parshall 2023. Parshall, Karen. 2023. A Convergence of Paths: Arthur Cayley, Charles Hermite, James Joseph Sylvester, and the Early Development of a Theory of Invariants. *Revue d'histoire des mathématiques*, to appear.

Poincaré 1879a. Poincaré, Henri. 1879. Sur quelques propriétés des formes quadratiques. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 89: 344–346.

Poincaré 1879b. Poincaré, Henri. 1879. Sur les formes quadratiques. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 89: 897–899.

Poincaré 1880a. Poincaré, Henri. 1880. Sur les formes cubiques ternaires. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 90: 1336–1339-.

Poincaré 1880b. Poincaré, Henri. 1880. Sur la réduction simultanée d'une forme quadratique et d'une forme linéaire.*Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 91: 844–846.

Poincaré 1880c. Poincaré, Henri. 1880. Sur un mode nouveau de représentation géométrique des formes quadratiques définies ou indéfinies. *Journal de l'École polytechnique* 47: 177–245.

Poincaré 1881a. Poincaré, Henri. 1881. Sur les fonctions fuchsiennes. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 92: 333–335.

Poincaré 1881b. Poincaré, Henri. 1881. Sur la représentation des nombres par des formes. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 92: 777-779.

Poincaré 1881c. Poincaré, Henri. 1881. Sur les formes cubiques ternaires et quaternaires. Première partie. *Journal de l'Ecole polytechnique* 50: 190–253.

Poincaré 1882a. Poincaré, Henri. 1882. Sur les invariants arithmétiques. In *Comptes rendus de la 10e session - Alger 1881*, 109–117. Paris: Association française pour l'avancement des sciences.

Poincaré 1882b. Poincaré, Henri. 1882. Sur les applications de la Géométrie non-euclidienne à la théorie des formes quadratiques. In *Comptes rendus de la 10e session - Alger 1881*, 132–138. Paris: Association française pour l'avancement des sciences.

Poincaré 1882c. Poincaré, Henri. 1882. Sur les formes cubiques ternaires et quaternaires. Seconde partie. *Journal de l'Ecole polytechnique* 51: 45–91.

Poincaré 1882d. Poincaré, Henri. 1882. Sur une extension de la notion arithmétique de genre. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 94: 67–69, 124–127.

Poincaré 1884. Poincaré, Henri. 1884. Sur une généralisation des fractions continues. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 99: 1014–1016.

Poincaré 1885. Poincaré, Henri. 1885. Sur la représentation des nombres par les formes. *Bulletin de la Société Mathématique de France* 13: 162-194.

Poincaré 1886a. Poincaré, Henri. 1886. Mémoire sur la réduction simultanée d'une forme quadratique et d'une forme linéaire. *Journal de l'Ecole polytechnique* 56: 79-142.

Poincaré 1886b. Poincaré, Henri. 1886. Sur les fonctions fuchsiennes et les formes quadratiques ternaires indéfinies. *Comptes rendus des séances hebdomadaires de l'Académie des sciences* 102: 735–737.

Poincaré 1886c. Poincaré, Henri. 1886. *Notice sur les travaux scientifiques*. Paris: Gauthier-Villars.

Poincaré 1887. Poincaré, Henri. 1887. Les fonctions fuchsiennes et l'arithmétique. *Journal de mathématiques pures et appliquées* 4e s., 3: 405–464.

Poincaré 1891a. Poincaré, Henri. 1891. Sur la distribution des nombres premiers. *Comptes rendus hebdomadaires des séances de l'Académie des Sciences* 113: 819.

Poincaré 1891b. Poincaré, Henri. 1891. Extension aux nombres premiers complexes des théorèmes de M. Tchebicheff. *Journal de mathématiques pures et appliquées* 4e s., 8: 25–68.

Poincaré 1889. Poincaré, Henri. 1889. *Leçons sur la théorie mathématique de la lumière*. Paris : Georges Carré.

Poincaré 1897. Poincaré, Henri. 1897. Les rapports de l'analyse pure et de la physique mathématiques. *Acta mathematica* 21: 331–341.

Poincaré 1901. Poincaré, Henri. 1901. Sur les propriétés arithmétiques courbes algébriques. *Journal de mathématiques pures et appliquées* 5e s. 7(2): 161–233.

Poincaré 1905a. Poincaré, Henri. 1905. Sur les invariants arithmétiques. *Journal für die reine und angewandte Mathematik* 129 (2): 89–150.

Poincaré 1905b. Poincaré, Henri. 1905. La Valeur de la science. Paris: Flammarion. English transl. by George Bruce Halsted. New York and Garrison: The Science Press, 1913.

Poincaré 1908a. Poincaré, Henri. 1908. L'Avenir des mathématiques. *Bulletin des sciences mathématiques* 2e s. 32: 168–190; eng. transl. The future of mathematics, *The Monist* 20 (1910): 76–92.

Poincaré 1908b. Poincaré, Henri. 1908. L'invention mathématique. *L'Enseignement Mathématique* 10, 357–371.

Poincaré 1921. Poincaré, Henri. 1921. Analyse des travaux scientifiques. *Acta Mathematica* 38: 3–135.

Poincaré 1950. Poincaré, Henri. 1950. *Œuvres*, Tome 5, ed. Albert Châtelet. Paris: Gauthier-Villars.

Poincaré 1986. Poincaré, Henri. 1986. La correspondance d'Henri Poincaré avec des mathématiciens (A-J), [ed. P. Dugac]. *Cahiers du séminaire d'histoire des mathématiques* 7: 59–219.

Rey 1994. Rey, Roselyne. 1994. La Classification des sciences (1750-1850). *Revue de synthèse* ser. IV, 1-2: 5–12.

Rynasiewicz & Renn 2005. Rynasiewicz, Robert, and Jürgen Renn. 2005. The turning point for Einstein's Annus mirabilis. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 37 (1):5–35.

Saint-Gervais 2011. de Saint-Gervais, Henri Paul. 2011. *Uniformisation des surfaces de Riemann : Retour sur un théorème centenaire*. Lyon: ENS Éditions.

Schappacher 1991. Schappacher, Norbert. 1991. Développement de la loi de groupe sur une cubique. *Séminaire de théorie des nombres de Paris 1988/1989*: 159–184. Basel, etc: Birkhäuser.

Schneider 2000. Schneider, Martina. 2000. *Poincarés Beitrag zum Studium rationaler Punkte auf elliptischen Kurven: Eine Verknüpfung von Funktionentheorie, Geometrie und Zahlentheorie*. Unpublished Diplomarbeit, Fachbereich Mathematik, Bergische Universität Gesamthochschule Wuppertal.

Scholz 1989. Scholz, Erhard. 1989. *Symmetrie, Gruppe, Dualität*. Basel: Birkhäuser.

Seeber 1824. Seeber, Ludwig August. 1824. Versuch zur Erklärung des inneren Baues der festen Körper. *Annalen der Physik*16 (3): 229–248, 349–372.

Selling 1877. Selling, Eduard. 1877. Des formes quadratiques binaires et ternaires. *Journal de mathématiques pures et appliquées* 3e s. 3: 21–60.

Siegmund-Schulze 1993. Siegmund-Schulze, Reinhard. 1993. Mathematische Berichterstattung in Hitlerdeutschland: der Niedergang des *Jahrbuchs über die Fortschritte der Mathematik*. Göttingen: Vandenhoeck & Ruprecht.

Smith 1861–1865. Smith, Henry John Stephen. 1861-1865. Report on the Theory of Numbers, part III-V. *Report of the British Association for the Advancement of Science* 1861: 292–340; 1862: 503–526; 1865: 768–786.

Tort 1989. Tort, Patrick. 1989. *La Raison classificatoire*. Paris: Aubier.

Weil 1955. Weil, André. 1955. Poincaré and arithmetic. In *Livre du centenaire de la naissance de Henri Poincaré*, 206–212. Paris: Gauthier-Villars.