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# On Prescribed-time Cooperative State Estimation of LTI Plants <sup>\*</sup>

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**Abstract:** The problem of prescribed-time cooperative state estimation is addressed in this paper. A network of strongly connected (linear homogenous) distributed observers are designed under the assumption that each observer can receive a time-varying scalar signal broadcasted by a supervisor. By a proper parameter tuning, the observer gains go infinity and estimation errors correspondingly reach zeros at the settling time prescribed by users.

*Keywords:* Cooperative state estimation, linear homogeneity, prescribed-time stability, finite-time stability, nonlinear systems.

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## 1. INTRODUCTION

Recent years, cyber-physical systems (CPS) are widely applied in industrial scenarios, whose output components are usually geographically dispersed in a large region. In most cases, no single sensor is able to obtain the global output information of CPS due to the geographic dispersion. To reconstruct the state of CPS, traditional Luenburger state observers based on global outputs are unable to be designed. In view of this problem, the cooperative state estimation technique based on consensus protocols is developed, where multiple observers are coordinated in a network, and each of them use local and neighboring estimations to recover the state of the plant. This new observer is known as the distributed observer.

The seminal result on cooperative state estimation is proposed by Park and Martins (2012). Under the assumption of global observability, a group of distributed Luenburger state observers are designed to cooperatively reconstruct the states of a discrete-time plant. For continuous-time plants, the distributed observers are designed by Kim et al. (2016). The work mentioned above mainly focus on the design of linear distributed observers, whose state estimation is accomplished with time goes into infinity. In this situation, it is hard to guarantee the performance of next-step control utilizing the estimated states. Some scholars also concentrate on the convergence rate of the observers. Han et al. (2018) propose a scheme that the plant states are estimated with guaranteed convergence rates. However, it can only increase the decay rates of estimation errors, but cannot change the fact that estimation errors are asymptotically stable. Based on that, nonlinear (weighted homogeneous) distributed observers are introduced by Silm et al. (2018), under which the estimation errors stabilize in a finite time. It is known that

the settling time of a finite-time stable system is dependent on the initial states of the system (Levant (1998), Andrieu et al. (2008)) which may also influence the control performance because real-time states may cannot be provided in a desired time. It is necessary to develop some new technique such that the cooperative estimation achieved with the settling time independent of system states, or even prescribed by users.

Indeed, for centralized systems, the research on control systems with state-independent settling time is not a new topic. There are two main branches of the existing research. On the one hand, as an extension of finite-time stabilization, fixed-time stabilization is proposed by Polyakov (2012), where the settling time is bounded for all initial states. On the other hand, a prescribed-time approach is proposed by Song et al. (2017), which employs a finite-time feedback input with time-varying gains that tends to infinity as time approaches the prescribed settling time. However, there are plenty of existing results regarding the prescribed-time control for distributed systems. For example, prescribed-time consensus of linear/nonlinear multi-agent systems (Jing and Wang (2018), Yong et al. (2012), Colunga et al. (2018)); prescribed-time containment control for multi-agent systems (Wang et al. (2018)); prescribed-time leader-following consensus of multi-agent systems (Wang and Song (2018)); cluster synchronization of complex networks (Liu et al. (2018)); lag consensus of second order leader-following multi-agent systems (Ren et al. (2021)); prescribed-time bipartite consensus tracking (Ning et al. (2020)); prescribed-time consensus over time-varying graph via time scaling (Kan et al. (2017), Yucelen et al. (2018), Arabi et al. (2021), Tran et al. (2021), Kurtoglu and Yucelen (2021)).

The centralized fixed-time and prescribed-time state observers are developed in Lopez-Ramirez et al. (2018) and Holloway and Krstic (2019), respectively. To address the cooperative state estimation, we propose to design these kind of observers which work distributively. To the best

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of our knowledge, such research has not received much attention; the only known paper (see Gong et al. (2021)) presents a prescribed-time observer for multi-agent system with SISO agents, but it is hard to demonstrate the distributed nature of the observer in the SISO case. Based on the above discussion, this paper develops a class of prescribed-time distributed observers for cooperative state estimation of linear MIMO plants, where each observer shares the results of local estimates to its neighbors under a strongly connected graph. The key idea of our paper is to upgrade the classical linear distributed observer to a homogeneous one following the scheme proposed in Wang et al. (2021). We show that such an upgrade is possible if all agent obtain synchronously a scalar time-varying signal which scales dynamically the gains of observers to guarantee a prescribed-time convergence (see Song et al. (2017) for more details). There are several ways to generate such a signal. For example, if all agents may have synchronized clocks then they can generate the required signal, simultaneously. Another approach by Li et al. (2022) assumes that there is a supervisor broadcasting the scalar signal simultaneously to all agents. Inspired by generalized homogeneity, in this paper we design a dynamics of the supervisor in such a way that the cooperative state estimation is achieved in a prescribed time. Below we show that this approach fits the concept proposed in Song et al. (2017), Holloway and Krstic (2019), which, in fact, *implicitly* uses the homogeneity (dilation symmetry) principles as well.

This paper is organized as follows. Section 2 gives some basic knowledge on graph theory, stability definitions and homogeneous systems. The problem to be studied is formulated in Section 3; The basic idea upgrading the classical linear distributed observers to homogeneous ones is proposed in Section 4; The main result regarding the design of prescribed-time observers and the convergence analysis are in Section 5; Finally, in Section 6, the effectiveness of the designed homogeneous observers is illustrated by some numerical simulations.

*Notation.*  $\mathbb{N}_+$  is the set of positive integers;  $\mathbb{R}$  is the set of real numbers;  $\mathbb{R}_{\geq 0}$  is the set of non-negative real numbers;  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times n}$  denote the  $n \times 1$  real vector and the  $n \times n$  real matrix, respectively;  $\|\cdot\|$  is the norm in  $\mathbb{R}^n$ ;  $I_N$  is the  $N \times N$  identity matrix;  $\mathbf{1}_N$  is the  $N$ -dimension vector whose components are all ones;  $\mathbf{0}$  is the matrix/vector composed of zeros with proper dimension;  $\exp(\cdot) = e^{(\cdot)}$ , with  $e$  is the Euler number;  $\text{diag}\{\sigma_i\}_{i=1}^N$  is the  $N \times N$  diagonal matrix with elements  $\sigma_i$ ;  $\lambda_{\max}(\cdot)$  and  $\lambda_{\min}(\cdot)$  denote the maximum and minimum eigenvalue of a matrix, respectively;  $P \succ 0$  ( $\prec 0$ ) for  $P \in \mathbb{R}^{n \times n}$  means that the matrix  $P$  is symmetric and positive (negative) definite;  $\otimes$  represents the Kronecker product;  $C(X, Y)$  denotes the space of continuous functions  $X \mapsto Y$ , where  $X, Y$  are subsets of normed vector spaces;  $C^\rho(X, Y)$  is the space of functions continuously differentiable at least up to the order  $\rho$ .

## 2. PRELIMINARIES

### 2.1 Graph Theory

In graph theory, a fixed directed graph  $\mathcal{G}$  is usually characterized by the *node set*  $\mathcal{V}$ , the *edge set*  $\mathcal{E}$  and the

*adjacency matrix*  $\mathcal{A}$ . To be specific, the node set  $\mathcal{V} = \overline{1, N}$ , which contains all the nodes at the graph labeled by  $i = 1 \dots N$ ; the edge set  $\mathcal{E} = \{(i, j) | i, j \in \mathcal{V}\}$ ,  $(i, j) \in \mathcal{E}$  if node  $j$  is able to transfer its local information to node  $i$ , the number of incoming edges of node  $i$  is  $n_i$ ;  $\mathcal{A} = \{a_{ij}\}$ , where  $a_{ij} = 1$  for  $(i, j) \in \mathcal{E}$  and  $a_{ij} = 0$  for  $(i, j) \notin \mathcal{E}$ . A *directed path* from node  $i$  to node  $j$  is a sequence of node  $\{i_1 \dots i_s\}$ , where  $i_1 = i$ ,  $i_s = j$  and  $(i_\kappa, i_{\kappa+1}) \in \mathcal{E}$ ,  $\kappa \in \overline{1, s-1}$ . The directed graph  $\mathcal{G}$  is *strongly connected* if there exists at least one directed path between each pair of the nodes. The *Laplacian matrix* associated to graph  $\mathcal{G}$  is defined as  $\mathcal{L} = \{l_{ij}\}$ , where  $l_{ij} = -a_{ij}$  for  $i \neq j$  and  $l_{ij} = \sum_{k=1}^N a_{ik}$  for  $i = j$ .

*Lemma 1.* [Ren and Beard (2005)] *For a strongly connected graph, the associated Laplacian matrix has a simple eigenvalue zero and all other eigenvalues have positive real parts.*

*Lemma 2.* [Yu et al. (2009)] *Assume  $\mathcal{L}$  be the Laplacian matrix corresponding to a strongly connected graph  $\mathcal{G}$ . Then  $\mathcal{L}$  can be similarly transformed by  $T$  and its inverse, which is given as  $T^{-1}\mathcal{L}T = \text{diag}\{0, \Delta\}$ , where  $\Delta$  is a Jordan block related to nonzero eigenvalues of  $\mathcal{L}$ ,  $T = (\mathbf{1}_N, Y_1)$  and  $T^{-1} = (\zeta, Y_2)^\top$ ,  $Y_1$  is composed of the eigenvectors of  $\mathcal{L}$  which associated to nonzero eigenvalues,  $Y_2^\top Y_1 = I_{N-1}$ ,  $\zeta = (\zeta_1 \dots \zeta_N)^\top$  is the left zero eigenvector of  $\mathcal{L}$ , which yields*

$$\zeta_i > 0, \mathbf{1}_N^\top \zeta = 1 \text{ and } \zeta^\top \mathcal{L} = \mathbf{0}. \quad (1)$$

### 2.2 Stability Notations

A system is considered with the following presentation:

$$\dot{\zeta}(t) = f(t, \zeta(t)), \quad t \geq 0, \quad (2)$$

where  $\zeta(t) \in \mathbb{R}^d$ ,  $d \in \mathbb{N}_+$  is the state vector; the vector field  $f: \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^d$  is assumed to be piece-wise continuous on time variable for any fixed state and continuous on state variable for almost all instances of time, such that  $f(\cdot, \mathbf{0}) = \mathbf{0}$  (the origin is an equilibrium). A solution of (2) is denoted as  $S(t, \zeta_0)$  with  $\zeta_0 = \zeta(0)$ .

To present the notion of the prescribed-time stability, some classes of functions are needed to be introduced.

*Definition 1.* [Khalil (1996)] *A continuous function  $\vartheta(\rho): \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$  belongs to*

- *$\mathcal{K}$  function if it is strictly increasing with  $\vartheta(0) = 0$ ;*
- *$\mathcal{GK}$  function if it is strictly increasing on  $\rho \in [\rho_0, +\infty)$  and  $\vartheta(\rho) = 0$  on  $\rho \in [0, \rho_0]$ .*

*Definition 2.* [Khalil (1996), Hong et al. (2010)] *A continuous function  $\varpi(\rho, t): \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$  belongs to*

- *$\mathcal{KL}$  function if it belongs to  $\mathcal{K}$  function for each fixed  $t \in \mathbb{R}_{> 0}$ , and, for each fixed  $\rho \in \mathbb{R}_{\geq 0}$ , it has  $\lim_{t \rightarrow +\infty} \varpi(\rho, t) = 0$ ;*
- *$\mathcal{GKL}$  function if it is a  $\mathcal{GK}$  function for each fixed  $t \in \mathbb{R}_{> 0}$ , and, for each fixed  $\rho \in \mathbb{R}_{\geq 0}$  there exists  $T(\rho) \in \mathbb{R}_{> 0}$  such that  $\lim_{t \rightarrow T(\rho)} \varpi(\rho, t) = 0$ .*

*Definition 3.* *The origin of the system (2) is said to be*

- *globally asymptotically stable [Khalil (1996)] if  $\exists \varpi \in \mathcal{KL}$  such that  $\|S(t, \zeta_0)\| \leq \varpi(\|\zeta_0\|, t)$ ,  $\forall t \in \mathbb{R}_{\geq 0}$  and  $\forall \zeta_0 \in \mathbb{R}^d$ ;*

- globally uniformly finite-time stable [Bhat and Bernstein (2000)] if it is globally uniformly asymptotically stable with  $\varpi \in \mathcal{GKL}$ ;
- globally uniformly fixed-time stable [Polyakov (2012)] if it is globally uniformly finite-time stable and  $\exists T_{\max} > 0$  such that  $T(\varrho) \leq T_{\max}, \forall \varrho \in \mathbb{R}_{>0}$ ;
- globally uniformly prescribed-time stable [Song et al. (2017)] if it is fixed-time stable and  $S(t, \varsigma_0) \neq \mathbf{0}$  for all  $t \in [0, T_{\max})$  if  $\varsigma_0 \neq \mathbf{0}$ .

In the view of the latter definition, the prescribed-time stability is a version of the finite-time stability, where the settling time is a constant (prescribed) value.

### 2.3 Dilation in $\mathbb{R}^n$

Homogeneity is an invariance of an object with respect to a class of transformations called dilations. Choosing a proper dilation group  $\mathbf{d}(s)$ ,  $s \in \mathbb{R}$  is vital for the homogeneity-based analysis,  $\mathbf{d}(s)$  is supposed to satisfy the limit property:  $\lim_{s \rightarrow \pm\infty} \|\mathbf{d}(s)x\| = \exp(\pm\infty)$  for  $\forall x \neq \mathbf{0}$ . Examples of dilations are as follows: *standard dilation* (introduced by L. Euler 18th century), defined as  $\mathbf{d}(s) = \exp(s)I$ ,  $s \in \mathbb{R}$ ; *weighted dilation* (Zubov (1958)) defined as  $\mathbf{d}(s) = \text{diag}\{\exp(r_1 s) \dots \exp(r_n s)\} \in \mathbb{R}^{n \times n}$ , where  $r_i > 0$ ,  $i \in \overline{1, n}$ ; *linear dilation* (Polyakov (2019)) defined as  $\mathbf{d}(s) = \exp(G_{\mathbf{d}} s)$ ,  $s \in \mathbb{R}^n$  where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix known as the generator of the dilation; *geometric dilation* (Khomenuk (1961), Kawski (1995)) which is defined as a flow generated by an unstable  $C^1$  vector field.

### 2.4 Homogeneous Systems

*Definition 4.* A vector field  $f: \mathbb{R}^n \mapsto \mathbb{R}^n$  (resp. a function  $h: \mathbb{R}^n \mapsto \mathbb{R}$ ) is said to be  $\mathbf{d}$ -homogeneous if there exist a  $\mu \in \mathbb{R}$  such that

$$f(\mathbf{d}(s)x) = \exp(\mu s)\mathbf{d}(s)f(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n,$$

$$\text{(resp. } h(\mathbf{d}(s)x) = \exp(\mu s)h(x), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n)$$

where  $\mathbf{d}$  is a dilation and the scalar  $\mu \in \mathbb{R}$  is known as the homogeneous degree of  $f$  (resp.  $h$ ).

One of the main advantages of homogenous systems is that whose finite-time stability can be achieved by an asymptotic stability scheme.

*Lemma 3.* [Bhat and Bernstein (1997)] Let  $f$  be a continuous  $\mathbf{d}$ -homogeneous vector field of degree  $\mu \in \mathbb{R}$ . The evolution system  $\dot{x} = f(x)$  is globally finite-time stable if  $\mu < 0$  and the system is asymptotically stable at the equilibrium.

Below we show that in the case of a discontinuous vector field  $f$ , for  $\mu < 0$  the homogeneous system may also be *partially* prescribed-time stable. Namely, one part of coordinates may converge to zero in a prescribed time dependent on the initial state of another part of coordinates.

## 3. PROBLEM STATEMENT

Consider a linear plant with multiple outputs:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & (3a) \\ y_i(t) = C_i x(t), \quad i \in \overline{1, N} & (3b) \end{cases}$$

where  $x(t) \in \mathbb{R}^n$  is the plant state;  $A \in \mathbb{R}^{n \times n}$ ;  $B \in \mathbb{R}^{n \times q}$ ;  $u \in \mathbb{R}^q$  is the control input;  $y_i \in \mathbb{R}^{p_i}$  is the  $i$ th portion of the measured output with  $C_i \in \mathbb{R}^{p_i \times n}$ . The plant (3) satisfies the following assumptions:

*Assumption 4.* The pair  $(A, B)$  is controllable and  $A$  is a nilpotent matrix.

*Remark 1.* The assumption  $A$  is a nilpotent matrix implies that the vector field  $x \mapsto Ax$  is linear homogeneous (Polyakov (2020)). This assumption is not conservative in our case and the explanation is as follows. Since  $(A, B)$  is controllable,  $\exists u_0 = K_0 x$ ,  $K_0 \in \mathbb{R}^{q \times n}$  such that  $A + BK_0$  is nilpotent (Zimenko et al. (2020)). Without loss of generality, let  $u = u_0 + u^{new}$ , the linear plant (3a) becomes  $\dot{x}(t) = A^{new}x(t) + Bu^{new}(t)$ , where  $A^{new} = A + BK_0$  is nilpotent.

*Assumption 5.* [Global Observability, Park and Martins (2012)] The pair  $(A, C)$  is observable, where

$$C = (C_1^\top \dots C_N^\top)^\top \in \mathbb{R}^{p \times n}, \quad p = \sum_{i=1}^N p_i.$$

*Remark 2.* According to pole placement theory in Chen (1984), the global observability implies that  $\exists \hat{H} \in \mathbb{R}^{n \times p}$  and  $P_1 \in \mathbb{R}^{n \times n}$  such that

$$P_1 \succ 0, \quad P_1 A + A P_1 + P_1 \hat{H} C + C^\top \hat{H}^\top P_1 \prec 0. \quad (4)$$

The goal of this work is to design a network of linear homogeneous observers  $\dot{\hat{x}}_i = f(y_i, \xi, \hat{x}_{j_1} \dots \hat{x}_{j_{n_i}})$ ,  $i \in \overline{1, N}$ ,  $\xi \in \mathbb{R}$ ,  $j_k \in \mathcal{V} : (i, j_k) \in \mathcal{E}$  to cooperatively reconstruct the state of (3a) in a prescribed time  $t_p \in \mathbb{R}_{\geq 0}$ , i.e.,  $\lim_{t \rightarrow t_p} (\hat{x}_i(t) - x(t)) = \mathbf{0}$ ,  $i \in \overline{1, N}$ . The design bases on the following assumption.

*Assumption 6.* All observers at the network are capable to receive a scalar signal  $\xi(t)$  broadcasted by a supervisor.

*Remark 3.* The existence of the scalar signal  $\xi(t)$  does not destroy the local distributed working style of the observer since  $\xi(t)$  is independent of any states or outputs of the whole system. The reason to introduce such a scalar  $\xi(t)$  is to regulate the gains in the homogeneous observers to be infinity at the prescribed time  $t_p$ . This is an essential part to achieve prescribed-time state estimation. Such a combination, i.e., utilizing the external  $\xi(t)$  and the local plant output as well as information from neighbors for state estimation, is named as the distributed prescribed-time cooperative estimation.

To achieve our purpose, we need to design properly a distributed observer

$$\dot{\hat{x}}_i = r(\hat{x}_i, y_i, \xi), \quad t \geq 0, \quad r: \mathbb{R}^n \times \mathbb{R}^{p_i} \times \mathbb{R} \mapsto \mathbb{R}^n \quad (5)$$

for (3) and a dynamic of the broadcasting scalar signal

$$\dot{\xi} = g(\xi), \quad t \geq 0, \quad g: \mathbb{R} \mapsto \mathbb{R} \quad (6)$$

such that

- the dynamic of the observation error

$$e = ((\hat{x}_1 - x)^\top \dots (\hat{x}_N - x)^\top)^\top,$$

system has the following presentation

$$\dot{e} = \varphi(e, \xi), \quad (7)$$

where  $\varphi: \mathbb{R}^{nN} \times \mathbb{R} \mapsto \mathbb{R}^{nN}$ ;

- the vector field  $(e, \xi) \mapsto \begin{pmatrix} \varphi(e, \xi) \\ g(\xi) \end{pmatrix}$  is linear homogeneous of a given degree  $\mu \in \mathbb{R}$ ;

- the augmented system (7), (6) is globally uniformly asymptotically stable and the error subsystem (7) is globally uniformly prescribed-time stable for  $\mu < 0$ .

#### 4. BASIC IDEA

Let us start from the classical linear distributed observer. Firstly, a useful lemma is introduced as follows:

*Lemma 7.* [Liu et al. (2017)] *The matrix defined as*

$$\Lambda = \begin{pmatrix} G & R \\ R^\top & F - \nu I_n \end{pmatrix}$$

*is negative definite for a sufficiently large  $\nu$  if  $G < 0$  and  $F$  is symmetric.*

*Theorem 8.* [Liu et al. (2017)] *Assume a group of linear distributed observers connected under a strongly connected topology  $\mathcal{G}$ , and whose dynamics are*

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x}_i + Bu + H_i(C_i\hat{x}_i - y_i) + \nu \sum_{j=1}^N a_{ij}(\hat{x}_j - \hat{x}_i), \\ H_i &\in \mathbb{R}^{n \times p_i}, \nu \in \mathbb{R}, i \in \overline{1, N}. \end{aligned} \quad (8)$$

*The error system (7) with  $\xi = 0$  is asymptotically stable if*

$$H_i = \frac{\hat{H}_i}{\zeta_i}, \quad \nu > \nu_l = \frac{\lambda_{\max}\{M_4 - M_2^\top M_1^{-1} M_2\}}{\lambda_{\min}\{P_2 \Delta + \Delta^\top P_2\}}, \quad (9)$$

*where*

$$\begin{aligned} M_1 &= P_1 \hat{A} + \hat{A}^\top P_1, \quad M_2 = P_1 \hat{B} + \hat{C}^\top (P_2 \otimes I_n), \\ M_4 &= (P_2 \otimes I_n) \Phi + \Phi^\top (P_2 \otimes I_n), \quad \hat{A} = A + \sum_{i=1}^N \zeta_i H_i C_i, \\ \hat{B} &= (\zeta^\top \otimes I_n) (\tilde{A} + \tilde{H} \tilde{C}) (Y_1 \otimes I_n), \\ \hat{C} &= (Y_2^\top \otimes I_n) (\tilde{A} + \tilde{H} \tilde{C}) (\mathbf{1}_N \otimes I_n), \\ \Phi &= (Y_2^\top \otimes I_n) (\tilde{A} + \tilde{H} \tilde{C}) (Y_1 \otimes I_n), \\ \tilde{A} &= I_N \otimes A, \quad \tilde{H} = \text{diag}\{H_i\}_{i \in \overline{1, N}}, \quad \tilde{C} = \text{diag}\{C_i\}_{i \in \overline{1, N}}, \end{aligned}$$

*the vector  $\zeta$  with component  $\zeta_i \neq 0$  is defined in (1),  $\Delta$ ,  $Y_1$  and  $Y_2$  are given in Lemma 2,  $\hat{H}_i$  is the component of  $\hat{H} = (\hat{H}_1 \dots \hat{H}_N)$ , and  $\hat{H}$  is defined in (4),  $P_1$  is given in (4) and  $P_2$  satisfies*

$$P_2 > 0, \quad P_2(-\Delta) + (-\Delta)^\top P_2 < 0. \quad (10)$$

It is obvious that the linear distributed observers (8) (correspondingly, the augmented system (7)) are homogeneous of degree  $\mu = 0$ . A scheme for upgrading a linear centralized Luenburger state observer to a linear homogeneous one is developed by Wang et al. (2021), which suggest us to replace the linear term  $H_i(C_i\hat{x}_i - y_i)$  with a homogeneous one  $\exp((G_0 + I_n)\mu \ln \xi) H_i(C_i\hat{x}_i - y_i)$ , where  $G_0 \in \mathbb{R}^{n \times n}$  is a properly designed matrix such that the dilation generator  $G_d := \mu G_0 + I_n$  anti-Hurwitz, and  $\xi$  is a *scalar signal* defined in a special way to guarantee finite/fixed-time convergence. In the case of observers working as the multi-agent system, the signal should be received by all individuals simultaneously.

As shown before, to guarantee the prescribed time state estimation this signal can be selected as a certain function of time. In practice, its computation may be realized by means of clock synchronization for all agents or by broadcasting of the required signal to all agents simultaneously.

#### 5. PRESCRIBED-TIME OBSERVER DESIGN AND CONVERGENCE ANALYSIS

Given  $\mu \neq 0$  and  $G_0 \in \mathbb{R}^{n \times n}$  satisfying the following identities

$$AG_0 = (G_0 + I_n)A, \quad CG_0 = \mathbf{0}. \quad (11)$$

From Polyakov (2020), it is known that such a  $G_0$  can always be found provided that  $(A, C)$  is observable and  $A$  is nilpotent.

*Theorem 9.* *Assume a network of homogenous observers, which interact among each other under a strongly connected graph  $\mathcal{G}$ , and the dynamics are as follows:*

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x}_i + Bu + \exp((G_0 + I_n)\mu \ln \xi) H_i(C_i\hat{x}_i - y_i) \\ &+ \nu \xi^\mu \sum_{j=1}^N a_{ij}(\hat{x}_j - \hat{x}_i), \quad H_i \in \mathbb{R}^{n \times p_i}, \nu \in \mathbb{R}, i \in \overline{1, N} \end{aligned} \quad (12)$$

*where  $G_0 \in \mathbb{R}^{n \times n}$  is obtained from (11), and  $\xi$  is calculated according to*

$$\begin{aligned} \dot{\xi}(t) &= -\gamma \xi^{\mu+1}(t), \quad t \in [0, t_1], \\ t_1 &= \frac{-\xi^{-\mu}(0)}{\gamma \mu}, \quad \xi(0) > 0, \quad \gamma \in \mathbb{R} \end{aligned} \quad (13)$$

*and we assign  $\xi(t) = 1, \forall t \geq t_1$ . Let  $\gamma > 0, \mu < 0, H_i = \frac{\hat{H}_i}{\zeta_i}$  and  $\nu > \nu_l$ , where  $\zeta_i \neq 0, \hat{H}_i$  and  $\nu_l$  are defined in (1), (4) and (9), then*

- *the differential equation  $\varphi(e, \xi)$  in (7) has the following presentation*

$$\varphi = \left[ \tilde{A} + (I_N \otimes \exp((G_0 + I_n)\mu \ln \xi)) \tilde{H} \tilde{C} - \nu \xi^\mu (\mathcal{L} \otimes I_n) \right] e, \quad (14)$$

*which together (13) is  $\tilde{\mathbf{d}}$ -homogeneous of degree  $\mu \in \mathbb{R}$ ,*

$$\tilde{\mathbf{d}}(s) = \begin{pmatrix} I_N \otimes \mathbf{d}(s) & \mathbf{0} \\ \mathbf{0} & \exp(s) \end{pmatrix},$$

*where  $s \in \mathbb{R}$  and the dilation  $\mathbf{d}(s)$  is generated by  $G_d = \mu G_0 + I_n$ ;*

- *the error system (7) is globally uniformly prescribed-time stable with the settling time  $T_{max} = \frac{-\xi^{-\mu}(0)}{\gamma \mu}$ .*

*Remark 4.* Users are able to arbitrarily prescribe the settling time  $t_p$  by choosing proper  $\gamma > 0, \xi(0) > 0$  or/and  $\mu < 0$ .

#### 6. SIMULATION RESULTS

Let the system matrixes of the LTI plant (3) be

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and output matrixes corresponding to  $y_i, i = 1, 2, 3$  are  $C_1 = (1 \ 1 \ 0), C_2 = (0 \ 0 \ 1)$  and  $C_3 = (1 \ 0 \ 0)$ . From (4), we have  $\hat{H}_1 = (0.9999 \ -1.0000 \ -2.0000)^\top, \hat{H}_2 = (-1.0000 \ 1.0000 \ -1.0000)^\top, \hat{H}_3 = (-1.9999 \ -0.9999 \ 2.9999)^\top$ .

To recover the state of the LTI plant, three observers are employed to form the distributed prescribed-time observer (12), whose communication graph is as Fig.1 shows. Corresponding Laplacian matrix can be calculated and the associated zero-eigenvector is  $\zeta = (0.3333 \ 0.3333 \ 0.3333)^\top$ . Then, from Theorem 9, we can obtain  $H_1 = 3 \times \hat{H}_1, H_2 = 3 \times \hat{H}_2, H_3 = 3 \times \hat{H}_3$ . Meanwhile, we let  $\nu = 15$ .

In addition, from the identity (11), we have

$$G_0 = \begin{pmatrix} -0.0136 & 0.1568 & 0.0326 \\ 0.1568 & -0.3193 & -0.0061 \\ 0.0326 & -0.0061 & 0.0364 \end{pmatrix}.$$

Then, the generator of dilation  $G_d = \mu G_0 + I_3$  can be specified by letting  $\mu = -0.4$ .

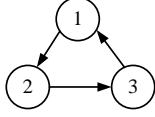


Fig. 1. The communication graph of observers

The estimation error  $e = (e_1^\top \dots e_3^\top)^\top$ , where  $e_i = \hat{x}_i - x$  is a vector contains the estimation error between the  $i_{th}$  observer and the state of the LTI plant. In order to make the simulation more clear, we introduce  $\psi_1 = (e_1(1) \ e_2(1) \ e_3(1))^\top$ ,  $\psi_2 = (e_1(2) \ e_2(2) \ e_3(2))^\top$ ,  $\psi_3 = (e_1(3) \ e_2(3) \ e_3(3))^\top$ , in which each  $\psi_i$  contains the estimation errors of all observers respect to  $i_{th}$  state of the LTI plant.

Let the initial state of the observers be zeros and the initial state of the LTI plant be  $x(0) = (1.0000 \ 1.5000 \ 2.5000)^\top$ . The control input  $u = Kx$  with  $K = (-0.0000 \ -2.3104 \ -0.0000)$  is such that the closed-loop LTI plant Lyapunov stable. Based on all the above parameters and the simulation step  $h = 0.01s$ , the MATLAB platform and the implicit Euler method are adopted for simulation. Theoretically, the states of the prescribed time observer reach the state of the LTI plant at the prescribed time  $t_p = 2.5s$  and the external signal  $\xi(t) \rightarrow 0$  as  $t \rightarrow t_p$ . However, in practice this reaching cannot be exact, for example, due to discretization issues. In fact, the sampled-time implementation of the prescribed-time observer implies instability due to infinite growth of observers gains as  $t \rightarrow t_p$  (resp. as  $\xi \rightarrow 0$ ). That is why the signal is taken  $\xi(t) = 1$  after the reaching phase. Since  $\exp((G_0 + I_n)\mu \ln \xi) = I_n$  if  $\xi = 1$  then the prescribed-time observer coincides with the original linear observer for  $t \geq t_p$ . Such a simple switching approach allows an acceleration the convergence of the observer (see Fig. 2).

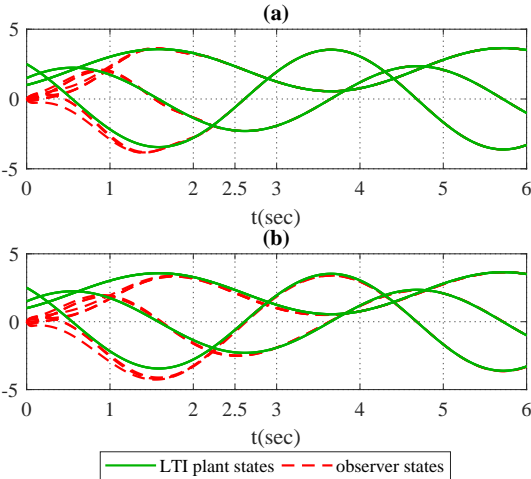


Fig. 2. Trajectories of LTI plant states and states of (a) prescribed-time observers; (b) linear observers.

Fig.3-Fig.5 show the comparison of  $\psi_i$ ,  $i \in \overline{1,3}$  between the asymptotic stabilization by linear observers and the prescribed-time stabilization by homogeneous observers. It is clear that errors settle at the prescribed time.

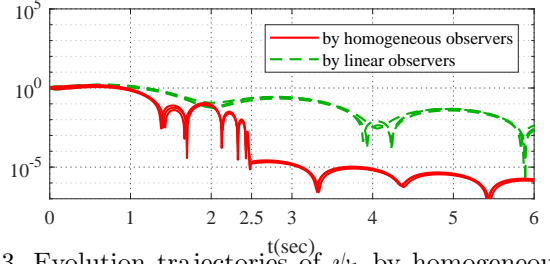


Fig. 3. Evolution trajectories of  $\psi_1$  by homogeneous observers and linear observers.

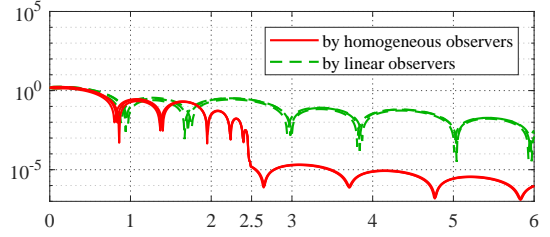


Fig. 4. Evolution trajectories of  $\psi_2$  by homogeneous observers and linear observers.

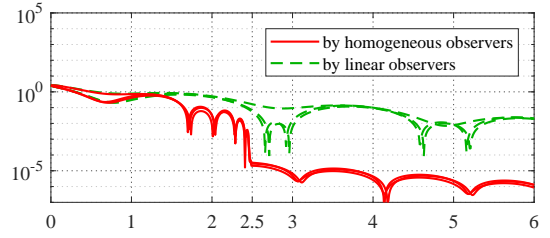


Fig. 5. Evolution trajectories of  $\psi_3$  by homogeneous observers and linear observers.

## 7. CONCLUSION

In this paper, a prescribed-time cooperative state estimation problem is addressed by a scheme of distributed homogeneous observers, which is proposed by upgrading the classical linear distributed observers with the concept of homogeneity. This novel scheme demonstrates a possibility to extend the design framework of prescribed-time observers from centralized to distributed one. The prescribed-time algorithms have certain difficulties in practical/digital implementation. The homogeneity may allow to overcome them in future using methods of consistent discretization.

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