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► **To cite this version:**

Y Zhou, H Ríos, M Mera, A Polyakov, G Zheng, et al.. Trajectory Tracking in Unicycle Mobile Robots: A Homogeneity-based Control Approach. IFAC 2023 - 22nd World Congress of the International Federation of Automatic Control, Jul 2023, Yokohama, Japan. hal-04059857

**HAL Id: hal-04059857**

**<https://hal.science/hal-04059857>**

Submitted on 5 Apr 2023

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# Trajectory Tracking in Unicycle Mobile Robots: A Homogeneity–based Control Approach <sup>★</sup>

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**Abstract:** This paper contributes to the design of a family of homogeneous controllers to deal with the trajectory tracking problem in unicycle mobile robots (UMRs). The control strategy is based on a cascade control paradigm, which includes the position and orientation tracking error subsystems. The proposed homogeneous controllers provide different types of convergence to zero for the tracking error dynamics, *i.e.*, asymptotic, exponential and finite–time convergence. Simulation results illustrate the performance of the proposed family of homogeneous controllers.

*Keywords:* Unicycle Mobile Robots, Trajectory Tracking, Homogeneity, Finite–Time.

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## 1. INTRODUCTION

The Unicycle mobile robot (UMR) as a well–known and widespread nonlinear system has attracted much interest for a long history. Non–holonomic vehicles are a class of systems that cannot be stabilized asymptotically via smooth static state feedback Brockett et al. (1983). In general, there are two solutions: non–smooth feedback control (see, *e.g.*, Bloch and Drakunov (1996) and Astolfi (1996)) and smooth time–varying feedback control (see, *e.g.*, Samson (1995) and Tian and Li (2002)).

Homogeneity is a dilation symmetry. Homogeneous systems have some important properties such as faster convergence, better robustness and less overshoots (for more details, see Hong (2001), Bhat and Bernstein (2005), Andrieu et al. (2008) and Polyakov (2020)). Many homogeneity–based controllers for UMRs proposed in the literature guarantee just exponential stabilization (see, *e.g.*, Kimura et al. (2013), Kimura et al. (2015), and M’Closkey and Murray (1997)).

From a practical perspective, a trajectory tracking task is more important than stabilization for a UMR (see, *e.g.*, Rochel et al. (2022), Maghenem et al. (2017), Zhai and Song (2019) ). Stabilization of the UMR is challenging because the linearized system around the origin loses

controllability. But the linearized system around some moving trajectories could be controllable, which make the tracking control design more flexible De Luca et al. (2001).

Finite–time control on the unicycle model has attracted a lot of attention in recent decades due to its fast convergence and robustness properties. There are two major approaches about the finite–time control design: direct Lyapunov method and homogeneity–based method.

Since any asymptotically stable homogeneous system with negative degree is finite–time stable Bhat and Bernstein (2005), many finite–time controllers based on homogeneity have been proposed (see, *e.g.*, Kimura et al. (2016), Wu et al. (2018) and Li and Tian (2007), Ding et al. (2010)).

*Motivation:* The majority of homogeneity–based tracking control designs focus on the system with a negative degree of homogeneity. The different convergence rates of a homogeneous system can be determined by simply adjusting the homogeneity degree. Thus, it motivates us to develop a homogeneous control algorithm providing different convergence rates. The cascade structure is widely used in UMR control design owing to the underactuated property of UMR. In Li and Tian (2007), Ding et al. (2010), homogeneity–based algorithms for the finite–time trajectory tracking problem are investigated using a cascade structure. However, these algorithms require a nonzero desired angular velocity, therefore they are unable to track some trajectories, *e.g.*, a simple straight line. It motivates us to design a homogeneity control based on a new cascade structure.

In this paper, we aim to develop a homogeneous tracking control for a UMR kinematic. The homogeneous control

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<sup>★</sup> This work was supported in part by the SEP–CONACYT–ANUIES–ECOS NORD Project 315597 and ECOS NORD Project M20M04. Y. Zhou acknowledges the support of CSC Grant, China. H. Ríos was supported in part by CONACYT IxM CVU 270504 Project 922, and in part by TecNM Projects. M. Mera was supported in part by Project IPN–SIP 20230170. Andrey Polyakov acknowledges the support of the National Natural Science Foundation of China under Grant 62050410352.

design, and stability analysis, are based on an implicit homogeneous Lyapunov function called canonical homogeneous norm. Considering the underactuated characteristic of the system, a cascade control scheme is adopted.

*Main Contribution:* The trajectory tracking problem in UMRs is solved by means of a family of homogeneous controllers. The control design and stability analysis are based on homogeneity properties and implicit Lyapunov function. It is able to provide different types of convergence to zero, *i.e.*, asymptotic, exponential, and finite-time convergence. The control utilizes a flatness-based cascade structure that allows it to track a wider range of trajectories.

The following is how this paper is structured. The problem is modeled in Section II. Section III presents useful knowledge of homogeneous systems. Section IV summarizes the main results of this research on cascade system design, homogeneous control design, and stability analysis. The simulation validations are presented in Section V. The concluding remarks are given in Section VI. Finally, some supporting results about Input-to-State Stability are given in the Appendix.

*Notation:* Let  $\mathbb{R}$  be the set of real numbers,  $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha \geq 0\}$ ;  $\|\cdot\|$  be a norm in  $\mathbb{R}^n$ ,  $\mathbf{0}$  denotes the zero vector from  $\mathbb{R}^n$ ;  $P \succ 0$  ( $\prec 0, \succeq 0, \preceq 0$ ) for  $P \in \mathbb{R}^{n \times n}$  means that the matrix  $P$  is symmetric and positive (negative) definite (semidefinite);  $\lambda_{\min}(P)$  and  $\lambda_{\max}(P)$  represent the minimal and maximal eigenvalue of a matrix  $P = P^\top$ ; for  $P \succeq 0$  the square root of  $P$  is a matrix  $M = P^{\frac{1}{2}}$  such that  $M^2 = P$ . The conical Euclidean norm for  $x \in \mathbb{R}^n$  is denoted as  $\|x\|_2 = \sqrt{x^\top x}$ . Let denote by  $\mathcal{K}$  the set of continuous increasing functions map  $\mathbb{R}_+$  to  $\mathbb{R}_+$ . The set of unbounded  $\mathcal{K}$  functions is denoted by  $\mathcal{K}_\infty$ . Let continuous function  $\beta(\cdot, \cdot) \in \mathcal{KL}$  if it is  $\mathcal{K}$  with respect to the first argument and strictly decreasing to zero with respect to the second argument.  $L^\infty$  is the set of the essentially bounded measure functions.  $L^1_{loc}$  is the set of local Lebesgue measure functions.

## 2. PROBLEM STATEMENT

Let us consider the kinematic model of a UMR:

$$\begin{cases} \dot{r} = v\xi, & r(0) = r_0, \\ \dot{\theta} = \omega, & \theta(0) = \theta_0, \end{cases} \quad (1)$$

where  $r = [x, y]^\top \in \mathbb{R}^2$  is the planar position vector,  $v \in \mathbb{R}_+$  is the velocity magnitude,  $\xi = [\cos(\theta), \sin(\theta)]^\top$  is the unit vector that defines the direction of the velocity,  $\theta \in \mathbb{R}$  is the angle between the vector  $r$  and the  $x$ -axis,  $\omega \in \mathbb{R}$  is the angular velocity. We assume that  $x, y$ , and  $\theta$  are the only available measurements of the system, which may be noised.

The system is unable to track any arbitrary trajectory due to the non-holonomic constraint

$$\dot{x} \sin(\theta) - \dot{y} \cos(\theta) = 0. \quad (2)$$

However, the system (1) is a differential flatness system, and a feasible trajectory can be defined according to the flat output and its derivatives. Selecting the flat output as  $(x, y)$ . For given desired position  $x_d, y_d \in C^2(\mathbb{R}_+, \mathbb{R})$ , according to the system we have can determine the desired velocity magnitude  $v_d$ :

$$\dot{r}_d = v_d \xi_d \Rightarrow v_d = \|\dot{r}_d\|_2 = \sqrt{\dot{x}_d^2 + \dot{y}_d^2},$$

where  $r_d = [x_d, y_d]^\top$ ,  $\xi_d = [\cos(\theta_d), \sin(\theta_d)]^\top$ . The desired trajectory should satisfy the non-holonomic constraint (2). Then the desired angle and desired angular velocity can be determined from the derivative of the  $x_d$  and  $y_d$ ,

$$\theta_d = \arctan\left(\frac{\dot{y}_d}{\dot{x}_d}\right), \quad \dot{\theta}_d = \omega_d = \frac{\ddot{y}_d \dot{x}_d - \dot{y}_d \ddot{x}_d}{\dot{x}_d^2 + \dot{y}_d^2}.$$

We define a *feasible trajectory* for the system (1) as a pair  $r_d \in C^2(\mathbb{R}_+, \mathbb{R}^2)$ ,  $\theta_d \in C^2(\mathbb{R}_+, \mathbb{R})$  satisfying

$$\begin{cases} \dot{r}_d = v_d \xi_d, \\ \dot{x}_d \sin(\theta_d) - \dot{y}_d \cos(\theta_d) = 0. \end{cases} \quad (3)$$

The conventional tracking problem for the perturbation-free system (1) can be formulated as follows: design a control law of the inputs  $v$  and  $\omega$  such that for any desired feasible trajectory  $r_d, \theta_d$  one holds

$$\|r(t) - r_d(t)\|_2 \rightarrow 0, \quad \|\theta(t) - \theta_d(t)\|_2 \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In practice, some additional properties may be requested for the control system, *e.g.*, stability of the error dynamics<sup>1</sup>, specification of the convergence rate, minimization of overshoots, robustness.

In this paper, we utilize the concept of generalized homogeneity (see below) allowing additional properties mentioned above to be guaranteed. In particular, we investigate the possibility of homogeneous finite-time<sup>2</sup> stabilization of the error dynamics.

## 3. PRELIMINARIES

The homogeneity (dilation symmetry) based analysis and design of control system requires a dilation to be specified. By definition, a dilation  $\mathbf{d}(s)$ , with  $s \in \mathbb{R}$ , is a one-parameter group of transformations satisfying the limit property  $\lim_{s \rightarrow s^\infty} \|\mathbf{d}(s)x\| = \exp(s^\infty)$ ,  $s^\infty = \pm\infty, \forall x \neq \mathbf{0}$  Husch (1970).

The linear dilation in  $\mathbb{R}^n$  is defined as follows Polyakov (2019)

$$\mathbf{d}(s) = \exp(sG_{\mathbf{d}}) := \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}, \quad s \in \mathbb{R}^n,$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz<sup>3</sup> matrix called the generator of the dilation  $\mathbf{d}$ .

**Definition 1.** A vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (resp. a function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ ) is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if

$$\begin{aligned} f(\mathbf{d}(s)x) &= \exp(\mu s) \mathbf{d}(s) f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}, \\ (\text{resp.}, h(\mathbf{d}(s)x) &= \exp(\mu s) h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}), \end{aligned}$$

where  $\mathbf{d}$  is a linear dilation in  $\mathbb{R}^n$ .

An essential characteristic of a homogeneous system is the existence of a homogeneous Lyapunov function. The

<sup>1</sup> Asymptotic convergence does not imply stability in the general case.

<sup>2</sup> The system  $\dot{e} = f(t, e), t > t_0$  is globally uniformly finite-time stable if it is uniformly Lyapunov stable and there exists a locally bounded function  $T : \mathbb{R}^n \mapsto \mathbb{R}_+$  such that  $\forall e_0 \in \mathbb{R}^n, e(t_0) = e_0 \Rightarrow e(t) = \mathbf{0}, \forall t \geq t_0 + T(e_0)$ .

<sup>3</sup> A matrix  $G_{\mathbf{d}}$  is anti-Hurwitz if  $-G_{\mathbf{d}}$  is Hurwitz.

analysis in this study is based on the implicit homogeneous Lyapunov function known as the canonical homogeneous norm.

**Definition 2.** *Polyakov (2019).* The function  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$  defined as  $\|\mathbf{0}\|_{\mathbf{d}} = 0$  and

$$\|u\|_{\mathbf{d}} = \exp(s_u), \text{ where } s_u \in \mathbb{R} : \|\mathbf{d}(-s_u)u\| = 1,$$

is called the canonical  $\mathbf{d}$ -homogeneous norm in  $\mathbb{R}^n$ , where  $\mathbf{d}$  is a linear strictly monotone dilation<sup>4</sup>.

### 3.1 Homogeneous Stabilization of Linear Plants

Let us consider the linear plant

$$\dot{x} = Ax + Bu, \quad t > 0, \quad (4)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ .

**Theorem 1.** *Zimenko et al. (2020).* Let the pair  $(G_0, Y_0) \in G_0 \times \mathbb{R}^{m \times n}$  be a solution of the linear matrix equation

$$AG_0 + BY_0 = G_0A + A, \quad G_0B = \mathbf{0}, \quad (5)$$

such that  $G_0 - I_n$  is invertible and the matrix  $G_{\mathbf{d}} = I_n + \mu G_0$  is anti-Hurwitz. Let the homogeneous control be defined as follows

$$u = u_{hom}(x) := K_0x + \|x\|_{\mathbf{d}}^{1+\mu} K \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x, \quad (6)$$

where  $K_0 = Y_0(G_0 - I_n)^{-1}$ ,  $K = YX^{-1}$  with the pair  $(X, Y) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{m \times n}$  being a solution of the system

$$\begin{cases} XA_0^T + A_0X + Y^T B^T + BY + \rho(XG_{\mathbf{d}}^T + G_{\mathbf{d}}X) = \mathbf{0}, \\ XG_{\mathbf{d}}^T + G_{\mathbf{d}}X > 0, \quad X \succ 0, \quad A_0 = A + BK_0, \end{cases} \quad (7)$$

and the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is induced by the norm  $\|x\| = \sqrt{x^T X^{-1}x}$ . Then, the closed-loop system (4) and (6) is  $\mathbf{d}$ -homogeneous of degree  $\mu$  and

- Globally uniformly finite-time stable for  $\mu < 0$ ;
- Globally uniformly exponentially stable for  $\mu = 0$ ;
- Globally uniformly nearly fixed-time stable for  $\mu > 0$ .

Moreover,  $\|\cdot\|_{\mathbf{d}}$  is a Lyapunov function of the system:

$$\frac{d}{dt} \|x\|_{\mathbf{d}} = -\rho \|x\|_{\mathbf{d}}^{1+\mu}. \quad (8)$$

If the pair  $\{A, B\}$  is controllable then any solution of the system (5) satisfies conditions of the latter theorem for  $\mu \leq 1/k$ , where  $k$  is a minimal number such that  $\text{rank}[B, AB, \dots, A^{k-1}B] = n$  (see Nekhoroshikh et al. (2021) for more details).

## 4. HOMOGENEITY-BASED CONTROL FOR THE UMR

### 4.1 Cascade Control Design

The main aim of this paper is to develop control algorithms for trajectory tracking by the unicycle vehicle being an under-actuated system. A widely used model for UMR tracking problems in the absence of disturbances is proposed in Kanayama et al. (1990). However, the model is not straightforward in the control design. Thus, this section presents a cascade control paradigm following the flatness idea.

<sup>4</sup> A dilation is strictly monotone if the function  $s \mapsto \|\mathbf{d}(s)x\|$  is strictly monotone for any  $x \neq \mathbf{0}$ . dilation in  $\mathbb{R}^n$ . Any liner dilation in  $\mathbb{R}^n$  is strictly monotone under a proper selection of a weighted Euclidean norm Polyakov (2019).

Let us define the tracking errors as

$$e_r = r - r^*, \quad (9)$$

where  $r^*$  is a feasible nominal trajectory such that

$$r^*(t) \rightarrow r_d(t) \quad \text{as } t \rightarrow +\infty.$$

Following the flatness of the system, the angle error and the angular error can be determined using the position error and its derivatives.

Then, the tracking error system is described as follows:

$$\dot{e}_r = v\xi - \dot{r}^*, \quad \dot{e}_\theta = \omega - \dot{\theta}^*. \quad (10)$$

where  $\theta^*$  is an artificial angle command driven by the position subsystem.

Following the back-stepping ideas, we assume that there exists virtual controllers  $u_r^*$  and  $u_\theta^*$  dependent on  $e_r$  and  $e_\theta$ , which stabilize the error subsystems  $\dot{e}_r = u_r^* \dot{e}_\theta = u_\theta^*$ , respectively. In this case, we can rewrite the error dynamics as follows

$$\begin{cases} \dot{e}_r = u_r^* + \delta_1, \\ \dot{e}_\theta = u_\theta^* + \delta_2, \end{cases} \quad (11)$$

where

$$\delta_1 = v\xi - \dot{r}^* - u_r^*, \quad \delta_2 = \omega - \dot{\theta}^* - u_\theta^*. \quad (12)$$

Therefore, to stabilize the error dynamics of  $e_r$ , we need to define the control inputs  $v$  and  $\omega$  such that

$$\delta = (\delta_1, \delta_2)^T \rightarrow 0.$$

The entire system (11) is under-actuated, the position subsystem is under-actuated by the input  $v$ , but the orientation subsystem is fully-actuated. Hence, the idea of the cascade structure is that the position subsystem determines the desired velocity vector  $v\xi^*$ , the  $v$  is the actual input of the system, then the direction command  $\xi^*$  is tracked by the orientation subsystem. To track the velocity direction  $\xi^*$ , the angular command  $\omega^*$  should be properly defined.

Let us select the control inputs as follows:

$$v = \|\dot{r}^* + u_r^*\|_2, \quad \omega = \omega^* + u_\theta^*, \quad (13)$$

where

$$\dot{\theta}^* = \omega^* := \frac{(\dot{r}^* + u_r^*)^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{d(\dot{r}^* + u_r^*)}{dt}}{\|\dot{r}^* + u_r^*\|_2^2}, \quad (14)$$

$$\theta^*(0) = \arctan \left( \frac{[0,1](\dot{r}^*(0) + u_r^*(0))}{[1,0](\dot{r}^*(0) + u_r^*(0))} \right), \quad (15)$$

where  $r^*(0)$  and  $u_r^*(0)$  denote the values of  $r^*$  and  $u_r^*$  at the initial instant of time  $t = 0$ .

**Lemma 1.** Let the functions  $t \mapsto \dot{r}^*(t)$  and  $t \mapsto u_r^*(t)$  be continuously differentiable on  $\mathbb{R}_+$ , the function  $t \mapsto \omega^*(t)$  be locally Lebesgue integrable, the set

$$\Delta = \{t \in \mathbb{R}_+ : \dot{r}^*(t) + u_r^*(t) = \mathbf{0}\}, \quad (16)$$

be finite and  $0 \notin \Delta$ . Then

- the differential equation (14), (15) has the unique solution  $t \mapsto \theta^*(t)$  satisfying

$$\theta^*(t) = \theta^*(0) + \int_0^t \omega^*(\tau) d\tau, \quad t \in \mathbb{R}_+, \quad (17)$$

that is absolutely continuous on  $\mathbb{R}_+$  and continuously differentiable on  $\mathbb{R}_+ \setminus \{\Delta\}$ ;

- the vector-valued function  $t \mapsto \xi^*(t) := \begin{bmatrix} \cos(\theta^*(t)) \\ \sin(\theta^*(t)) \end{bmatrix}^T$ , satisfies the identity

$$\xi^*(t) = \frac{\dot{r}^*(t) + u_r^*(t)}{\|\dot{r}^*(t) + u_r^*(t)\|_2}, \quad \forall t \in \mathbb{R}_+; \quad (18)$$

- the function  $t \mapsto e_\theta(t)$  is absolutely continuous on  $\mathbb{R}_+$  provided that  $\omega$  is locally Lebesgue integrable.

If the conditions of Lemma 1 are fulfilled and  $v$  and  $\omega$  are defined as in (13) then the set  $\Delta$  defines some instances of time, where  $\theta^*$  has a singularity. In the control design, it is assumed that the measure of the set  $\Delta$  is zero. It is not conventional. Since  $\Delta$  does not contain any equilibrium, the assumption holds in most cases. So, the error equation (11) is fulfilled almost everywhere with

$$\delta_1 = \|r^* + u_r^*\| (\xi - \xi^*), \quad \delta_2 = u_\theta^* + \omega^*.$$

Since  $\|\xi\| = 1$  and

$$\xi - \xi^* = \begin{bmatrix} \cos(\theta) - \cos(\theta^*) \\ \sin(\theta) - \sin(\theta^*) \end{bmatrix} = 2 \sin\left(\frac{e_\theta}{2}\right) \begin{bmatrix} -\sin\frac{1}{2}(\theta + \theta^*) \\ \cos\frac{1}{2}(\theta + \theta^*) \end{bmatrix}$$

then

$$\|\delta_1\|_2 \leq \|r^* + u_r^*\|_2 (2\|e_\theta\|_2), \quad (19)$$

for almost all  $t \in \mathbb{R}_+$ .

**Remark 1.** Notice that the function  $\theta^*$  is defined by the formula (14) just for simplicity of the mathematical constructions. Since  $\theta^*(0) = \tilde{\theta}(0)$  and  $\frac{d\tilde{\theta}(t)}{dt} = \dot{\theta}^*(t)$  almost every on  $\mathbb{R}_+$ , then in practice, a computation of  $\theta^*(t)$  can be realized using an "arctan" formula

$$\theta^*(t) = \arctan\left(\frac{[0 \ 1] (\dot{r}^*(t) + u_r^*(t))}{[1 \ 0] (\dot{r}^*(t) + u_r^*(t))}\right). \quad (20)$$

The coupling term  $\delta \rightarrow 0$  as  $e_\theta \rightarrow 0$ . However,  $\delta$  may be non-vanishing in the general case, so below we use the generalized homogeneity to analyse the robustness of the control system with respect to the coupling term. In the following section, we show that  $u_r^*$  and  $u_\theta^*$  can always be selected such that the error dynamics of the system (11) is locally homogeneous in a certain sense.

## 4.2 Locally Homogeneous Error Dynamics

Based on the above scheme, we construct a homogeneous cascade system in this section. The orientation command (15) calls for the derivative of  $u_r^*$ . To guarantee continuous differentiability of  $u_r^*$ , let us define the virtual control  $u_r^*$  as follows

$$\begin{aligned} u_r^* &= u, \\ \dot{u} &= u_{hom}(e^*) := \|e^*\|_{\mathbf{d}^*}^{1+\mu} K \mathbf{d}^* (-\ln \|e^*\|_{\mathbf{d}^*}) e^*, \end{aligned} \quad (21)$$

where  $e^* = [e_r^\top, u^\top]^\top$  and the homogeneous feedback law  $u_{hom}$  is designed according to Theorem 1, where  $A = \begin{bmatrix} \mathbf{0} & I_2 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ ,  $B = \begin{bmatrix} \mathbf{0} \\ I_2 \end{bmatrix}$ ,  $K_0 = \mathbf{0}$ ,  $\mu \in [-1, 1/2]$ ,  $\mathbf{d}^*$  is a dilation in  $\mathbb{R}^4$  generated by  $G_{\mathbf{d}^*} = \begin{bmatrix} (1-\mu)I_2 & \mathbf{0} \\ \mathbf{0} & I_2 \end{bmatrix}$ ,  $K = YX^{-1}$  and the pair  $(X, Y)$  is a solution of (7) with some  $\rho > 0$ , and  $\|\cdot\|_{\mathbf{d}^*}$  is the canonical homogeneous norm induced by the norm  $\|e^*\| = \sqrt{(e^*)^\top X^{-1} e^*}$ .

Let the virtual control of the orientation be defined as:

$$u_\theta^* = -\alpha |e_\theta|^\nu e_\theta, \quad \alpha > 0, \nu \geq -1. \quad (22)$$

The closed-loop system can be represented as follows

$$\begin{aligned} \dot{e} &= f(e) + \delta, \\ f(e) &:= [f_1^\top(e^*) \ f_2^\top(e_\theta)]^\top, \\ f_1(e^*) &= Ae^* + Bu_{hom}(e^*), \quad f_2(e_\theta) = u_\theta^*(e_\theta), \end{aligned} \quad (23)$$

where  $\delta = [\delta_1^\top, \mathbf{0}, \delta_2]^\top$ ,  $e = [e^{\ast\top}, e_\theta]^\top$ . Notice that the vector field  $f$  is homogeneous. Indeed, the vector field  $f_1$  is  $\mathbf{d}^*$ -homogeneous of degree  $\mu$ . The vector field  $f_2$  is  $e^s$ -homogeneous of degree  $\nu$ . If  $\nu\mu > 0$  or  $\nu = \mu = 0$ , then the vector field  $f$  is  $\mathbf{d}$ -homogeneous of degree  $\mu$  for the dilation in  $\mathbb{R}^5$  defined as follows  $\mathbf{d}(s) = \begin{bmatrix} \mathbf{d}^*(s) & \mathbf{0} \\ \mathbf{0} & \exp(\gamma s) \end{bmatrix}$ , where  $\gamma = \frac{\mu}{\nu}$  for  $\nu \neq 0$  and  $\gamma = 1$  for  $\nu = 0$ . Simple computations show that  $f(\mathbf{d}(s)e) = \exp(\mu s) \mathbf{d}(s) f(e)$ , for all  $e \in \mathbb{R}^5$ ,  $\forall s \in \mathbb{R}$ . Therefore, the system (23) can be regarded as a homogeneous system with perturbations; however, the perturbation  $\delta$  is dependent on the subsystem states. The stability and robustness of (23) is investigated below.

## 4.3 Stability of the Error Dynamics

The system (23) can be treated as two homogeneous subsystems and the term  $\delta = \delta(\dot{r}^*, \omega^*, u_r^*, u_\theta^*, e_\theta)$  defines an interconnection of these systems. For  $\delta = 0$ , the system (23) is  $\mathbf{d}$ -homogeneous with degree  $\mu$ . Since  $\delta_2$  is independent of  $e^*$  then stability and robustness properties of the subsystems can be analyzed based on ISS of the cascade systems (see Appendix and Sontag et al. (1989), Sontag (1998) and Angeli et al. (2000)).

**Theorem 2.** The control inputs  $v$  and  $\omega$  for the system (1) be defined by the formulas (13), (14), (21), (22). If  $\sup_{t \in \mathbb{R}_+} \|r^*\| = \bar{v} < +\infty$  and  $\omega^* \in L_{loc}^1$ ; then, the error system (23) is:

- Globally finite-time stable for  $\nu < 0, \mu < 0$ ;
- Globally asymptotically stable  $-1 \leq \mu, \nu \leq 0$ ;
- Globally asymptotically stable and locally exponentially stable for  $\mu = \nu = 0$ .

The control scheme is summarized as follows:

$$\begin{aligned} v &= \|r^* + u_r\|_2, \quad u_r = f \|e^*\|_{\mathbf{d}^*}^{1+\mu} K \mathbf{d}^* (-\ln \|e^*\|_{\mathbf{d}^*}) e, \\ \omega &= \dot{\theta}^* + u_\theta, \quad u_\theta^* = -\alpha |e_\theta|^\nu e_\theta, \\ \dot{\theta}^* &= \frac{(\dot{r}^* + u_r^*)^\top \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{d(\dot{r}^* + u_r^*)}{dt}}{\|\dot{r}^* + u_r^*\|_2^2}. \end{aligned}$$

Note that when  $\mu = \nu = 0$ , the controllers are linear; when  $\mu = \nu = -1$ , the controllers are sliding-mode controllers. The simulation below demonstrates the parameters tuning and performance validation.

## 5. SIMULATION RESULTS

A numerical simulation illustrates the performance of the controllers proposed in this paper. The simulation is done in Simulink with the sampling step  $10^{-4}$ . The lemniscate trajectory

$$x_d(t) = \frac{\cos(\phi t)}{1 + \sin^2(\phi t)}, \quad y_d(t) = \frac{\sin(\phi t) \cos(\phi t)}{1 + \sin^2(\phi t)},$$

with  $\phi = 0.8$ , is used as a benchmark for mobile robot trajectory tracking. In this case, We set  $r^* = r_d$ . The orientation control parameters are  $\nu = -0.5$  and  $\alpha = -8$ . The position control parameters are obtained by solving the system (7) in MATLAB using the packages Yalmip and SEDUMI for  $\mu = -0.5$  and  $\rho = 2$ :  $K = [-19.6938I_2, -5I_2]$ ,

$$P = \begin{bmatrix} 30.4741I_2 & 4.6422I_2 \\ 4.6422I_2 & 1.5474I_2 \end{bmatrix}, \quad G_{\mathbf{d}^*} = \begin{bmatrix} 1.5I_2 & \mathbf{0} \\ \mathbf{0} & I_2 \end{bmatrix}.$$

The initial state of the system is  $r(0) = [1.5, 0.25]^\top$ ,  $\theta = 0$ . In this example, we select  $r^*(t) = r_d(t)$ . The simulation results are depicted in Fig. 1, 2, 3. As seen in Fig. 1, there is a reaching phase due to an initial error  $e_r(0) = [0.5, 0.25]^\top$ , but the vehicle quickly reaches the target trajectory and then follows it. Fig. 2 confirms the finite-time convergence (see the logarithmic scales for the error). Variations of the stabilization error for values less than  $10^{-10}$  are caused by computational errors. The control signals  $v$  and  $\omega$  are presented in Fig. 3.

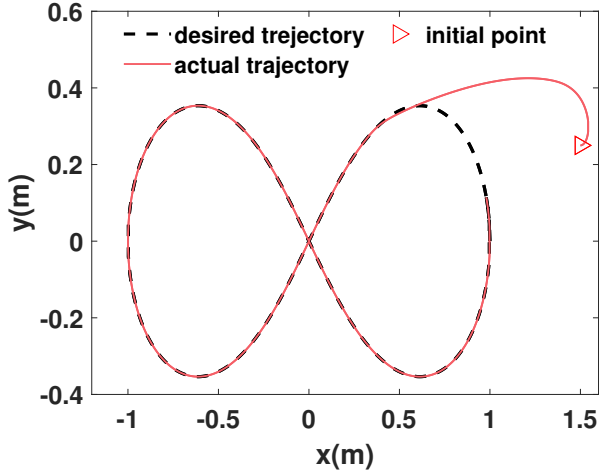


Fig. 1. Lemniscate trajectory tracking.

## 6. CONCLUSION

In this paper, a family of homogeneous controllers is proposed to deal with the trajectory tracking problem in UMRs. The proposed approach is based on a cascade control paradigm. In comparison to earlier kinematic finite-time tracking techniques, the proposed control strategy does not require a non-zero angular velocity for the desired trajectory; hence, it is able to track a broader range of feasible trajectories. Different combinations of homogeneous degrees, for the position and orientation tracking error subsystems, exhibit different convergence properties, *i.e.*, globally and locally asymptotic, locally exponential, and globally finite-time convergence. The simulation results confirm the efficiency of the proposed homogeneous control approach.

### Appendix A.

#### A.1 Input-to-State Stability

Consider a nonlinear system with control inputs:

$$\dot{x} = f(x, u), \quad (\text{A.1})$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^m$  is the input and  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  denotes a Lipschitz function, and  $x(t_0) = x_0$ .

**Definition 3.** *Chaillet et al. (2014a).* The system (A.1) is said to be *Strongly iISS* if it is both *iISS* and *ISS*, with respect to small inputs. In other words, there exist  $R > 0, \beta \in \mathcal{KL}$  and  $\mu_1, \mu_2, \mu \in \mathcal{K}_\infty$  such that, for all

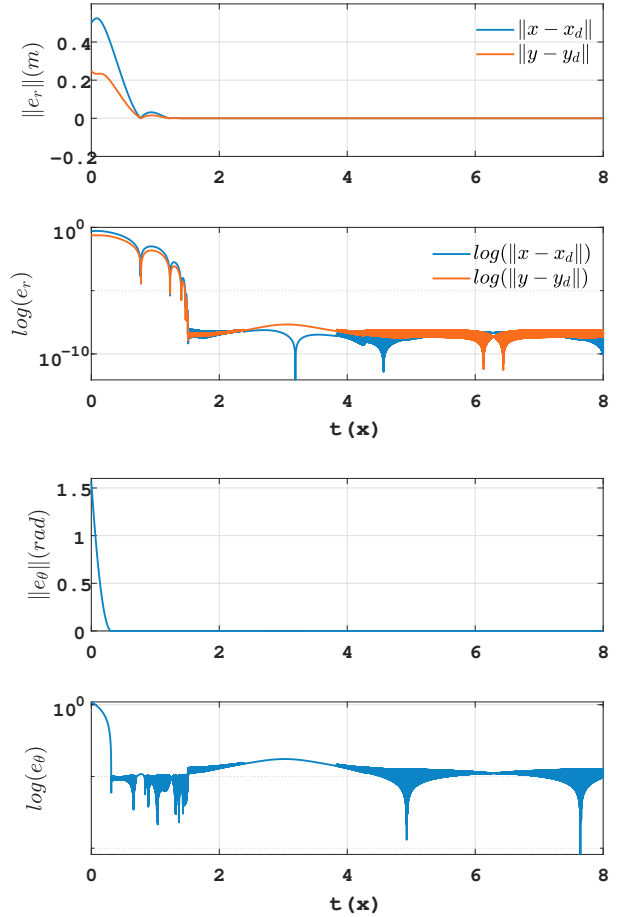


Fig. 2. Position error and orientation error.

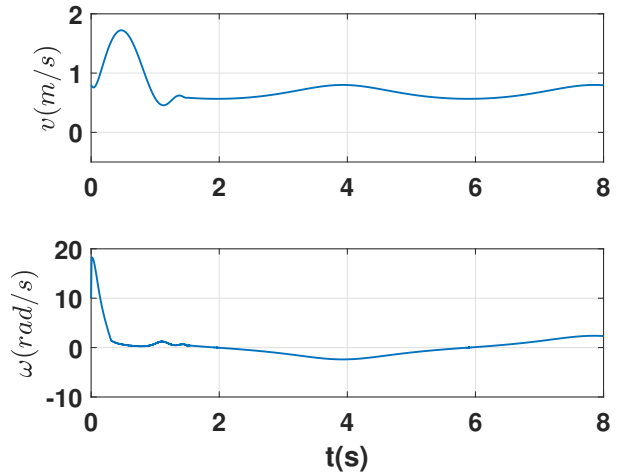


Fig. 3. Control input  $v$  and  $w$ .

$u \in \mathcal{U}^m$ , all  $x_0 \in \mathbb{R}^n$  and all  $t \geq 0$ , its solution satisfies the following two properties:

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \mu_1 \int_0^t \mu_2(\|u(s)\|) ds,$$

$$\|u\| < R \Rightarrow \|x(t)\| \leq \beta(\|x_0\|, t) + \mu(\|u\|).$$

The constant  $R$  is called an input threshold and the function  $\mu$  is referred to as a supply gain for (A.1),

$\mathcal{U}^m$  denotes the set of all measurable locally essentially bounded functions  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ .

Consider a cascade system

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_2),\end{aligned}\tag{A.2}$$

where  $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mapsto \mathbb{R}^{n_1}$  and  $f_2 : \mathbb{R}^{n_2} \times \mathbb{R}^{m_2} \mapsto \mathbb{R}^{n_2}$  denote two locally Lipschitz functions satisfying  $f_1(0, 0) = 0$  and  $f_2(0, 0) = 0$ .

**Corollary 1.** *Chaillet et al. (2014b). If the system  $\dot{x}_1 = f_1(x_1, x_2)$  is Strongly iISS and the origin of  $\dot{x}_2 = f_2(x_2)$  is globally asymptotically stable, then the origin of the cascade (A.2) is globally asymptotically stable.*

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