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# Generalized Homogeneous Unit Control

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**Abstract:** The methodology of the unit sliding mode control design (known since 1970s) for linear MIMO (Multiply Inputs Multiply Outputs) systems is revised based on the concept of the generalized homogeneity. The conventional restriction about a consistency of the number of control inputs with the dimension of the sliding surface is eliminated. A simple procedure for control parameters tuning is developed. The robustness of the homogeneous unit controller with respect to bounded matched perturbations is proven. The theoretical results are supported by numerical simulations.

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## 1. INTRODUCTION

The sliding mode control (SMC) seems to be the first robust control design methodology known since 1960s (see Utkin (1977) and references therein). The unit control proposed by Gutman and Leitmann (1976) is one of famous SMC algorithms supported with a very simple rule of control parameters tuning (see Utkin (1992), Edwards and Spurgeon (1998)). In the SISO (Single Input Single Output) case, the unit control simply coincides with the conventional relay SMC. The difference can be discovered for linear MIMO (Multiply Inputs Multiply Outputs) systems. Indeed, in this case the unit control has discontinuities only on a set being an intersection of several  $(n - 1)$ -dimensional linear surfaces, while the relay SMC is discontinuous on each such hyperplane. Therefore, the discontinuity set of the unit control has a smaller dimension than the discontinuity set of the conventional relay SMC. The latter may simplify a practical implementation and a chattering attenuation of SMC system (see e.g., Shtessel et al. (2014)). The main well-known restriction of the conventional SMC design for linear MIMO systems is a consistency of dimensions of the sliding surface and the control input. Namely, to implement the unit control for the linear MIMO system, the sliding surface  $Cx = 0$  must be of the dimension  $n - p$  with  $1 \leq p \leq m$  (i.e.,  $C \in \mathbb{R}^{p \times n}$ ), where  $n$  is a dimension of the state vector  $x \in \mathbb{R}^n$  and  $m$  is the number of control inputs. The aim of this paper is eliminate this restriction and to develop a unit control design methodology ensuring sliding mode on a linear surface of a dimension from 0 to  $n - 1$ . For SISO case, this problem is treated by the so-called high order sliding mode (HOSM) algorithms (see Levant (2003), Shtessel et al. (2014)). The design procedure for such controllers is rather complicated and has to be supported with a non-trivial stability analysis (see, e.g. Cruz-Zavala and Moreno (2017) and references therein). An extension of the HOSM control methodology to MIMO systems requires even more cumbersome constructions (see e.g., Plestan et al. (2008), Pisano (2012), García-Mathey and Moreno (2022)). This paper suggests a rather simple methodology of a unit control design based on the theory of the so-called generalized homogeneous systems (see, e.g., Polyakov (2020)).

The homogeneity is a dilation symmetry, which is well-known in mathematics due to the papers of Zubov (1958), Folland (1975), Kawski (1991), Fischer and Ruzhansky (2016). A Lie symmetry in general and the dilation symmetry in particular is a feature of many physical process (see, e.g. Noether (1918)). A lot of PDE models of mathematical physics are homogeneous in a generalized sense. Such systems have the so-called self-similar solutions known since 1940s (see, e.g., Stanyukovich (1960) for more details). The standard (or Euler) homogeneity is a symmetry of a mathematical object (a function, a set, a differential equation, etc) with respect to the standard dilation  $x \mapsto e^s x$  being simply a multiplication of a vector  $x$  by a positive scalar  $e^s$ , where  $s \in \mathbb{R}$ . The generalized dilation (see Zubov (1958), Kawski (1991)) is defined as  $x \mapsto \mathbf{d}(s)x$ , where  $\mathbf{d}(s)$  is a family of operators (parameterized by  $s \in \mathbb{R}$ ) having a topological characterization typical for a dilation (see Husch (1970)). The generalized homogeneity is widely utilized in control and systems theory for control design (Bhat and Bernstein (2005), Levant (2005), Orlov (2005), Andrieu et al. (2008), Zimenko et al. (2020b)), state estimation (Perruquetti et al. (2008), Cruz-Zavala et al. (2011), Lopez-Ramirez et al. (2018)) and stability/controllability/robustness analysis (Hermes (1986), Rosier (1992), Grüne (2000), Hong (2001)). In fact, the conventional unit control is a standard homogeneous control algorithm (as explained in Section 4). This paper shows that using the generalized homogeneity, the methodology of the unit control can be adapted to sliding surfaces  $Cx = 0$  which may have any dimension from 0 to  $n - 1$ , but all other advantages of the unit control known before are preserved.

*Notation.*  $\mathbb{R}$  is the field of reals;  $\|\cdot\|$  denotes the weighted Euclidean norm  $\mathbb{R}^n$ , namely,  $\|x\| = \sqrt{x^\top P x}$  with  $x \in \mathbb{R}^n$  and  $P = P^\top \in \mathbb{R}^{n \times n} : P \succ 0$ , where the inequality  $P \succ 0$  means that the matrix  $P$  is positive definite;  $|\cdot|$  denotes the Euclidean norm in  $\mathbb{R}^n$ , i.e.,  $|x| = \sqrt{x^\top x}$ ;  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix;  $\text{rank } M$  denotes the rank of a matrix  $M$ ;  $\mathbf{0}$  is a zero element of a vector space (i.e.,  $\mathbf{0} \in \mathbb{R}^n$  means that  $\mathbf{0}$  is the zero vector, but  $\mathbf{0} \in \mathbb{R}^{n \times m}$  is the zero matrix).

## 2. PROBLEM STATEMENT

Let us consider the linear MIMO system

$$\dot{x} = Ax + B(u + \gamma(t, x)), \quad t > 0, \quad x(0) = x_0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $u(t) \in \mathbb{R}^m$  is the control input, the matrices  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$  are known and  $\gamma : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^m$  is an unknown bounded measurable function, which models the matched system uncertainties and disturbances. The system (1) has Filippov solutions for any locally bounded measurable feedback law  $u = u(t, x)$  (see Filippov (1988)).

*Assumption 1.* Let us assume that

$$|\gamma(t, x)| \leq \bar{\gamma}, \quad \forall t \geq 0, \quad \forall x \in \mathbb{R}^n, \quad (2)$$

where  $\bar{\gamma} \geq 0$  is a *known* constant.

For the system (1) we consider the classical problem of the sliding mode control design (see Utkin (1992), Edwards and Spurgeon (1998)). Namely, the *control aim* is to synthesize a feedback control law such that any trajectory of the closed-loop system reaches a desired linear surface

$$Cx = 0, \quad C \in \mathbb{R}^{p \times n}, \quad \text{rank } C = p, \quad (3)$$

is a finite time and slides on it after then. The so-called unit control (see Gutman and Leitmann (1976), Utkin (1992), Edwards and Spurgeon (1998))

$$\tilde{u} = \hat{K}x + K \frac{Cx}{|Cx|}, \quad (4)$$

solves this problem provided that  $p = m$ , the matrix  $CB$  is invertible and the matrices  $\hat{K} \in \mathbb{R}^{m \times n}$  and  $K \in \mathbb{R}^{m \times p}$  are properly selected:  $\hat{K} = -(CB)^{-1}CA$  and  $K = -(\delta + |CB|\bar{\gamma})(CB)^{-1}$  with  $\delta > 0$ . We relax the restriction  $p = m$ ,  $\det(CB) \neq 0$  by the following mild assumption.

*Assumption 2.* There exists  $\tilde{K} \in \mathbb{R}^{m \times n}$  such that

$$C(A + B\tilde{K})(I_n - C^\top(CC^\top)^{-1}C) = 0 \quad (5)$$

and the pair  $\{A_\sigma, B_\sigma\}$  is controllable, where

$$A_\sigma = C(A + B\tilde{K})C^\top(CC^\top)^{-1}, \quad B_\sigma = CB. \quad (6)$$

If  $CB$  is invertible then the latter assumption is always fulfilled for  $\tilde{K} = -(CB)^{-1}CA$ . However, this assumption is fulfilled in many other cases, for example, if  $C = I_n$  and the pair  $\{A, B\}$  is controllable. Assumption 2 simply means that the dynamics of the sliding variable  $\sigma = Cx$  is completely controllable. Indeed, below we show that, under Assumption 2, we have  $\dot{\sigma} = A_\sigma\sigma + B_\sigma(u - \tilde{K}x + \gamma)$ , so an influence of any other (possibly uncontrollable) component of the system state may be rejected by a control law. Notice that the controllability of the pair  $\{A_\sigma, B_\sigma\}$  is necessary for a finite-time reaching of the sliding surface  $Cx = 0$  for arbitrary initial state  $x_0 \in \mathbb{R}^n$ .

Assumption 2 can be easily checked by solving the linear algebraic equation (5) with respect to the variable  $\tilde{K}$ . The absence of solution would simply mean that the selected sliding surface is not appropriate for the unit SMC design. The sliding surface  $Cx = 0$  is usually designed such that the closed-loop system is, at least, exponentially stable. Any method of a stable sliding surface design (see, e.g., Utkin (1992), Edwards and Spurgeon (1998), Shtessel et al. (2014)) can be utilized taking into account Assumption 2.

The sliding mode control is known to be robust with respect to bounded matched disturbances. The Assumption

1 is conventional for the SMC theory (see, e.g., Utkin (1992), Edwards and Spurgeon (1998), Shtessel et al. (2014)). Therefore, the control aim is to design a unit sliding mode control under the classical Assumption 1 and the relaxed Assumption 2. To solve the problem, the term  $\frac{Cx}{|Cx|}$  in (4) is going to be slightly modified using the concept of a generalized homogeneity.

## 3. PRELIMINARIES: LINEAR HOMOGENEITY

The linear dilation in  $\mathbb{R}^n$  is given by (see Polyakov (2019))

$$\mathbf{d}(s) = e^{sG_d} = \sum_{i=0}^{\infty} \frac{(sG_d)^i}{i!}, \quad s \in \mathbb{R}, \quad (7)$$

where  $G_d \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix<sup>1</sup> being a *generator of linear dilation*. The latter implies that  $\mathbf{d}$  satisfies the limit property,  $\|\mathbf{d}(s)x\| \rightarrow 0$  as  $s \rightarrow -\infty$  and  $\|\mathbf{d}(s)x\| \rightarrow +\infty$  as  $s \rightarrow +\infty$ , required for a group  $\mathbf{d}$  to be a dilation in  $\mathbb{R}^n$  (see, e.g., Kawski (1991)). The linear dilation introduces an alternative norm topology in  $\mathbb{R}^n$  by means of the canonical homogeneous norm.

*Definition 1.* (Polyakov (2019)) The function  $\|\cdot\|_d : \mathbb{R}^n \mapsto \mathbb{R}_+$  given by  $\|x\|_d = 0$  for  $x = \mathbf{0}$  and

$$\|x\|_d = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (8)$$

is called the *canonical homogeneous norm* in  $\mathbb{R}^n$ , where  $\mathbf{d}$  is a linear monotone dilation<sup>2</sup>.

Notice that  $\|x\| = 1$  is equivalent to  $\|x\|_d = 1$ . Moreover,  $\|\mathbf{d}(-\ln \|x\|_d)x\| = 1$ , so the operator  $\mathbf{d}(-\ln \|x\|_d)x$  is the so-called *d-homogeneous projector of  $x$  on the unit sphere* (Polyakov, 2020, page 159). For the standard dilation  $\mathbf{d}(s) = e^s I_n$ ,  $s \in \mathbb{R}$  we have  $\|\cdot\| = \|\cdot\|_d$  and  $\mathbf{d}(-\ln \|x\|_d)x = \frac{x}{\|x\|}$  is the standard homogeneous projector of  $x$  on the unit sphere.

*Theorem 1.* (Polyakov (2019)). If  $\|x\| = \sqrt{x^\top P x}$  with a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  satisfying  $PG_d + G_d^\top P \succ 0$ ,  $P \succ 0$  then  $\mathbf{d}$  is monotone and the canonical homogeneous norm  $\|\cdot\|_d$  is continuous on  $\mathbb{R}^n$ , smooth on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ ,

$$\frac{\partial \|x\|_d}{\partial x} = \frac{\|x\|_d x^\top \mathbf{d}^\top(-\ln \|x\|_d) P \mathbf{d}(-\ln \|x\|_d)}{x^\top \mathbf{d}^\top(-\ln \|x\|_d) P G_d \mathbf{d}(-\ln \|x\|_d) x}, \quad \forall x \neq \mathbf{0}. \quad (9)$$

Moreover,  $\underline{\sigma}(\|x\|) \leq \|x\|_d \leq \bar{\sigma}(\|x\|)$ ,  $\forall x \in \mathbb{R}^n$ , with

$$\underline{\sigma}(r) = \begin{cases} r^{1/\alpha} & \text{if } r \geq 1, \\ r^{1/\beta} & \text{if } r < 1, \end{cases} \quad \bar{\sigma}(r) = \begin{cases} r^{1/\beta} & \text{if } r \geq 1, \\ r^{1/\alpha} & \text{if } r < 1, \end{cases}$$

where  $\alpha = 0.5\lambda_{\max}(P^{1/2}G_d P^{-1/2} + P^{-1/2}G_d^\top P^{1/2}) > 0$

and  $\beta = 0.5\lambda_{\min}(P^{1/2}G_d P^{-1/2} + P^{-1/2}G_d^\top P^{1/2}) > 0$ .

Below the canonical homogeneous norm is utilized as a Lyapunov function for a homogeneous unit control design.

*Definition 2.* (Kawski (1991)). A vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  (resp. a function  $h : \mathbb{R}^n \mapsto \mathbb{R}$ ) is said to be *d-homogeneous of degree  $\mu \in \mathbb{R}$*  if  $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s)f(x)$  (resp.  $h(\mathbf{d}(s)x) = e^{\mu s} h(x)$ ), for all  $x \in \mathbb{R}^n$ ,  $s \in \mathbb{R}$ .

If  $f$  is *d-homogeneous of degree  $\mu$*  then solutions of  $\dot{x} = f(x)$  are symmetric (Kawski (1991)):  $x(e^{-\mu s}t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(t, x_0)$ , where  $x(t, z)$  denotes a solution with  $x(0) = z$ .

By definition, the system  $\dot{x} = f(x)$  is *d-homogeneous* if the vector field is *d-homogeneous*.

<sup>1</sup> A matrix  $G_d \in \mathbb{R}^{n \times n}$  is anti-Hurwitz if  $-G_d$  is Hurwitz.

<sup>2</sup> A dilation in  $\mathbb{R}^n$  is monotone if for any  $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  the function  $s \mapsto \|\mathbf{d}(s)x\|$ ,  $s \in \mathbb{R}$  is strictly increasing.

The homogeneity degree specifies the convergence rate of a stable homogeneous system.

*Theorem 2.* (Bhat and Bernstein (2005)) Let a continuous vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  be  $\mathbf{d}$ -homogeneous of degree  $\mu < 0$ . If the system  $\dot{x} = f(x)$  is locally asymptotically stable then it is globally uniformly finite-time stable.

SMC theory implicitly uses the same result for ensuring a finite-time reaching of a sliding surface, Orlov (2005).

The homogeneous control systems are robust (ISS) with respect to a rather large class of perturbations (Hong (2001), Andrieu et al. (2008)).

#### 4. HOMOGENEOUS UNIT CONTROL

The conventional unit control (4) has two components. In the context of the homogeneity theory, the linear component is aimed at a homogenization of the unperturbed dynamics of the sliding variable  $\sigma = Cx$ ,  $C \in \mathbb{R}^{p \times n}$ , while the second (stabilizing) component is discontinuous and homogeneous. Indeed, for (1), (4) we have

$$\dot{\sigma} = (CA + CB\hat{K})x + CBK \frac{\sigma}{|\sigma|} + CB\gamma.$$

Since, in the conventional case,  $p = m$  and  $\det(CB) \neq 0$  then in order to eliminate the linear term in the above equation, the matrix  $\hat{K}$  is traditionally selected as  $\hat{K} = -(CB)^{-1}CA$ . In the latter case, the unperturbed system  $\dot{\sigma} = CBK \frac{\sigma}{|\sigma|}$  is obviously standard homogeneous of negative degree ( $\mu = -1$ ). Moreover, the perturbed system ( $\gamma \neq 0$ ) is globally uniformly finite-time stable under proper selection of control parameters (e.g.,  $K = -(\delta + |CB|\bar{\gamma})(CB)^{-1}$ ,  $\delta > 0$ ). We follow this homogeneity-based interpretation in order to design a similar two component feedback law for the case  $p \neq m$ . Namely, the linear component should homogenize (in a generalized sense) the dynamics of the sliding variable, while the discontinuous one should be uniformly bounded and generalized homogeneous of degree  $\mu = -1$ . Moreover, the dynamics of the sliding variable has to be globally uniformly finite-time stable even in the perturbed case.

Notice that the linear vector field  $\sigma \mapsto A_\sigma \sigma$  may be standard homogeneous of non-zero degree if and only if  $A_\sigma = \mathbf{0}$ . This explains the structure of the matrix  $\hat{K} = -(CB)^{-1}CA$  in the classical unit control (4). To relax this condition, the generalized homogeneity can be utilized. Indeed, according to (Polyakov, 2020, Lemma 7.5) and (Zimenko et al., 2020b, Lemma 1)), a linear vector field  $\sigma \mapsto A_\sigma \sigma$  is  $\mathbf{d}$ -homogeneous of non-zero degree if and only if the matrix  $A_\sigma$  is nilpotent. Therefore, the matrix  $A_\sigma$  may be non-zero if  $p \neq m$ . On the other hand, the operator  $\frac{\sigma}{|\sigma|}$  is the standard homogeneous projector on the unit sphere (*that is why the feedback law (4) is called the unit control*). Such a projector for a linear dilation  $\mathbf{d}$  can be defined as  $\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma$ , where  $\|\cdot\|_{\mathbf{d}}$  is a  $\mathbf{d}$ -homogeneous norm (see Section 3).

Following the above explanations we define the control law:

$$u = \hat{K}x + K\mathbf{d}(-\ln \|Cx\|_{\mathbf{d}})Cx, \quad \hat{K} = \tilde{K} + K_0C, \quad (10)$$

where  $\sigma = Cx$  is the sliding variable, the matrix  $\tilde{K} \in \mathbb{R}^{m \times n}$  is defined in Assumption 2, the matrix  $K_0 \in \mathbb{R}^{m \times p}$  is required for  $\mathbf{d}$ -homogenization of a linear component

of the dynamics of  $\sigma$ , the matrix  $K \in \mathbb{R}^{m \times p}$  has to be properly designed for the finite-time stabilization of the dynamics of the sliding variable,  $\mathbf{d}$  is a linear dilation in  $\mathbb{R}^p$  and  $\|\cdot\|_{\mathbf{d}}$  is a canonical homogeneous norm to be defined. Since, by definition of the canonical homogeneous norm, we have  $\|\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma\| = 1$  then (10) is a unit control as well. However, its design is based on the concept of a generalized (non-standard) homogeneity, so, in order to distinguish the conventional and suggested algorithms, we call the feedback law (10) by *the homogeneous unit control*.

If  $\tilde{K}$  satisfies Assumption 2 then using the identity

$$C(I_n - C^\top(CC^\top)^{-1}C) = \mathbf{0}$$

we derive

$$\begin{aligned} CA + CB\hat{K} &= CA + CB\tilde{K} \\ &= (CA + CB\tilde{K})(I_n - C^\top(CC^\top)^{-1}C) \\ &\quad + CAC^\top(CC^\top)^{-1}C + CB\tilde{K}C^\top(CC^\top)^{-1}C \\ &= C(A + B\tilde{K})C^\top(CC^\top)^{-1}C = A_\sigma C \end{aligned}$$

where  $A_\sigma = C(A + B\tilde{K})C^\top(CC^\top)^{-1}$ . Therefore, the sliding variable  $\sigma = Cx$  of the closed-loop system (1), (10) satisfies the equation

$$\dot{\sigma} = (A_\sigma + B_\sigma K_0)\sigma + B_\sigma K\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma + B_\sigma \gamma, \quad (11)$$

where  $B_\sigma = CB$ . Since the pair  $\{A_\sigma, B_\sigma\}$  is assumed to be controllable then the matrix  $A_0 = A_\sigma + B_\sigma K_0$  can be designed nilpotent by means of a proper selection of  $K_0$  and the system (11) can be homogeneously stabilized to zero in a finite time by a proper selection of the gain  $K$ .

*Theorem 3.* If Assumptions 1 and 2 are fulfilled for some  $\bar{\gamma} \geq 0$  and  $C \in \mathbb{R}^{p \times m}$  then

- 1) the linear algebraic equation

$$A_\sigma G_0 - G_0 A_\sigma + B_\sigma Y_0 = A, \quad G_0 B_\sigma = \mathbf{0} \quad (12)$$

always has a solution  $Y_0 \in \mathbb{R}^{m \times p}$ ,  $G_0 \in \mathbb{R}^{p \times p}$ ; moreover, for any solution (12), the matrix

$$G_{\mathbf{d}} = I_n - G_0 \quad (13)$$

is anti-Hurwitz, the matrix

$$A_0 = A_\sigma - B_\sigma Y_0 G_{\mathbf{d}}^{-1} \quad (14)$$

is nilpotent and satisfies the identity

$$A_0 G_{\mathbf{d}} = (G_{\mathbf{d}} - I_n)A_0, \quad G_{\mathbf{d}} B_\sigma = B_\sigma; \quad (15)$$

- 2) the linear algebraic system

$$\begin{aligned} (A_0 + \rho G_{\mathbf{d}})X + X(A_0 + \rho G_{\mathbf{d}})^\top + B_\sigma Y + Y^\top B_\sigma^\top + \bar{\gamma}X &= -\bar{\gamma}B_\sigma B_\sigma^\top, \\ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top &\succ 0, \quad X = X^\top \succ 0 \end{aligned} \quad (16)$$

has a solution  $X \in \mathbb{R}^{n \times n}$ ,  $Y \in \mathbb{R}^{m \times n}$  for any  $\rho > 0$ ;

- 3) the system (11) with the control (10) is globally uniformly finite-time stable and

$$\sigma(t) = \mathbf{0}, \quad \forall t \geq \frac{\|\sigma(0)\|_{\mathbf{d}}}{\rho},$$

provided that  $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$  with  $s \in \mathbb{R}$ ,

$$K_0 = -Y_0 G_{\mathbf{d}}^{-1}, \quad K = YX^{-1}, \quad (17)$$

and  $\|\cdot\|_{\mathbf{d}}$  is the canonical homogeneous norm induced by the weighted Euclidean norm  $\|\sigma\| = \sqrt{\sigma^\top X^{-1}\sigma}$ .

The homogeneous unit control (10) is discontinuous only on the sliding surface  $Cx = 0$  and rejects bounded matched perturbations  $|\gamma(t, x)| \leq \bar{\gamma}$ . The following simple *algorithm* of its design can be proposed.

- *Step 1:* given  $C \in \mathbb{R}^{p \times m}$  solve the linear algebraic equation (5) with respect to the variable  $\tilde{K} \in \mathbb{R}^{m \times n}$ .
- *Step 2:* check the controllability of the pair  $\{A_\sigma, B_\sigma\}$ , where  $A_\sigma \in \mathbb{R}^{p \times p}$  and  $B_\sigma \in \mathbb{R}^{p \times n}$  are given by (6).
- *Step 3:* solve the linear algebraic equation (12) with respect to  $Y_0 \in \mathbb{R}^{m \times p}$  and  $G_0 \in \mathbb{R}^{p \times p}$ .
- *Step 4:* solve the linear matrix inequality (16) with respect to  $Y \in \mathbb{R}^{m \times p}$  and  $X \in \mathbb{R}^{p \times p}$ .
- *Step 5:* compute the parameters  $K_0 \in \mathbb{R}^{m \times p}$  and  $K \in \mathbb{R}^{m \times p}$  of the homogeneous unit control (10) by the formulas (17), (13).

*Remark 1.* (On sliding motion equation). Using the equivalent control method (see Utkin (1992)) for  $t \geq \|Cx(0)\|_{\mathbf{d}}$  we derive the sliding motion equation in the form:

$$\dot{x}(t) = (A + B\tilde{K})x(t), \quad Cx(t) = \mathbf{0}, \quad (18)$$

where  $\tilde{K} \in \mathbb{R}^{m \times n}$  is defined in Assumption 2. If the control aim is to stabilize the system (1) then the sliding surface  $Cx = 0$  has to be selected such that the system (18) is globally asymptotically stable. The simplest condition to  $C$  in this case is given by the following matrix inequality

$$\tilde{P}(A+B\tilde{K})+(A+B\tilde{K})^\top \tilde{P}-\beta C^\top C < 0, \quad \tilde{P} \succ 0, \quad \beta > 0. \quad (19)$$

Indeed, considering the Lyapunov function candidate  $V = x^\top P x$  we derive

$$\begin{aligned} \dot{V} &= 2x^\top \tilde{P}(A+B\tilde{K})x \\ &= x^\top (\tilde{P}(A+B\tilde{K})+(A+B\tilde{K})^\top \tilde{P}-\beta C^\top C)x < 0, \end{aligned}$$

where the identity  $Cx = 0$  and the inequality (19) are utilized on the last step. Therefore, to design a stabilizing homogeneous unit control (10) for the system (1) it is sufficient to find a matrix  $C$  satisfying Assumption 2 and the matrix inequality (19).

*Remark 2.* (On chattering attenuation). Following the conventional approach to the chattering reduction (see, e.g., Utkin (1992)), the discontinuous term in (10) can be "smoothed" using the saturation function:

$$u_\alpha = \hat{K}x + K\mathbf{d}(-\ln \max\{\alpha, \|Cx\|_{\mathbf{d}}\})Cx, \quad \alpha \geq 0 \quad (20)$$

coincides with (10) for  $\alpha = 0$ . On the one hand, the parameter  $\alpha > 0$  simply defines an attractive set of the sliding variable  $\sigma$ . Indeed, in the view of Theorem 3 for the closed-loop system (1), (20) we have

$$\frac{d\|\sigma(t)\|_{\mathbf{d}}}{dt} \leq -\rho \quad \text{if} \quad \|\sigma(t)\|_{\mathbf{d}} \geq \alpha.$$

The latter means that the sliding variable  $\sigma$  converges to the homogeneous ball  $\|\sigma\|_{\mathbf{d}} \leq \alpha$  in a finite time. On the other hand, the nonlinear term in (20) becomes linear  $K\mathbf{d}(-\ln \alpha)Cx$  for  $\|x\|_{\mathbf{d}} \leq \alpha$ , so the feedback law  $u_\alpha$  is continuous on  $\mathbb{R}^n$  if  $\alpha > 0$ . Therefore, in practice the parameter  $\alpha$  can be utilized for an attenuation of the chattering caused by discontinuity of SMC.

*Remark 3.* (On computation of  $\|\cdot\|_{\mathbf{d}}$ ). Since the canonical homogeneous norm is defined implicitly by the formula (8), a computational algorithm is required for practical implementation of the homogeneous unit control given by (10). In Polyakov (2019), (Polyakov, 2020, Chapter 8), Zimenko et al. (2020a) a scheme for an approximation of  $\|\cdot\|_{\mathbf{d}}$  by an explicit homogeneous function is presented. Issues of numerical on-line computation of  $\|\cdot\|_{\mathbf{d}}$  are studied in Polyakov et al. (2015), Polyakov et al. (2016) based on a bisection method. This numerical method is shown to be very efficient in practice (see e.g., Wang et al. (2021)).

*Remark 4.* (On exactness of the reaching time estimate). If  $\bar{\gamma} = 0$  and  $\rho = 1$  then  $T(x_0) = \|Cx_0\|_{\mathbf{d}}$  is the *exact* reaching time of the sliding surface  $Cx = 0$  by a trajectory of the closed loop system (1), (10). Indeed, repeating the proof of Theorem 3 we derive

$$\frac{d\|\sigma(t)\|_{\mathbf{d}}}{dt} = -1, \quad t > 0,$$

i.e.,  $\sigma(t) \neq \mathbf{0}$  for all  $t \in [0, \|\sigma(0)\|_{\mathbf{d}}]$  and  $\sigma(t) = \mathbf{0}$  for all  $t \geq \|\sigma(0)\|_{\mathbf{d}}$ . The reaching time can be tuned by the parameter  $\rho > 0$ . The larger  $\rho$ , faster the reaching of the sliding surface (see the estimate of the reaching time in Theorem 3). Theoretically, the reaching time may be adjusted arbitrarily small. However, a selection of a too large  $\rho$  may lead to large feedback gains and a large chattering in practice. Therefore, in practice,  $\rho$  could be one more tuning parameter for the chattering attenuation.

## 5. NUMERICAL EXAMPLE

For

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

the pair  $\{A, B\}$  is not controllable, so the first two components of the state vector cannot be affected by the control. However, the selected matrix  $C$  means that the sliding mode needs to be enforced in the last three components of the state vector simultaneously. Since the dimensions of  $C$  and  $B$  are not consistent then we cannot use the conventional scheme of the unit control design. Let us design the homogeneous unit control according to Theorem 3.

The equation (5) has the unique solution

$$\tilde{K} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this case, we derive

$$A_\sigma = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \quad B_\sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The pair  $\{A_\sigma, B_\sigma\}$  is controllable, so Assumption 2 is fulfilled. A solution of (12) is given by

$$G_0 = \begin{pmatrix} -1 & 0 & 0 \\ -0.5 & 0 & 0 \\ 0.5 & 0 & 0 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} -0.5 & -2.5 & -0.5 \\ -1.5 & 1.5 & -3.5 \end{pmatrix}$$

so the generator of the dilation  $\mathbf{d}$  is defined as

$$G_{\mathbf{d}} = I_n - G_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix}.$$

### 5.1 Disturbance-free case

First, let us consider the disturbance-free case  $\gamma \equiv \mathbf{0}$ . Solving the linear system (16) for  $\bar{\gamma} = 0$  and  $\rho = 1$  we derive

$$K = \begin{pmatrix} 4.75 & -2 & 1 \\ -4.75 & 1 & -2 \end{pmatrix}, \quad P = \begin{pmatrix} 4.6 & -1.2 & 1.2 \\ -1.2 & 0.5538 & -0.2462 \\ 1.2 & -0.2462 & 0.5538 \end{pmatrix}.$$

The numerical simulation of the closed-loop system (1), (10) with  $x_0 = (1, 0, 2, 0, 1)^\top$  was done in MATLAB

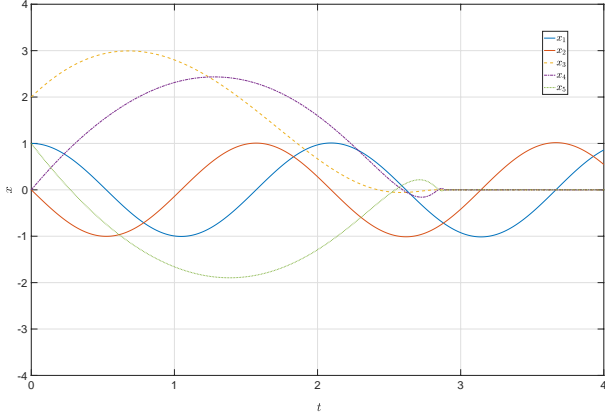


Fig. 1. The evolution of the state in the disturbance-free case

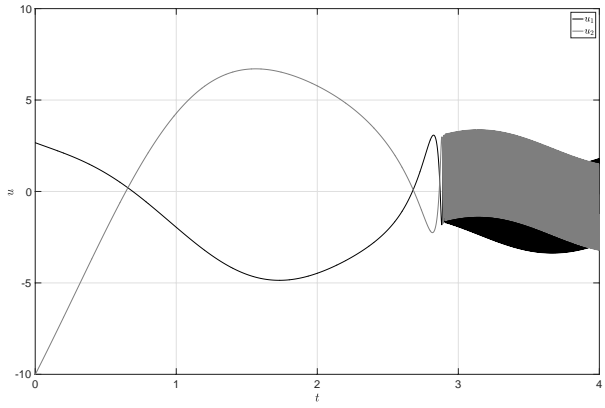


Fig. 2. The control signals in the disturbance-free case

using the explicit Euler method with the constant step  $h = 0.001$ . The simulation results for the disturbance-free case ( $\gamma \equiv \mathbf{0}$ ) are depicted on Fig. 1 and Fig. 2. Notice that a chattering attenuation scheme (see Remark 2) was not utilized for the mentioned simulations. The chattering magnitude of the state (estimated from the simulation results) is about  $2 \cdot 10^{-3}$ . It was checked on simulations that the scheme of the chattering reduction by a saturation (with  $\alpha = 0.005$ ) proposed in Remark 2 allows an asymptotic convergence of the sliding variable to zero even for the numerical solution.

In the disturbance-free case, the canonical homogeneous norm defines the reaching time. Indeed, according to Remark 4 the reaching time of the trajectory of the closed loop system is  $\|Cx_0\|_d \approx 2.8860$ . The numerical simulations show the same reaching time (up to the computation precision).

### 5.2 The case of matched perturbations

The homogeneous unit control design in the perturbed case uses Assumption 1 to define the maximum magnitude of the matched perturbations. Solving the linear system (16) for  $\bar{\gamma} = 0.5$  we derive the control parameters

$$K = \begin{pmatrix} 5.8379 & -2.5103 & 1.1938 \\ -5.8379 & 1.1938 & -2.5103 \end{pmatrix},$$

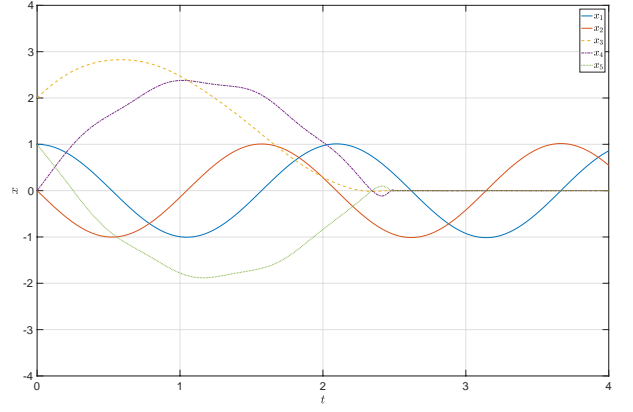


Fig. 3. The evolution of the state in the perturbed case

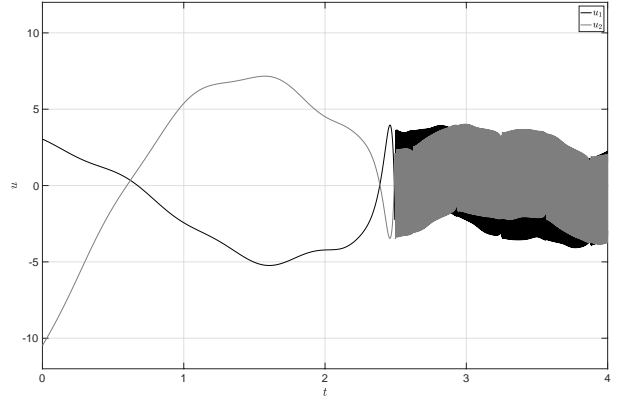


Fig. 4. The control signals in the perturbed case

$$P = \begin{pmatrix} 5.3112 & -1.3265 & 1.3265 \\ -1.3265 & 0.5413 & -0.2751 \\ 1.3265 & -0.2751 & 0.5413 \end{pmatrix}.$$

According to Theorem 3 the homogeneous unit control rejects bounded matched disturbances  $\gamma$  of the magnitude  $\bar{\gamma} = 0.5$ . The numerical simulations for the matched disturbances modeled by

$$\gamma(t) = 0.5(\sin(10t), \cos(10t))^T$$

are presented on Fig. 3 and Fig. 4.

## 6. CONCLUSIONS

A methodology of a unit (sliding mode) control design has been revised. It is shown that a linear sliding surface  $Cx = 0$  can be selected independently of the number of control inputs provided that the dynamics of the sliding variable  $\sigma = Cx$  is controllable. The design is essentially based on the so-called homogeneous norm, which defines the homogeneous projector on the unit sphere. The mentioned homogeneous projector forms the discontinuous component of the unit control (in both classical and revised cases), while the linear (continuous) component of the unit control is utilized (as usual) for a homogenization of the dynamics of the sliding variable. The simplicity of the control design and parameters tuning is demonstrated on the numerical example. Extension of the presented methodology to infinite dimensional unit control (see, e.g. Utkin and Orlov (2019)) is the interesting problem for the future research.

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