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A New Finite–Time Sliding–Mode Controller for a Class of Second–Order Non–Linear Systems [★]

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Abstract: This paper contributes to the design of a new finite–time sliding–mode controller for a class of uncertain second–order non–linear systems. The proposed strategy takes into account the design of a family of sliding variables, which establishes a second–order sliding–mode and ensures the convergence of the trajectories of the system to the origin in a finite time and despite the effect of some uncertainties on the system. The straightforward structure of the controller is simple to tune and implement. The global, uniform and finite–time stability of the closed–loop dynamics is demonstrated by means of Lyapunov functions. Furthermore, the performance of the proposed approach is validated through some simulation results.

Keywords: Robust Control; Sliding–Mode Control; Second–Order Systems.

1. INTRODUCTION

The problem of controllers design, in the presence of unknown parameters or signal uncertainties, remains as a challenging problem. Therefore, robust control techniques are needed in order to design a controller that counteracts the effect of such disturbances. One of the most effective and used control techniques is the sliding–mode control due to its simplicity and robustness (see, *e.g.*, Edwards and Spurgeon (1998), Utkin et al. (1999) and Shtessel et al. (2014)).

It is well–known that, in order to compensate the effect of disturbances, the sliding–mode control laws employ a discontinuous term, which can usually produce the so–called chattering phenomena. There exist several approaches to eliminate or attenuate such a phenomena: approximations of the discontinuous function by saturation/sigmoid functions, *i.e.*, the boundary layer method (see, *e.g.*, Slotine and Sastry (1983) and Chen et al. (2002)); or order augmentation by adding integrators at the control input (see, *e.g.*, Sira–Ramírez (1993) and Levant (2001)). However, the capability to reject disturbances could be degraded in some cases.

An alternative method to attenuate the chattering is the use of second–order sliding–mode controllers (Levant, 1993). The main idea of a second–order sliding–mode is to increase the relative degree of the sliding variable, with respect to the control input, from one to two such that the sliding variable, and its derivative, can be kept in zero. Among these discontinuous second–order sliding–mode controllers, we can find the “twisting” controller (Levant, 1993), the “sub–optimal” controller (Bartolini et al., 2003), the “terminal” controller (Man et al., 1994) and the “quasi–continuous” controller (Levant, 2005b).

Recently, some other second/arbitrary–order sliding–mode controllers have been proposed in the literature. For instance, in Polyakov (2012) the author presents a family of non–linear controllers exhibiting fixed–time convergence. Particularly, for second–order systems, some modifications of the second–order finite–time nested controller (Levant, 2005a) are provided to adjust the settling time of the closed–loop system. In Ding and Li (2017), the authors propose a novel second–order sliding–mode control method for non–linear systems with disturbances bounded by positive functions. A Lyapunov–based design for high–order sliding–modes controllers is proposed by Cruz–Zavala and Moreno (2017) for arbitrary order uncertain non–linear systems. Despite the interesting Lyapunov characterization, the gain calculation is complex. In Incremona et al. (2017), the authors provide a general design for second–order sliding–mode control algorithms for non–linear systems in the presence of control and state constraints. In Ding et al. (2019) a state saturated like second–order sliding–mode algorithm is proposed for some class of non–

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linear systems. The proposed controller is based on the saturation technique and the backstepping-like method. In Cruz-Zavala and Moreno (2019), a family of high-order sliding-modes controllers, with a discontinuous integral action, is proposed for a class of SISO non-linear systems with arbitrary order ensuring finite-time convergence to zero of the tracking error. Similarly, a family of homogeneous discontinuous integral controllers is proposed in Mercado-Uribe and Moreno (2020) for uncertain minimum phase systems with arbitrary relative degree ensuring finite-time convergence to zero of the output tracking error. It is important to recall that one of the first controllers with a discontinuous integral action is the super-twisting algorithm (Levant, 1993). However, these integral controllers only compensate Lipschitz disturbances and this may limit the class of possible perturbations affecting the system.

In this paper, a straightforward and novel second-order sliding-mode controller is proposed to solve a stabilization problem for a class of second-order non-linear systems with bounded disturbances. The proposed control approach possesses the following features:

- (1) The designed control law guarantees global, uniform, and finite-time convergence of the trajectories of the system to the origin.
- (2) The considered uncertainties are completely compensated.
- (3) The controller parameters selection is simple and this facilitates its implementability.

Contrary to most of the convergence analysis given in the literature, in this paper, the finite-time convergence proof is carried out by means of Lyapunov functions in a straightforward way. Some simulation results highlight the feasibility and performance of the proposed second-order sliding-mode controller.

The rest of the paper is organized as follows. The problem statement is given in Section 2 while some preliminaries are provided in Section 3. The proposed discontinuous controller is presented in Section 4. Some simulation results are presented in Section 5. Finally, the concluding remarks are provided in Section 6.

Notation: Define the function $[s]^\gamma = |s|^\gamma \text{sign}(s)$, for $\gamma \in \mathbb{R}_{\geq 0}$ and any $s \in \mathbb{R}$. Recall that the function $[s]^\gamma$ satisfies $d[s]^\gamma/ds = \gamma|s|^{\gamma-1}$.

2. PROBLEM STATEMENT

Consider the following class of second-order non-linear systems

$$\dot{x}_1 = x_2, \quad (1a)$$

$$\dot{x}_2 = f(t, x) + bu + \xi(t, x), \quad (1b)$$

where $x_1, x_2 \in \mathbb{R}$ are the position and velocity, respectively; $f: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ represents the nominal part of the system dynamics, $u \in \mathbb{R}$ is the control input, $b \in \mathbb{R}_{>0}$ is the input coefficient (for mechanical systems, this coefficient normally depends on the torque constant and the actuator inertia), and the term $\xi: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ describes the external forces affecting the system and possible unmodeled dynamics. The uncertain function satisfies the following boundedness condition

$$|\xi(t, x)| < c, \quad (2)$$

for some known constant $c \in \mathbb{R}_{>0}$.

The aim of this work is to design a robust control law such that the trajectories of the system (1) converge to zero in a finite time and despite the presence of external disturbances and unmodeled dynamics.

3. PRELIMINARIES

Consider the system

$$\dot{x} = f(t, x), \quad t \in \mathbb{R}_{\geq 0}, \quad x(0) = x_0, \quad (3)$$

where $x \in \mathbb{R}^n$ is the state vector. The function $f: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be locally bounded uniformly in t . For f locally measurable but discontinuous with respect to x , the solutions are understood in the sense of Filippov Filippov (1988). That is, $x(t, x_0)$ is a solution to (3) if it is absolutely continuous, and if it satisfies almost everywhere the differential inclusion

$$\dot{x} \in K[f](t, x) = \overline{\text{co}} \bigcap_{\varepsilon > 0} \bigcap_{\mu N = 0} f(t, B(x, \varepsilon) \setminus N),$$

where $K[f](t, x)$ is an upper semi-continuous, non-empty, compact and convex valued map, $\overline{\text{co}}$ represents the convex closure of a set, $B(x, \varepsilon) = \{v \in \mathbb{R}^n : \|x - v\| < \varepsilon\}$ and μ is the Lebesgue measure. Note that the intersections are taken over all the sets N of Lebesgue measure zero, over all $\varepsilon > 0$.

Definition 1. (Orlov, 2004), (Zimenko et al., 2018). *Let the origin be an equilibrium of (3). Then, the system (3) is Globally Uniformly Finite-Time Stable (FTS) if it satisfies:*

- *Lyapunov Stability: there exists a function $\alpha \in \kappa$ such that $\|x(t, x_0)\| \leq \alpha(\|x_0\|)$, for all $t \geq 0$, for any $x_0 \in \mathbb{R}^n$; and*
- *Finite-Time Attractive: there exists a locally bounded function $T: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_{\geq 0}$ such that $x(t, x_0) = 0$, for all $t \geq T(x_0)$ and any $x_0 \in \mathbb{R}^n$.*

4. FINITE-TIME CONTROLLER DESIGN

Based on Polyakov (2012), we propose the following family of sliding variables

$$s = x_2 + \left[[x_2]^\alpha + k_2 g(x_1) \right]^{\frac{1}{\alpha}}, \quad (4)$$

with positive design controller parameters $\alpha > 1$ and $k_2 > 0$, and any function $g \in \mathcal{C}^1$ such that $g(0) = 0$, $x_1 g(x_1) > 0$, for all $x_1 \neq 0$, $|g'(x_1)| \leq g^+$, for all $x_1 \in \mathbb{R}$, and $g'(0) > 0$. Then, the control input u is designed as

$$u = -\frac{1}{b} \left(f(t, x) + k_1 [s]^0 \right), \quad (5)$$

where $k_1 > 0$ represents the rest of design controller parameters, which will be designed further on.

The following theorem provides the main result of this work.

Theorem 1. *Let the control law (5), with the sliding variable (4), be applied to the uncertain system (1), with the functions h and b satisfying (2). If the controller parameters are designed as*

$$\alpha > 1, \quad k_2 > 0, \quad k_1 \geq \frac{k_2 g^+ + \alpha c}{\alpha},$$

then, the system (1) is Globally Uniformly FTS.

Remarks:

- For a fixed $\alpha > 1$, the controller parameter k_2 is proportional to the convergence rate to zero of the sliding variable s , and the states x_1 and x_2 , respectively.
- For a fixed $k_2 > 0$, the controller parameter α is inversely proportional to the convergence rate to zero of the sliding variable s , and the states x_1 and x_2 , respectively.
- It is possible to consider that the input coefficient b is uncertain but satisfies a boundedness condition, *i.e.*, $0 < k_m \leq b \leq k_M$, with some known constants $k_m, k_M \in \mathbb{R}_{>0}$. Then, one only needs to redesign the gain k_1 in terms of c, k_m, k_M, k_2 and α . Moreover, it is also possible to consider that the uncertain functions are bounded by known positive functions.
- Assuming that the nominal part of the system is such that $|f(t, x)| \leq L$; then, the proposed controller is clearly bounded, and hence, one may take into account input saturation constraints straightforwardly.
- Some examples for the selection of function g are: $g(x_1) = \arctan(x_1)$, $g(x_1) = x_1/(x_1^2 + 1)$ and $g(x_1) = x_1$. Note that for these options, $|g'(x_1)| \leq g^+ = 1$ and $g'(0) = 1$.

5. SIMULATION RESULTS

The corresponding simulations have been done in Matlab with the Euler discretization method, sample time equal to 0.001[s].

Consider the pendulum system that is given as follows:

$$\ddot{\theta} = \frac{1}{J}u - \frac{mgL}{2J} \sin(\theta) - \frac{V_s}{J}\dot{\theta} - \frac{1}{J}F_s(\dot{\theta}) + d(t), \quad (6)$$

where

$$F_s(\dot{\theta}) = \begin{cases} P_s[\dot{\theta}]^0, & \text{if } \dot{\theta} \neq 0, \\ P_a, & \text{if } \dot{\theta} = 0, \end{cases}$$

and $m = 1.1[\text{kg}]$ is the pendulum mass, $L = 0.9[\text{m}]$ is the pendulum longitude, $J = 0.891[\text{kgm}^2]$ is the arm inertia, $V_s = 0.18[\text{kgm}^2/\text{s}]$ is the the viscous friction coefficient, $P_s = 0.45[\text{kgm}^2/\text{s}]$ is the coulomb friction coefficient, $P_a = 0.15[\text{Nm}]$ is an applied force, $g = 9.815[\text{m}^2/\text{s}]$ is the gravitational constant and d represents the external disturbances.

The system (6) can be rewritten as in (1) with $x_1 = \theta$ and $x_2 = \dot{\theta}$. Consider that the initial conditions are given as $x_1(0) = \pi/3$, $x_2(0) = 2$ while the external disturbance force is given as $d(t) = 1 + 2(\sin(2t) + \cos(5t))$. Then, we have that

$$f(t, x) = -\frac{mgL}{2J} \sin(\theta) - \frac{V_s}{J}\dot{\theta},$$

$$\xi(t, x) = -\frac{1}{J}F_s(\dot{\theta}) + d(t),$$

and hence, $c = P_s/J + 3$. Therefore, we design the controller parameters, with the function $g(x_1) = \arctan(x_1)$, following the statement of Theorem 1, *i.e.*, $\alpha = 1.5$, $k_2 = 3$, $g^+ = 1$, and then, $k_1 = 5.50$.

The corresponding results are depicted by Figs. 1, 2 and 3. These results show that the proposed second-order

sliding-mode controller provides finite-time convergence to zero despite the considered external disturbance force and the unmodeled dynamics by means of a discontinuous control action. Particularly, Fig. 1 illustrates how the trajectories of the system converge to the sliding surface $s = 0$ in a finite time; and then, once on the sliding surface, the trajectories are steered to zero in a finite time. It is worth mentioning that the proposed finite-time controller only requires for 2 parameters tuning, *i.e.*, α and k_2 , and thus, its synthesis is very simple.

Finally, in order to illustrate the performance of the proposed controller depending on the controller parameter α , we provide several results for $\alpha = \{1.1, 1.5, 2, 2.5, 3\}$, $k_2 = 10$, and then, $k_1 = \{12.59, 10.17, 8.50, 7.50, 6.83\}$. The results are depicted by Figs. 4 and 5. It is clear that, for a fixed k_2 , α is inversely proportional to the convergence rate.

6. CONCLUSIONS

This paper contributes to the design of a second-order sliding-mode controller for a class of uncertain second-order non-linear systems. The proposed strategy takes into account the design of a family of sliding variables, which ensures the convergence of the system trajectories in a finite time despite some uncertainties on the system. The straightforward structure of the controller is simple to tune and to implement. The global, uniform, and finite-time stability of the closed-loop dynamics is demonstrated by means of Lyapunov functions in a straightforward way. Furthermore, the performance of the proposed approach is validated through simulation results using a pendulum system. The future work may be focused on extending the result to the case of state constraints or the design of high-order sliding-mode controllers.

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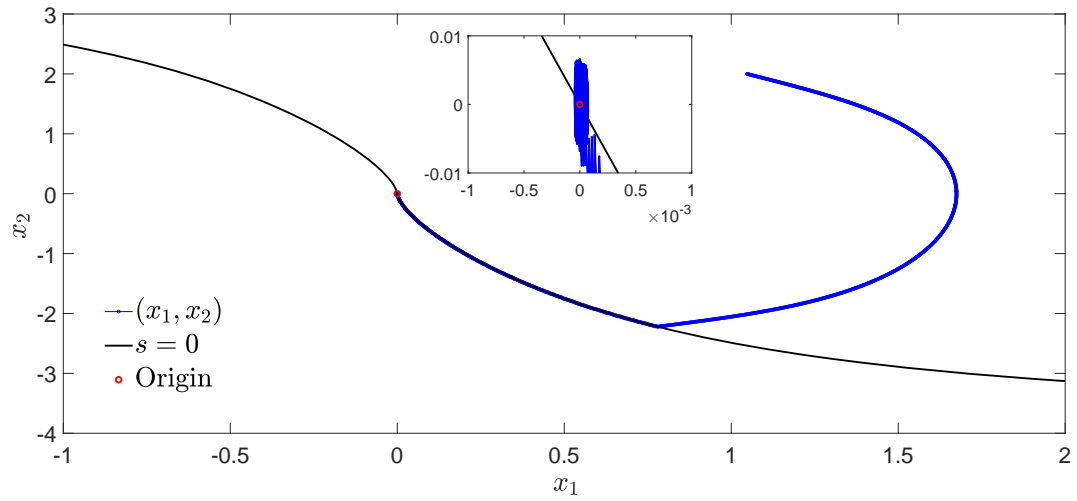


Fig. 1. Phase Portrait

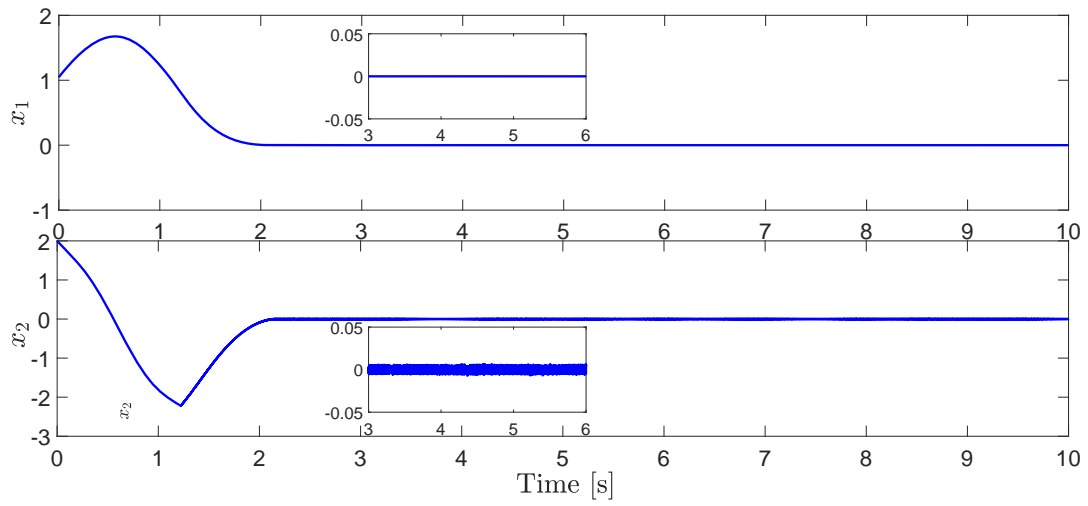


Fig. 2. Trajectories of the System

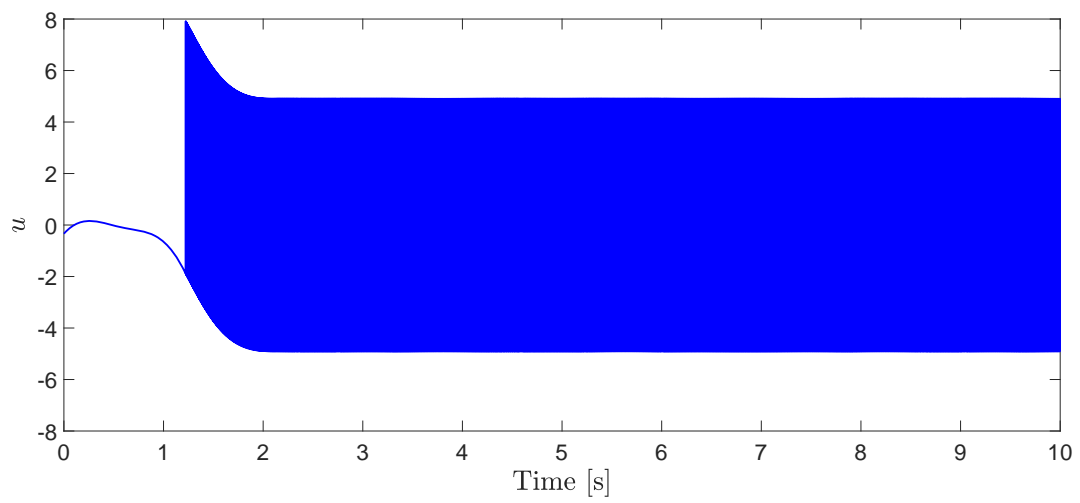


Fig. 3. Control Signals

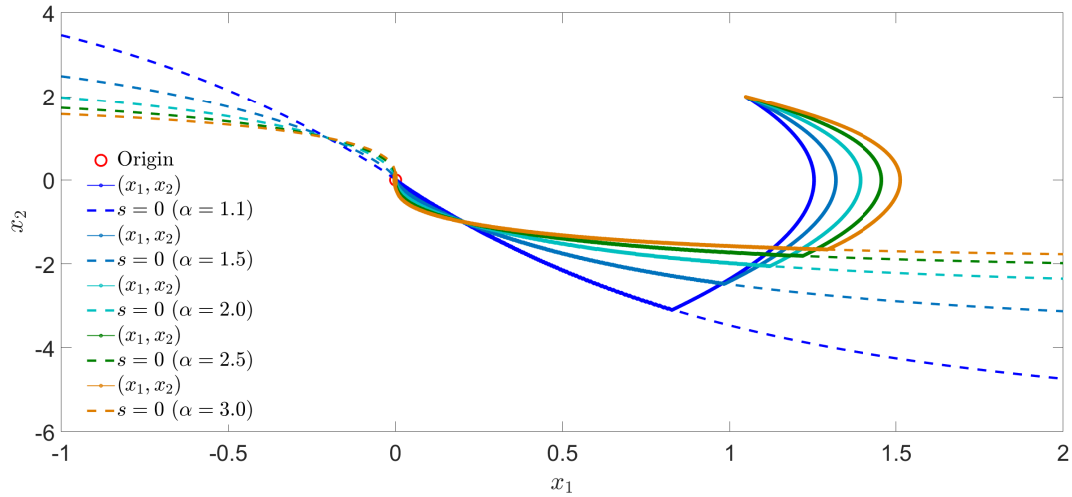


Fig. 4. Phase Portraits – Performance depending on α

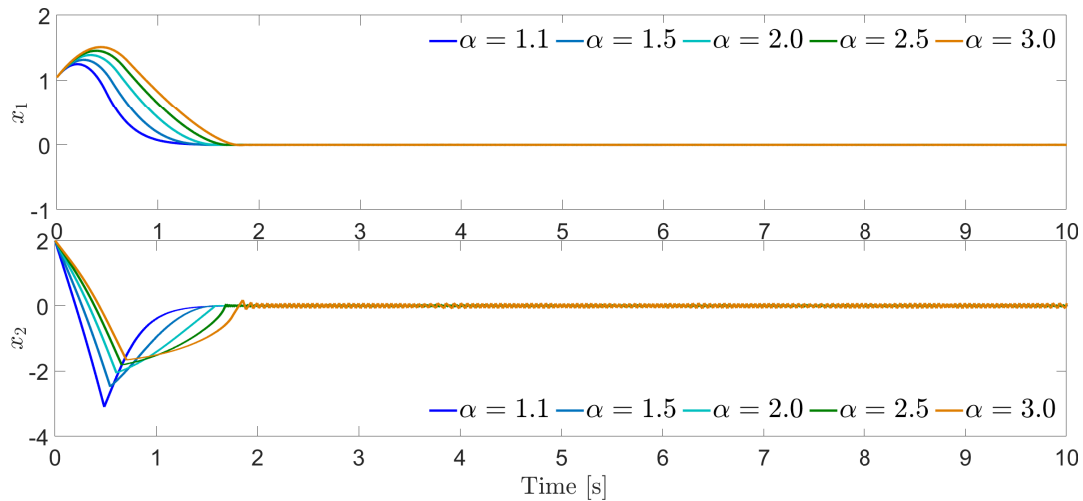


Fig. 5. Trajectories of the System – Performance depending on α

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