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SUPER-RESOLUTION RECONSTRUCTION FROM TRUNCATED FOURIER TRANSFORM

M. ISAEV, R.G. NOVIKOV, G.V. SABININ

ABSTRACT. We present recent theoretical and numerical results on recovering a compactly supported function v on \mathbb{R}^d , $d \geq 1$, from its Fourier transform $\mathcal{F}v$ given within the ball B_r . We proceed from known results on the prolate spheroidal wave functions and on the Radon transform. The most interesting point of our numerical examples consists in super-resolution, that is, in recovering details beyond the diffraction limit, that is, details of size less than π/r , where r is the radius of the ball mentioned above. This short review is based on the works Isaev, Novikov (2022 J. Math. Pures Appl. 163 318–333) and Isaev, Novikov, Sabinin (2022 Inverse Problems 38 105002).

1. INTRODUCTION

It is well known that there is a diffraction limit for recovering the structure of an object from results of its probing by waves of length λ . This limit is that the details of the object smaller than $\lambda/2$ are blurred or not recovered by standard numerical methods. Our recent results of [1, 2] give a mathematical approach to overcoming this limit, thereby achieving *super-resolution*, without significant a priori assumptions on the unknown structure. In particular, the details in numerical examples of [2] are distinguished even of the size $\lambda/12$ (that is, 6 times smaller than the diffraction limit) and even with 20% random noise in the data.

In [1, 2] the original problem is modeled as a problem of recovering a function with compact support on \mathbb{R}^d from its Fourier transform restricted to the ball B_r of fixed radius $r = 2\pi/\lambda$ (that is, from the truncated Fourier transform); see Problem 1 below. This model arises, for example, as a reconstruction problem in monochromatic acoustic tomography in the Born approximation.

Thus, we consider the Fourier transform \mathcal{F} defined by the formula

$$\mathcal{F}[v](p) = \hat{v}(p) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ipq} v(q) dq, \quad p \in \mathbb{R}^d, \quad (1)$$

where v is a complex-valued test function on \mathbb{R}^d , $d \geq 1$.

Let $B_\rho := \{q \in \mathbb{R}^d : |q| < \rho\}$, $\rho > 0$.

We now state formally the aforementioned problem.

Problem 1. Find $v \in L^2(\mathbb{R}^d)$, where $\text{supp } v \subset B_\sigma$, from $\hat{v} = \mathcal{F}v$ given on the ball B_r (possibly with some noise), for fixed $r, \sigma > 0$.

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Problem 1 arises in different areas and has been extensively studied in the literature. However, solving Problem 1 is complicated considerably by the fact it is exponentially unstable, for fixed $r, \sigma > 0$. Nevertheless, there exist several techniques to approach this problem theoretically and numerically; see, [1, 2] and references therein.

The conventional approach for solving Problem 1 is based on the following approximation

$$v \approx v_{\text{naive}} := \mathcal{F}^{-1}[w](q) = \int_{B_r} e^{-ipq} w(p) dp \quad q \in B_\sigma, \quad (2)$$

where \mathcal{F}^{-1} is the standard inverse Fourier transform and w is such that $w|_{B_r}$ coincides with the data of Problem 1 and $w|_{\mathbb{R}^d \setminus B_r} \equiv 0$.

Formula (2) leads to a stable and accurate reconstruction for sufficiently large r , which is carefully studied in the literature and widely used for industrial applications. Nevertheless, it has the well-known diffraction limit: small details (especially less than π/r) are blurred.

A new approach for *super-resolution* in comparison with the resolution of (2) was recently developed in [1, 2]; see Sections 3 and 4 below. Some preliminaries are presented in Section 2.

2. PRELIMINARIES

For convenience, we consider the scaling of v with respect to the size of its support:

$$v_\sigma(q) := v(\sigma q), \quad q \in \mathbb{R}^d. \quad (3)$$

Note that $\text{supp } v_\sigma \subset B_1$. Let

$$c := r\sigma \quad (4).$$

The data in Problem 1 (for the case without noise) can be presented as follows (for more details, see [1]):

$$\hat{v}(rx) = \frac{\sigma}{2\pi} \mathcal{F}_c[v_\sigma](x) \quad \text{for } d = 1, \quad (5)$$

$$\hat{v}(rx\theta) = \left(\frac{\sigma}{2\pi}\right)^d \mathcal{F}_c[\mathcal{R}_\theta[v_\sigma]](x) \quad \text{for } d \geq 2, \quad (6)$$

where $x \in [-1, 1]$, $\theta \in \mathbb{S}^{d-1}$, $c = r\sigma$, $v_\sigma(q) = v(\sigma q)$, and the operators \mathcal{F}_c and \mathcal{R}_θ are defined by

$$\mathcal{F}_c[f](x) := \int_{-1}^1 e^{icxy} f(y) dy, \quad x \in [-1, 1], \quad (7)$$

$$\mathcal{R}_\theta[u](y) := \int_{q \in \mathbb{R}^d, q\theta=y} u(q) dq, \quad y \in \mathbb{R}, \quad (8)$$

where f is a test function on $[-1, 1]$ and u is a test function of \mathbb{R}^d .

Recall that $\mathcal{R}_\theta[u] \equiv \mathcal{R}[u](\cdot, \theta)$, where \mathcal{R}_θ is defined by (8) and \mathcal{R} is the classical Radon transform of [4]. In fact, presentation (6) follows from the projection theorem of the Radon transform theory.

The operator \mathcal{F}_c defined by (7) is a variant of band-limited Fourier transform. This operator is one of the key objects of the theory of *prolate spheroidal wave functions*. In particular, the operator \mathcal{F}_c has the following singular value decomposition in $L^2([-1, 1])$:

$$\mathcal{F}_c[f](x) = \sum_{j \in \mathbb{N}} \mu_{j,c} \psi_{j,c}(x) \int_{-1}^1 \psi_{j,c}(y) f(y) dy, \quad (9)$$

where $(\psi_{j,c})_{j \in \mathbb{N}}$ are the prolate spheroidal wave functions (these functions reduce to the Legendre polynomials as $c \rightarrow 0$ and reduce to the Hermite functions as $c \rightarrow \infty$). In addition, it is known that the eigenvalues $\{\mu_{j,c}\}_{j \in \mathbb{N}}$ satisfy the following properties:

$$0 < |\mu_{j+1,c}| < |\mu_{j,c}| \text{ for all } j \in \mathbb{N} = \{0, 1, 2, \dots\}; \quad (10)$$

$$\left\lfloor \frac{2c}{\pi} \right\rfloor - 1 \leq \left| \{j \in \mathbb{N}, |\mu_{j,c}| \geq \sqrt{\pi/c}\} \right| \leq \left\lceil \frac{2c}{\pi} \right\rceil + 1, \quad (11)$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and the ceiling functions, respectively, and $|\cdot|$ in (11) is the number of elements in a set;

$$\mu_{j,c} \text{ decay superexponentially as } j \rightarrow \infty. \quad (12)$$

The functions $(\psi_{j,c})_{j \in \mathbb{N}}$ are certain of wave functions introduced by Niven in [3] for solving the Helmholtz equation in prolate spheroidal coordinates. Originally, $(\psi_{j,c})_{j \in \mathbb{N}}$ are defined as the eigenfunctions of the spectral problem

$$\frac{d}{dx} \left[(1-x^2) \frac{d\psi}{dx} \right] + c^2 x^2 \psi = \chi \psi, \quad \psi \in C^2([-1, 1]). \quad (13)$$

The fact that $(\psi_{j,c})_{j \in \mathbb{N}}$ are the eigenfunctions of the finite Fourier transform \mathcal{F}_c defined by (7) was pointed out by Slepian and Pollak in [5] as a special case of more general integral relations satisfied by Niven's wave functions. As mentioned in [5]

"These functions ... possess properties that make them ideally suited for the study of certain questions regarding the relationship between functions and their Fourier transforms."

Surprisingly, to our knowledge, these functions were not used in the context of Problem 1 before the recent work [1], at least, for $d > 1$.

3. RECONSTRUCTION FORMULAS FROM [1]

The formulas of [1] for solving Problem 1 are as follows, where we use the notations of Section 2.

For $d = 1$, the reconstruction is given by

$$v_\sigma = \frac{2\pi}{\sigma} \mathcal{F}_c^{-1}[\hat{v}_r], \quad (14)$$

where $\hat{v}_r(x) = \hat{v}(rx)$, $x \in [-1, 1]$, and

$$\mathcal{F}_c^{-1}[g](y) = \sum_{j \in \mathbb{N}} \frac{1}{\mu_{j,c}} \psi_{j,c}(y) \int_{-1}^1 \psi_{j,c}(x) g(x) dx, \quad (15)$$

where g is a test function from the range of \mathcal{F}_c acting on $L^2([-1, 1])$.

For $d \geq 2$, the reconstruction is given by

$$v_\sigma = \left(\frac{2\pi}{\sigma}\right)^d \mathcal{R}^{-1}[f_{r,\sigma}], \quad (16)$$

where

$$f_{r,\sigma}(y, \theta) = \begin{cases} \mathcal{F}_c^{-1}[\hat{v}_{r,\theta}](y), & \text{if } y \in [-1, 1], \\ 0, & \text{otherwise;} \end{cases}$$

$$\hat{v}_{r,\theta}(x) = \hat{v}(rx\theta), \quad x \in [-1, 1], \quad \theta \in \mathbb{S}^{d-1}.$$

In addition, in view of property (12), for the case of noisy data in Problem 1, the operator \mathcal{F}_c^{-1} is approximated by the finite rank operator $\mathcal{F}_{n,c}^{-1}$ defined by

$$\mathcal{F}_{n,c}^{-1}[g](y) := \sum_{j=0}^n \frac{1}{\mu_{j,c}} \psi_{j,c}(y) \int_{-1}^1 \psi_{j,c}(x) g(x) dx. \quad (17)$$

It is important to remark that:

- the operator $\mathcal{F}_{n,c}^{-1}$ is correctly defined on $L^2([-1, 1])$ for any $n \in \mathbb{N}$;
- the function $\mathcal{F}_{n,c}^{-1}[g]$ is *the quasi-solution* in the sense of Ivanov of the equation $\mathcal{F}_c[f] = g \in L^2([-1, 1])$ on the span of the first $n+1$ functions $(\psi_{j,c})_{j \leq n}$;
- the rank n is a regularisation parameter and its choice is crucial for both theoretical results and numerical applications.

For formulations of mathematical theorems related with formulas (14)-(17), see [1]. These theoretical results include Hölder-logarithmic stability estimates for the reconstruction based on these formulas.

4. NUMERICAL RESULTS OF [2]

The numerical implementation of [2] of the approach of [1] includes, in particular, different principles for choosing the aforementioned regularisation parameter n such as *residual minimisation* and the *Morozov discrepancy principle*. One of the most interesting points of the results of [2] lies in examples of super-resolution, that is, recovering details of size less than π/r , where r is the band-limiting radius of Problem 1. In [2] we also obtain a better reconstruction in the sense of L^2 -norm than the conventional reconstruction based on formula (2).

We illustrate numerical results of [2] by Figures 1 and 2. In these numerical examples we use the values $\sigma = 1$ and $c = r = 10$; see [2] for a detailed description of the reconstructions.

Figure 1 shows one of the numerical examples of [2] for $d = 1$. In this example, the preimage v equals the sum of characteristic functions of two disjoint intervals at distance $\pi/(2r)$, which is twice smaller than the diffraction limit. Figure 1 shows our super-resolution reconstruction based on formulas (14), (17) in comparison with preimage v and naive Fourier inversion v_{naive} based on formula (2), where the data are noiseless.

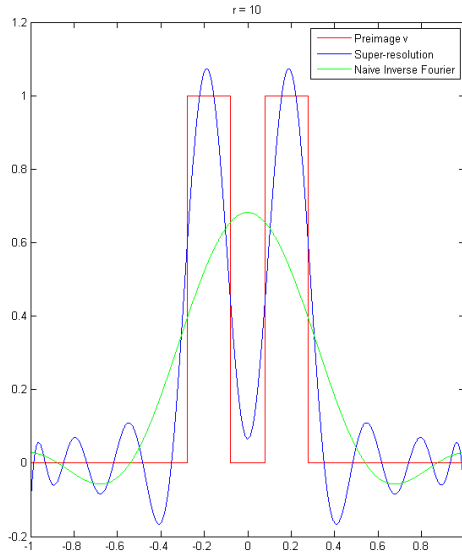


FIGURE 1. Example of our super-resolution reconstruction in comparison with preimage v and naive Fourier inversion, for $d = 1$.

Figure 2 shows one of the numerical examples of [2] for $d = 2$. In this example, the preimage v equals the sum of characteristic functions of three disjoint square parts. The distances between the parts of v are significantly less than $\pi/r \approx 0.314$: two bottom squares are at the distance 0.1, while the top square and any of the bottom squares are at the distance 0.05, which is 6 times smaller than the diffraction limit. Figure 2 shows our super-resolution reconstruction based on formulas (16)-(17) in comparison with preimage v and naive Fourier inversion v_{naive} based on formula (2), where the data are noiseless.

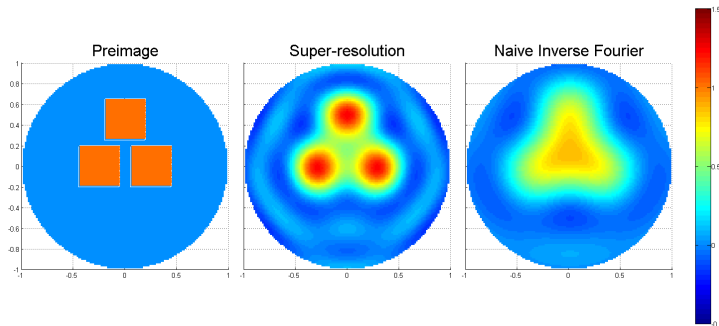


FIGURE 2. Example of our super-resolution reconstruction in comparison with preimage v and naive Fourier inversion, for $d = 2$.

Our work [2] also contains many other numerical results and examples. In particular, in spite of the exponential instability of Problem 1, we achieve super-resolution even for noisy data by an appropriate choice of the regularisation parameter n . In fact, our reconstruction with $n = n_0 = \lfloor \frac{2\epsilon}{\pi} \rfloor$ behaves similarly to the conventional reconstruction based on formula (2). Taking n properly larger than n_0 leads to better results. Moreover, for $d \geq 2$, our approach works well even for a considerable level of random noise: for example, the reconstruction given in Figure 2 remains reliable even if we add 20% random noise to the data.

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