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# A remark on the distributional Jacobian

Petru Mironescu<sup>(1)</sup>

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## Abstract

We investigate the existence of the distributional Jacobian for maps  $u \in W_{loc}^{1,1}(\mathbb{R}^N; \mathbb{R}^N)$ . In 2D, we present necessary and sufficient existence assumptions. In dimensions 3 and higher, we exhibit sufficient existence conditions weaker than the standard ones.

## 1 Introduction

This note complements the expository text [2] by H. Brezis, J. Mawhin, and the author, presenting a brief – and personal – history of the (distributional) Jacobian. Given a mapping  $u \in W_{loc}^{1,1}(\Omega; \mathbb{R}^N)$ , where  $N \geq 2$  and  $\Omega \subset \mathbb{R}^N$  is an open set, and indices  $1 \leq i, j \leq N$ , we denote by  $C_{i,j}$  the cofactor of  $\partial_j u_i$  in the Jacobian  $\det(\nabla u)$  of  $u$ . Note that  $C_{i,j}$  is a measurable function defined a.e., but this function need not belong to  $L_{loc}^1(\Omega; \mathbb{R})$ . Assuming extra regularity assumptions on  $u$ , e.g.,

$$u \in W_{loc}^{1,N-1}(\Omega; \mathbb{R}^N) \cap L_{loc}^\infty(\Omega; \mathbb{R}^N) \quad (1.1)$$

or

$$u \in W_{loc}^{1,N^2/(N+1)}(\Omega; \mathbb{R}^N), \quad (1.2)$$

we have  $u_i C_{i,j} \in L_{loc}^1(\Omega; \mathbb{R})$ , and then we may define

$$T_i := \sum_{j=1}^N \partial_j (u_i C_{i,j}) \in \mathcal{D}'(\Omega; \mathbb{R}). \quad (1.3)$$

Moreover, under the assumptions (1.1) or (1.2),  $T_i$  is independent of  $i$ , i.e.,

$$\sum_{j=1}^N \partial_j (u_i C_{i,j}) = \sum_{j=1}^N \partial_j (u_\ell C_{\ell,j}) \text{ in } \mathcal{D}'(\Omega; \mathbb{R}), \quad \forall 1 \leq i, \ell \leq N, \quad (1.4)$$

and we may thus define

$$\text{Det}(\nabla u) := T_i, \quad \forall 1 \leq i \leq N, \quad (1.5)$$

the *distributional Jacobian* made popular by the seminal work of J. Ball [1].

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The question we address here is the validity of (1.4) for  $u \in W_{loc}^{1,1}(\Omega; \mathbb{R}^N)$  under ‘minimal’ extra assumptions, i.e, we look for ‘minimal’ assumptions allowing us to define the distributional Jacobian as in (1.5). For the sake of simplicity of the statements, we consider only the case  $i = 1$ ,  $\ell = 2$ . Given the local nature of our problem, we may assume that  $\Omega = (-1, 1)^N$ , and discard ‘loc’. It will be convenient to use the language of differential forms, and to consider the following more general situation. Let  $2 \leq k \leq N$  and

$$u = (u_1, \dots, u_k) \in W^{1,1}(\Omega; \mathbb{R}^k). \quad (1.6)$$

We want to find ‘minimal’ assumptions implying that

$$d\omega_1 + d\omega_2 = 0 \text{ in } \mathcal{D}'(\Omega; \Lambda^k), \quad (1.7)$$

where  $\Lambda^k$  stands for the space of  $k$ -forms and

$$\omega_1 := u_1 du_2 \wedge du_3 \wedge \dots \wedge du_k, \quad \omega_2 := u_2 du_1 \wedge du_3 \wedge \dots \wedge du_k \quad (1.8)$$

are  $(k-1)$ -forms with measurable coefficients. (Our original question corresponds to  $N = k \geq 2$ .)

Note that, formally, (1.7) is clear, since, for smooth  $u$ ,

$$\omega_1 + \omega_2 = d(u_1 u_2 du_3 \wedge \dots \wedge du_k),$$

and thus

$$d\omega_1 + d\omega_2 = d^2(u_1 u_2 du_3 \wedge \dots \wedge du_k) = 0.$$

In order to give a meaning to (1.7), we require

$$\omega_1, \omega_2 \in L^1(\Omega; \Lambda^{k-1}). \quad (1.9)$$

In the easier case  $k = 2$ , the minimal conditions (1.6) and (1.9) are sufficient for the validity of (1.7). In fact, more can be said in this case.

**Proposition 1.** Let  $N \geq 1$ . Let  $u_1, u_2 \in W^{1,1}(\Omega; \mathbb{R})$ . Assume that  $u_1 du_2 + u_2 du_1 \in L^1(\Omega; \Lambda^1)$ . Then:

1.  $u_1 u_2 \in W^{1,1}(\Omega; \mathbb{R})$ .
2.  $d(u_1 u_2) = u_1 du_2 + u_2 du_1$ . In particular:
  - (a) We have  $d(u_1 du_2 + u_2 du_1) = 0$  in  $\mathcal{D}'(\Omega; \Lambda^2)$ .
  - (b) Under the stronger assumptions  $u_1 du_2 \in L^1(\Omega; \Lambda^1)$  and  $u_2 du_1 \in L^1(\Omega; \Lambda^1)$ , we have  $d(u_1 du_2) + d(u_2 du_1) = 0$  in  $\mathcal{D}'(\Omega; \Lambda^2)$ . Consequently, (1.7) with  $k = 2$  holds under the assumptions (1.6) and (1.9).
3.  $\|u_1 u_2\|_{L^1} \leq C(\|u_1\|_{W^{1,1}} \|u_2\|_{W^{1,1}} + \|u_1 du_2 + u_2 du_1\|_{L^1})$  (with  $C$  depending only on  $N$ ).

**Corollary 1.** Let  $\Omega \subset \mathbb{R}^2$  be an open set. The distributional Jacobian  $\text{Det}(\nabla u)$  is well-defined (via (1.5)) for maps  $u = (u_1, u_2): \Omega \rightarrow \mathbb{R}^2$  satisfying

$$u_1, u_2 \in W_{loc}^{1,1}(\Omega; \mathbb{R}), \quad u_1 du_2, u_2 du_1 \in L_{loc}^1(\Omega; \Lambda^1).$$

When  $k \geq 3$ , the necessary conditions (1.6) and (1.9) are no more sufficient for the validity of (1.7).

**Proposition 2.** Let  $N \geq k \geq 3$ . There exists some  $u = (u_1, \dots, u_k) \in W^{1,1}(\Omega; \mathbb{R}^k)$  such that:

- a)  $u_j \in C^\infty(\overline{\Omega}; \mathbb{R})$ ,  $j \neq 2, 3$ .
- b)  $u_j \in W^{1,1}(\Omega; \mathbb{R}) \cap L^\infty$  and  $du_j \in L^{2,\infty}(\Omega; \Lambda^1)$ ,  $j = 2, 3$ . (So that  $u_2$  and  $u_3$  are ‘almost’ in  $W^{1,2}$ .)
- c)  $\omega_1, \omega_2 \in L^{2,\infty}(\Omega; \Lambda^{k-1})$ .
- d)  $d\omega_1 + d\omega_2 \neq 0$  in  $\mathcal{D}'(\Omega; \Lambda^k)$ .

Finally, we present an existence result for the distributional Jacobian when  $k \geq 3$ . Motivated by the example in Proposition 2 (see also Remark 1 in the next section) and the validity of (1.7) under the assumption (1.1), we consider the following assumption, weaker than (1.1):

$$du_j \in L^{k-1}([\varepsilon < |u_j| < M]), \quad \forall 0 < \varepsilon < M < \infty, \quad \forall 1 \leq j \leq k. \quad (1.10)$$

**Proposition 3.** Assume (1.6), (1.9), and (1.10). Then (1.7) holds.

**Corollary 2.** Let  $N \geq 3$  and let  $\Omega \subset \mathbb{R}^N$  be an open set. Let  $u \in W_{loc}^{1,1}(\Omega; \mathbb{R}^N)$  satisfy

$$|u| |du|^{N-1} \in L^1(\Omega; \mathbb{R}) \quad (1.11)$$

(where  $|\cdot|$  stands for the Euclidean length). Then the distributional Jacobian  $\text{Det}(\nabla u)$  is well-defined via (1.5).

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## 2 Proofs

*Proof of Proposition 1.* In what follows,  $C$  denotes a generic constant depending only on  $N$ .

When  $N = 1$ , the assumption  $u_1 u_2' + u_2 u_1' \in L^1$  is always satisfied, part 2 is simply the chain rule  $(u_1 u_2)' = u_1 u_2' + u_2 u_1'$ , and the estimate in part 3 follows from the embedding  $W^{1,1} \hookrightarrow L^\infty$ .

Let  $N \geq 2$ . Using the case  $N = 1$  and Fubini, we find that parts 1 and 2 hold *provided we know that*  $u_1 u_2 \in L^1$ .

In particular, when  $N = 2$ , parts 1, 2, and 3 follow from the above considerations and the embedding  $W^{1,1} \hookrightarrow L^2$ , which implies that  $\|u_1 u_2\|_{L^1} \leq C \|u_1\|_{W^{1,1}} \|u_2\|_{W^{1,1}}$ .

The case  $N \geq 3$  follows by induction on  $N$ . With no loss of generality, we may assume that  $u_1$  and  $u_2$  are absolutely continuous on a.e. line segment of the form

$$L_{(x_1, \dots, x_{N-1})} := \{(x_1, \dots, x_{N-1}, t); -1 < t < 1\}.$$

Pick some  $t_0 \in (-1, 1)$  such that the partial functions

$$(-1, 1)^{N-1} \ni (x_1, \dots, x_{N-1}) \mapsto v_i(x_1, \dots, x_{N-1}) := u_i(x_1, \dots, x_{N-1}, t_0), \quad i = 1, 2,$$

satisfy  $v_1, v_2 \in W^{1,1}((-1, 1)^{N-1}; \mathbb{R})$  and the estimates

$$\|v_i\|_{W^{1,1}} \leq C \|u_i\|_{W^{1,1}}, \quad i = 1, 2, \quad (2.1)$$

$$\|v_1 dv_2 + v_2 dv_1\|_{L^1} \leq C \|u_1 du_2 + u_2 du_1\|_{L^1}. \quad (2.2)$$

By (2.1), (2.2), and the induction assumption, we then have  $v_1 v_2 \in L^1((-1, 1)^{N-1}; \mathbb{R})$ , and the estimate

$$\|v_1 v_2\|_{L^1} \leq C (\|u_1\|_{W^{1,1}} \|u_2\|_{W^{1,1}} + \|u_1 du_2 + u_2 du_1\|_{L^1}). \quad (2.3)$$

On the other hand, for a.e.  $(x_1, \dots, x_{N-1}) \in (-1, 1)^{N-1}$ , the Leibniz rule applies, on  $L(x_1, \dots, x_{N-1})$ , to the absolutely continuous partial maps

$$(-1, 1) \ni t \mapsto w_i(t) := u_i(x_1, \dots, x_{N-1}, t), \quad i = 1, 2,$$

which implies, for such  $(x_1, \dots, x_{N-1})$ , that

$$\|w_1 w_2\|_{L^1((-1, 1))} \leq C (|w_1(t_0) w_2(t_0)| + \|w_1 w_2' + w_2 w_1'\|_{L^1((-1, 1))}). \quad (2.4)$$

Combining (2.3) and (2.4), we find that item 3 holds, and in particular, that  $u_1 u_2 \in L^1(\Omega; \mathbb{R})$ . As explained above, this also implies the validity of items 1 and 2.  $\square$

*Proof of Proposition 2.* Set, for  $x \in \Omega$ ,  $\rho = \rho(x) := (x_1^2 + x_2^2)^{1/2}$ , and let

$$u_1(x) := 1, \quad u_2(x) := \frac{x_1}{\rho} e^{x_3}, \quad u_3(x) := \frac{x_2}{\rho} e^{x_3}, \quad u_j(x) := x_j, \quad 4 \leq j \leq k.$$

Properties a) and b) are straightforward.

We now prove c). Clearly,  $\omega_2 = 0$ . On the other hand, we have

$$du_2 = \frac{x_2^2}{\rho^3} e^{x_3} dx_1 - \frac{x_1 x_2}{\rho^3} e^{x_3} dx_2 + \frac{x_1}{\rho} e^{x_3} dx_3 \text{ in } \mathcal{D}'(\Omega; \Lambda^1),$$

$$du_3 = -\frac{x_1 x_2}{\rho^3} e^{x_3} dx_1 + \frac{x_1^2}{\rho^3} e^{x_3} dx_2 + \frac{x_2}{\rho} e^{x_3} dx_3 \text{ in } \mathcal{D}'(\Omega; \Lambda^1),$$

and therefore, with  $\omega_0 := dx_4 \wedge \dots \wedge dx_k$  (and the convention  $\omega_0 := 1$  if  $k = 3$ )

$$\omega_1 = -\left( \frac{x_1}{\rho^2} e^{2x_3} dx_2 \wedge dx_3 + \frac{x_2}{\rho^2} dx_3 \wedge dx_1 \right) \wedge \omega_0 \text{ in } \mathcal{D}'(\Omega; \Lambda^{k-1}). \quad (2.5)$$

Thus c) holds.

Finally, using (2.5) and the fact that

$$\partial_1 \left( \frac{x_1}{\rho^2} \right) + \partial_2 \left( \frac{x_2}{\rho^2} \right) = 2\pi \delta_{(0,0)} \text{ in } \mathcal{D}'(\mathbb{R}^2; \mathbb{R}),$$

we find that

$$d\omega_1 + d\omega_2 = d\omega_1 = -2\pi \delta_{(0,0)} \otimes e^{2x_3} dx_1 \wedge \dots \wedge dx_k \text{ in } \mathcal{D}'(\Omega; \Lambda^k),$$

and thus d) holds.  $\square$

**Remark 1.** A similar example shows that, in order to insure the validity of (1.4), the condition (1.1) cannot be weakened to  $u_j \in L_{loc}^\infty(\Omega; \mathbb{R})$  and  $du_j \in L_{loc}^{N-1, \infty}(\Omega; \Lambda^1)$ ,  $1 \leq j \leq N$ . Indeed, if we set  $r = r(x) := (x_1^2 + \dots + x_{k-1}^2)^{1/2}$  and let

$$u_1(x) := 1, u_j(x) := \frac{x_{j-1}}{r} e^{x_k}, 2 \leq j \leq k,$$

then (by repeating the proof of Proposition 2) we have  $u_j \in L^\infty(\Omega; \mathbb{R})$ ,  $du_j \in L^{k-1, \infty}(\Omega; \Lambda^1)$ ,  $1 \leq j \leq k$ ,  $\omega_1, \omega_2 \in L^{k-1, \infty}(\Omega; \Lambda^{k-1})$ , and  $d\omega_1 + d\omega_2 \neq 0$  in  $\mathcal{D}'(\Omega; \Lambda^k)$ .

*Proof of Proposition 3.* Consider a sequence  $(\Phi_\ell)_{\ell \geq 2} \subset C^\infty(\mathbb{R}; \mathbb{R})$  such that  $\Phi_\ell(t) = 0$  if  $|t| < 1/(2\ell)$ ,  $\Phi_\ell(t) = (\ell + 1) \operatorname{sgn} t$  if  $|t| > \ell + 1$ ,  $\Phi_\ell(t) = t$  if  $1/\ell \leq |t| \leq \ell$ , and  $|\Phi'_\ell| \leq 2$ ,  $\forall \ell$ . Set  $\tilde{u}_j = \tilde{u}_{j, \ell} := \Phi_\ell(u_j)$ . The strategy of the proof consists of proving (1.7) for  $\tilde{u}_j$  instead of  $u_j$ , and then letting  $\ell \rightarrow \infty$ .

Since  $\tilde{u}_1, \tilde{u}_2 \in W^{1,1} \cap L^\infty$ , the Leibniz rule implies that  $d(\tilde{u}_1 \tilde{u}_2) = \tilde{u}_1 d\tilde{u}_2 + \tilde{u}_2 d\tilde{u}_1$ , and thus

$$(\tilde{u}_1 d\tilde{u}_2 + \tilde{u}_2 d\tilde{u}_1) \wedge d\tilde{u}_3 \wedge \dots \wedge d\tilde{u}_k = d(\tilde{u}_1 \tilde{u}_2) \wedge d\tilde{u}_3 \wedge \dots \wedge d\tilde{u}_k \text{ in } \mathcal{D}'(\Omega; \Lambda^{k-1}). \quad (2.6)$$

We claim that, under the only assumption (1.10), we have

$$d(\tilde{u}_1 \tilde{u}_2) \wedge d\tilde{u}_3 \wedge \dots \wedge d\tilde{u}_k = d((\tilde{u}_1 \tilde{u}_2) d\tilde{u}_3 \wedge \dots \wedge d\tilde{u}_k) \text{ in } \mathcal{D}'(\Omega; \Lambda^k) \quad (2.7)$$

(and this also holds for  $N = k - 1$ ).

Indeed, (2.7) follows by approximation with smooth maps, using the fact that, by the assumption (1.10) and the chain and Leibniz rules, we have  $\tilde{u}_1 \tilde{u}_2, \tilde{u}_3, \dots, \tilde{u}_k \in W^{1, k-1}$ .

Combining (2.6) and (2.7), we find that

$$d(\tilde{u}_1 d\tilde{u}_2 \wedge d\tilde{u}_3 \wedge \dots \wedge d\tilde{u}_k) + d(\tilde{u}_2 d\tilde{u}_1 \wedge d\tilde{u}_3 \wedge \dots \wedge d\tilde{u}_k) = 0 \text{ in } \mathcal{D}'(\Omega; \Lambda^k). \quad (2.8)$$

We obtain (1.7) by letting  $\ell \rightarrow \infty$  in (2.8). (Recall that  $\tilde{u}_j = \tilde{u}_{j, \ell}$ .) Passing to the limits is justified via the assumption (1.9) and the uniform inequalities

$$\begin{aligned} |\tilde{u}_{1, \ell} d\tilde{u}_{2, \ell} \wedge d\tilde{u}_{3, \ell} \wedge \dots \wedge d\tilde{u}_{k, \ell}| &\leq 2^{k-1} |u_1 du_2 \wedge du_3 \wedge \dots \wedge du_k|, \forall \ell, \\ |\tilde{u}_{2, \ell} d\tilde{u}_{1, \ell} \wedge d\tilde{u}_{3, \ell} \wedge \dots \wedge d\tilde{u}_{k, \ell}| &\leq 2^{k-1} |u_2 du_1 \wedge du_3 \wedge \dots \wedge du_k|, \forall \ell, \end{aligned}$$

inequalities that follow from the properties of  $\Phi_\ell$  and the chain rule.  $\square$

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