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# A brief history of the Jacobian

Haïm Brezis<sup>(1)</sup>, Jean Mawhin<sup>(2)</sup>, Petru Mironescu<sup>(3)</sup>✉

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## Abstract

In his pioneering work, Jacobi discovered two remarkable identities related to the Jacobian. The first one asserts that the Jacobian has a divergence structure. The second one, that some vector fields involving the cofactors of the Jacobian are divergence free. We illustrate the fundamental impact of these properties on research, from the times of Jacobi to our days.

## 0 Introduction

Given a mapping  $u : \Omega \subset \mathbb{R}^N \rightarrow \mathbb{R}^N$  of class  $C^1$ , recall that its Jacobian is given by

$$\det(\nabla u) = \sum_{j=1}^N (\partial_j u_i) C_{i,j}, \quad i \in \{1, \dots, N\}, \quad (0.1)$$

where  $C_{i,j}$  denotes the cofactor of  $\partial_j u_i$  in the matrix  $(\partial_\ell u_k)_{\ell,k}$ .

This short expository text is a tribute to two remarkable (families of) identities related to the Jacobian of  $u$ , with  $u$  of class  $C^2$ , which appear frequently in the literature, usually without credit. The first one asserts that

$$\det(\nabla u) = \sum_{j=1}^N \partial_j (u_i C_{i,j}), \quad i \in \{1, \dots, N\}, \quad (0.2)$$

and in particular implies that the Jacobian has a **divergence structure**. The second one is

$$\sum_{j=1}^N \partial_j C_{i,j} = 0, \quad i \in \{1, \dots, N\}. \quad (0.3)$$

These (equivalent) identities were obtained by Jacobi himself and rediscovered, under possibly different but equivalent forms, a number of times; see Section 1 below.

The distributional Jacobian – which grew out of the identity (0.2) – plays an important role in topics from the calculus of variations, arising, e.g., in nonlinear elasticity, and developed in the seminal works of Morrey [37], Reshetnyak [44], and Ball [5]. In this context, the proofs use the continuity of the mapping  $u \mapsto \det(\nabla u)$ , property which relies

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heavily on the divergence structure of the Jacobian. Section 2 contains some optimal versions of the continuity properties of the (distributional) Jacobian. In Sections 3 and 4, we present a partial collection (reflecting our personal taste!) of topics, motivated, e.g., by liquid crystals, Ginzburg-Landau theories, surfaces of constant mean curvature, which involve (0.2) and (0.3) and illustrate their important influence on current research.

# 1 The origins of a formula

Carl Gustav Jacobi (1804–1851) is famous for his pioneering work on elliptic functions, but most students in mathematics learn his name through the concept of **Jacobian** of a  $C^1$  mapping of  $\mathbb{R}^N$  into  $\mathbb{R}^N$ , which is the determinant of the matrix of the first order partial derivatives of its components. The term ‘Jacobian’ refers to an important memoir (in Latin) of Jacobi on **functional determinants** (his terminology) published in 1841 [29]. Jacobians occur in an essential way in fundamental questions of analysis like the implicit functions theorem and the change of variables in multiple integrals. The name ‘Jacobian’ was coined in 1853 by James Joseph Sylvester (1814–1897) [48, p. 476]: ‘where  $J$  indicates the Jacobian of the given functions  $f$  and  $g$  in respect to the variables  $x$  and  $l$ , meaning thereby the so-called Functional Determinant of Jacobi to  $f$  and  $g$  in respect of  $x$  and  $l$ ’. The memoir [29] of Jacobi provided the first systematic treatment of the principal properties of the Jacobians in arbitrary finite dimension.

A few years later, in 1844–45, Jacobi published a substantial memoir [30] on the **method of the multiplier**, which extends to systems of ordinary differential equations the method of the integrating factor for scalar equations. Consider an autonomous system  $x' = X(x)$  of  $N$  ordinary differential equations, i.e.,

$$x'_i = X_i(x_1, \dots, x_N), \quad i \in \{1, \dots, N\}, \quad (1.1)$$

and  $(N - 1)$  functionally independent first integrals of (1.1),

$$\theta_i = \theta_i(x_1, \dots, x_N), \quad i \in \{1, \dots, N - 1\}.$$

Let  $\theta = \theta(x_1, \dots, x_N)$  be a  $C^1$  function, and write

$$\det(\nabla\theta, \nabla\theta_1, \dots, \nabla\theta_{N-1}) = \sum_{j=1}^N (\partial_j\theta) \Delta_j.$$

Clearly, the cofactor  $\Delta_j = \Delta_j(x_1, \dots, x_N)$  of  $\partial_j\theta$  depends only on  $\theta_1, \dots, \theta_{N-1}$ . Note that

$$\sum_{j=1}^N (\partial_j\theta_i) \Delta_j = 0, \quad i \in \{1, \dots, N - 1\}, \quad (1.2)$$

and, since each  $\theta_i$  is a first integral,

$$\sum_{j=1}^N (\partial_j\theta_i) X_j = 0, \quad i \in \{1, \dots, N - 1\}. \quad (1.3)$$

Assuming that  $X(x) \neq 0$ , we find, from (1.2), (1.3), and the functional independence of  $\theta_1, \dots, \theta_{N-1}$ , that there exists some  $M(x) \neq 0$  such that  $M(x)X_j = \Delta_j$ ,  $j \in \{1, \dots, N\}$ . Granted (0.3), one obtains the differential equation satisfied by  $M$ ,

$$\sum_{j=1}^N \partial_j(MX_j) = 0. \quad (1.4)$$

The paramount importance of  $M$  (the celebrated **Jacobi multiplier**) comes from the fact that, if  $M, M'$  are solutions of (1.4), the quotient  $M/M'$  is a first integral of (1.1).

The validity of (0.3) was stated and proved by Jacobi on p. 203 of the first part of [30]. With the notation used in the introduction, the result, that Jacobi named ‘**fundamental lemma**’, is the following.

**Lemma 1.1.** *If  $u : \Omega \subset \mathbb{R}^N \rightarrow \mathbb{R}^N$  is of class  $C^2$ , then*

$$\sum_{j=1}^N \partial_j C_{i,j} = 0, \quad i \in \{1, \dots, N\}.$$

Jacobi’s proof relies on differentiation of functional determinants defined classically as alternate sums of products of the first partial derivatives of the  $u_i$ . At the beginning of his proof [30, p. 203], Jacobi observed that the formula (0.2) can be deduced from the formula (0.3). Indeed, the elementary identity

$$\sum_{j=1}^N \partial_j(u_i C_{i,j}) = \sum_{j=1}^N (\partial_j u_i) C_{i,j} + u_i \left( \sum_{j=1}^N \partial_j C_{i,j} \right) = \det(\nabla u) + u_i \left( \sum_{j=1}^N \partial_j C_{i,j} \right),$$

$i \in \{1, \dots, N\},$

combined with (0.3) yields (0.2).

On the other hand, note that (by construction)  $C_{i,j}$  is independent of  $u_i$ ,  $\forall i, \forall j$ . Therefore, if we define  $\tilde{u}$  by  $\tilde{u}_j := u_j$  for  $j \neq i$  and  $\tilde{u}_i := 1$  we have (with obvious notation)  $\tilde{C}_{i,j} = C_{i,j}$ ,  $\forall j$ . Applying (0.2) to  $\tilde{u}$  yields

$$0 = \det(\nabla \tilde{u}) = \sum_{j=1}^N \partial_j \tilde{C}_{i,j} = \sum_{j=1}^N \partial_j C_{i,j},$$

so that (0.3) holds.

An excellent analysis of Lemma 1.1 and of Jacobi’s proof is proposed by T. Muir on p. 230–235 of the second volume of his monumental treatise on the history of determinants [38].

In his lectures on dynamics delivered in 1842–43, posthumously published in 1866, reproduced (slightly revised) in 1884 as a supplementary volume of his *Gesammelte Werke* [31], and translated in English in [32], Jacobi developed in Lecture 13 the theory of functional determinants, essentially following [29], and, in Chapter 14, the theory of the multiplier. At this occasion, in [31, p. 104–106] (or [32, p. 112–114]), he proved again Lemma 1.1, without isolating it as a separate statement.

The multiplier rapidly became a classical tool in analysis and analytical mechanics and, in particular, it inspired to Henri Poincaré (1854–1912) his theory of the **integral invariants** [43]. The Section V of Chapter XXV of the monumental treatise of rational

mechanics of Paul Appell (1855–1930) [4] is devoted to the Jacobi multiplier. Lemma 1.1 is proved in [4, p. 460–461] following the argument of Jacobi sketched above.

Lemma 1.1 has also found applications outside of the fields of ordinary differential equations and analytical mechanics. In 1910, Jacques Hadamard (1865–1963) extended, in an Appendix to Tannery’s textbook on real functions [25], the notion of **Kronecker index** to continuous non vanishing mappings  $u$  on the smooth boundary  $\partial\Omega$  of a bounded open set  $\Omega \subset \mathbb{R}^N$ . At this occasion, he stated and used in [25, p. 455–456] the following generalization of the formulas (0.2) and (0.3)), that we formulate keeping the notation and assumptions of Lemma 1.1.

**Lemma 1.2.** *For any  $h = h(y) = h(y_1, \dots, y_N) : \mathbb{R}^N \rightarrow \mathbb{R}^N$  of class  $C^1$ , one has*

$$\sum_{j=1}^N \partial_j \left( \sum_{k=1}^N (h_k \circ u) C_{k,j} \right) = \left[ \sum_{k=1}^N (\partial_k h_k) \circ u \right] \cdot \det(\nabla u), \quad (1.5)$$

where, on the left-hand side,  $\partial_j = \partial/\partial x_j$ , while, on the right-hand side,  $\partial_k = \partial/\partial y_k$ .

Notice that if, in (1.5), we take  $h_k(y) := \delta_{kj} y_j$ ,  $k, j \in \{1, \dots, N\}$ , we recover the formulas (0.2), while, if we take  $h_k(y) := \delta_{kj}$ ,  $k, j \in \{1, \dots, N\}$ , we recover the formulas (0.3).

Hadamard’s ‘proof’ of this ‘easy to verify’ identity consists in a footnote asserting that the formula is ‘well-known when the  $h_k$ ’s are constant and plays a role in the theory of multipliers’ (referring to the classical treatises on analysis by Jordan, Goursat, etc.). It is doubtful that the majority of readers of Tannery’s textbook containing Hadamard’s paper [25] were able to reconstruct a proof of (1.5) from this rough information. We sketch an argument at the end of this section.

The formulas (0.3) reappear almost fifty years later in the analytical definition of the **Brouwer degree** (an extension of Kronecker’s index; see, e.g., Dinca and Mawhin [22]) given by Erhard Heinz (1924–2017) in [27, p. 232]. They are just referred there as ‘well-known relations’ and stated without proof. They are essential to justify Heinz’ definition of the Brouwer degree of  $u$  in  $\Omega$  as some integral over  $\Omega$  of an expression depending on  $u$  and  $\det(\nabla u)$ .

As noticed in Brezis and Nguyen [19], one also finds the formulas (0.3) (or rather the version (1.6) below) in the monograph of Charles B. Morrey (1907–1984) on the calculus of variations and partial differential equations [37, Lemma 4.4.6, p. 122], which has led the authors of [19] to attribute them to Morrey. The proof in [37] is by induction on  $N$ , and the motivation of Morrey is the study of **strongly quasi-convex functions**.

Another identity involving Jacobians is fundamental in the analytical proof by Dunford and Schwartz [23, p. 467] of the **Brouwer fixed point theorem** [23, p. 468–470].

**Lemma 1.3.** *If  $f : \Omega \rightarrow \mathbb{R}^{N-1}$  is of class  $C^2$  and, for each  $j \in \{1, \dots, N\}$ ,  $E_j$  denotes the determinant whose columns are the  $(N-1)$  partial derivatives  $\partial_1 f, \dots, \widehat{\partial_j f}, \dots, \partial_N f$ , then*

$$\sum_{j=1}^N (-1)^{j-1} \partial_j E_j = 0. \quad (1.6)$$

(As usual,  $\widehat{\partial_j f}$  indicates that the factor  $\partial_j f$  is missing.)

*Proof.* It turns out that (1.6) is actually equivalent to (0.3). Indeed, if  $u$  is as in Lemma 1.1 and we set  $f := (u_1, \dots, \widehat{u_i}, \dots, u_N) : \Omega \rightarrow \mathbb{R}^{N-1}$ , then  $E_j = (-1)^{i+j} C_{i,j}$ ,  $j \in \{1, \dots, N\}$ , and thus (1.6) implies (0.3). On the other hand, if  $f$  is as in Lemma 1.3 and we let  $u : \Omega \rightarrow \mathbb{R}^N$  be any  $C^2$  mapping whose first  $(N-1)$  components are those of  $f$ , then  $C_{N,j} = (-1)^{N+j} E_j$ ,  $j \in \{1, \dots, N\}$ , and therefore (0.3) implies (1.6).  $\square$

The modern ‘textbook’ analytical proofs of the Brouwer fixed point theorem (see, e.g., Kannai [33] or Evans [24, Section 8.1.4]) follow the lines of [23] and rely on the formulas (0.3) or cousin formulas, as (1.6).

Finally, let us mention – as suggested to us by Ball – that the formulas (0.3) establish a connection between Jacobians and **null Lagrangians**. More specifically, they imply that  $L(P) := \det P$ , where  $P$  is an  $N \times N$  matrix, is a null Lagrangian. (More generally, (0.3) implies that  $L(P, z) := \eta(z) \det P$ , with  $\eta : \mathbb{R}^N \rightarrow \mathbb{R}$  of class  $C^1$ , is a null Lagrangian.) This is well explained in [24, Section 8.1.4], where the reader may find a proof of (0.3). For more insights on the structure of null Lagrangians, see, e.g., Olver and Sivaloganathan [41] and the survey by Iwaniec [28].

This confirms the ubiquity of (0.3).

Some final comments. The somewhat mysterious character of the formulas (0.2) (or, equivalently, (0.3)) disappears when they are written in the language of exterior differential calculus, where they take the form of the trivial results

$$d \left[ (-1)^{i-1} u_i du_1 \wedge \dots \wedge \widehat{du_i} \wedge \dots \wedge du_N \right] = du_1 \wedge \dots \wedge du_N, \quad i \in \{1, \dots, N\},$$

where  $d$  denotes the exterior differential,  $\wedge$  the exterior product, and  $\widehat{du_i}$  means that the factor  $du_i$  is missing. On the other hand, the identities (0.3) correspond to the trivial property

$$d(du_1 \wedge \dots \wedge \widehat{du_i} \wedge \dots \wedge du_N) = 0, \quad i \in \{1, \dots, N\},$$

which follows from the fact that  $d^2 u_j = 0$ ,  $\forall j$ . This is the reason why the formulas (0.2) and (0.3) do not appear explicitly in the analytical presentation of the Kronecker index and of the Brouwer degree in terms of differential forms, given for example in [22].

Here are some more detailed explanations. It is well-known that

$$du_1 \wedge \dots \wedge du_N = \det(\nabla u) dx_1 \wedge \dots \wedge dx_N. \quad (1.7)$$

On the other hand, one may check that

$$du_1 \wedge \dots \wedge \widehat{du_i} \wedge \dots \wedge du_N = \sum_{j=1}^N (-1)^{i+j} C_{i,j} dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_N. \quad (1.8)$$

Using (1.7), (1.8), the exterior differentiation rules, and the fact that  $d^2 u_j = 0$ ,  $\forall j$ , we find that

$$\begin{aligned} \det(\nabla u) dx_1 \wedge \dots \wedge dx_N &= du_1 \wedge \dots \wedge du_N \\ &= d[(-1)^{i-1} u_i du_1 \wedge \dots \wedge \widehat{du_i} \wedge \dots \wedge du_N] \\ &= d \left[ \sum_{j=1}^N (-1)^{j-1} u_i C_{i,j} dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_N \right] \\ &= \sum_{j=1}^N \partial_j (u_i C_{i,j}) dx_1 \wedge \dots \wedge dx_N, \end{aligned}$$

whence (0.2).

Similarly, the formula (1.5) of Hadamard amounts to the straightforward identity

$$d \left[ \left( \sum_{k=1}^N (-1)^{k-1} h_k \circ u \right) du_1 \wedge \dots \wedge \widehat{du_k} \wedge \dots \wedge du_N \right] = \sum_{k=1}^N (\partial_k h_k) \circ u du_1 \wedge \dots \wedge du_N.$$

## 2 The distributional Jacobian

For simplicity, we assume throughout the rest of this paper that  $\Omega$  is smooth and bounded.

Taking advantage of the divergence structure (0.2) of the Jacobian, Ball [5] introduced in 1976 the very useful concept of **Jacobian in the sense of distributions**, nowadays rather known as **distributional Jacobian**. [5] deals only with the cases  $N = 2$  and  $N = 3$  (see (6.8) and (6.10) in [5]), but the definitions readily extend to any  $N \geq 2$ . More precisely, given a vector-field  $u : \Omega \rightarrow \mathbb{R}^N$ , set

$$\text{Det}(\nabla u) = T_1 := \sum_{j=1}^N \partial_j (u_1 C_{1,j}) \quad (2.1)$$

whenever the right-hand side of (2.1) is meaningful as a distribution, and, in particular, if  $u_1 C_{1,j} \in L^1(\Omega)$ ,  $\forall j$ . This is, e.g., the case if:

$$u \in W^{1,N-1}(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N) \quad (2.2)$$

or

$$u \in W^{1,N^2/(N+1)}(\Omega; \mathbb{R}^N) \quad (2.3)$$

(thanks to the Sobolev embedding  $W^{1,N^2/(N+1)}(\Omega) \hookrightarrow L^{N^2}(\Omega)$ ).

As noted in [5, Corollary 6.2.1], if  $u$  satisfies (2.2) or (2.3), one has

$$\text{Det}(\nabla u) = T_i := \sum_{j=1}^N \partial_j (u_i C_{i,j}), \quad i \in \{2, \dots, N\}. \quad (2.4)$$

The identity (2.4) easily follows from (0.2) and a standard approximation argument.

Another sufficient condition for the validity of (2.4) is  $|u| |\nabla u|^{N-1} \in L^1(\Omega)$  – which is weaker than (2.2) and (2.3). This assertion is a consequence of a more general result due to Mironescu [35].

In recent years, it has become customary to use a more symmetric definition of the distributional Jacobian:

$$\text{Det}(\nabla u) = \frac{1}{N} \sum_{i=1}^N T_i = \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N \partial_j (u_i C_{i,j}), \quad (2.5)$$

or, equivalently,

$$\text{Det}(\nabla u) = \frac{1}{N} \text{div} D, \quad (2.6)$$

where  $D = D(u) = (D_1, \dots, D_N)$  and

$$D_i = D_i(u) := \det(\partial_1 u, \dots, \partial_{i-1} u, u, \partial_{i+1} u, \dots, \partial_N u), \quad i \in \{1, \dots, N\}. \quad (2.7)$$

Note that, at least when  $u$  satisfies (2.2) or (2.3), the formulas (2.1) and (2.6) yield the same distributional Jacobian.

Clearly, (0.2) implies that, when  $u \in C^2(\Omega; \mathbb{R}^N)$ , we have

$$\text{Det}(\nabla u) = \det(\nabla u). \quad (2.8)$$

One can investigate the validity of (2.8) under weaker assumption on  $u$ , assuming at least that  $u \in W_{loc}^{1,1}(\Omega; \mathbb{R}^N)$  (so that the right-hand side of (2.8) is a function defined a.e.). An approximation argument shows that (2.8) still holds when  $u \in C^1(\Omega; \mathbb{R}^N)$ , and thus the distributional Jacobian  $\text{Det}(\nabla u)$  generalizes the classical Jacobian  $\det(\nabla u)$ . Moreover, the equality (2.8) persists when  $u \in W^{1,N}(\Omega; \mathbb{R}^N)$ ; this follows, again, by approximation. However, Ball noticed that, when  $u \in W^{1,N-1}(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ , (2.8) **may fail**. His example [5, Example 6.1] is the following. Let  $\Omega = \mathbb{B}^N := \{x \in \mathbb{R}^N; |x| < 1\}$  and

$$u(x) := \frac{R(|x|)}{|x|}x, \quad (2.9)$$

where  $R$  is a smooth real-valued function on  $[0, 1]$  such that  $R(0) \neq 0$ . Note that  $u$  satisfies (2.2). (In fact,  $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ ,  $\forall p < N$ .) Ball pointed out that ‘ $\text{Det}(\nabla u)$  has an atom at  $x = 0$ ’, but he didn’t elaborate; a more detailed computation yields

$$\text{Det}(\nabla u) = \frac{R^{N-1}(|x|)R'(|x|)}{|x|^{N-1}} + |\mathbb{B}^N| R^N(0) \delta_0 \text{ in } \mathcal{D}'(\Omega), \quad (2.10)$$

where  $|\mathbb{B}^N|$  denotes the measure of the unit ball. (We will return to this phenomenon in Section 3 below.)

However, we have the following noticeable result, obtained by Müller [39].

**Theorem 2.1.** *Assume that (2.3) holds.*

*If, in addition,  $\text{Det}(\nabla u) \in L^1$ , then  $\text{Det}(\nabla u) = \det(\nabla u)$ .*

*More generally, if (2.3) holds and  $\text{Det}(\nabla u)$  is a Radon measure, then  $\det(\nabla u)$  is (the density of) the absolutely continuous part of  $\text{Det}(\nabla u)$  (with respect to the Lebesgue measure).*

The distributional Jacobian enjoys remarkable continuity (‘stability’) properties, which rely heavily on its divergence structure, and are relevant for variational problems (which require passing to the weak limits). Here are some typical examples taken from Brezis and Nguyen [19].

**Theorem 2.2.** *Assume that*

$$p \geq N - 1 \text{ and } \frac{N-1}{p} + \frac{1}{q} = 1. \quad (2.11)$$

*For all  $u, v \in W^{1,p}(\Omega; \mathbb{R}^N) \cap L^q(\Omega; \mathbb{R}^N)$  and  $\varphi \in C_c^1(\Omega; \mathbb{R})$ , we have*

$$|\langle \text{Det}(\nabla u), \varphi \rangle - \langle \text{Det}(\nabla v), \varphi \rangle| \leq C_N \|u - v\|_{L^q} (\|\nabla u\|_{L^p} + \|\nabla v\|_{L^p})^{N-1} \|\nabla \varphi\|_{L^\infty}, \quad (2.12)$$

$$\begin{aligned} |\langle \text{Det}(\nabla u), \varphi \rangle - \langle \text{Det}(\nabla v), \varphi \rangle| &\leq C_N \|\nabla u - \nabla v\|_{L^p} (\|\nabla u\|_{L^p} + \|\nabla v\|_{L^p})^{N-2} \\ &\quad \times (\|u\|_{L^q} + \|v\|_{L^q}) \|\nabla \varphi\|_{L^\infty}. \end{aligned} \quad (2.13)$$



Estimate (2.12) illuminates the fact that, if  $u^k \rightharpoonup u$  weakly in  $W^{1,r}(\Omega; \mathbb{R}^N)$  for some  $r > N^2/(N+1)$ , then

$$\text{Det}(\nabla u^k) \rightarrow \text{Det}(\nabla u) \text{ in } \mathcal{D}'(\Omega). \quad (2.14)$$

(Indeed, it suffices to apply (2.12) with  $p = N^2/(N+1)$  and  $q = N^2$  and to use the fact that  $W^{1,r}(\Omega) \hookrightarrow L^{N^2}(\Omega)$ , with compact injection.) This result is due to Reshetnyak [44] when  $r = N$ . His proof works in the full range  $r > N^2/(N+1)$ , and also yields (2.14) in the limiting case  $r = N^2/(N+1)$ , under the extra assumption that  $u^k \rightarrow u$  strongly in  $L^{N^2}(\Omega; \mathbb{R}^N)$ . However, it lacks the quantitative estimate (2.12).

Dacorogna and Murat [21, proof of Theorem 1] have shown that, when  $r = N^2/(N+1)$ , the extra assumption that  $u^k \rightarrow u$  strongly in  $L^{N^2}(\Omega; \mathbb{R}^N)$  cannot be omitted; they have constructed a sequence  $(u^k) \subset W^{1, N^2/(N+1)}(\Omega; \mathbb{R}^N)$  such that  $u^k \rightharpoonup 0$  weakly in  $W^{1, N^2/(N+1)}$ , while  $\text{Det}(\nabla u^k)$  does not converge to 0 in  $\mathcal{D}'(\Omega)$  (see also [19, Proposition 1] for a more precise conclusion).

It turns out that the space  $W^{1, N^2/(N+1)}$  is the **largest** Sobolev space in the scale  $W^{1,p}$ ,  $1 \leq p < \infty$ , for which the distributional Jacobian makes sense. This is a consequence of the following result (see [21, proof of Theorem 1] and [19, proof of Lemma 5, Case 1]).

**Theorem 2.3.** *Given any  $1 \leq p < N^2/(N+1)$ , there exists a sequence  $(u^k) \subset C^\infty(\overline{\Omega}; \mathbb{R}^N)$  such that, as  $k \rightarrow \infty$ ,  $u^k \rightarrow 0$  strongly in  $W^{1,p}$  and*

$$\int_{\Omega} \det(\nabla u^k) \varphi \rightarrow \infty \text{ for some } \varphi \in C_c^\infty(\Omega; \mathbb{R}).$$

A more general natural question investigated in Brezis and Nguyen [19] is the existence of a largest fractional Sobolev space  $W^{s,p}$  in which  $\text{Det}(\nabla u)$  is well-defined and stable. The answer is given by the following result [19, Theorem 3, Theorem 4].

**Theorem 2.4.** *1. If  $u, v \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$  and  $\varphi \in C_c^1(\Omega; \mathbb{R})$ , then*

$$\begin{aligned} |\langle \det(\nabla u), \varphi \rangle - \langle \det(\nabla v), \varphi \rangle| &\leq C_{N,\Omega} |u - v|_{W^{(N-1)/N, N}} \\ &\quad \times \left( |u|_{W^{(N-1)/N, N}}^{N-1} + |v|_{W^{(N-1)/N, N}}^{N-1} \right) \|\nabla \varphi\|_{L^\infty}. \end{aligned} \quad (2.15)$$

*In particular, there exists a unique continuous mapping*

$$W^{(N-1)/N, N}(\Omega; \mathbb{R}^N) \ni u \mapsto \text{Det}(\nabla u) \in \mathcal{D}'(\Omega)$$

*such that  $\text{Det}(\nabla u) = \det(\nabla u)$  when  $u \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ .*

*2. Let  $s > 0$  and  $1 \leq p \leq \infty$ . If the mapping*

$$C^\infty(\overline{\Omega}; \mathbb{R}^N) \ni u \mapsto \det(\nabla u) \in \mathcal{D}'(\Omega)$$

*is continuous with respect to the  $W^{s,p}$ -convergence, then  $W^{s,p}(\Omega) \hookrightarrow W^{(N-1)/N, N}(\Omega)$ .*

We recall that, when  $0 < s < 1$  and  $1 \leq p < \infty$ ,

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega); |u|_{W^{s,p}} < \infty\},$$

where

$$|u|_{W^{s,p}} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}$$

is the Gagliardo seminorm.

The proof of (2.15) relies on the identity (2.17) below, with roots in Bourgain, Brezis, and Mironescu [9] and Hang and Lin [26]. Let  $U \in C^\infty(\overline{\Omega} \times [0, 1], \mathbb{R}^N)$  be an extension of  $u$ , and let  $\Phi \in C_c^1(\Omega \times [0, 1], \mathbb{R})$  be an extension of  $\varphi$ . Let, for each  $j \in \{1, \dots, N+1\}$ ,  $E_j = E_j(U)$  denote the determinant whose columns are the  $N$  partial derivatives  $\partial_1 U, \dots, \widehat{\partial_j U}, \dots, \partial_{N+1} U$ . By Lemma 1.3, we have

$$\sum_{j=1}^{N+1} (-1)^{N+j} \partial_j E_j = 0. \quad (2.16)$$

Combining (2.16) and the divergence theorem, we find that

$$\int_{\Omega} \det(\nabla u) \varphi = \int_{\Omega \times (0,1)} \sum_{j=1}^{N+1} \partial_j ((-1)^{N+j} E_j \Phi) = \int_{\Omega \times (0,1)} \sum_{j=1}^{N+1} (-1)^{N+j} E_j \partial_j \Phi. \quad (2.17)$$

The estimate (2.15) follows from (2.17) via estimates of the quantities  $E_j(U) - E_j(V)$ , where  $U$ , respectively  $V$ , are well-chosen extensions of  $u$ , respectively  $v$ .

### 3 The Jacobian of sphere-valued maps

The distributional Jacobian plays a major role in the study of maps from  $\Omega \subset \mathbb{R}^N$  into  $\mathbb{S}^k$ ,  $1 \leq k \leq N-1$ . The analysis of such maps combines analytical, geometrical, and topological tools. We first present, in Sections 3.1–3.3, some typical results in the case where  $k = N-1$ , in which geometry plays a moderate role. In Section 3.4, we present the counterparts of these results in the general case  $1 \leq k \leq N-1$ . (For more details and further applications, we refer to [17, Chapter 4].)

#### 3.1 Vanishing of the Jacobian and approximation by smooth maps

Note that  $\text{Det}(\nabla u)$  is well-defined for every  $u \in W^{1,N-1}(\Omega; \mathbb{S}^{N-1})$  (since  $u \in L^\infty(\Omega; \mathbb{R}^N)$ ). We claim that

$$\text{Det}(\nabla u) = 0, \quad \forall u \in W^{1,N}(\Omega; \mathbb{S}^{N-1}). \quad (3.1)$$

Indeed, on the one hand we know (see Section 2) that

$$\text{Det}(\nabla u) = \det(\nabla u), \quad \forall u \in W^{1,N}(\Omega; \mathbb{R}^N). \quad (3.2)$$

On the other hand, by differentiating the identity  $|u|^2 \equiv 1$ , we find that  $u \cdot \partial_j u = 0$  a.e. on  $\Omega$ ,  $\forall j = 1, \dots, N$ ,  $\forall u \in W^{1,1}(\Omega; \mathbb{S}^{N-1})$ . Thus, for a.e.  $x \in \Omega$ , the vectors  $\partial_j u(x)$ ,  $j = 1, \dots, N$ , are contained in a hyperplane, so that  $\det(\nabla u) = 0$  a.e. on  $\Omega$ . (Alternatively, one could establish (3.1): (i) first, for smooth  $\mathbb{S}^{N-1}$ -valued maps; (ii) next, for arbitrary maps  $u \in W^{1,N}(\Omega; \mathbb{S}^{N-1})$ , using an approximation argument by smooth  $\mathbb{S}^{N-1}$ -valued maps, due to Schoen and Uhlenbeck [45] and relying on the embedding  $W^{1,N} \subset \text{VMO}$ ; see also [17, Chapter 4].)

It follows from (3.1) that

$$\text{Det}(\nabla u) = 0, \forall u \in W^{1,p}(\Omega; \mathbb{S}^{N-1}), \forall p \geq N. \quad (3.3)$$

The restriction  $p \geq N$  in (3.3) is optimal. Indeed, as we are going to see in Section 3.2 below, (3.3) fails when  $N - 1 \leq p < N$ . Note however that

$$\text{Det}(\nabla u) = 0, \forall u \in W^{1,N-1}(\Omega; \mathbb{S}^{N-1}) \cap C(\overline{\Omega}; \mathbb{S}^{N-1}), \quad (3.4)$$

and, more generally,

$$\text{Det}(\nabla u) = 0, \forall u \in \overline{C^\infty(\overline{\Omega}; \mathbb{S}^{N-1})}^{W^{1,N-1}}. \quad (3.5)$$

Surprisingly, the converse of (3.5) is also true; its proof is non-trivial. More specifically, we have the following

**Theorem 3.1.** *Let  $N - 1 \leq p < N$ . Assume that  $u \in W^{1,p}(\Omega; \mathbb{S}^{N-1})$  satisfies*

$$\text{Det}(\nabla u) = 0. \quad (3.6)$$

*Then*

$$u \in \overline{C^\infty(\overline{\Omega}; \mathbb{S}^{N-1})}^{W^{1,p}}. \quad (3.7)$$

Theorem 3.1 is due to Bethuel [6] when  $p = N - 1$  and to Bethuel, Coron, Demengel, and Hélein [7] when  $N - 1 < p < N$ ; see also Ponce and Van Schaftingen [42].

## 3.2 The Jacobian as a sum of Dirac masses

We now turn to the example considered in (2.9) and to formula (2.10), which implies in particular that, if  $u(x) = \frac{x-a}{|x-a|}$  for some  $a \in \Omega$ , then

$$\text{Det}(\nabla u) = |\mathbb{B}^N| \delta_a \text{ in } \mathcal{D}'(\Omega). \quad (3.8)$$

We start with an extension of (3.8).

**Theorem 3.2.** *Let  $u \in W^{1,N-1}(\Omega; \mathbb{S}^{N-1}) \cap C(\Omega \setminus \{a\})$ . Then*

$$\text{Det}(\nabla u) = |\mathbb{B}^N| \deg(u, a) \delta_a \text{ in } \mathcal{D}'(\Omega), \quad (3.9)$$

*where  $\deg(u, a)$  is the (Brouwer) degree of  $u$  computed on a small sphere around  $a$ .*

*More generally, if  $u \in W^{1,N-1}(\Omega; \mathbb{S}^N) \cap C(\Omega \setminus \{a_1, \dots, a_\ell\})$ , where  $a_1, \dots, a_\ell \in \Omega$ , then*

$$\text{Det}(\nabla u) = |\mathbb{B}^N| \sum_{j=1}^{\ell} \deg(u, a_j) \delta_{a_j} \text{ in } \mathcal{D}'(\Omega). \quad (3.10)$$

Theorem 3.2 is due to Brezis, Coron, and Lieb [16]. Its proof relies heavily on the Kronecker formula which relates the degree to the Jacobian:

$$\deg f = \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} \det(\nabla f), \quad \forall f \in C^1(\mathbb{S}^{N-1}; \mathbb{S}^{N-1}), \quad (3.11)$$

where  $\det(\nabla f)$  is the Jacobian of  $f$  viewed as a map between manifolds. (See [22, Section 1.1, Section 1.2] for various formulas for  $\deg f$  and further discussions.) Formula (3.10) implies that the Jacobian of sphere-valued maps conveys information about the location and ‘topological strength’ of their singularities – a fact that was used and applied in numerous subsequent works.

In the spirit of (3.10), the Jacobian of a **general** map  $u \in W^{1,N-1}(\Omega; \mathbb{S}^{N-1})$  is given by the following

**Theorem 3.3.** *Let  $u \in W^{1,N-1}(\Omega; \mathbb{S}^{N-1})$ . Then there exist points  $P_j, N_j \in \overline{\Omega}$  (not necessarily distinct) such that*

$$\sum_j |P_j - N_j| < \infty \quad (3.12)$$

and

$$\text{Det}(\nabla u) = |\mathbb{B}^N| \sum_j (\delta_{P_j} - \delta_{N_j}) \text{ in } \mathcal{D}'(\Omega). \quad (3.13)$$

Conversely, given points  $P_j, N_j \in \overline{\Omega}$  satisfying (3.12), there exists some  $u \in W^{1,N-1}(\Omega; \mathbb{S}^{N-1})$  such that (3.13) holds.

Note that (3.10) and (3.13) are consistent, since, in  $\mathcal{D}'(\Omega)$ , we may always write  $\sum_{j=1}^\ell d_j \delta_{a_j}$  as  $\sum_j (\delta_{P_j} - \delta_{N_j})$  for some finite collection of  $P_j$ ’s and  $N_j$ ’s, by repeating the  $a_j$ ’s according to their multiplicity  $|d_j|$  and adding arbitrary points on  $\partial\Omega$ .

Theorem 3.3 was announced in [12] and is inspired by an earlier result of Bourgain, Brezis, and Mironescu [9, Theorem 1], [10, Theorem 1] on  $\mathbb{S}^1$ -valued maps. It is a special case of Theorem 3.6 in Section 3.4. A sketch of proof of Theorem 3.3 is presented in [17, Chapter 4].

### 3.3 Least energy with prescribed singularities/Jacobian

In this subsection we are concerned with a question originally raised in the context of liquid crystals (see [16] and [17, Chapter 4]): study the least energy  $S_1$  required to generate a configuration having singularities located at given points  $a_j, j = 1, \dots, \ell$ , in  $\Omega \subset \mathbb{R}^3$ , with given degrees  $d_j \in \mathbb{Z}, j = 1, \dots, \ell$ . In a simplified model, the least energy is

$$S_1 := \inf \left\{ \int_{\Omega} |\nabla v|^2; v \in H^1(\Omega; \mathbb{S}^2) \cap C(\Omega \setminus \{a_1, \dots, a_\ell\}) \text{ such that} \right. \\ \left. \deg(v, a_j) = d_j, \forall j \right\}. \quad (3.14)$$

In view of Theorem 3.2, it is reasonable to consider also the quantity

$$S_2 := \inf \left\{ \int_{\Omega} |\nabla v|^2; v \in H^1(\Omega; \mathbb{S}^2), \text{Det}(\nabla v) = (4\pi/3) \sum_{j=1}^\ell d_j \delta_{a_j} \text{ in } \mathcal{D}'(\Omega) \right\}. \quad (3.15)$$

Clearly,  $S_2 \leq S_1$ , and in fact we can prove that  $S_2 = S_1$  [17, Section 4.6]. Going one step further, it is natural to introduce, in full generality, the quantity

$$S := \inf \left\{ \int_{\Omega} |\nabla v|^{N-1}; v \in W^{1,N-1}(\Omega; \mathbb{S}^{N-1}), \text{Det}(\nabla v) = |\mathbb{B}^N| \sum_j (\delta_{P_j} - \delta_{N_j}) \right. \\ \left. \text{in } \mathcal{D}'(\Omega) \right\}, \quad (3.16)$$

where  $\Omega \subset \mathbb{R}^N$  and the given points  $P_j, N_j \in \overline{\Omega}$  satisfy (3.12).

There is a remarkable explicit formula for  $S$ .

**Theorem 3.4.** *We have*

$$S = c_{N-1} \inf \left\{ \sum_j |\tilde{P}_j - \tilde{N}_j|; \tilde{P}_j, \tilde{N}_j \in \overline{\Omega} \text{ satisfy (3.12) and} \right. \\ \left. \sum_j (\delta_{\tilde{P}_j} - \delta_{\tilde{N}_j}) = \sum_j (\delta_{P_j} - \delta_{N_j}) \text{ in } \mathcal{D}'(\Omega) \right\}, \quad (3.17)$$

where  $c_{N-1} := (N-1)^{(N-1)/2} |\mathbb{S}^{N-1}|$ .

In the special case where the collections  $(P_j)$ , respectively  $(N_j)$ , consist of a finite number  $m$  of points, formula (3.17) becomes

$$S = c_{N-1} \min_{\sigma \in S_m} \sum_{j=1}^m d(P_j, N_{\sigma(j)}), \quad (3.18)$$

where  $S_m$  denotes the set of permutations of  $\{1, \dots, m\}$  and  $d$  is the pseudometric given by

$$d(P, N) := \min\{|P - N|, \text{dist}(P, \partial\Omega) + \text{dist}(N, \partial\Omega)\}, \quad \forall P, N \in \overline{\Omega}.$$

Formula (3.18) was established by Brezis, Coron, and Lieb [16] using tools from Optimal Transport. Formula (3.17), which is a generalization of (3.18), is due to Brezis, Mironescu, and Ponce [18] when  $N = 2$  and Brezis and Mironescu in the general case; a sketch of proof is presented in [17, Chapter 4].

We point out that Theorem 3.4 asserts that the **physical** quantity  $S$  (or rather  $S_1$ ) coincides with a **geometrical** quantity, i.e., the length (with respect to the pseudometric  $d$ ) of a ‘minimal connection’ connecting the points  $P_j$  to the points  $N_j$ . Its geometrical nature will become more transparent in the next section, where it will be identified with the least area spanned by a given ‘contour’ (in the special situation considered in this subsection, the ‘contour’ corresponds to the points  $P_j, N_j$ ).

### 3.4 Maps with values into lower-dimensional spheres

In this subsection, we consider maps  $u : \Omega \rightarrow \mathbb{R}^{k+1}$  (and, in particular, maps with values into  $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ ), where  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 2$ , and  $1 \leq k \leq N-1$ .

We first introduce the appropriate concept of Jacobian. As in Section 1, it will be convenient to use the language of differential forms, or more precisely the one of  $\ell$ -currents, i.e., distributions with values into  $\ell$ -forms. (We use below the standard terminology

about currents. If needed, we refer, e.g., to [17, Chapter 4] for precise definitions.) Given  $u \in W^{1,k}(\Omega; \mathbb{R}^{k+1}) \cap L^\infty(\Omega; \mathbb{R}^{k+1})$ , we set

$$Ju := \frac{1}{k+1} dD(u) \in \mathcal{D}'(\Omega; \Lambda^{k+1}), \quad (3.19)$$

where  $\Lambda^\ell$  denotes the space of  $\ell$ -forms on  $\mathbb{R}^N$  and

$$D(u) := \sum_{j=1}^{k+1} (-1)^{j-1} u_j du_1 \wedge \dots \widehat{du_j} \wedge \dots \wedge du_{k+1} \in L^1(\Omega; \Lambda^k). \quad (3.20)$$

For many purposes, it is more convenient to consider, instead of the  $(k+1)$ -current  $Ju$ , the  $(N-k-1)$ -current

$$\text{Jac}u := *Ju \quad (3.21)$$

(where  $*$  is the Hodge  $*$  operator). This is consistent with Sections 1 and 2, since, when  $k = N-1$ , we have  $\text{Det}(\nabla u) = \text{Jac}u$ .

Our goal is to present far-reaching extensions of the results in Sections 3.1–3.3, involving  $\text{Jac}u$  when  $u \in W^{1,k}(\Omega; \mathbb{S}^k)$ , with arbitrary  $N > k$ . We start with an analogue of Theorem 3.2.

**Theorem 3.5.** *Let  $\Gamma \subset \Omega$  be a smooth connected oriented  $(N-k-1)$ -submanifold without boundary (in  $\Omega$ ). Let  $u \in W^{1,k}(\Omega; \mathbb{S}^k) \cap C(\Omega \setminus \Gamma)$ . Then, with  $m := \deg(u, \Gamma)$ , we have*

$$\text{Jac}u = |\mathbb{B}^{k+1}| m \Gamma \text{ in } \mathcal{D}'(\Omega; \Lambda^{N-k-1}). \quad (3.22)$$

Here,  $\Gamma$  is identified, as usual, with an  $(N-k-1)$ -current, and  $m \in \mathbb{Z}$  is defined as follows. For any  $x \in \Gamma$ , let  $N = N(x)$  be the  $(k+1)$ -dimensional normal space to  $\Gamma$  at  $x$ , oriented by  $\Gamma$ . Then  $m$  is the Brouwer degree of  $u$  restricted to the sphere  $B(x, \varepsilon)$  of  $N$ , for sufficiently small  $\varepsilon$ . (One may prove that the definition does not depend on  $x$  or  $\varepsilon$ .)

In order to generalize Theorem 3.3, it is convenient to introduce a distinguished class of currents. Given an integer  $\ell$  with  $0 \leq \ell \leq N-2$ , set

$$\mathcal{F}^\ell := \{T; T \text{ is an } \ell\text{-current such that } T = \partial M \text{ for some } (\ell+1)\text{-rectifiable current } M \text{ on } \Omega\}. \quad (3.23)$$

The next result provides a complete geometric description of  $\text{Jac}u$  for a general  $u \in W^{1,k}(\Omega; \mathbb{S}^k)$ .

**Theorem 3.6.** *For any  $u \in W^{1,k}(\Omega; \mathbb{S}^k)$ ,  $1 \leq k \leq N-1$ , we have*

$$\frac{1}{|\mathbb{B}^{k+1}|} \text{Jac}u \in \mathcal{F}^{N-k-1}. \quad (3.24)$$

*Conversely, given any  $T \in \mathcal{F}^{N-k-1}$ , there exists some  $u \in W^{1,k}(\Omega; \mathbb{S}^k)$  such that*

$$\text{Jac}u = |\mathbb{B}^{k+1}| T. \quad (3.25)$$

Theorem 3.6, due to Alpert, Baldo, and Orlandi [2] (see also [1]), is a far-reaching generalization of Theorem 3.3.

In the general setting considered in this subsection, Theorem 3.4 reads as follows.

**Theorem 3.7.** *Given any  $T \in \mathcal{F}^{N-k-1}$ , set*

$$S := \inf \left\{ \int_{\Omega} |\nabla v|^k; v \in W^{1,k}(\Omega; \mathbb{S}^k), \text{Jac } v = |\mathbb{B}^{k+1}| T \right\}. \quad (3.26)$$

*Then*

$$S = c_k \inf \{|M|; M \subset \Omega \text{ is an } (N-k)\text{-rectifiable current such that } \partial M = T\}, \quad (3.27)$$

*where  $c_k := k^{k/2} |\mathbb{S}^k|$ .*

Here,  $|M|$  denotes the mass of  $M$  (i.e., its  $(N-k)$ -dimensional Hausdorff measure, counted with multiplicities).

We emphasize that Theorem 3.7 asserts that the **least ‘energy’** required to produce singularities on the given ‘contour’  $T$  coincides, up to a constant, with the **least ‘surface area’** spanned by  $T$  (as in the Plateau problem).

Theorem 3.7 confirms a conjecture in [16], corresponding roughly speaking to the special case where  $N = 3$ ,  $k = 1$ , and  $T$  is a closed curve  $\Gamma \subset \mathbb{R}^3$  (as in Theorem 3.5). The conjecture was settled by Almgren, Browder, and Lieb [3] when the singular set is nice. In their remarkable paper, they explained that the upper bound  $\leq$  in (3.27) could be obtained by adapting the dipole construction of [16]; the main new idea in [3] was to point out the importance, in this context, of the coarea formula, that the authors used to derive the lower bound  $\geq$  in (3.27). In full generality, Theorem 3.7 is due to Molnar [36]; her proof relies heavily on techniques developed in [2] and [3].

Finally, we mention that, in the spirit of Theorem 2.4, one may investigate the existence of the largest fractional Sobolev spaces in which one can define a ‘robust’ distributional Jacobian, and identify the possible distributional Jacobians arising in this setting. These questions were settled by Bourgain, Brezis, and Mironescu [11] when  $k = N - 1$ , respectively Bousquet and Mironescu [8] in the general case.

## 4 Improved regularity in $W^{1,N}(\mathbb{R}^N; \mathbb{R}^N)$

We conclude our brief excursion into the enchanted world of Jacobians with several remarkable properties of maps  $u \in W^{1,N}(\Omega; \mathbb{R}^N)$ , where  $N \geq 2$  and  $\Omega \subset \mathbb{R}^N$  is an open set. Their common feature is the crucial role played by the Jacobian and its algebraic properties.

If  $u \in W^{1,N}(\mathbb{R}^N; \mathbb{R}^N)$ , then, clearly,  $\det(\nabla u) \in L^1(\mathbb{R}^N)$ . The following ‘microscopic’ but rich of consequences improvement of this property is due to Coifman, Lions, Meyer, and Semmes [20].

**Theorem 4.1.** *Let  $u \in W^{1,N}(\mathbb{R}^N; \mathbb{R}^N)$ . Then  $\det(\nabla u) \in \mathcal{H}^1(\mathbb{R}^N)$ .*

Here,  $\mathcal{H}^1(\mathbb{R}^N) \subset L^1(\mathbb{R}^N)$  denotes the **Hardy space**

$$\mathcal{H}^1(\mathbb{R}^N) := \{f \in L^1(\mathbb{R}^N); \exists g \in L^1(\mathbb{R}^N) \text{ such that } |f * \varphi_\varepsilon(x)| \leq g(x), \forall x \in \mathbb{R}^N, \forall \varepsilon > 0\},$$

where  $\varphi$  is a bump function. (The definition is independent of  $\varphi$ .)

Slightly smaller than  $L^1(\mathbb{R}^N)$ , the space  $\mathcal{H}^1(\mathbb{R}^N)$  is of paramount importance in harmonic analysis (see, e.g., Stein [47, Chapter III]) and enjoys properties that are ‘barely’ missed by  $L^1(\mathbb{R}^N)$ , e.g.,

$$-\Delta v \in \mathcal{H}^1(\mathbb{R}^N) \implies v \in W_{loc}^{2,1}(\mathbb{R}^N) \quad (4.1)$$



(see, e.g., [47, Chapter III, Section 3, Theorem 3]).

Another noticeable property of  $\mathcal{H}^1(\mathbb{R}^N)$  is given by ‘Stein’s lemma’ [46]:

$$[f \in \mathcal{H}^1(\mathbb{R}^N), f \geq 0 \text{ a.e. in } \Omega] \implies [f \ln(2+f) \in L_{loc}^1(\Omega)]. \quad (4.2)$$

The proof of Theorem 4.1 relies on Jacobi’s ‘fundamental Lemma’ 1.1, combined with the following remarkable result, also from [20].

**Theorem 4.2.** *Let  $1 < p < \infty$ . If  $E \in L^p(\mathbb{R}^N; \mathbb{R}^N)$  and  $B \in L^{p'}(\mathbb{R}^N; \mathbb{R}^N)$  satisfy  $\operatorname{div} E = 0$  and  $\operatorname{curl} B = 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ , then  $E \cdot B \in \mathcal{H}^1(\mathbb{R}^N)$ .*

One obtains Theorem 4.1 from Theorem 4.2 by letting  $p := N/(N-1)$ , and, with the notation of Section 1,  $E := (C_{1,j})_{1 \leq j \leq N}$  and  $B := \nabla u_1$ . Indeed, we have

$$\det(\nabla u) = \sum_{j=1}^N \partial_j u_1 C_{1,j} = B \cdot E,$$

and, clearly,  $B$  and  $E$  satisfy  $B \in L^N(\mathbb{R}^N; \mathbb{R}^N)$ ,  $E \in L^{N/(N-1)}(\mathbb{R}^N; \mathbb{R}^N)$ ,  $\operatorname{curl} B = 0$  in  $\mathcal{D}'(\mathbb{R}^N)$  and, by Lemma 1.1,  $\operatorname{div} E = 0$  in  $\mathcal{D}'(\mathbb{R}^N)$ .

Theorem 4.1 has three noticeable consequences.

**Corollary 4.3.** *Let  $\Omega \subset \mathbb{R}^2$  be a smooth bounded domain. If  $u \in H^1(\Omega; \mathbb{R}^2)$ , then the problem*

$$\begin{cases} -\Delta v = \det(\nabla u) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (4.3)$$

*has a (unique) solution  $v \in H_0^1(\Omega) \cap C(\overline{\Omega})$ , satisfying*

$$\|\nabla v\|_{L^2} \leq C_\Omega \|\nabla u_1\|_{L^2} \|\nabla u_2\|_{L^2} \text{ and } \|v\|_{L^\infty} \leq C'_\Omega \|\nabla u_1\|_{L^2} \|\nabla u_2\|_{L^2}. \quad (4.4)$$

**Corollary 4.4.** *Let  $\Omega \subset \mathbb{R}^N$  be a smooth bounded domain. If  $u \in W^{1,N}(\Omega; \mathbb{R}^N)$ , then  $\det(\nabla u) \in W^{-1,N/(N-1)}(\Omega; \mathbb{R})$  and*

$$|\langle f, \det(\nabla u) \rangle| \leq C_\Omega \|f\|_{L^N} \|\nabla u\|_{L^N}^N, \quad \forall f \in W_0^{1,N}(\Omega; \mathbb{R}), \quad \forall u \in W^{1,N}(\Omega; \mathbb{R}^N). \quad (4.5)$$

*If, in addition,  $\det(\nabla u) \geq 0$  a.e. on  $\Omega$ , then*

$$f \det(\nabla u) \in L^1(\Omega) \text{ and } \int_\Omega f \det(\nabla u) = \langle f, \det(\nabla u) \rangle, \quad \forall f \in W_0^{1,N}(\Omega; \mathbb{R}). \quad (4.6)$$

**Corollary 4.5.** *Let  $u \in W_{loc}^{1,N}(\Omega; \mathbb{R}^N)$ . If  $\det(\nabla u) \geq 0$  a.e. in  $\Omega$ , then  $\det(\nabla u) \ln(2+\det(\nabla u)) \in L_{loc}^1(\Omega)$ .*

Corollary 4.3 is originally due to Wentz [50]; see also Brezis and Coron [15], where it plays a central role in the solution of Rellich’s conjecture. For optimal constants in (4.4), see Topping [49] and the references therein. ‘Modern’ proofs of Corollary 4.3 derive it from Theorem 4.1, (4.1), and the embedding  $W_{loc}^{2,1}(\mathbb{R}^2) \subset C(\mathbb{R}^2)$ .

Assertion (4.5) in Corollary 4.4 is a consequence of Theorem 4.1, the embedding  $W^{1,N}(\mathbb{R}^N) \subset \operatorname{BMO}(\mathbb{R}^N)$  (the John-Nirenberg space of **bounded mean oscillation functions**), and of Fefferman’s theorem asserting that  $[\mathcal{H}^1(\mathbb{R}^N)]^* = \operatorname{BMO}(\mathbb{R}^N)$  [47, Chapter IV]. Assertion (4.6) in Corollary 4.4 relies on a result of Brezis and Browder [13, 14].

Corollary 4.5, originally due to Müller [40], is a straightforward consequence of Theorem 4.1 and (4.2). For a review of subsequent results in the spirit of Corollary 4.5 (results that do not follow from Theorem 4.1), we refer the reader to the surveys of Iwaniec [28] and Mironescu [34].



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