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Non-local approximations of the gradient

Haïm Brezis⁽¹⁾, Petru Mironescu⁽²⁾

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Abstract

We revisit the proofs of a few basic results concerning non-local approximations of the gradient. A typical such result asserts that, if (ρ_ε) is a radial approximation to the identity in \mathbb{R}^N and u belongs to a homogeneous Sobolev space $\dot{W}^{1,p}$, then

$$V_\varepsilon(x) := N \int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N,$$

converges in L^p to the distributional gradient ∇u as $\varepsilon \rightarrow 0$.

We highlight the crucial role played by the representation formula $V_\varepsilon = (\nabla u) * F_\varepsilon$, where F_ε is an approximation to the identity defined via ρ_ε . This formula allows to unify the proofs of a significant number of results in the literature, by reducing them to standard properties of the approximations to the identity.

We also highlight the effectiveness of a symmetric nonlocal integration by parts formula.

Relaxations of the assumptions on u and ρ_ε , allowing, e.g., heavy tails kernels or a distributional definition of V_ε , are also discussed. In particular, we show that heavy tails kernels may be treated as perturbations of approximations to the identity.

1 A representation formula and applications

Let $(\rho_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ be a family functions on \mathbb{R}^N such that:

$$\rho_\varepsilon \text{ is non-negative, integrable, radial, } \forall \varepsilon, \tag{1}$$

$$\int_{\mathbb{R}^N} \rho_\varepsilon = 1, \quad \forall \varepsilon, \tag{2}$$

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| > \delta} \rho_\varepsilon(h) dh = 0, \quad \forall \delta > 0. \tag{3}$$

Following Mengesha and Spector [7] (with roots in Bourgain, Brezis, and Mironescu [1], Gilboa and Osher [6], Du, Gunzburger, Lehouck, and Zhou [5]; see also a detailed list of references in [7, p. 254]), we set, for any measurable function $u \in L^1_{loc}(\mathbb{R}^N)$, and assuming that the integral below exists,

$$V_\varepsilon(x) = V_{\varepsilon,u}(x) = V_{\varepsilon,u,\rho_\varepsilon}(x) := N \int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N. \tag{4}$$

V_ε may be seen as a non-local approximation of the gradient. Indeed (see Remark 2), (i) when u is C^1 and bounded, we have the pointwise convergence $V_\varepsilon(x) \rightarrow \nabla u(x)$ as $\varepsilon \rightarrow 0$, $\forall x \in \mathbb{R}^N$; (ii) when u is C^1 and compactly supported, we have $V_\varepsilon \rightarrow \nabla u$ uniformly.

In what follows, we revisit the proofs of a few results establishing the validity of the convergence

$$V_\varepsilon \rightarrow Du \text{ as } \varepsilon \rightarrow 0, \quad (5)$$

in various functional settings. Many of these results were originally obtained, in slightly different forms, in [7].

Before presenting the main results and methods, we make some easy observations concerning the existence of V_ε . Set

$$W_\varepsilon(x) = W_{\varepsilon,u}(x) = W_{\varepsilon,u,\rho_\varepsilon}(x) := \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N.$$

Clearly, the following holds.

Lemma 1. Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. Then V_ε is well-defined a.e. and is measurable. Moreover, we have $|V_\varepsilon| \leq N W_\varepsilon$ a.e., and thus $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$.

Remark 1. In the above statements, the condition $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$ seems constraining. However, under the following assumption:

$$\text{for every } \varepsilon, \text{ there exist some } \delta_\varepsilon, R_\varepsilon > 0 \text{ such that } \rho_\varepsilon(h) = 0 \text{ if } |h| < \delta_\varepsilon \text{ or if } |h| > R_\varepsilon, \quad (6)$$

we have $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$, $\forall u \in L^1_{loc}(\mathbb{R}^N)$.

This is especially relevant for Propositions 5, 6, 7, 8, and 18 below.

We next present a sufficient condition for having $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ (and thus, by Lemma 1, $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$). For $1 \leq p < \infty$, set

$$I_{\varepsilon,p} = I_{\varepsilon,p,u} = I_{\varepsilon,p,u,\rho_\varepsilon} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dx dh.$$

Lemma 2. Assume (2). Let $1 \leq p < \infty$ and $u \in L^1_{loc}(\mathbb{R}^N)$. Then $\|W_\varepsilon\|_{L^p(\mathbb{R}^N)}^p \leq I_{\varepsilon,p}$.

Consequently, if $I_{\varepsilon,p} < \infty$, then V_ε is well-defined a.e., measurable, and $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq N [I_{\varepsilon,p}]^{1/p}$.

In the above and in what follows, the L^p -norms of vector fields $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are computed with respect to the Euclidean norm $|\cdot|$, i.e.,

$$\|F\|_{L^p(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} |F(x)|^p dx.$$

Similarly, the mass of a measure $F \in \mathcal{M}(\mathbb{R}^N; \mathbb{R}^N)$ is computed with respect to the Euclidean norm, i.e.,

$$\|F\|_{\mathcal{M}(\mathbb{R}^N)} = \sup \left\{ \sum_{j=1}^N \int_{\mathbb{R}^N} \zeta_j dF_j; \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), |\zeta(x)| \leq 1, \forall x \right\}.$$

Proof of Lemma 2. The conclusion follows by integrating in x the inequality

$$\begin{aligned} [W_\varepsilon(x)]^p &\leq \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dh \left(\int_{\mathbb{R}^N} \rho_\varepsilon(h) dh \right)^{p-1} \\ &= \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dh, \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

□

We now recall a few sufficient conditions for having $I_{\varepsilon,p} < \infty$. Set

$$\begin{aligned} \dot{W}^{1,p} &:= \{u \in \mathcal{D}'(\mathbb{R}^N); Du \in L^p(\mathbb{R}^N)\} = \{u \in L^1_{loc}(\mathbb{R}^N); Du \in L^p(\mathbb{R}^N)\}, \quad 1 \leq p < \infty, \\ \dot{B}V &:= \{u \in \mathcal{D}'(\mathbb{R}^N); Du \in \mathcal{M}(\mathbb{R}^N)\} = \{u \in L^1_{loc}(\mathbb{R}^N); Du \in \mathcal{M}(\mathbb{R}^N)\}. \end{aligned}$$

In what follows, for $u \in \dot{W}^{1,p}$, we denote the distributional gradient ∇u .

Let $K_{p,N} := \int_{\mathbb{S}^{N-1}} |h_j|^p d\sigma(h)$ (which does not depend on $j \in \llbracket 1, N \rrbracket$).

We have the following

Lemma 3. [1] Assume (1).

1. Let $1 \leq p < \infty$ and $u \in \dot{W}^{1,p}$. Then $I_{\varepsilon,p} \leq K_{p,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p$.

In particular, $V_\varepsilon \in L^p(\mathbb{R}^N)$ and $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq N [K_{p,N}]^{1/p} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}^{1/p} \|\nabla u\|_{L^p(\mathbb{R}^N)}$.

2. Let $u \in \dot{B}V$. Then $I_{\varepsilon,1} \leq K_{1,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}$.

In particular, $V_\varepsilon \in L^1(\mathbb{R}^N)$ and $\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \leq N K_{1,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}$.

We next present a crucial identity that illuminates the validity of (5): (10), and its avatar (11). Although (10) was probably known to experts (it is implicit in [7, proof of Lemma 3.3] and related to several identities in [5]), its intimate connection to (5) seems to have remained relatively unnoticed.

Assume (1). Let $f_\varepsilon : (0, \infty) \rightarrow [0, \infty)$ be a measurable function such that $\rho_\varepsilon(x) = f_\varepsilon(|x|)$ for a.e. $x \in \mathbb{R}^N$. Set

$$F_\varepsilon(h) := N \int_{|h|}^{\infty} \frac{f_\varepsilon(t)}{t} dt, \quad \forall h \in \mathbb{R}^N \setminus \{0\}. \quad (7)$$

Let us note that

$$\begin{aligned} \|F_\varepsilon\|_{L^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} F_\varepsilon = N |\mathbb{S}^{N-1}| \int_0^\infty r^{N-1} \int_r^\infty \frac{f_\varepsilon(t)}{t} dt dr = |\mathbb{S}^{N-1}| \int_0^\infty t^{N-1} f_\varepsilon(t) dt \\ &= \int_{\mathbb{R}^N} \rho_\varepsilon(h) dh = \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}, \end{aligned} \quad (8)$$

so that, in particular,

$$F_\varepsilon \in L^1(\mathbb{R}^N). \quad (9)$$

Lemma 4. Assume (1). Let F_ε be as in (7).

1. Let $1 \leq p < \infty$ and $u \in \dot{W}^{1,p}$. Then

$$V_\varepsilon = (\nabla u) * F_\varepsilon \text{ a.e.} \quad (10)$$

2. Let $u \in B\dot{V}$. Then

$$V_\varepsilon = (Du) * F_\varepsilon \text{ a.e.} \quad (11)$$

Proof. Step 1. Proof of (10) when $u \in C^\infty(\mathbb{R}^N)$ and $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$. In this case, we actually prove that

$$V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x), \quad \forall x \in \mathbb{R}^N.$$

For this purpose, we note that

$$F_\varepsilon \text{ is compactly supported,} \quad (12)$$

$$\nabla F_\varepsilon(h) = -N \frac{h}{|h|^2} \rho_\varepsilon(h), \quad \forall h \in \mathbb{R}^N \setminus \{0\}, \quad (13)$$

$$F_\varepsilon(h) = O(|\ln |h||) \text{ as } h \rightarrow 0. \quad (14)$$

Using (12)–(14), we find, via an integration by parts, that

$$\begin{aligned} V_\varepsilon(x) &= - \int_{\mathbb{R}^N} [u(x+h) - u(x)] \nabla F_\varepsilon(h) dh = - \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^N \setminus \bar{B}(0,\delta)} [u(x+h) - u(x)] \nabla F_\varepsilon(h) dh \\ &= \int_{\mathbb{R}^N} \nabla u(x+h) F_\varepsilon(h) dh = \int_{\mathbb{R}^N} \nabla u(x-h) F_\varepsilon(h) dh \\ &= [(\nabla u) * F_\varepsilon](x), \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

Step 2. Proof of (10) and (11) when $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. Let η be a radial non-increasing normalized bump function. By Step 1, we have

$$V_{\varepsilon, u * \eta_\delta}(x) = (\nabla(u * \eta_\delta)) * F_\varepsilon(x) = (Du) * (F_\varepsilon * \eta_\delta)(x), \quad \forall x \in \mathbb{R}^N. \quad (15)$$

On the one hand, as $\delta \rightarrow 0$, the right-hand side of (15) converges (possibly along a subsequence) a.e. to $(Du) * F_\varepsilon(x)$. (This follows by combining the Young inequality with the fact that $F_\varepsilon * \eta_\delta \rightarrow F_\varepsilon$ in L^1 .)

In order to obtain (10), respectively (11), it suffices to prove that

$$V_{\varepsilon, u * \eta_\delta, \rho_\varepsilon}(x) \rightarrow V_{\varepsilon, u, \rho_\varepsilon}(x) \text{ as } \delta \rightarrow 0 \text{ for a.e. } x \in \mathbb{R}^N. \quad (16)$$

Property (16) is obtained via dominated convergence, using the standard inequality

$$|u * \eta_\delta(y)| \leq \mathcal{M}_1 u(y), \quad \forall 0 < \delta < 1, \quad \forall y \in \mathbb{R}^N, \quad (17)$$

(see, e.g., [11, eq (17), p. 57]), where $\mathcal{M}_1 u$ is the centered truncated maximal function of u ,

$$\mathcal{M}_1 u(x) := \sup \left\{ \int_{B_r(x)} |u|; \quad 0 < r \leq 1 \right\}.$$

(Here, we use the fact that η is radial, non-increasing, and supported in the unit ball.) Using (17) and the extra assumptions on ρ_ε , we obtain the domination

$$\frac{|u * \eta_\delta(x+h) - u * \eta_\delta(x)|}{|h|} \rho_\varepsilon(h) \leq [\mathcal{M}_1 u(x+h) + \mathcal{M}_1 u(x)] g(h), \quad \forall 0 < \delta < 1, \quad (18)$$

with $g(h) := \rho_\varepsilon(h)/|h|$ bounded and compactly supported. The right-hand side of (18) is in $L^1(\mathbb{R}^N)$ since $\mathcal{M}_1 u \in L^1_{loc}(\mathbb{R}^N)$ (and thus, in particular, $\mathcal{M}_1 u$ is finite a.e.). The latter property follows by combining the Sobolev embeddings $\dot{W}^{1,p}$, $B\dot{V} \hookrightarrow L^{N/(N-1)}_{loc}(\mathbb{R}^N)$ with the fact that, by the maximal function theorem, we have $\mathcal{M}_1 u \in L^r_{loc}(\mathbb{R}^N)$ when $u \in L^r_{loc}(\mathbb{R}^N)$ for some $r > 1$. We obtain that the convergence in (16) holds on the full measure set

$$\{x \in \mathbb{R}^N; \mathcal{M}_1 u(x) < \infty \text{ and } x \text{ is a Lebesgue point of } u\}.$$

Step 3. Proof of (10) and (11) in the general case. For fixed ε , we approximate ρ_ε in L^1 with a sequence $(\rho_{\varepsilon,j})_j$ of kernels $\rho_{\varepsilon,j} \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying (1). By Step 2, the corresponding associated kernels $F_{\varepsilon,j}$ satisfy

$$V_{\varepsilon,u,\rho_{\varepsilon,j}} = (Du) * F_{\varepsilon,j} \text{ a.e.} \quad (19)$$

Let us note that $F_{\varepsilon,j} \rightarrow F_\varepsilon$ in $L^1(\mathbb{R}^N)$ as $j \rightarrow \infty$. (This follows from a straightforward variant of (8).)

We obtain (10), respectively (11), by letting $j \rightarrow \infty$ in (19). Passing to the limits is justified, on the left-hand side, by Lemma 3, and, on the right-hand side, by the Young inequality combined with the fact that $F_{\varepsilon,j} \rightarrow F_\varepsilon$ in $L^1(\mathbb{R}^N)$. \square

Using Lemma 4, (8), and the Young inequality, we obtain the following

Lemma 5. Assume (1)–(2).

1. Let $1 \leq p < \infty$ and $u \in \dot{W}^{1,p}$. Then $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq \|\nabla u\|_{L^p(\mathbb{R}^N)}$.
2. Let $u \in B\dot{V}$. Then $\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \leq \|Du\|_{\mathcal{M}(\mathbb{R}^N)}$.

This is an improvement of Lemma 3, since $N [K_{p,N}]^{1/p} > 1$ when $N \geq 2$. Indeed, the Jensen inequality yields

$$K_{p,N} \geq \left(\int_{\mathbb{S}^{N-1}} |h_j| d\sigma(h) \right)^p = \left(\frac{1}{N} \int_{\mathbb{S}^{N-1}} \sum_{k=1}^N |h_k| d\sigma(h) \right)^p > \left(\frac{1}{N} \int_{\mathbb{S}^{N-1}} d\sigma(h) \right)^p = \frac{1}{N^p}.$$

We next present two direct consequences of Lemma 4, originally obtained, with different arguments, in [7].

Proposition 1. [7, Theorem 1.1 (b)] Assume (1)–(3). Let $u \in \dot{W}^{1,p}(\mathbb{R}^N)$. Then

$$V_\varepsilon \rightarrow \nabla u \text{ in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0.$$

Proposition 2. [7, Theorem 1.2] Assume (1)–(3). Let $u \in B\dot{V}$. Then

$$V_\varepsilon \rightharpoonup Du \text{ *weakly in } \mathcal{M}(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0 \quad (20)$$

and

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)}.$$

Proof of Propositions 1 and 2. By Lemma 6 below, (F_ε) is an approximation to the identity. We conclude by combining this fact with Lemma 4. \square

Lemma 6. Under the assumptions (1)–(3), (F_ε) is an approximation to the identity.

Proof. If $\delta > 0$ is fixed, then (7) and (3) yield

$$\begin{aligned}
\int_{|h|>\delta} F_\varepsilon(h) dh &= N |\mathbb{S}^{N-1}| \int_\delta^\infty r^{N-1} \int_r^\infty \frac{f_\varepsilon(t)}{t} dt dr \\
&= N |\mathbb{S}^{N-1}| \int_\delta^\infty \int_\delta^t r^{N-1} dr \frac{f_\varepsilon(t)}{t} dt \leq |\mathbb{S}^{N-1}| \int_\delta^\infty t^{N-1} f_\varepsilon(t) dt \\
&= \int_{|h|>\delta} \rho_\varepsilon(h) dh \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\end{aligned} \tag{21}$$

The conclusion of the lemma follows from (8) and (21). \square

We next present two a.e. versions of the above results.

Proposition 3. Assume (1)–(3). Let $u \in \dot{W}^{1,p}$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \rightarrow \nabla u(x)$ as $\varepsilon \rightarrow 0$.

Proposition 4. Assume (1)–(3). Let $u \in \dot{B}V$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \rightarrow \nabla^{ac} u(x)$ as $\varepsilon \rightarrow 0$.

These results clearly follow from Lemmas 4 and 6 and the following measure-theoretical

Lemma 7. Let (F_ε) be a an approximation to the identity, with F_ε radial and non-increasing. Then

1. For every $1 \leq p < \infty$ and $G \in L^p(\mathbb{R}^N)$, we have $G * F_\varepsilon(x) \rightarrow G(x)$ at each Lebesgue point of G .
2. For every finite Borel measure ν singular with respect to the Lebesgue measure in \mathbb{R}^N , we have $\nu * F_\varepsilon \rightarrow 0$ a.e.

Proof of Lemma 7. The assumptions on F_ε imply that there exist (unique) non-negative Borel measures μ_ε on $(0, \infty)$ such that

$$F_\varepsilon(x) = \mu_\varepsilon((|x|, \infty)), \text{ for a.e. } x \in \mathbb{R}^N, \tag{22}$$

$$\frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N d\mu_\varepsilon(t) = 1, \forall \varepsilon, \tag{23}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{|\mathbb{S}^{N-1}|}{N} \int_\delta^\infty t^N d\mu_\varepsilon(t) = 1, \forall \delta > 0. \tag{24}$$

Proof of item 1. We have

$$\begin{aligned}
|G * F_\varepsilon(x) - G(x)| &= \left| \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} [G(x - r\omega) - G(x)] ds(\omega) \mu_\varepsilon((r, \infty)) dr \right| \\
&\leq \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) \mu_\varepsilon((r, \infty)) dr \\
&= \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) \int_r^\infty d\mu_\varepsilon(t) dr \\
&= \int_0^\infty \int_0^t r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) dr d\mu_\varepsilon(t)
\end{aligned} \tag{25}$$

$$\begin{aligned}
&= \int_0^\infty \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t) \\
&= \int_0^\infty |B(x,t)| \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t) \\
&= \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t).
\end{aligned}$$

We complete the proof by combining (23)–(25) with the fact that

$$\lim_{t \rightarrow 0} \int_{B(x,t)} |G(y) - G(x)| dy = 0 \text{ at each Lebesgue point } x \text{ of } G$$

and the straightforward inequality

$$\begin{aligned}
\int_{B(x,t)} |G(y) - G(x)| dy &\leq \frac{1}{|B(x,t)|^{1/p}} \|G\|_{L^p(\mathbb{R}^N)} + |G(x)| \\
&\leq \frac{1}{|B(x,\delta)|^{1/p}} \|G\|_{L^p(\mathbb{R}^N)} + |G(x)|, \forall t \geq \delta.
\end{aligned}$$

Proof of item 2. We have

$$\begin{aligned}
\nu * F_\varepsilon(x) &= \int_{\mathbb{R}^N} \mu_\varepsilon(|x-y|, \infty) d\nu(y) = \int_{\mathbb{R}^N} \int_{|x-y|}^\infty d\mu_\varepsilon(t) d\nu(y) \\
&= \int_0^\infty \int_{|x-y| < t} d\nu(y) d\mu_\varepsilon(t) = \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \frac{\nu(B(x,t))}{|B(x,t)|} d\mu_\varepsilon(t).
\end{aligned} \tag{26}$$

We complete the proof combining (23), (24), and (26) with the fact that (by the Lebesgue-Besicovitch differentiation theorem) we have

$$\lim_{t \rightarrow 0} \frac{\nu(B(x,t))}{|B(x,t)|} = 0 \text{ for a.e. } x \in \mathbb{R}^N,$$

and the upper bound

$$\frac{\nu(B(x,t))}{|B(x,t)|} \leq \frac{\nu(\mathbb{R}^N)}{|B(x,\delta)|}, \forall t \geq \delta. \quad \square$$

For other results in the spirit of Propositions 3 and 4, see Spector [10, Theorem 1.2] and Brezis and Nguyen [2, Theorems 1 and 2].

Remark 2. Here are two additional quick consequences of the fact that, under the assumptions (1)–(3), (F_ε) is an approximation to the identity. It is straightforward that the pointwise equality $V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x)$, $\forall x \in \mathbb{R}^N$, holds if $u \in C_c^1(\mathbb{R}^N)$, and therefore for such u we have $V_\varepsilon \rightarrow \nabla u$ uniformly in \mathbb{R}^N as $\varepsilon \rightarrow 0$. Similarly, this equality holds when $u \in (C^1 \cap L^\infty)(\mathbb{R}^N)$, and in this case we have $V_\varepsilon(x) \rightarrow \nabla u(x)$, as $\varepsilon \rightarrow 0$, $\forall x \in \mathbb{R}^N$.

2 An integration by parts formula and applications

The representation formula (10) naturally leads to the following formal calculation, with $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, $\check{f}(x) := f(-x)$, $\forall x \in \mathbb{R}^N$, and $\{e_j\}_{1 \leq j \leq N}$ the canonical basis of \mathbb{R}^N :

$$\begin{aligned} \int_{\mathbb{R}^N} V_{\varepsilon, u} \cdot \zeta &= \sum_j \int_{\mathbb{R}^N} [V_{\varepsilon, u} \cdot e_j] \zeta_j = \sum_j [V_{\varepsilon, u} \cdot e_j] * \check{\zeta}_j(0) = \sum_j [(\partial_j u) * F_\varepsilon] * \check{\zeta}_j(0) \\ &= \sum_j u * [F_\varepsilon * \partial_j \check{\zeta}_j](0) = \sum_j u * V_{\varepsilon, \check{\zeta}_j} \cdot e_j(0) \\ &= \int_{\mathbb{R}^N} u(x) \sum_j [V_{\varepsilon, \check{\zeta}_j} \cdot e_j](-x). \end{aligned} \quad (27)$$

Combining (27) with the (formal) identity $V_{\varepsilon, \check{f}}(-x) = -V_{\varepsilon, f}(x)$, we obtain the formal identity

$$\int_{\mathbb{R}^N} V_{\varepsilon, u} \cdot \zeta = - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx, \quad (28)$$

and its more symmetric avatar

$$\int_{\mathbb{R}^N} [V_{\varepsilon, u} \cdot e_j] \psi = - \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \psi}(x) \cdot e_j] dx, \quad \forall j, \forall \psi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}). \quad (29)$$

Similar “nonlocal integration by parts” identities were known in the literature (see, e.g., [5], [7, Theorem 1.4], and, in a slightly different setting, Šilhavý [12, Section 6]). As we will see below, (28) holds under mild assumptions on u (this is to be compared with the more restrictive assumptions in Lemma 4). The importance of such identities is that they provide a first direction for generalizing the results in Section 1, consisting of weakening the *assumption* $u \in \dot{W}^{1,p}$ (respectively $u \in \dot{B}V$), widely used in Section 1, to a reasonable one allowing V_ε to be well-defined a.e. and to obtain the *property* $u \in \dot{W}^{1,p}$ (respectively $u \in \dot{B}V$) as a *conclusion*.

We first formalize the validity of (28).

Lemma 8. Let $\varepsilon > 0$ be fixed. Assume (1). Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$.

1. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} V_{\varepsilon, u} \cdot \zeta = - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N). \quad (30)$$

2. If ρ_ε is compactly supported, then (30) holds.

Proof. We first note the following equalities, valid (thanks to the Fubini theorem applied to the first line) for every $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$:

$$\begin{aligned} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h \cdot \zeta(x)}{|h|} \rho_\varepsilon(h) dh \right) dx \\ &= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h \cdot \zeta(x)}{|h|} \rho_\varepsilon(h) dx \right) dh \\ &= N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} u(x) \frac{h \cdot [\zeta(x-h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dx \right) dh \\ &= -N \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} u(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dx \right) dh. \end{aligned} \quad (31)$$

We next claim that we may apply the Fubini theorem to the last integral in (31). By linearity, in item 1 we may assume that either $u \in L^1(\mathbb{R}^N)$ or $u \in L^\infty(\mathbb{R}^N)$.

Proof of the claim when $u \in L^1(\mathbb{R}^N)$. In this case, we have

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |u(x)| \frac{|h \cdot [\zeta(x+h) - \zeta(x)]|}{|h|^2} \rho_\varepsilon(h) dx \right) dh \leq \|\nabla \zeta\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty.$$

Proof of the claim when $u \in L^\infty(\mathbb{R}^N)$. By Lemma 3 item 1, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |u(x)| \frac{|h \cdot [\zeta(x+h) - \zeta(x)]|}{|h|^2} \rho_\varepsilon(h) dx \right) dh \\ & \leq K(1, N) \|u\|_{L^\infty(\mathbb{R}^N)} \sum_{j=1}^N \|\nabla \zeta_j\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty. \end{aligned}$$

Proof of the claim when ρ_ε is compactly supported. Let $r, R > 0$ be such that $\text{supp } \zeta \subset B(0, r)$ and $\text{supp } \rho_\varepsilon \subset B(0, R)$. Set $v := u \chi_{B(0, r+R)} \in L^1(\mathbb{R}^N)$. Then

$$u(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) = v(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h), \quad \forall x, h \in \mathbb{R}^N,$$

and we then argue as in the case where $u \in L^1(\mathbb{R}^N)$.

Applying the Fubini theorem in (31), we find that

$$\begin{aligned} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= -N \int_{\mathbb{R}^N} u(x) \left(\int_{\mathbb{R}^N} \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dh \right) dx \\ &= -N \int_{\mathbb{R}^N} u(x) \left(\int_{\mathbb{R}^N} \sum_{j=1}^N \frac{\zeta_j(x+h) - \zeta_j(x)}{|h|} \frac{h_j}{|h|} \rho_\varepsilon(h) dh \right) dx \\ &= - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx, \end{aligned}$$

so that (30) holds. □

Here are two quick consequences of (30), in the spirit of [7, Theorems 1.5 and 1.6].

Proposition 5. Assume (1)–(3). Let $1 < p < \infty$. Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, $\forall \varepsilon$.

1. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad (\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p}). \quad (32)$$

2. If there exists some $R < \infty$ such that

$$\text{supp } \rho_\varepsilon \subset B(0, R), \quad \forall 0 < \varepsilon < \varepsilon_0, \quad (33)$$

then (32) holds.

Proposition 6. Assume (1)–(3). Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, $\forall \varepsilon$.

1. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{B}\dot{V}\text{)}. \quad (34)$$

2. If (33) holds, then (34) holds.

Open Problem 1. Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. Is it true that (32), respectively (34), hold, without assuming the support assumption (33)?

Proof of Propositions 5 and 6. In view of Lemma 3 item 2 and Propositions 1 and 2, it suffices to prove the following. If $\ell := \liminf_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} < \infty$, then $u \in \dot{W}^{1,p}$ if $1 < p < \infty$, respectively $u \in \dot{B}\dot{V}$ if $p = 1$. Clearly, this holds provided

$$\int_{\mathbb{R}^N} u \operatorname{div} \zeta \leq \ell \|\zeta\|_{L^q(\mathbb{R}^N)}, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), \quad (35)$$

where q is the conjugate exponent of p . In turn, (35) holds provided

$$-\int_{\mathbb{R}^N} u \operatorname{div} \zeta = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N). \quad (36)$$

In order to complete the proof, it suffices to establish (36) under the assumptions of Proposition 5, respectively 6 (with no boundedness assumption on $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)}$).

In view of (30), in order to obtain (36) it suffices to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} u V_{\varepsilon, \zeta_j} \cdot e_j = \int_{\mathbb{R}^N} u \partial_j \zeta_j, \quad 1 \leq j \leq N. \quad (37)$$

When $u \in L^\infty(\mathbb{R}^N)$, this follows from Proposition 1 applied to ζ_j with $p = 1$.

When $u \in L^1(\mathbb{R}^N)$, we note the domination

$$|u V_{\varepsilon, \zeta_j} \cdot e_j| \leq \|\nabla \zeta_j\|_{L^\infty(\mathbb{R}^N)} |u| \in L^1(\mathbb{R}^N).$$

We conclude by dominated convergence, using the fact that $V_{\varepsilon, \zeta_j} \cdot e_j$ converges to $\partial_j \zeta_j$ pointwise as $\varepsilon \rightarrow 0$ (see Remark 2).

The argument for $u \in L^1_{loc}(\mathbb{R}^N)$ under the support condition (33) is similar.

The proof of Propositions 5 and 6 is complete. \square

One may consider versions of Propositions 5 and 6 for families of functions instead of a fixed function. Here are, for example, two versions of [7, Theorem 3.7].

Proposition 7. Assume (1)–(3). Let $1 < p < \infty$. Let, for every ε , $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^p(\mathbb{R}^N). \quad (38)$$

1. If (u_ε) is bounded in $(L^1 + L^\infty)(\mathbb{R}^N)$, then there exists some $u \in \dot{W}^{1,p}$ such that, up to a subsequence $\varepsilon_k \rightarrow 0$,

$$u_\varepsilon \rightharpoonup u \text{ *weakly in } \mathcal{M}_{loc}(\mathbb{R}^N), \quad (39)$$

$$\|\nabla u\|_{L^p(\Omega)} \leq \liminf_{\varepsilon \rightarrow 0} \|V_{\varepsilon, u_\varepsilon}\|_{L^p(\Omega)}, \quad \text{for every open set } \Omega \subset \mathbb{R}^N. \quad (40)$$

2. If (u_ε) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (33) holds, then (39)–(40) hold.

Proposition 8. Assume (1)–(3). Let, for every ε , $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^1(\mathbb{R}^N). \quad (41)$$

1. If (u_ε) is bounded in $(L^1 + L^\infty)(\mathbb{R}^N)$, then there exists some $u \in BV$ such that, up to a subsequence $\varepsilon_k \rightarrow 0$,

$$u_\varepsilon \rightharpoonup u \text{ *weakly in } \mathcal{M}_{loc}(\mathbb{R}^N), \quad (42)$$

$$\|Du\|_{\mathcal{M}(\Omega)} \leq \liminf_{\varepsilon \rightarrow 0} \|V_{\varepsilon, u_\varepsilon}\|_{L^1(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \quad (43)$$

2. If (u_ε) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (33) holds, then (42)–(43) hold.

Remark 3. Note that, by Lemma 2, (38) (respectively, (41)) holds if $I_{\varepsilon, p, u_\varepsilon} \leq C < \infty$, $\forall 0 < \varepsilon < \varepsilon_0$ (respectively, $I_{\varepsilon, 1, u_\varepsilon} \leq C < \infty$, $\forall 0 < \varepsilon < \varepsilon_0$).

Proofs of Propositions 7 and 8. We present the argument for Proposition 8; the proof of Proposition 7 is similar. Consider a (signed) Radon measure μ on \mathbb{R}^N such that, up to a subsequence, $u_{\varepsilon_k} \rightharpoonup \mu$ *weakly in $\mathcal{M}_{loc}(\mathbb{R}^N)$. Fix some $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$. We then have

$$\begin{aligned} \int_{\mathbb{R}^N} \operatorname{div} \zeta \, d\mu &= \lim_k \int_{\mathbb{R}^N} \operatorname{div} \zeta \, u_{\varepsilon_k} \\ &= \lim_k \int_{\mathbb{R}^N} \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] u_{\varepsilon_k} + \lim_k \int_{\mathbb{R}^N} \left(\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \right) u_{\varepsilon_k} \\ &:= \lim_k A_k + \lim_k B_k. \end{aligned} \quad (44)$$

Step 1. We have $B_k \rightarrow 0$. We have to treat three cases: (i) (u_{ε_k}) is bounded in $L^1(\mathbb{R}^N)$; (ii) (u_{ε_k}) is bounded in $L^\infty(\mathbb{R}^N)$; (iii) (u_{ε_k}) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (33) holds.

Step 1.1. Proof in case (i). We use the fact that, by Remark 2, $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$ uniformly in \mathbb{R}^N , together with the boundedness of (u_{ε_k}) in $L^1(\mathbb{R}^N)$.

Step 1.2. Proof in case (ii). By Proposition 1, we have $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$ in $L^1(\mathbb{R}^N)$. We combine this fact with the boundedness of (u_{ε_k}) in $L^\infty(\mathbb{R}^N)$.

Step 1.3. Proof in case (iii). By Remark 2, we have $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$ uniformly in \mathbb{R}^N . By the support condition (33), there exists some $r > 0$ such that, for each ε , $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] = 0$ in $\mathbb{R}^N \setminus B(0, r)$. We find that

$$|B_k| \leq \left\| \operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \right\|_{L^\infty(\mathbb{R}^N)} \|u_{\varepsilon_k}\|_{L^1(B(0, r))} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Step 2. Conclusion. By Lemma 8, we have

$$A_k = - \int_{\mathbb{R}^N} V_{\varepsilon_k, u_{\varepsilon_k}} \cdot \zeta. \quad (45)$$

Let $\Omega \subset \mathbb{R}^N$ be an open set. Combining (44), Step 1, (45), and the assumption (41), we find that, when $\zeta \in C_c^\infty(\Omega; \mathbb{R}^N)$,

$$\int_{\Omega} \operatorname{div} \zeta \, d\mu \leq \|\zeta\|_{L^\infty(\Omega)} \liminf_{\varepsilon} \|V_{\varepsilon, u_\varepsilon}\|_{L^1(\Omega)}.$$

It follows that $\mu \in BV$ and that (43) holds. \square

Remark 4. In view of [1, Theorem 4] and Ponce [8, Theorem 1.2], it is likely, but not known, that, in Propositions 7 and 8, the boundedness assumptions on u_ε can be removed, and that the $*$ -weak convergence in \mathcal{M}_{loc} can be improved to strong L^p_{loc} convergence. In this direction, we formulate below two open questions.

Open Problem 2. Let $1 \leq p < \infty$. Let, for every ε , $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^p(\mathbb{R}^N), \quad (46)$$

$$\int_{B(0,1)} u_\varepsilon = 0, \quad \forall \varepsilon. \quad (47)$$

1. Is it true that (u_ε) is bounded in $L^p_{loc}(\mathbb{R}^N)$, or at least in $L^1_{loc}(\mathbb{R}^N)$?
2. Is it true that (u_ε) is relatively compact in $L^p_{loc}(\mathbb{R}^N)$, or at least in $L^1_{loc}(\mathbb{R}^N)$?

Note the natural condition (47). Such a “normalization” condition is needed since V_ε “does not see constants”; therefore, in order to have *a priori* estimates, one has to “kill the constants”.

3 A distributional approach

A natural generalization of the approach in the previous section (based on the identity (30)), consistent with the spirit of the theory of distributions, was initiated in [7]. It consists of taking the identity (30) as a *definition* of V_ε . More precisely, instead of assuming that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, we assume that (30) holds for every $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ and some function $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ that is not, *a priori*, given by (4). This is a *distributional version* of V_ε given by (4) and, by the proof of (30), it coincides with V_ε provided that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. One could even go one step beyond and define the *distribution* V_ε through the formula $V_\varepsilon(\zeta)$ =the right-hand side of (30). (See also, for similar approaches in different but related settings, Shieh and Spector [9], Comi and Stefani [4], Bruè, Calzi, Comi, and Stefani [3].)

Repeating the end of the proof of the Propositions 5 and 6, we obtain, e.g., the following

Proposition 9. Assume (1)–(3). Let $1 < p < \infty$. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (30) holds, $\forall \varepsilon, \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p}). \quad (48)$$

And the usual variants for $p = 1$ or $u \in L^1_{loc}(\mathbb{R}^N)$, under the support condition (33).

4 Heavy tails kernels

The results in this section are in the spirit of [4].

Let ρ_ε satisfy (1) and the following variants of (2)–(3):

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| < 1} \rho_\varepsilon(h) dh = 1, \quad (49)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| > \delta} \frac{\rho_\varepsilon(h)}{|h|} dh = 0, \quad \forall \delta > 0. \quad (50)$$

Note that these assumptions are weaker than (2)–(3) and that they allow *heavy tails* kernels, which are not integrable at infinity. Here is a special case, considered, e.g., in [4], of kernels satisfying (49)–(50):

$$\rho_\varepsilon(h) := \frac{2^{1-\varepsilon}}{N \pi^{N/2}} \frac{\Gamma\left(\frac{N-\varepsilon}{2} + 1\right)}{\Gamma\left(\frac{\varepsilon}{2}\right)} \frac{1}{|h|^{N-\varepsilon}}, \quad 0 < \varepsilon < N + 2. \quad (51)$$

Indeed, the validity of (50) is straightforward, while (49) follows from the fact that

$$\frac{2^{1-\varepsilon}}{N \pi^{N/2}} \frac{\Gamma\left(\frac{N-\varepsilon}{2} + 1\right)}{\Gamma\left(\frac{\varepsilon}{2}\right)} \sim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{N \pi^{N/2}} \Gamma\left(\frac{N}{2} + 1\right) = \frac{\varepsilon}{|\mathbb{S}^{N-1}|},$$

combined with the identity

$$\frac{\varepsilon}{|\mathbb{S}^{N-1}|} \int_{|h|<1} \frac{1}{|h|^{N-\varepsilon}} dh = 1.$$

The results in the previous sections can be easily adapted to kernels satisfying (49)–(50). The price to pay is that the natural function setting is $W^{1,p}(\mathbb{R}^N)$, respectively $BV(\mathbb{R}^N)$, rather than $\dot{W}^{1,p}$, respectively \dot{BV} . Here are some results from the previous sections adapted to the assumptions (1) and (49)–(50).

Proposition 10. Assume (1) and (49)–(50). Let $1 \leq p < \infty$. Let $u \in W^{1,p}(\mathbb{R}^N)$. Then

$$V_\varepsilon \rightarrow \nabla u \text{ in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0.$$

Proposition 11. Assume (1) and (49)–(50). Let $u \in BV(\mathbb{R}^N)$. Then

$$V_\varepsilon \rightharpoonup Du \text{ *weakly in } \mathcal{M}(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0 \quad (52)$$

and

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)}. \quad (53)$$

In the next two results, we assume that, for some $1 < q \leq \infty$, we have

$$\lim_{\varepsilon \rightarrow 0} \|\rho_\varepsilon(h)/|h|\|_{L^q(|h|>1)} = 0. \quad (54)$$

Proposition 12. Assume (1) and (49)–(50). Let $1 \leq p < \infty$. Assume that (54) holds when q is the conjugate exponent of p . Let $u \in W^{1,p}(\mathbb{R}^N)$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \rightarrow \nabla u(x)$ as $\varepsilon \rightarrow 0$.

Proposition 13. Assume (1) and (49)–(50). Assume that (54) holds when $q = \infty$. Let $u \in BV(\mathbb{R}^N)$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \rightarrow \nabla^{ac} u(x)$ as $\varepsilon \rightarrow 0$.

Remark 5. Given any $q > 1$, the kernel in (51) satisfies (54). Therefore, Propositions 12 and 13 apply to these kernels.

Remark 6. Propositions 12 and 13 have straightforward versions, in which the assumption on u is $u \in L^r(\mathbb{R}^N) \cap \dot{W}^{1,p}$, respectively $u \in L^r(\mathbb{R}^N) \cap BV$ for some $r \in [1, \infty)$ (and then, in (54), q is the conjugate exponent of r).

Proposition 14. Assume (1) and (49)–(50). Let $1 < p < \infty$. If $u \in L^p(\mathbb{R}^N)$ and $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ for every ε , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}(\mathbb{R}^N)\text{)}. \quad (55)$$

Proposition 15. Assume (1) and (49)–(50). If $u \in L^1(\mathbb{R}^N)$ and $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ for every ε , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N)\text{)}. \quad (56)$$

Proposition 16. Assume (1) and (49)–(50). Let $1 < p < \infty$. If $u \in L^p(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (30) holds, $\forall \varepsilon, \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}(\mathbb{R}^N)\text{)}. \quad (57)$$

Proposition 17. Assume (1) and (49)–(50). If $u \in L^1(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (30) holds, $\forall \varepsilon, \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N)\text{)}. \quad (58)$$

We prove only Propositions 10 and 12; the other results are obtained from the corresponding ones in the previous sections using similar arguments.

Proof of Proposition 10. Set

$$\rho_\varepsilon^1 := \rho_\varepsilon \chi_{B(0,1)}, \rho_\varepsilon^2 := \rho_\varepsilon - \rho_\varepsilon^1, V_\varepsilon^1 := V_{\varepsilon, u, \rho_\varepsilon^1}, V_\varepsilon^2 := V_{\varepsilon, u, \rho_\varepsilon^2}, \kappa_\varepsilon(h) := \frac{\rho_\varepsilon^2(h)}{|h|}.$$

By Proposition 1 and the assumptions (1) and (49)–(50), we have $V_\varepsilon^1 \rightarrow \nabla u$ in $L^p(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$. On the other hand, we have the straightforward inequality

$$|V_\varepsilon^2(x)| \leq |u| * \kappa_\varepsilon(x) + |u(x)| \|\kappa_\varepsilon\|_{L^1(\mathbb{R}^N)}, \forall x \in \mathbb{R}^N. \quad (59)$$

Combining (50), (59) with $\delta = 1$, and the Young inequality, we find that $V_\varepsilon^2 \rightarrow 0$ in $L^p(\mathbb{R}^N)$ as $\varepsilon \rightarrow 0$. \square

Proof of Proposition 12. It suffices to note that (by (59) and (54)) we have $V_\varepsilon^2 \rightarrow 0$ pointwise as $\varepsilon \rightarrow 0$. \square

Remark 7. The fact that ρ is radial when $|h| > 1$ is not relevant for the above results.

5 Γ -convergence

One can associate Γ -convergence results with the above convergence statements. We present one result of this type, in the spirit of [7, Theorem 1.7].

Let $1 \leq p < \infty$. Set, for $0 < \varepsilon < \varepsilon_0$ and $u \in L^1_{loc}(\mathbb{R}^N)$,

$$J_{\varepsilon,p}(u) := \begin{cases} \|V_{\varepsilon,u}\|_{L^p(\mathbb{R}^N)}, & \text{if } W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N) \\ \infty, & \text{otherwise} \end{cases},$$

$$J_{0,p}(u) := \begin{cases} \|\nabla u\|_{L^p(\mathbb{R}^N)}, & \text{if } u \in \dot{W}^{1,p} \\ \infty, & \text{otherwise} \end{cases}, \quad \forall 1 < p < \infty,$$

$$J_{0,1}(u) := \begin{cases} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}, & \text{if } u \in B\dot{V} \\ \infty, & \text{otherwise} \end{cases}.$$

Proposition 18. Assume (1)–(3). Let $1 \leq p < \infty$.

1. For $1 \leq q < \infty$, $J_{\varepsilon,p}$ Γ -converges to $J_{0,p}$ in $L^q(\mathbb{R}^N)$.
2. Under the support assumption (33), $J_{\varepsilon,p}$ Γ -converges to $J_{0,p}$ in $L^1_{loc}(\mathbb{R}^N)$.

Proof. *Proof of item 2.* Let $(u_\varepsilon)_{0 < \varepsilon < \varepsilon_0} \subset L^1_{loc}(\mathbb{R}^N)$ be a family such that $u_\varepsilon \rightarrow u$ in $L^1_{loc}(\mathbb{R}^N)$ and $\liminf_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon) < \infty$. By Proposition 7 and the proof of Propositions 5 and 6 (see, more specifically, (35) and (36)), we find that $u \in \dot{W}^{1,p}$ if $1 < p < \infty$ (respectively $u \in B\dot{V}$ if $p = 1$) and $J_{0,p}(u) \leq \liminf_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon)$.

In the opposite direction, we do not need the support assumption (33). Let $u \in \dot{W}^{1,p}$ if $1 < p < \infty$, respectively $u \in B\dot{V}$ if $p = 1$. Let η be a normalized bump function. By Proposition 1 applied to $u * \eta_{1/j}$, where $j \geq 1$ is an integer, there exists a sequence $(\varepsilon_j)_{j \geq 1}$ such that

$$J_{\varepsilon,p}(u * \eta_{1/j}) \leq J_{0,p}(u * \eta_{1/j}) + \frac{1}{j}, \quad \forall j \geq 1, \quad \forall 0 < \varepsilon < \varepsilon_j.$$

With no loss of generality, we may assume that $\varepsilon_j \rightarrow 0$ and $\varepsilon_{j+1} < \varepsilon_j$. If we set

$$u_\varepsilon := u * \eta_{1/j}, \quad \forall \varepsilon_{j+1} \leq \varepsilon < \varepsilon_j,$$

then $u_\varepsilon \rightarrow u$ in $L^1_{loc}(\mathbb{R}^N)$ when $\varepsilon \rightarrow 0$ and

$$\limsup_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon) \leq \limsup_{j \rightarrow \infty} J_{0,p}(u * \eta_{1/j}) = J_{0,p}(u).$$

Proof of item 1. The proof is similar to the one of item 2. It suffices to note that Propositions 5, 6, and 7 still hold if we replace $(L^1 + L^\infty)(\mathbb{R}^N)$ with $L^q(\mathbb{R}^N)$. \square

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