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# Non-local approximations of the gradient

Haïm Brezis<sup>(1)</sup>, Petru Mironescu<sup>(2)</sup>

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## Abstract

We revisit the proofs of a few basic results concerning non-local approximations of the gradient. A typical such result asserts that, if  $(\rho_\varepsilon)$  is a radial approximation to the identity in  $\mathbb{R}^N$  and  $u$  belongs to a homogeneous Sobolev space  $\dot{W}^{1,p}$ , then

$$V_\varepsilon(x) := N \int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N,$$

converges in  $L^p$  to the distributional gradient  $\nabla u$  as  $\varepsilon \rightarrow 0$ .

We highlight the crucial role played by the representation formula  $V_\varepsilon = (\nabla u) * F_\varepsilon$ , where  $F_\varepsilon$  is an approximation to the identity defined via  $\rho_\varepsilon$ . This formula allows to unify the proofs of a significant number of results in the literature, by reducing them to standard properties of the approximations to the identity.

We also highlight the effectiveness of a symmetric nonlocal integration by parts formula.

Relaxations of the assumptions on  $u$  and  $\rho_\varepsilon$ , allowing, e.g., heavy tails kernels or a distributional definition of  $V_\varepsilon$ , are also discussed. In particular, we show that heavy tails kernels may be treated as perturbations of approximations to the identity.

## 1 A representation formula and applications

Let  $(\rho_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$  be a family functions on  $\mathbb{R}^N$  such that:

$$\rho_\varepsilon \text{ is non-negative, integrable, radial, } \forall \varepsilon, \tag{1}$$

$$\int_{\mathbb{R}^N} \rho_\varepsilon = 1, \quad \forall \varepsilon, \tag{2}$$

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| > \delta} \rho_\varepsilon(h) dh = 0, \quad \forall \delta > 0. \tag{3}$$

Following Mengesha and Spector [7] (with roots in Bourgain, Brezis, and Mironescu [1], Gilboa and Osher [6], Du, Gunzburger, Lehouck, and Zhou [5]; see also a detailed list of references in [7, p. 254]), we set, for any measurable function  $u \in L^1_{loc}(\mathbb{R}^N)$ , and assuming that the integral below exists,

$$V_\varepsilon(x) = V_{\varepsilon,u}(x) = V_{\varepsilon,u,\rho_\varepsilon}(x) := N \int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N. \tag{4}$$

$V_\varepsilon$  may be seen as a non-local approximation of the gradient. Indeed (see Remark 2), (i) when  $u$  is  $C^1$  and bounded, we have the pointwise convergence  $V_\varepsilon(x) \rightarrow \nabla u(x)$  as  $\varepsilon \rightarrow 0$ ,  $\forall x \in \mathbb{R}^N$ ; (ii) when  $u$  is  $C^1$  and compactly supported, we have  $V_\varepsilon \rightarrow \nabla u$  uniformly.

In what follows, we revisit the proofs of a few results establishing the validity of the convergence

$$V_\varepsilon \rightarrow Du \text{ as } \varepsilon \rightarrow 0, \quad (5)$$

in various functional settings. Many of these results were originally obtained, in slightly different forms, in [7].

Before presenting the main results and methods, we make some easy observations concerning the existence of  $V_\varepsilon$ . Set

$$W_\varepsilon(x) = W_{\varepsilon,u}(x) = W_{\varepsilon,u,\rho_\varepsilon}(x) := \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|}{|h|} \rho_\varepsilon(h) dh, \quad x \in \mathbb{R}^N.$$

Clearly, the following holds.

**Lemma 1.** Let  $u \in L^1_{loc}(\mathbb{R}^N)$  be such that  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ . Then  $V_\varepsilon$  is well-defined a.e. and is measurable. Moreover, we have  $|V_\varepsilon| \leq N W_\varepsilon$  a.e., and thus  $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ .

**Remark 1.** In the above statements, the condition  $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$  seems constraining. However, under the following assumption:

$$\text{for every } \varepsilon, \text{ there exist some } \delta_\varepsilon, R_\varepsilon > 0 \text{ such that } \rho_\varepsilon(h) = 0 \text{ if } |h| < \delta_\varepsilon \text{ or if } |h| > R_\varepsilon, \quad (6)$$

we have  $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$ ,  $\forall u \in L^1_{loc}(\mathbb{R}^N)$ .

This is especially relevant for Propositions 5, 6, 7, 8, and 18 below.

We next present a sufficient condition for having  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  (and thus, by Lemma 1,  $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ ). For  $1 \leq p < \infty$ , set

$$I_{\varepsilon,p} = I_{\varepsilon,p,u} = I_{\varepsilon,p,u,\rho_\varepsilon} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dx dh.$$

**Lemma 2.** Assume (2). Let  $1 \leq p < \infty$  and  $u \in L^1_{loc}(\mathbb{R}^N)$ . Then  $\|W_\varepsilon\|_{L^p(\mathbb{R}^N)}^p \leq I_{\varepsilon,p}$ .

Consequently, if  $I_{\varepsilon,p} < \infty$ , then  $V_\varepsilon$  is well-defined a.e., measurable, and  $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq N [I_{\varepsilon,p}]^{1/p}$ .

In the above and in what follows, the  $L^p$ -norms of vector fields  $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$  are computed with respect to the Euclidean norm  $|\cdot|$ , i.e.,

$$\|F\|_{L^p(\mathbb{R}^N)}^p = \int_{\mathbb{R}^N} |F(x)|^p dx.$$

Similarly, the mass of a measure  $F \in \mathcal{M}(\mathbb{R}^N; \mathbb{R}^N)$  is computed with respect to the Euclidean norm, i.e.,

$$\|F\|_{\mathcal{M}(\mathbb{R}^N)} = \sup \left\{ \sum_{j=1}^N \int_{\mathbb{R}^N} \zeta_j dF_j; \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), |\zeta(x)| \leq 1, \forall x \right\}.$$

*Proof of Lemma 2.* The conclusion follows by integrating in  $x$  the inequality

$$\begin{aligned} [W_\varepsilon(x)]^p &\leq \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dh \left( \int_{\mathbb{R}^N} \rho_\varepsilon(h) dh \right)^{p-1} \\ &= \int_{\mathbb{R}^N} \frac{|u(x+h) - u(x)|^p}{|h|^p} \rho_\varepsilon(h) dh, \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

□

We now recall a few sufficient conditions for having  $I_{\varepsilon,p} < \infty$ . Set

$$\begin{aligned} \dot{W}^{1,p} &:= \{u \in \mathcal{D}'(\mathbb{R}^N); Du \in L^p(\mathbb{R}^N)\} = \{u \in L^1_{loc}(\mathbb{R}^N); Du \in L^p(\mathbb{R}^N)\}, \quad 1 \leq p < \infty, \\ \dot{B}V &:= \{u \in \mathcal{D}'(\mathbb{R}^N); Du \in \mathcal{M}(\mathbb{R}^N)\} = \{u \in L^1_{loc}(\mathbb{R}^N); Du \in \mathcal{M}(\mathbb{R}^N)\}. \end{aligned}$$

In what follows, for  $u \in \dot{W}^{1,p}$ , we denote the distributional gradient  $\nabla u$ .

Let  $K_{p,N} := \int_{\mathbb{S}^{N-1}} |h_j|^p d\sigma(h)$  (which does not depend on  $j \in \llbracket 1, N \rrbracket$ ).

We have the following

**Lemma 3.** [1] Assume (1).

1. Let  $1 \leq p < \infty$  and  $u \in \dot{W}^{1,p}$ . Then  $I_{\varepsilon,p} \leq K_{p,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|\nabla u\|_{L^p(\mathbb{R}^N)}^p$ .

In particular,  $V_\varepsilon \in L^p(\mathbb{R}^N)$  and  $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq N [K_{p,N}]^{1/p} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}^{1/p} \|\nabla u\|_{L^p(\mathbb{R}^N)}$ .

2. Let  $u \in \dot{B}V$ . Then  $I_{\varepsilon,1} \leq K_{1,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}$ .

In particular,  $V_\varepsilon \in L^1(\mathbb{R}^N)$  and  $\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \leq N K_{1,N} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}$ .

We next present a crucial identity that illuminates the validity of (5): (10), and its avatar (11). Although (10) was probably known to experts (it is implicit in [7, proof of Lemma 3.3] and related to several identities in [5]), its intimate connection to (5) seems to have remained relatively unnoticed.

Assume (1). Let  $f_\varepsilon : (0, \infty) \rightarrow [0, \infty)$  be a measurable function such that  $\rho_\varepsilon(x) = f_\varepsilon(|x|)$  for a.e.  $x \in \mathbb{R}^N$ . Set

$$F_\varepsilon(h) := N \int_{|h|}^{\infty} \frac{f_\varepsilon(t)}{t} dt, \quad \forall h \in \mathbb{R}^N \setminus \{0\}. \quad (7)$$

Let us note that

$$\begin{aligned} \|F_\varepsilon\|_{L^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} F_\varepsilon = N |\mathbb{S}^{N-1}| \int_0^\infty r^{N-1} \int_r^\infty \frac{f_\varepsilon(t)}{t} dt dr = |\mathbb{S}^{N-1}| \int_0^\infty t^{N-1} f_\varepsilon(t) dt \\ &= \int_{\mathbb{R}^N} \rho_\varepsilon(h) dh = \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}, \end{aligned} \quad (8)$$

so that, in particular,

$$F_\varepsilon \in L^1(\mathbb{R}^N). \quad (9)$$

**Lemma 4.** Assume (1). Let  $F_\varepsilon$  be as in (7).

1. Let  $1 \leq p < \infty$  and  $u \in \dot{W}^{1,p}$ . Then

$$V_\varepsilon = (\nabla u) * F_\varepsilon \text{ a.e.} \quad (10)$$

2. Let  $u \in B\dot{V}$ . Then

$$V_\varepsilon = (Du) * F_\varepsilon \text{ a.e.} \quad (11)$$

*Proof. Step 1. Proof of (10) when  $u \in C^\infty(\mathbb{R}^N)$  and  $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ .* In this case, we actually prove that

$$V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x), \quad \forall x \in \mathbb{R}^N.$$

For this purpose, we note that

$$F_\varepsilon \text{ is compactly supported,} \quad (12)$$

$$\nabla F_\varepsilon(h) = -N \frac{h}{|h|^2} \rho_\varepsilon(h), \quad \forall h \in \mathbb{R}^N \setminus \{0\}, \quad (13)$$

$$F_\varepsilon(h) = O(|\ln |h||) \text{ as } h \rightarrow 0. \quad (14)$$

Using (12)–(14), we find, via an integration by parts, that

$$\begin{aligned} V_\varepsilon(x) &= - \int_{\mathbb{R}^N} [u(x+h) - u(x)] \nabla F_\varepsilon(h) dh = - \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^N \setminus \bar{B}(0,\delta)} [u(x+h) - u(x)] \nabla F_\varepsilon(h) dh \\ &= \int_{\mathbb{R}^N} \nabla u(x+h) F_\varepsilon(h) dh = \int_{\mathbb{R}^N} \nabla u(x-h) F_\varepsilon(h) dh \\ &= [(\nabla u) * F_\varepsilon](x), \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

*Step 2. Proof of (10) and (11) when  $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ .* Let  $\eta$  be a radial non-increasing normalized bump function. By Step 1, we have

$$V_{\varepsilon, u * \eta_\delta}(x) = (\nabla(u * \eta_\delta)) * F_\varepsilon(x) = (Du) * (F_\varepsilon * \eta_\delta)(x), \quad \forall x \in \mathbb{R}^N. \quad (15)$$

On the one hand, as  $\delta \rightarrow 0$ , the right-hand side of (15) converges (possibly along a subsequence) a.e. to  $(Du) * F_\varepsilon(x)$ . (This follows by combining the Young inequality with the fact that  $F_\varepsilon * \eta_\delta \rightarrow F_\varepsilon$  in  $L^1$ .)

In order to obtain (10), respectively (11), it suffices to prove that

$$V_{\varepsilon, u * \eta_\delta, \rho_\varepsilon}(x) \rightarrow V_{\varepsilon, u, \rho_\varepsilon}(x) \text{ as } \delta \rightarrow 0 \text{ for a.e. } x \in \mathbb{R}^N. \quad (16)$$

Property (16) is obtained via dominated convergence, using the standard inequality

$$|u * \eta_\delta(y)| \leq \mathcal{M}_1 u(y), \quad \forall 0 < \delta < 1, \quad \forall y \in \mathbb{R}^N, \quad (17)$$

(see, e.g., [11, eq (17), p. 57]), where  $\mathcal{M}_1 u$  is the centered truncated maximal function of  $u$ ,

$$\mathcal{M}_1 u(x) := \sup \left\{ \int_{B_r(x)} |u|; \quad 0 < r \leq 1 \right\}.$$

(Here, we use the fact that  $\eta$  is radial, non-increasing, and supported in the unit ball.) Using (17) and the extra assumptions on  $\rho_\varepsilon$ , we obtain the domination

$$\frac{|u * \eta_\delta(x+h) - u * \eta_\delta(x)|}{|h|} \rho_\varepsilon(h) \leq [\mathcal{M}_1 u(x+h) + \mathcal{M}_1 u(x)] g(h), \quad \forall 0 < \delta < 1, \quad (18)$$

with  $g(h) := \rho_\varepsilon(h)/|h|$  bounded and compactly supported. The right-hand side of (18) is in  $L^1(\mathbb{R}^N)$  since  $\mathcal{M}_1 u \in L^1_{loc}(\mathbb{R}^N)$  (and thus, in particular,  $\mathcal{M}_1 u$  is finite a.e.). The latter property follows by combining the Sobolev embeddings  $\dot{W}^{1,p}$ ,  $B\dot{V} \hookrightarrow L^{N/(N-1)}_{loc}(\mathbb{R}^N)$  with the fact that, by the maximal function theorem, we have  $\mathcal{M}_1 u \in L^r_{loc}(\mathbb{R}^N)$  when  $u \in L^r_{loc}(\mathbb{R}^N)$  for some  $r > 1$ . We obtain that the convergence in (16) holds on the full measure set

$$\{x \in \mathbb{R}^N; \mathcal{M}_1 u(x) < \infty \text{ and } x \text{ is a Lebesgue point of } u\}.$$

*Step 3. Proof of (10) and (11) in the general case.* For fixed  $\varepsilon$ , we approximate  $\rho_\varepsilon$  in  $L^1$  with a sequence  $(\rho_{\varepsilon,j})_j$  of kernels  $\rho_{\varepsilon,j} \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$  satisfying (1). By Step 2, the corresponding associated kernels  $F_{\varepsilon,j}$  satisfy

$$V_{\varepsilon,u,\rho_{\varepsilon,j}} = (Du) * F_{\varepsilon,j} \text{ a.e.} \quad (19)$$

Let us note that  $F_{\varepsilon,j} \rightarrow F_\varepsilon$  in  $L^1(\mathbb{R}^N)$  as  $j \rightarrow \infty$ . (This follows from a straightforward variant of (8).)

We obtain (10), respectively (11), by letting  $j \rightarrow \infty$  in (19). Passing to the limits is justified, on the left-hand side, by Lemma 3, and, on the right-hand side, by the Young inequality combined with the fact that  $F_{\varepsilon,j} \rightarrow F_\varepsilon$  in  $L^1(\mathbb{R}^N)$ .  $\square$

Using Lemma 4, (8), and the Young inequality, we obtain the following

**Lemma 5.** Assume (1)–(2).

1. Let  $1 \leq p < \infty$  and  $u \in \dot{W}^{1,p}$ . Then  $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \leq \|\nabla u\|_{L^p(\mathbb{R}^N)}$ .
2. Let  $u \in B\dot{V}$ . Then  $\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \leq \|Du\|_{\mathcal{M}(\mathbb{R}^N)}$ .

This is an improvement of Lemma 3, since  $N [K_{p,N}]^{1/p} > 1$  when  $N \geq 2$ . Indeed, the Jensen inequality yields

$$K_{p,N} \geq \left( \int_{\mathbb{S}^{N-1}} |h_j| d\sigma(h) \right)^p = \left( \frac{1}{N} \int_{\mathbb{S}^{N-1}} \sum_{k=1}^N |h_k| d\sigma(h) \right)^p > \left( \frac{1}{N} \int_{\mathbb{S}^{N-1}} d\sigma(h) \right)^p = \frac{1}{N^p}.$$

We next present two direct consequences of Lemma 4, originally obtained, with different arguments, in [7].

**Proposition 1.** [7, Theorem 1.1 (b)] Assume (1)–(3). Let  $u \in \dot{W}^{1,p}(\mathbb{R}^N)$ . Then

$$V_\varepsilon \rightarrow \nabla u \text{ in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0.$$

**Proposition 2.** [7, Theorem 1.2] Assume (1)–(3). Let  $u \in B\dot{V}$ . Then

$$V_\varepsilon \rightharpoonup Du \text{ *weakly in } \mathcal{M}(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0 \quad (20)$$

and

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)}.$$

*Proof of Propositions 1 and 2.* By Lemma 6 below,  $(F_\varepsilon)$  is an approximation to the identity. We conclude by combining this fact with Lemma 4.  $\square$

**Lemma 6.** Under the assumptions (1)–(3),  $(F_\varepsilon)$  is an approximation to the identity.

*Proof.* If  $\delta > 0$  is fixed, then (7) and (3) yield

$$\begin{aligned}
\int_{|h|>\delta} F_\varepsilon(h) dh &= N |\mathbb{S}^{N-1}| \int_\delta^\infty r^{N-1} \int_r^\infty \frac{f_\varepsilon(t)}{t} dt dr \\
&= N |\mathbb{S}^{N-1}| \int_\delta^\infty \int_\delta^t r^{N-1} dr \frac{f_\varepsilon(t)}{t} dt \leq |\mathbb{S}^{N-1}| \int_\delta^\infty t^{N-1} f_\varepsilon(t) dt \\
&= \int_{|h|>\delta} \rho_\varepsilon(h) dh \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\end{aligned} \tag{21}$$

The conclusion of the lemma follows from (8) and (21).  $\square$

We next present two a.e. versions of the above results.

**Proposition 3.** Assume (1)–(3). Let  $u \in \dot{W}^{1,p}$ . Then, for a.e.  $x \in \mathbb{R}^N$ , we have  $V_\varepsilon(x) \rightarrow \nabla u(x)$  as  $\varepsilon \rightarrow 0$ .

**Proposition 4.** Assume (1)–(3). Let  $u \in \dot{B}V$ . Then, for a.e.  $x \in \mathbb{R}^N$ , we have  $V_\varepsilon(x) \rightarrow \nabla^{ac} u(x)$  as  $\varepsilon \rightarrow 0$ .

These results clearly follow from Lemmas 4 and 6 and the following measure-theoretical

**Lemma 7.** Let  $(F_\varepsilon)$  be a an approximation to the identity, with  $F_\varepsilon$  radial and non-increasing. Then

1. For every  $1 \leq p < \infty$  and  $G \in L^p(\mathbb{R}^N)$ , we have  $G * F_\varepsilon(x) \rightarrow G(x)$  at each Lebesgue point of  $G$ .
2. For every finite Borel measure  $\nu$  singular with respect to the Lebesgue measure in  $\mathbb{R}^N$ , we have  $\nu * F_\varepsilon \rightarrow 0$  a.e.

*Proof of Lemma 7.* The assumptions on  $F_\varepsilon$  imply that there exist (unique) non-negative Borel measures  $\mu_\varepsilon$  on  $(0, \infty)$  such that

$$F_\varepsilon(x) = \mu_\varepsilon((|x|, \infty)), \text{ for a.e. } x \in \mathbb{R}^N, \tag{22}$$

$$\frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N d\mu_\varepsilon(t) = 1, \forall \varepsilon, \tag{23}$$

$$\lim_{\varepsilon \rightarrow 0} \frac{|\mathbb{S}^{N-1}|}{N} \int_\delta^\infty t^N d\mu_\varepsilon(t) = 1, \forall \delta > 0. \tag{24}$$

*Proof of item 1.* We have

$$\begin{aligned}
|G * F_\varepsilon(x) - G(x)| &= \left| \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} [G(x - r\omega) - G(x)] ds(\omega) \mu_\varepsilon((r, \infty)) dr \right| \\
&\leq \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) \mu_\varepsilon((r, \infty)) dr \\
&= \int_0^\infty r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) \int_r^\infty d\mu_\varepsilon(t) dr \\
&= \int_0^\infty \int_0^t r^{N-1} \int_{\mathbb{S}^{N-1}} |G(x - r\omega) - G(x)| ds(\omega) dr d\mu_\varepsilon(t)
\end{aligned} \tag{25}$$

$$\begin{aligned}
&= \int_0^\infty \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t) \\
&= \int_0^\infty |B(x,t)| \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t) \\
&= \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \int_{B(x,t)} |G(y) - G(x)| dy d\mu_\varepsilon(t).
\end{aligned}$$

We complete the proof by combining (23)–(25) with the fact that

$$\lim_{t \rightarrow 0} \int_{B(x,t)} |G(y) - G(x)| dy = 0 \text{ at each Lebesgue point } x \text{ of } G$$

and the straightforward inequality

$$\begin{aligned}
\int_{B(x,t)} |G(y) - G(x)| dy &\leq \frac{1}{|B(x,t)|^{1/p}} \|G\|_{L^p(\mathbb{R}^N)} + |G(x)| \\
&\leq \frac{1}{|B(x,\delta)|^{1/p}} \|G\|_{L^p(\mathbb{R}^N)} + |G(x)|, \forall t \geq \delta.
\end{aligned}$$

*Proof of item 2.* We have

$$\begin{aligned}
\nu * F_\varepsilon(x) &= \int_{\mathbb{R}^N} \mu_\varepsilon(|x-y|, \infty) d\nu(y) = \int_{\mathbb{R}^N} \int_{|x-y|}^\infty d\mu_\varepsilon(t) d\nu(y) \\
&= \int_0^\infty \int_{|x-y| < t} d\nu(y) d\mu_\varepsilon(t) = \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \frac{\nu(B(x,t))}{|B(x,t)|} d\mu_\varepsilon(t).
\end{aligned} \tag{26}$$

We complete the proof combining (23), (24), and (26) with the fact that (by the Lebesgue-Besicovitch differentiation theorem) we have

$$\lim_{t \rightarrow 0} \frac{\nu(B(x,t))}{|B(x,t)|} = 0 \text{ for a.e. } x \in \mathbb{R}^N,$$

and the upper bound

$$\frac{\nu(B(x,t))}{|B(x,t)|} \leq \frac{\nu(\mathbb{R}^N)}{|B(x,\delta)|}, \forall t \geq \delta. \quad \square$$

For other results in the spirit of Propositions 3 and 4, see Spector [10, Theorem 1.2] and Brezis and Nguyen [2, Theorems 1 and 2].

**Remark 2.** Here are two additional quick consequences of the fact that, under the assumptions (1)–(3),  $(F_\varepsilon)$  is an approximation to the identity. It is straightforward that the pointwise equality  $V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x)$ ,  $\forall x \in \mathbb{R}^N$ , holds if  $u \in C_c^1(\mathbb{R}^N)$ , and therefore for such  $u$  we have  $V_\varepsilon \rightarrow \nabla u$  uniformly in  $\mathbb{R}^N$  as  $\varepsilon \rightarrow 0$ . Similarly, this equality holds when  $u \in (C^1 \cap L^\infty)(\mathbb{R}^N)$ , and in this case we have  $V_\varepsilon(x) \rightarrow \nabla u(x)$ , as  $\varepsilon \rightarrow 0$ ,  $\forall x \in \mathbb{R}^N$ .



## 2 An integration by parts formula and applications

The representation formula (10) naturally leads to the following formal calculation, with  $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ ,  $\check{f}(x) := f(-x)$ ,  $\forall x \in \mathbb{R}^N$ , and  $\{e_j\}_{1 \leq j \leq N}$  the canonical basis of  $\mathbb{R}^N$ :

$$\begin{aligned} \int_{\mathbb{R}^N} V_{\varepsilon, u} \cdot \zeta &= \sum_j \int_{\mathbb{R}^N} [V_{\varepsilon, u} \cdot e_j] \zeta_j = \sum_j [V_{\varepsilon, u} \cdot e_j] * \check{\zeta}_j(0) = \sum_j [(\partial_j u) * F_\varepsilon] * \check{\zeta}_j(0) \\ &= \sum_j u * [F_\varepsilon * \partial_j \check{\zeta}_j](0) = \sum_j u * V_{\varepsilon, \check{\zeta}_j} \cdot e_j(0) \\ &= \int_{\mathbb{R}^N} u(x) \sum_j [V_{\varepsilon, \check{\zeta}_j} \cdot e_j](-x). \end{aligned} \quad (27)$$

Combining (27) with the (formal) identity  $V_{\varepsilon, \check{f}}(-x) = -V_{\varepsilon, f}(x)$ , we obtain the formal identity

$$\int_{\mathbb{R}^N} V_{\varepsilon, u} \cdot \zeta = - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx, \quad (28)$$

and its more symmetric avatar

$$\int_{\mathbb{R}^N} [V_{\varepsilon, u} \cdot e_j] \psi = - \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \psi}(x) \cdot e_j] dx, \quad \forall j, \quad \forall \psi \in C_c^\infty(\mathbb{R}^N; \mathbb{R}). \quad (29)$$

Similar “nonlocal integration by parts” identities were known in the literature (see, e.g., [5], [7, Theorem 1.4], and, in a slightly different setting, Šilhavý [12, Section 6]). As we will see below, (28) holds under mild assumptions on  $u$  (this is to be compared with the more restrictive assumptions in Lemma 4). The importance of such identities is that they provide a first direction for generalizing the results in Section 1, consisting of weakening the *assumption*  $u \in \dot{W}^{1,p}$  (respectively  $u \in BV$ ), widely used in Section 1, to a reasonable one allowing  $V_\varepsilon$  to be well-defined a.e. and to obtain the *property*  $u \in \dot{W}^{1,p}$  (respectively  $u \in BV$ ) as a *conclusion*.

We first formalize the validity of (28).

**Lemma 8.** Let  $\varepsilon > 0$  be fixed. Assume (1). Let  $u \in L^1_{loc}(\mathbb{R}^N)$  be such that  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ .

1. If  $u \in (L^1 + L^\infty)(\mathbb{R}^N)$ , then

$$\int_{\mathbb{R}^N} V_{\varepsilon, u} \cdot \zeta = - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N). \quad (30)$$

2. If  $\rho_\varepsilon$  is compactly supported, then (30) holds.

*Proof.* We first note the following equalities, valid (thanks to the Fubini theorem applied to the first line) for every  $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ :

$$\begin{aligned} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= N \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h \cdot \zeta(x)}{|h|} \rho_\varepsilon(h) dh \right) dx \\ &= N \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{u(x+h) - u(x)}{|h|} \frac{h \cdot \zeta(x)}{|h|} \rho_\varepsilon(h) dx \right) dh \\ &= N \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} u(x) \frac{h \cdot [\zeta(x-h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dx \right) dh \\ &= -N \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} u(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dx \right) dh. \end{aligned} \quad (31)$$

We next claim that we may apply the Fubini theorem to the last integral in (31). By linearity, in item 1 we may assume that either  $u \in L^1(\mathbb{R}^N)$  or  $u \in L^\infty(\mathbb{R}^N)$ .

*Proof of the claim when  $u \in L^1(\mathbb{R}^N)$ .* In this case, we have

$$\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |u(x)| \frac{|h \cdot [\zeta(x+h) - \zeta(x)]|}{|h|^2} \rho_\varepsilon(h) dx \right) dh \leq \|\nabla \zeta\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty.$$

*Proof of the claim when  $u \in L^\infty(\mathbb{R}^N)$ .* By Lemma 3 item 1, we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |u(x)| \frac{|h \cdot [\zeta(x+h) - \zeta(x)]|}{|h|^2} \rho_\varepsilon(h) dx \right) dh \\ & \leq K(1, N) \|u\|_{L^\infty(\mathbb{R}^N)} \sum_{j=1}^N \|\nabla \zeta_j\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty. \end{aligned}$$

*Proof of the claim when  $\rho_\varepsilon$  is compactly supported.* Let  $r, R > 0$  be such that  $\text{supp } \zeta \subset B(0, r)$  and  $\text{supp } \rho_\varepsilon \subset B(0, R)$ . Set  $v := u \chi_{B(0, r+R)} \in L^1(\mathbb{R}^N)$ . Then

$$u(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) = v(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h), \quad \forall x, h \in \mathbb{R}^N,$$

and we then argue as in the case where  $u \in L^1(\mathbb{R}^N)$ .

Applying the Fubini theorem in (31), we find that

$$\begin{aligned} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= -N \int_{\mathbb{R}^N} u(x) \left( \int_{\mathbb{R}^N} \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) dh \right) dx \\ &= -N \int_{\mathbb{R}^N} u(x) \left( \int_{\mathbb{R}^N} \sum_{j=1}^N \frac{\zeta_j(x+h) - \zeta_j(x)}{|h|} \frac{h_j}{|h|} \rho_\varepsilon(h) dh \right) dx \\ &= - \sum_{j=1}^N \int_{\mathbb{R}^N} u(x) [V_{\varepsilon, \zeta_j}(x) \cdot e_j] dx, \end{aligned}$$

so that (30) holds. □

Here are two quick consequences of (30), in the spirit of [7, Theorems 1.5 and 1.6].

**Proposition 5.** Assume (1)–(3). Let  $1 < p < \infty$ . Let  $u \in L^1_{loc}(\mathbb{R}^N)$  be such that  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ ,  $\forall \varepsilon$ .

1. If  $u \in (L^1 + L^\infty)(\mathbb{R}^N)$ , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \quad (\text{with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p}). \quad (32)$$

2. If there exists some  $R < \infty$  such that

$$\text{supp } \rho_\varepsilon \subset B(0, R), \quad \forall 0 < \varepsilon < \varepsilon_0, \quad (33)$$

then (32) holds.

**Proposition 6.** Assume (1)–(3). Let  $u \in L^1_{loc}(\mathbb{R}^N)$  be such that  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ ,  $\forall \varepsilon$ .

1. If  $u \in (L^1 + L^\infty)(\mathbb{R}^N)$ , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{B}\dot{V}\text{)}. \quad (34)$$

2. If (33) holds, then (34) holds.

**Open Problem 1.** Let  $u \in L^1_{loc}(\mathbb{R}^N)$  be such that  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ . Is it true that (32), respectively (34), hold, without assuming the support assumption (33)?

*Proof of Propositions 5 and 6.* In view of Lemma 3 item 2 and Propositions 1 and 2, it suffices to prove the following. If  $\ell := \liminf_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} < \infty$ , then  $u \in \dot{W}^{1,p}$  if  $1 < p < \infty$ , respectively  $u \in \dot{B}\dot{V}$  if  $p = 1$ . Clearly, this holds provided

$$\int_{\mathbb{R}^N} u \operatorname{div} \zeta \leq \ell \|\zeta\|_{L^q(\mathbb{R}^N)}, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N), \quad (35)$$

where  $q$  is the conjugate exponent of  $p$ . In turn, (35) holds provided

$$-\int_{\mathbb{R}^N} u \operatorname{div} \zeta = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta, \quad \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N). \quad (36)$$

In order to complete the proof, it suffices to establish (36) under the assumptions of Proposition 5, respectively 6 (with no boundedness assumption on  $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)}$ ).

In view of (30), in order to obtain (36) it suffices to prove that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} u V_{\varepsilon, \zeta_j} \cdot e_j = \int_{\mathbb{R}^N} u \partial_j \zeta_j, \quad 1 \leq j \leq N. \quad (37)$$

When  $u \in L^\infty(\mathbb{R}^N)$ , this follows from Proposition 1 applied to  $\zeta_j$  with  $p = 1$ .

When  $u \in L^1(\mathbb{R}^N)$ , we note the domination

$$|u V_{\varepsilon, \zeta_j} \cdot e_j| \leq \|\nabla \zeta_j\|_{L^\infty(\mathbb{R}^N)} |u| \in L^1(\mathbb{R}^N).$$

We conclude by dominated convergence, using the fact that  $V_{\varepsilon, \zeta_j} \cdot e_j$  converges to  $\partial_j \zeta_j$  pointwise as  $\varepsilon \rightarrow 0$  (see Remark 2).

The argument for  $u \in L^1_{loc}(\mathbb{R}^N)$  under the support condition (33) is similar.

The proof of Propositions 5 and 6 is complete.  $\square$

One may consider versions of Propositions 5 and 6 for families of functions instead of a fixed function. Here are, for example, two versions of [7, Theorem 3.7].

**Proposition 7.** Assume (1)–(3). Let  $1 < p < \infty$ . Let, for every  $\varepsilon$ ,  $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  be such that  $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$ . Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^p(\mathbb{R}^N). \quad (38)$$

1. If  $(u_\varepsilon)$  is bounded in  $(L^1 + L^\infty)(\mathbb{R}^N)$ , then there exists some  $u \in \dot{W}^{1,p}$  such that, up to a subsequence  $\varepsilon_k \rightarrow 0$ ,

$$u_\varepsilon \rightharpoonup u \text{ *weakly in } \mathcal{M}_{loc}(\mathbb{R}^N), \quad (39)$$

$$\|\nabla u\|_{L^p(\Omega)} \leq \liminf_{\varepsilon \rightarrow 0} \|V_{\varepsilon, u_\varepsilon}\|_{L^p(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \quad (40)$$

2. If  $(u_\varepsilon)$  is bounded in  $L^1_{loc}(\mathbb{R}^N)$  and the support condition (33) holds, then (39)–(40) hold.

**Proposition 8.** Assume (1)–(3). Let, for every  $\varepsilon$ ,  $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  be such that  $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$ . Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^1(\mathbb{R}^N). \quad (41)$$

1. If  $(u_\varepsilon)$  is bounded in  $(L^1 + L^\infty)(\mathbb{R}^N)$ , then there exists some  $u \in BV$  such that, up to a subsequence  $\varepsilon_k \rightarrow 0$ ,

$$u_\varepsilon \rightharpoonup u \text{ *weakly in } \mathcal{M}_{loc}(\mathbb{R}^N), \quad (42)$$

$$\|Du\|_{\mathcal{M}(\Omega)} \leq \liminf_{\varepsilon \rightarrow 0} \|V_{\varepsilon, u_\varepsilon}\|_{L^1(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \quad (43)$$

2. If  $(u_\varepsilon)$  is bounded in  $L^1_{loc}(\mathbb{R}^N)$  and the support condition (33) holds, then (42)–(43) hold.

**Remark 3.** Note that, by Lemma 2, (38) (respectively, (41)) holds if  $I_{\varepsilon, p, u_\varepsilon} \leq C < \infty$ ,  $\forall 0 < \varepsilon < \varepsilon_0$  (respectively,  $I_{\varepsilon, 1, u_\varepsilon} \leq C < \infty$ ,  $\forall 0 < \varepsilon < \varepsilon_0$ ).

*Proofs of Propositions 7 and 8.* We present the argument for Proposition 8; the proof of Proposition 7 is similar. Consider a (signed) Radon measure  $\mu$  on  $\mathbb{R}^N$  such that, up to a subsequence,  $u_{\varepsilon_k} \rightharpoonup \mu$  \*weakly in  $\mathcal{M}_{loc}(\mathbb{R}^N)$ . Fix some  $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ . We then have

$$\begin{aligned} \int_{\mathbb{R}^N} \operatorname{div} \zeta \, d\mu &= \lim_k \int_{\mathbb{R}^N} \operatorname{div} \zeta \, u_{\varepsilon_k} \\ &= \lim_k \int_{\mathbb{R}^N} \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] u_{\varepsilon_k} + \lim_k \int_{\mathbb{R}^N} \left( \operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \right) u_{\varepsilon_k} \\ &:= \lim_k A_k + \lim_k B_k. \end{aligned} \quad (44)$$

*Step 1.* We have  $B_k \rightarrow 0$ . We have to treat three cases: (i)  $(u_{\varepsilon_k})$  is bounded in  $L^1(\mathbb{R}^N)$ ; (ii)  $(u_{\varepsilon_k})$  is bounded in  $L^\infty(\mathbb{R}^N)$ ; (iii)  $(u_{\varepsilon_k})$  is bounded in  $L^1_{loc}(\mathbb{R}^N)$  and the support condition (33) holds.

*Step 1.1. Proof in case (i).* We use the fact that, by Remark 2,  $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$  uniformly in  $\mathbb{R}^N$ , together with the boundedness of  $(u_{\varepsilon_k})$  in  $L^1(\mathbb{R}^N)$ .

*Step 1.2. Proof in case (ii).* By Proposition 1, we have  $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$  in  $L^1(\mathbb{R}^N)$ . We combine this fact with the boundedness of  $(u_{\varepsilon_k})$  in  $L^\infty(\mathbb{R}^N)$ .

*Step 1.3. Proof in case (iii).* By Remark 2, we have  $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \rightarrow 0$  uniformly in  $\mathbb{R}^N$ . By the support condition (33), there exists some  $r > 0$  such that, for each  $\varepsilon$ ,  $\operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] = 0$  in  $\mathbb{R}^N \setminus B(0, r)$ . We find that

$$|B_k| \leq \left\| \operatorname{div} \zeta - \sum_j [V_{\varepsilon_k, \zeta_j} \cdot e_j] \right\|_{L^\infty(\mathbb{R}^N)} \|u_{\varepsilon_k}\|_{L^1(B(0, r))} \rightarrow 0 \text{ as } k \rightarrow \infty.$$

*Step 2. Conclusion.* By Lemma 8, we have

$$A_k = - \int_{\mathbb{R}^N} V_{\varepsilon_k, u_{\varepsilon_k}} \cdot \zeta. \quad (45)$$

Let  $\Omega \subset \mathbb{R}^N$  be an open set. Combining (44), Step 1, (45), and the assumption (41), we find that, when  $\zeta \in C_c^\infty(\Omega; \mathbb{R}^N)$ ,

$$\int_{\Omega} \operatorname{div} \zeta \, d\mu \leq \|\zeta\|_{L^\infty(\Omega)} \liminf_{\varepsilon} \|V_{\varepsilon, u_\varepsilon}\|_{L^1(\Omega)}.$$

It follows that  $\mu \in BV$  and that (43) holds.  $\square$

**Remark 4.** In view of [1, Theorem 4] and Ponce [8, Theorem 1.2], it is likely, but not known, that, in Propositions 7 and 8, the boundedness assumptions on  $u_\varepsilon$  can be removed, and that the  $*$ -weak convergence in  $\mathcal{M}_{loc}$  can be improved to strong  $L^p_{loc}$  convergence. In this direction, we formulate below two open questions.

**Open Problem 2.** Let  $1 \leq p < \infty$ . Let, for every  $\varepsilon$ ,  $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  be such that  $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$ . Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^p(\mathbb{R}^N), \quad (46)$$

$$\int_{B(0,1)} u_\varepsilon = 0, \quad \forall \varepsilon. \quad (47)$$

1. Is it true that  $(u_\varepsilon)$  is bounded in  $L^p_{loc}(\mathbb{R}^N)$ , or at least in  $L^1_{loc}(\mathbb{R}^N)$ ?
2. Is it true that  $(u_\varepsilon)$  is relatively compact in  $L^p_{loc}(\mathbb{R}^N)$ , or at least in  $L^1_{loc}(\mathbb{R}^N)$ ?

Note the natural condition (47). Such a “normalization” condition is needed since  $V_\varepsilon$  “does not see constants”; therefore, in order to have *a priori* estimates, one has to “kill the constants”.

### 3 A distributional approach

A natural generalization of the approach in the previous section (based on the identity (30)), consistent with the spirit of the theory of distributions, was initiated in [7]. It consists of taking the identity (30) as a *definition* of  $V_\varepsilon$ . More precisely, instead of assuming that  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ , we assume that (30) holds for every  $\zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$  and some function  $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  that is not, *a priori*, given by (4). This is a *distributional version* of  $V_\varepsilon$  given by (4) and, by the proof of (30), it coincides with  $V_\varepsilon$  provided that  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ . One could even go one step beyond and define the *distribution*  $V_\varepsilon$  through the formula  $V_\varepsilon(\zeta)$ =the right-hand side of (30). (See also, for similar approaches in different but related settings, Shieh and Spector [9], Comi and Stefani [4], Bruè, Calzi, Comi, and Stefani [3].)

Repeating the end of the proof of the Propositions 5 and 6, we obtain, e.g., the following

**Proposition 9.** Assume (1)–(3). Let  $1 < p < \infty$ . If  $u \in (L^1 + L^\infty)(\mathbb{R}^N)$  and there exists some  $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  such that (30) holds,  $\forall \varepsilon, \forall \zeta \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^N)$ , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p}). \quad (48)$$

And the usual variants for  $p = 1$  or  $u \in L^1_{loc}(\mathbb{R}^N)$ , under the support condition (33).

### 4 Heavy tails kernels

The results in this section are in the spirit of [4].

Let  $\rho_\varepsilon$  satisfy (1) and the following variants of (2)–(3):

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| < 1} \rho_\varepsilon(h) dh = 1, \quad (49)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{|h| > \delta} \frac{\rho_\varepsilon(h)}{|h|} dh = 0, \quad \forall \delta > 0. \quad (50)$$

Note that these assumptions are weaker than (2)–(3) and that they allow *heavy tails* kernels, which are not integrable at infinity. Here is a special case, considered, e.g., in [4], of kernels satisfying (49)–(50):

$$\rho_\varepsilon(h) := \frac{2^{1-\varepsilon}}{N \pi^{N/2}} \frac{\Gamma\left(\frac{N-\varepsilon}{2} + 1\right)}{\Gamma\left(\frac{\varepsilon}{2}\right)} \frac{1}{|h|^{N-\varepsilon}}, \quad 0 < \varepsilon < N + 2. \quad (51)$$

Indeed, the validity of (50) is straightforward, while (49) follows from the fact that

$$\frac{2^{1-\varepsilon}}{N \pi^{N/2}} \frac{\Gamma\left(\frac{N-\varepsilon}{2} + 1\right)}{\Gamma\left(\frac{\varepsilon}{2}\right)} \sim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{N \pi^{N/2}} \Gamma\left(\frac{N}{2} + 1\right) = \frac{\varepsilon}{|\mathbb{S}^{N-1}|},$$

combined with the identity

$$\frac{\varepsilon}{|\mathbb{S}^{N-1}|} \int_{|h| < 1} \frac{1}{|h|^{N-\varepsilon}} dh = 1.$$

The results in the previous sections can be easily adapted to kernels satisfying (49)–(50). The price to pay is that the natural function setting is  $W^{1,p}(\mathbb{R}^N)$ , respectively  $BV(\mathbb{R}^N)$ , rather than  $\dot{W}^{1,p}$ , respectively  $\dot{BV}$ . Here are some results from the previous sections adapted to the assumptions (1) and (49)–(50).

**Proposition 10.** Assume (1) and (49)–(50). Let  $1 \leq p < \infty$ . Let  $u \in W^{1,p}(\mathbb{R}^N)$ . Then

$$V_\varepsilon \rightarrow \nabla u \text{ in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0.$$

**Proposition 11.** Assume (1) and (49)–(50). Let  $u \in BV(\mathbb{R}^N)$ . Then

$$V_\varepsilon \rightharpoonup Du \text{ *weakly in } \mathcal{M}(\mathbb{R}^N) \text{ as } \varepsilon \rightarrow 0 \quad (52)$$

and

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)}. \quad (53)$$

In the next two results, we assume that, for some  $1 < q \leq \infty$ , we have

$$\lim_{\varepsilon \rightarrow 0} \|\rho_\varepsilon(h)/|h|\|_{L^q(|h| > 1)} = 0. \quad (54)$$

**Proposition 12.** Assume (1) and (49)–(50). Let  $1 \leq p < \infty$ . Assume that (54) holds when  $q$  is the conjugate exponent of  $p$ . Let  $u \in W^{1,p}(\mathbb{R}^N)$ . Then, for a.e.  $x \in \mathbb{R}^N$ , we have  $V_\varepsilon(x) \rightarrow \nabla u(x)$  as  $\varepsilon \rightarrow 0$ .

**Proposition 13.** Assume (1) and (49)–(50). Assume that (54) holds when  $q = \infty$ . Let  $u \in BV(\mathbb{R}^N)$ . Then, for a.e.  $x \in \mathbb{R}^N$ , we have  $V_\varepsilon(x) \rightarrow \nabla^{ac} u(x)$  as  $\varepsilon \rightarrow 0$ .

**Remark 5.** Given any  $q > 1$ , the kernel in (51) satisfies (54). Therefore, Propositions 12 and 13 apply to these kernels.

**Remark 6.** Propositions 12 and 13 have straightforward versions, in which the assumption on  $u$  is  $u \in L^r(\mathbb{R}^N) \cap \dot{W}^{1,p}$ , respectively  $u \in L^r(\mathbb{R}^N) \cap BV$  for some  $r \in [1, \infty)$  (and then, in (54),  $q$  is the conjugate exponent of  $r$ ).

**Proposition 14.** Assume (1) and (49)–(50). Let  $1 < p < \infty$ . If  $u \in L^p(\mathbb{R}^N)$  and  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  for every  $\varepsilon$ , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}(\mathbb{R}^N)\text{)}. \quad (55)$$

**Proposition 15.** Assume (1) and (49)–(50). If  $u \in L^1(\mathbb{R}^N)$  and  $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  for every  $\varepsilon$ , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N)\text{)}. \quad (56)$$

**Proposition 16.** Assume (1) and (49)–(50). Let  $1 < p < \infty$ . If  $u \in L^p(\mathbb{R}^N)$  and there exists some  $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  such that (30) holds,  $\forall \varepsilon, \forall \zeta \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$ , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}(\mathbb{R}^N)\text{)}. \quad (57)$$

**Proposition 17.** Assume (1) and (49)–(50). If  $u \in L^1(\mathbb{R}^N)$  and there exists some  $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$  such that (30) holds,  $\forall \varepsilon, \forall \zeta \in C^\infty(\mathbb{R}^N; \mathbb{R}^N)$ , then

$$\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathcal{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathcal{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N)\text{)}. \quad (58)$$

We prove only Propositions 10 and 12; the other results are obtained from the corresponding ones in the previous sections using similar arguments.

*Proof of Proposition 10.* Set

$$\rho_\varepsilon^1 := \rho_\varepsilon \chi_{B(0,1)}, \rho_\varepsilon^2 := \rho_\varepsilon - \rho_\varepsilon^1, V_\varepsilon^1 := V_{\varepsilon, u, \rho_\varepsilon^1}, V_\varepsilon^2 := V_{\varepsilon, u, \rho_\varepsilon^2}, \kappa_\varepsilon(h) := \frac{\rho_\varepsilon^2(h)}{|h|}.$$

By Proposition 1 and the assumptions (1) and (49)–(50), we have  $V_\varepsilon^1 \rightarrow \nabla u$  in  $L^p(\mathbb{R}^N)$  as  $\varepsilon \rightarrow 0$ . On the other hand, we have the straightforward inequality

$$|V_\varepsilon^2(x)| \leq |u| * \kappa_\varepsilon(x) + |u(x)| \|\kappa_\varepsilon\|_{L^1(\mathbb{R}^N)}, \quad \forall x \in \mathbb{R}^N. \quad (59)$$

Combining (50), (59) with  $\delta = 1$ , and the Young inequality, we find that  $V_\varepsilon^2 \rightarrow 0$  in  $L^p(\mathbb{R}^N)$  as  $\varepsilon \rightarrow 0$ .  $\square$

*Proof of Proposition 12.* It suffices to note that (by (59) and (54)) we have  $V_\varepsilon^2 \rightarrow 0$  pointwise as  $\varepsilon \rightarrow 0$ .  $\square$

**Remark 7.** The fact that  $\rho$  is radial when  $|h| > 1$  is not relevant for the above results.

## 5 $\Gamma$ -convergence

One can associate  $\Gamma$ -convergence results with the above convergence statements. We present one result of this type, in the spirit of [7, Theorem 1.7].

Let  $1 \leq p < \infty$ . Set, for  $0 < \varepsilon < \varepsilon_0$  and  $u \in L^1_{loc}(\mathbb{R}^N)$ ,

$$J_{\varepsilon,p}(u) := \begin{cases} \|V_{\varepsilon,u}\|_{L^p(\mathbb{R}^N)}, & \text{if } W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N) \\ \infty, & \text{otherwise} \end{cases},$$

$$J_{0,p}(u) := \begin{cases} \|\nabla u\|_{L^p(\mathbb{R}^N)}, & \text{if } u \in \dot{W}^{1,p} \\ \infty, & \text{otherwise} \end{cases}, \quad \forall 1 < p < \infty,$$

$$J_{0,1}(u) := \begin{cases} \|Du\|_{\mathcal{M}(\mathbb{R}^N)}, & \text{if } u \in B\dot{V} \\ \infty, & \text{otherwise} \end{cases}.$$

**Proposition 18.** Assume (1)–(3). Let  $1 \leq p < \infty$ .

1. For  $1 \leq q < \infty$ ,  $J_{\varepsilon,p}$   $\Gamma$ -converges to  $J_{0,p}$  in  $L^q(\mathbb{R}^N)$ .
2. Under the support assumption (33),  $J_{\varepsilon,p}$   $\Gamma$ -converges to  $J_{0,p}$  in  $L^1_{loc}(\mathbb{R}^N)$ .

*Proof.* *Proof of item 2.* Let  $(u_\varepsilon)_{0 < \varepsilon < \varepsilon_0} \subset L^1_{loc}(\mathbb{R}^N)$  be a family such that  $u_\varepsilon \rightarrow u$  in  $L^1_{loc}(\mathbb{R}^N)$  and  $\liminf_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon) < \infty$ . By Proposition 7 and the proof of Propositions 5 and 6 (see, more specifically, (35) and (36)), we find that  $u \in \dot{W}^{1,p}$  if  $1 < p < \infty$  (respectively  $u \in B\dot{V}$  if  $p = 1$ ) and  $J_{0,p}(u) \leq \liminf_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon)$ .

In the opposite direction, we do not need the support assumption (33). Let  $u \in \dot{W}^{1,p}$  if  $1 < p < \infty$ , respectively  $u \in B\dot{V}$  if  $p = 1$ . Let  $\eta$  be a normalized bump function. By Proposition 1 applied to  $u * \eta_{1/j}$ , where  $j \geq 1$  is an integer, there exists a sequence  $(\varepsilon_j)_{j \geq 1}$  such that

$$J_{\varepsilon,p}(u * \eta_{1/j}) \leq J_{0,p}(u * \eta_{1/j}) + \frac{1}{j}, \quad \forall j \geq 1, \quad \forall 0 < \varepsilon < \varepsilon_j.$$

With no loss of generality, we may assume that  $\varepsilon_j \rightarrow 0$  and  $\varepsilon_{j+1} < \varepsilon_j$ . If we set

$$u_\varepsilon := u * \eta_{1/j}, \quad \forall \varepsilon_{j+1} \leq \varepsilon < \varepsilon_j,$$

then  $u_\varepsilon \rightarrow u$  in  $L^1_{loc}(\mathbb{R}^N)$  when  $\varepsilon \rightarrow 0$  and

$$\limsup_{\varepsilon \rightarrow 0} J_{\varepsilon,p}(u_\varepsilon) \leq \limsup_{j \rightarrow \infty} J_{0,p}(u * \eta_{1/j}) = J_{0,p}(u).$$

*Proof of item 1.* The proof is similar to the one of item 2. It suffices to note that Propositions 5, 6, and 7 still hold if we replace  $(L^1 + L^\infty)(\mathbb{R}^N)$  with  $L^q(\mathbb{R}^N)$ .  $\square$

## References

- [1] J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces. In J. L. Menaldi, E. Rofman, and A. Sulem, editors, *Optimal Control and Partial Differential Equations – Innovations and Applications*, pages 439–455, Amsterdam, 2001. IOS Press.
- [2] H. Brezis and H.-M. Nguyen. The BBM formula revisited. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 27(4):515–533, 2016.



- [3] E. Bruè, M. Calzi, G. E. Comi, and G. Stefani. A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics II. *C. R. Math. Acad. Sci. Paris*, 360:589–626, 2022.
- [4] G. E. Comi and G. Stefani. A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up. *J. Funct. Anal.*, 277(10):3373–3435, 2019.
- [5] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. *Math. Models Methods Appl. Sci.*, 23(3):493–540, 2013.
- [6] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.
- [7] T. Mengesha and D. Spector. Localization of nonlocal gradients in various topologies. *Calc. Var.*, 52:253–279, 2015.
- [8] A. C. Ponce. An estimate in the spirit of Poincaré’s inequality. *J. Eur. Math. Soc. (JEMS)*, 6(1):1–15, 2004.
- [9] T.-T. Shieh and D. Spector. On a new class of fractional partial differential equations. *Adv. Calc. Var.*, 8(4):321–336, 2015.
- [10] D. Spector.  $L^p$ -Taylor approximations characterize the Sobolev space  $W^{1,p}$ . *C. R. Math. Acad. Sci. Paris*, 353:327–332, 2015.
- [11] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.
- [12] M. Šilhavý. Fractional vector analysis based on invariance requirements (critique of coordinate approaches). *Contin. Mech. Thermodyn.*, 32(1):207–228, 2020.

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