Non-local approximations of the gradient

Petru Mironescu, Haïm Brezis

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Haïm Brezis[(1)], Petru Mironescu[(2)]

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Abstract

We revisit the proofs of a few basic results concerning non-local approximations of the gradient. A typical such result asserts that, if (ρ_ε) is a radial approximation to the identity in $\mathbb{R}^N$ and u belongs to a homogeneous Sobolev space $\dot{W}^{1,p}$, then

$$V_\varepsilon(x) := N \int\limits_{\mathbb{R}^N} \frac{u(x+h)-u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h)\, dh,\ x \in \mathbb{R}^N,$$

converges in L^p to the distributional gradient ∇u as $\varepsilon \to 0$.

We highlight the crucial role played by the representation formula $V_\varepsilon = (\nabla u) * F_\varepsilon$, where F_ε is an approximation to the identity defined *via* ρ_ε. This formula allows to unify the proofs of a significant number of results in the literature, by reducing them to standard properties of the approximations to the identity.

We also highlight the effectiveness of a symmetric nonlocal integration by parts formula.

Relaxations of the assumptions on u and ρ_ε, allowing, e.g., heavy tails kernels or a distributional definition of V_ε, are also discussed. In particular, we show that heavy tails kernels may be treated as perturbations of approximations to the identity.

1 A representation formula and applications

Let $(\rho_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be a family functions on $\mathbb{R}^N$ such that:

$$\rho_\varepsilon \text{ is non-negative, integrable, radial, } \forall\, \varepsilon, \tag{1}$$

$$\int\limits_{\mathbb{R}^N} \rho_\varepsilon = 1,\ \forall\, \varepsilon, \tag{2}$$

$$\lim_{\varepsilon \to 0} \int\limits_{|h|>\delta} \rho_\varepsilon(h)\, dh = 0,\ \forall\, \delta > 0. \tag{3}$$

Following Mengesha and Spector [7] (with roots in Bourgain, Brezis, and Mironescu [1], Gilboa and Osher [6], Du, Gunzburger, Lehouck, and Zhou [5]; see also a detailed list of references in [7, p. 254]), we set, for any measurable function $u \in L^1_{loc}(\mathbb{R}^N)$, and assuming that the integral below exists,

$$V_\varepsilon(x) = V_{\varepsilon,u}(x) = V_{\varepsilon,u,\rho_\varepsilon}(x) := N \int\limits_{\mathbb{R}^N} \frac{u(x+h)-u(x)}{|h|} \frac{h}{|h|} \rho_\varepsilon(h)\, dh,\ x \in \mathbb{R}^N. \tag{4}$$

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V_ε may be seen as a non-local approximation of the gradient. Indeed (see Remark 2), (i) when u is C^1 and bounded, we have the pointwise convergence $V_\varepsilon(x) \to \nabla u(x)$ as $\varepsilon \to 0$, $\forall\, x \in \mathbb{R}^N$; (ii) when u is C^1 and compactly supported, we have $V_\varepsilon \to \nabla u$ uniformly.

In what follows, we revisit the proofs of a few results establishing the validity of the convergence

$$V_\varepsilon \to Du \text{ as } \varepsilon \to 0, \tag{5}$$

in various functional settings. Many of these results were originally obtained, in slightly different forms, in [7].

Before presenting the main results and methods, we make some easy observations concerning the existence of V_ε. Set

$$W_\varepsilon(x) = W_{\varepsilon,u}(x) = W_{\varepsilon,u,\rho_\varepsilon}(x) := \int\limits_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|}{|h|}\rho_\varepsilon(h)\,dh,\ x \in \mathbb{R}^N.$$

Clearly, the following holds.

Lemma 1. Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. Then V_ε is well-defined a.e. and is measurable. Moreover, we have $|V_\varepsilon| \le N\, W_\varepsilon$ a.e., and thus $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$.

Remark 1. In the above statements, the condition $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$ seems constraining. However, under the following assumption:

$$\text{for every } \varepsilon, \text{ there exist some } \delta_\varepsilon, R_\varepsilon > 0 \text{ such that } \rho_\varepsilon(h) = 0 \text{ if } |h| < \delta_\varepsilon \text{ or if } |h| > R_\varepsilon, \tag{6}$$

we have $W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N)$, $\forall\, u \in L^1_{loc}(\mathbb{R}^N)$.

This is especially relevant for Propositions 5, 6, 7, 8, and 18 below.

We next present a sufficient condition for having $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ (and thus, by Lemma 1, $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$). For $1 \le p < \infty$, set

$$I_{\varepsilon,p} = I_{\varepsilon,p,u} = I_{\varepsilon,p,u,\rho_\varepsilon} := \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^p}\rho_\varepsilon(h)\,dxdh.$$

Lemma 2. Assume (2). Let $1 \le p < \infty$ and $u \in L^1_{loc}(\mathbb{R}^N)$. Then $\|W_\varepsilon\|^p_{L^p(\mathbb{R}^N)} \le I_{\varepsilon,p}$.

Consequently, if $I_{\varepsilon,p} < \infty$, then V_ε is well-defined a.e., measurable, and $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \le N\,[I_{\varepsilon,p}]^{1/p}$.

In the above and in what follows, the L^p-norms of vector fields $F : \mathbb{R}^N \to \mathbb{R}^N$ are computed with respect to the Euclidean norm $|\ |$, i.e.,

$$\|F\|^p_{L^p(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N} |F(x)|^p\,dx.$$

Similarly, the mass of a measure $F \in \mathscr{M}(\mathbb{R}^N;\mathbb{R}^N)$ is computed with respect to the Euclidean norm, i.e.,

$$\|F\|_{\mathscr{M}(\mathbb{R}^N)} = \sup\left\{\sum_{j=1}^N \int\limits_{\mathbb{R}^N} \zeta_j\,dF_j;\ \zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N),\ |\zeta(x)| \le 1,\ \forall\, x\right\}.$$

Proof of Lemma 2. The conclusion follows by integrating in x the inequality

$$[W_\varepsilon(x)]^p \le \int\limits_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^p}\rho_\varepsilon(h)\,dh \left(\int\limits_{\mathbb{R}^N}\rho_\varepsilon(h)\,dh\right)^{p-1}$$
$$= \int\limits_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|^p}{|h|^p}\rho_\varepsilon(h)\,dh,\ \forall\, x\in\mathbb{R}^N.$$

□

We now recall a few sufficient conditions for having $I_{\varepsilon,p}<\infty$. Set

$$\dot{W}^{1,p} := \{u\in\mathscr{D}'(\mathbb{R}^N);\ Du\in L^p(\mathbb{R}^N)\} = \{u\in L^1_{loc}(\mathbb{R}^N);\ Du\in L^p(\mathbb{R}^N)\},\ 1\le p<\infty,$$
$$B\dot{V} := \{u\in\mathscr{D}'(\mathbb{R}^N);\ Du\in\mathscr{M}(\mathbb{R}^N)\} = \{u\in L^1_{loc}(\mathbb{R}^N);\ Du\in\mathscr{M}(\mathbb{R}^N)\}.$$

In what follows, for $u\in\dot{W}^{1,p}$, we denote the distributional gradient ∇u.

Let $K_{p,N} := \fint_{\mathbb{S}^{N-1}} |h_j|^p\,d\sigma(h)$ (which does not depend on $j\in[\![1,N]\!]$).

We have the following

Lemma 3. [1] Assume (1).

1. Let $1\le p<\infty$ and $u\in\dot{W}^{1,p}$. Then $I_{\varepsilon,p}\le K_{p,N}\,\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}\,\|\nabla u\|^p_{L^p(\mathbb{R}^N)}$.

   In particular, $V_\varepsilon\in L^p(\mathbb{R}^N)$ and $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)}\le N\,[K_{p,N}]^{1/p}\,\|\rho_\varepsilon\|^{1/p}_{L^1(\mathbb{R}^N)}\,\|\nabla u\|_{L^p(\mathbb{R}^N)}$.

2. Let $u\in B\dot{V}$. Then $I_{\varepsilon,1}\le K_{1,N}\,\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}\,\|Du\|_{\mathscr{M}(\mathbb{R}^N)}$.

   In particular, $V_\varepsilon\in L^1(\mathbb{R}^N)$ and $\|V_\varepsilon\|_{L^1(\mathbb{R}^N)}\le N\,K_{1,N}\,\|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}\,\|Du\|_{\mathscr{M}(\mathbb{R}^N)}$.

We next present a crucial identity that illuminates the validity of (5)*:* (10)*, and its avatar* (11). Although (10) was probably known to experts (it is implicit in [7, proof of Lemma 3.3] and related to several identities in [5])), its intimate connection to (5) seems to have remained relatively unnoticed.

Assume (1). Let $f_\varepsilon:(0,\infty)\to[0,\infty)$ be a measurable function such that $\rho_\varepsilon(x)=f_\varepsilon(|x|)$ for a.e. $x\in\mathbb{R}^N$. Set

$$F_\varepsilon(h) := N\int\limits_{|h|}^\infty \frac{f_\varepsilon(t)}{t}\,dt,\ \forall\, h\in\mathbb{R}^N\setminus\{0\}. \tag{7}$$

Let us note that

$$\|F_\varepsilon\|_{L^1(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N} F_\varepsilon = N\,|\mathbb{S}^{N-1}|\int\limits_0^\infty r^{N-1}\int\limits_r^\infty \frac{f_\varepsilon(t)}{t}\,dtdr = |\mathbb{S}^{N-1}|\int\limits_0^\infty t^{N-1}\,f_\varepsilon(t)\,dt$$
$$= \int\limits_{\mathbb{R}^N}\rho_\varepsilon(h)\,dh = \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)}, \tag{8}$$

so that, in particular,

$$F_\varepsilon\in L^1(\mathbb{R}^N). \tag{9}$$

Lemma 4. Assume (1). Let F_ε be as in (7).

1. Let $1 \le p < \infty$ and $u \in \dot{W}^{1,p}$. Then

$$V_\varepsilon = (\nabla u) * F_\varepsilon \text{ a.e.} \tag{10}$$

2. Let $u \in B\dot{V}$. Then

$$V_\varepsilon = (Du) * F_\varepsilon \text{ a.e.} \tag{11}$$

Proof. Step 1. Proof of (10) *when* $u \in C^\infty(\mathbb{R}^N)$ *and* $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$. In this case, we actually prove that

$$V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x),\ \forall\, x \in \mathbb{R}^N.$$

For this purpose, we note that

$$F_\varepsilon \text{ is compactly supported,} \tag{12}$$

$$\nabla F_\varepsilon(h) = -N \frac{h}{|h|^2} \rho_\varepsilon(h),\ \forall\, h \in \mathbb{R}^N \setminus \{0\}, \tag{13}$$

$$F_\varepsilon(h) = O(|\ln |h||) \text{ as } h \to 0. \tag{14}$$

Using (12)–(14), we find, via an integration by parts, that

$$\begin{aligned} V_\varepsilon(x) &= -\int_{\mathbb{R}^N} [u(x+h) - u(x)]\, \nabla F_\varepsilon(h)\, dh = -\lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus \overline{B}(0,\delta)} [u(x+h) - u(x)]\, \nabla F_\varepsilon(h)\, dh \\ &= \int_{\mathbb{R}^N} \nabla u(x+h)\, F_\varepsilon(h)\, dh = \int_{\mathbb{R}^N} \nabla u(x-h)\, F_\varepsilon(h)\, dh \\ &= [(\nabla u) * F_\varepsilon](x),\ \forall\, x \in \mathbb{R}^N. \end{aligned}$$

Step 2. Proof of (10) *and* (11) *when* $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. Let η be a radial non-increasing normalized bump function. By Step 1, we have

$$V_{\varepsilon, u*\eta_\delta}(x) = (\nabla(u * \eta_\delta)) * F_\varepsilon(x) = (Du) * (F_\varepsilon * \eta_\delta)(x),\ \forall\, x \in \mathbb{R}^N. \tag{15}$$

On the one hand, as $\delta \to 0$, the right-hand side of (15) converges (possibly along a subsequence) a.e. to $(Du) * F_\varepsilon(x)$. (This follows by combining the Young inequality with the fact that $F_\varepsilon * \eta_\delta \to F_\varepsilon$ in L^1.)
In order to obtain (10), respectively (11), it suffices to prove that

$$V_{\varepsilon, u*\eta_\delta, \rho_\varepsilon}(x) \to V_{\varepsilon, u, \rho_\varepsilon}(x) \text{ as } \delta \to 0 \text{ for a.e. } x \in \mathbb{R}^N. \tag{16}$$

Property (16) is obtained *via* dominated convergence, using the standard inequality

$$|u * \eta_\delta(y)| \le \mathscr{M}_1 u(y), \forall\, 0 < \delta < 1,\ \forall\, y \in \mathbb{R}^N, \tag{17}$$

(see, e.g., [11, eq (17), p. 57]), where $\mathscr{M}_1 u$ is the centered truncated maximal function of u,

$$\mathscr{M}_1 u(x) := \sup \left\{ \fint_{B_r(x)} |u|;\ 0 < r \le 1 \right\}.$$

(Here, we use the fact that η is radial, non-increasing, and supported in the unit ball.) Using (17) and the extra assumptions on ρ_ε, we obtain the domination

$$\frac{|u * \eta_\delta(x+h) - u * \eta_\delta(x)|}{|h|} \rho_\varepsilon(h) \le [\mathscr{M}_1 u(x+h) + \mathscr{M}_1 u(x)]\, g(h),\ \forall\, 0 < \delta < 1, \tag{18}$$

with $g(h) := \rho_\varepsilon(h)/|h|$ bounded and compactly supported. The right-hand side of (18) is in $L^1(\mathbb{R}^N)$ since $\mathscr{M}_1 u \in L^1_{loc}(\mathbb{R}^N)$ (and thus, in particular, $\mathscr{M}_1 u$ is finite a.e.). The latter property follows by combining the Sobolev embeddings $\dot{W}^{1,p}$, $B\dot{V} \hookrightarrow L^{N/(N-1)}_{loc}(\mathbb{R}^N)$ with the fact that, by the maximal function theorem, we have $\mathscr{M}_1 u \in L^r_{loc}(\mathbb{R}^N)$ when $u \in L^r_{loc}(\mathbb{R}^N)$ for some $r > 1$. We obtain that the convergence in (16) holds on the full measure set

$$\{x \in \mathbb{R}^N;\ \mathscr{M}_1 u(x) < \infty \text{ and } x \text{ is a Lebesgue point of } u\}.$$

Step 3. Proof of (10) *and* (11) *in the general case.* For fixed ε, we approximate ρ_ε in L^1 with a sequence $(\rho_{\varepsilon,j})_j$ of kernels $\rho_{\varepsilon,j} \in C^\infty_c(\mathbb{R}^N \setminus \{0\})$ satisfying (1). By Step 2, the corresponding associated kernels $F_{\varepsilon,j}$ satisfy

$$V_{\varepsilon,u,\rho_{\varepsilon,j}} = (Du) * F_{\varepsilon,j} \text{ a.e.} \tag{19}$$

Let us note that $F_{\varepsilon,j} \to F_\varepsilon$ in $L^1(\mathbb{R}^N)$ as $j \to \infty$. (This follows from a straightforward variant of (8).)

We obtain (10), respectively (11), by letting $j \to \infty$ in (19). Passing to the limits is justified, on the left-hand side, by Lemma 3, and, on the right-hand side, by the Young inequality combined with the fact that $F_{\varepsilon,j} \to F_\varepsilon$ in $L^1(\mathbb{R}^N)$. □

Using Lemma 4, (8), and the Young inequality, we obtain the following

Lemma 5. Assume (1)–(2).

1. Let $1 \le p < \infty$ and $u \in \dot{W}^{1,p}$. Then $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)} \le \|\nabla u\|_{L^p(\mathbb{R}^N)}$.
2. Let $u \in B\dot{V}$. Then $\|V_\varepsilon\|_{L^1(\mathbb{R}^N)} \le \|Du\|_{\mathscr{M}(\mathbb{R}^N)}$.

This is an improvement of Lemma 3, since $N\,[K_{p,N}]^{1/p} > 1$ when $N \ge 2$. Indeed, the Jensen inequality yields

$$K_{p,N} \ge \left(\fint_{\mathbb{S}^{N-1}} |h_j|\, d\sigma(h) \right)^p = \left(\frac{1}{N} \fint_{\mathbb{S}^{N-1}} \sum_{k=1}^N |h_k|\, d\sigma(h) \right)^p > \left(\frac{1}{N} \fint_{\mathbb{S}^{N-1}} d\sigma(h) \right)^p = \frac{1}{N^p}.$$

We next present two direct consequences of Lemma 4, originally obtained, with different arguments, in [7].

Proposition 1. [7, Theorem 1.1 (b)] Assume (1)–(3). Let $u \in \dot{W}^{1,p}(\mathbb{R}^N)$. Then

$$V_\varepsilon \to \nabla u \text{ in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \to 0.$$

Proposition 2. [7, Theorem 1.2] Assume (1)–(3). Let $u \in B\dot{V}$. Then

$$V_\varepsilon \rightharpoonup Du \ *\text{-weakly in } \mathscr{M}(\mathbb{R}^N) \text{ as } \varepsilon \to 0 \tag{20}$$

and

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)}.$$

Proof of Propositions 1 and 2. By Lemma 6 below, (F_ε) is an approximation to the identity. We conclude by combining this fact with Lemma 4. □

Lemma 6. Under the assumptions (1)–(3), (F_ε) is an approximation to the identity.

Proof. If $\delta > 0$ is fixed, then (7) and (3) yield

$$\begin{aligned}\int\limits_{|h|>\delta} F_\varepsilon(h)\,dh &= N\,|\mathbb{S}^{N-1}|\int\limits_\delta^\infty r^{N-1}\int\limits_r^\infty \frac{f_\varepsilon(t)}{t}\,dtdr\\ &= N\,|\mathbb{S}^{N-1}|\int\limits_\delta^\infty\int\limits_\delta^t r^{N-1}\,dr\frac{f_\varepsilon(t)}{t}\,dt \le |\mathbb{S}^{N-1}|\int\limits_\delta^\infty t^{N-1}\,f_\varepsilon(t)\,dt \qquad (21)\\ &= \int\limits_{|h|>\delta}\rho_\varepsilon(h)\,dh \to 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

The conclusion of the lemma follows from (8) and (21). □

We next present two a.e. versions of the above results.

Proposition 3. Assume (1)–(3). Let $u \in \dot{W}^{1,p}$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \to \nabla u(x)$ as $\varepsilon \to 0$.

Proposition 4. Assume (1)–(3). Let $u \in B\dot{V}$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \to \nabla^{ac}u(x)$ as $\varepsilon \to 0$.

These results clearly follow from Lemmas 4 and 6 and the following measure-theoretical

Lemma 7. Let (F_ε) be a an approximation to the identity, with F_ε radial and non-increasing. Then

1. For every $1 \le p < \infty$ and $G \in L^p(\mathbb{R}^N)$, we have $G * F_\varepsilon(x) \to G(x)$ at each Lebesgue point of G.
2. For every finite Borel measure ν singular with respect to the Lebesgue measure in $\mathbb{R}^N$, we have $\nu * F_\varepsilon \to 0$ a.e.

Proof of Lemma 7. The assumptions on F_ε imply that there exist (unique) non-negative Borel measures μ_ε on $(0,\infty)$ such that

$$F_\varepsilon(x) = \mu_\varepsilon((|x|,\infty)), \text{ for a.e. } x \in \mathbb{R}^N, \qquad (22)$$

$$\frac{|\mathbb{S}^{N-1}|}{N}\int\limits_0^\infty t^N\,d\mu_\varepsilon(t) = 1,\ \forall\,\varepsilon, \qquad (23)$$

$$\lim_{\varepsilon\to 0}\frac{|\mathbb{S}^{N-1}|}{N}\int\limits_\delta^\infty t^N\,d\mu_\varepsilon(t) = 1,\ \forall\,\delta > 0. \qquad (24)$$

Proof of item 1. We have

$$\begin{aligned}|G * F_\varepsilon(x) - G(x)| &= \left|\int\limits_0^\infty r^{N-1}\int\limits_{\mathbb{S}^{N-1}}[G(x-r\omega)-G(x)]\,ds(\omega)\,\mu_\varepsilon((r,\infty))\,dr\right|\\ &\le \int\limits_0^\infty r^{N-1}\int\limits_{\mathbb{S}^{N-1}}|G(x-r\omega)-G(x)|\,ds(\omega)\,\mu_\varepsilon((r,\infty))\,dr\\ &= \int\limits_0^\infty r^{N-1}\int\limits_{\mathbb{S}^{N-1}}|G(x-r\omega)-G(x)|\,ds(\omega)\int\limits_r^\infty d\mu_\varepsilon(t)\,dr\\ &= \int\limits_0^\infty\int\limits_0^t r^{N-1}\int\limits_{\mathbb{S}^{N-1}}|G(x-r\omega)-G(x)|\,ds(\omega)\,dr\,d\mu_\varepsilon(t) \qquad (25)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_{B(x,t)} |G(y) - G(x)|\, dy\, d\mu_\varepsilon(t) \\
&= \int_0^\infty |B(x,t)| \fint_{B(x,t)} |G(y) - G(x)|\, dy\, d\mu_\varepsilon(t) \\
&= \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \fint_{B(x,t)} |G(y) - G(x)|\, dy\, d\mu_\varepsilon(t).
\end{aligned}
$$

We complete the proof by combining (23)–(25) with the fact that

$$
\lim_{t\to 0} \fint_{B(x,t)} |G(y) - G(x)|\, dy = 0 \text{ at each Lebesgue point } x \text{ of } G
$$

and the straightforward inequality

$$
\begin{aligned}
\fint_{B(x,t)} |G(y) - G(x)|\, dy &\le \frac{1}{|B(x,t)|^{1/p}} \|G\|_{L^p(\mathbb{R}^N)} + |G(x)| \\
&\le \frac{1}{|B(x,\delta)|^{1/p}} \|G\|_{L^p(\mathbb{R}^N)} + |G(x)|,\ \forall\, t \ge \delta.
\end{aligned}
$$

Proof of item 2. We have

$$
\begin{aligned}
\nu * F_\varepsilon(x) &= \int_{\mathbb{R}^N} \mu_\varepsilon((|x-y|,\infty))\, d\nu(y) = \int_{\mathbb{R}^N} \int_{|x-y|}^\infty d\mu_\varepsilon(t)\, d\nu(y) \\
&= \int_0^\infty \int_{|x-y|<t} d\nu(y)\, d\mu_\varepsilon(t) = \frac{|\mathbb{S}^{N-1}|}{N} \int_0^\infty t^N \frac{\nu(B(x,t))}{|B(x,t)|}\, d\mu_\varepsilon(t).
\end{aligned} \tag{26}
$$

We complete the proof combining (23), (24), and (26) with the fact that (by the Lebesgue-Besicovitch differentiation theorem) we have

$$
\lim_{t\to 0} \frac{\nu(B(x,t))}{|B(x,t)|} = 0 \text{ for a.e. } x \in \mathbb{R}^N,
$$

and the upper bound

$$
\frac{\nu(B(x,t))}{|B(x,t)|} \le \frac{\nu(\mathbb{R}^N)}{|B(x,\delta)|},\ \forall\, t \ge \delta. \qquad \square
$$

For other results in the spirit of Propositions 3 and 4, see Spector [10, Theorem 1.2] and Brezis and Nguyen [2, Theorems 1 and 2].

Remark 2. Here are two additional quick consequences of the fact that, under the assumptions (1)–(3), (F_ε) is an approximation to the identity. It is straightforward that the pointwise equality $V_\varepsilon(x) = (\nabla u) * F_\varepsilon(x)$, $\forall\, x \in \mathbb{R}^N$, holds if $u \in C_c^1(\mathbb{R}^N)$, and therefore for such u we have $V_\varepsilon \to \nabla u$ uniformly in $\mathbb{R}^N$ as $\varepsilon \to 0$. Similarly, this equality holds when $u \in (C^1 \cap L^\infty)(\mathbb{R}^N)$, and in this case we have $V_\varepsilon(x) \to \nabla u(x)$, as $\varepsilon \to 0$, $\forall\, x \in \mathbb{R}^N$.

2 An integration by parts formula and applications

The representation formula (10) naturally leads to the following formal calculation, with $\zeta \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$, $\check{f}(x) := f(-x)$, $\forall\, x \in \mathbb{R}^N$, and $\{e_j\}_{1\le j\le N}$ the canonical basis of $\mathbb{R}^N$:

$$\begin{aligned}\int_{\mathbb{R}^N} V_{\varepsilon,u}\cdot\zeta &= \sum_j \int_{\mathbb{R}^N} [V_{\varepsilon,u}\cdot e_j]\zeta_j = \sum_j [V_{\varepsilon,u}\cdot e_j] * \check{\zeta}_j(0) = \sum_j [(\partial_j u) * F_\varepsilon] * \check{\zeta}_j(0)\\ &= \sum_j u * \left[F_\varepsilon * \partial_j\check{\zeta}_j\right](0) = \sum_j u * V_{\varepsilon,\check{\zeta}_j}\cdot e_j(0)\\ &= \int_{\mathbb{R}^N} u(x)\sum_j [V_{\varepsilon,\check{\zeta}_j}\cdot e_j](-x).\end{aligned} \tag{27}$$

Combining (27) with the (formal) identity $V_{\varepsilon,\check{f}}(-x) = -V_{\varepsilon,f}(x)$, we obtain the formal identity

$$\int_{\mathbb{R}^N} V_{\varepsilon,u}\cdot\zeta = -\sum_{j=1}^N \int_{\mathbb{R}^N} u(x)\,[V_{\varepsilon,\zeta_j}(x)\cdot e_j]\,dx, \tag{28}$$

and its more symmetric avatar

$$\int_{\mathbb{R}^N} [V_{\varepsilon,u}\cdot e_j]\psi = -\int_{\mathbb{R}^N} u(x)\,[V_{\varepsilon,\psi}(x)\cdot e_j]\,dx, \forall\, j,\ \forall\,\psi\in C_c^\infty(\mathbb{R}^N;\mathbb{R}). \tag{29}$$

Similar "nonlocal integration by parts" identities were known in the literature (see, e.g., [5], [7, Theorem 1.4], and, in a slightly different setting, Šilhavý [12, Section 6]). As we will see below, (28) holds under mild assumptions on u (this is to be compared with the more restrictive assumptions in Lemma 4). The importance of such identities is that they provide a first direction for generalizing the results in Section 1, consisting of weakening the *assumption* $u \in \dot{W}^{1,p}$ (respectively $u \in B\dot{V}$), widely used in Section 1, to a reasonable one allowing V_ε to be well-defined a.e. and to obtain the *property* $u \in \dot{W}^{1,p}$ (respectively $u \in B\dot{V}$) *as a conclusion*.

We first formalize the validity of (28).

Lemma 8. Let $\varepsilon > 0$ be fixed. Assume (1). Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$.

1. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then
$$\int_{\mathbb{R}^N} V_{\varepsilon,u}\cdot\zeta = -\sum_{j=1}^N \int_{\mathbb{R}^N} u(x)\,[V_{\varepsilon,\zeta_j}(x)\cdot e_j]\,dx,\ \forall\,\zeta\in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N). \tag{30}$$
2. If ρ_ε is compactly supported, then (30) holds.

Proof. We first note the following equalities, valid (thanks to the Fubini theorem applied to the first line) for every $\zeta \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$:

$$\begin{aligned}\int\limits_{\mathbb{R}^N} V_\varepsilon\cdot\zeta &= N\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{u(x+h)-u(x)}{|h|}\frac{h\cdot\zeta(x)}{|h|}\rho_\varepsilon(h)\,dh\right)dx\\ &= N\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{u(x+h)-u(x)}{|h|}\frac{h\cdot\zeta(x)}{|h|}\rho_\varepsilon(h)\,dx\right)dh\\ &= N\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}u(x)\frac{h\cdot[\zeta(x-h)-\zeta(x)]}{|h|^2}\rho_\varepsilon(h)\,dx\right)dh\\ &= -N\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}u(x)\frac{h\cdot[\zeta(x+h)-\zeta(x)]}{|h|^2}\rho_\varepsilon(h)\,dx\right)dh.\end{aligned} \tag{31}$$

We next claim that we may apply the Fubini theorem to the last integral in (31). By linearity, in item 1 we may assume that either $u \in L^1(\mathbb{R}^N)$ or $u \in L^\infty(\mathbb{R}^N)$.

Proof of the claim when $u \in L^1(\mathbb{R}^N)$. In this case, we have

$$\int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} |u(x)| \frac{|h \cdot [\zeta(x+h) - \zeta(x)]|}{|h|^2} \rho_\varepsilon(h)\, dx \right) dh \leq \|\nabla \zeta\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty.$$

Proof of the claim when $u \in L^\infty(\mathbb{R}^N)$. By Lemma 3 item 1, we have

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} |u(x)| \frac{|h \cdot [\zeta(x+h) - \zeta(x)]|}{|h|^2} \rho_\varepsilon(h)\, dx \right) dh \\ &\quad \leq K(1,N) \|u\|_{L^\infty(\mathbb{R}^N)} \sum_{j=1}^N \|\nabla \zeta_j\|_{L^1(\mathbb{R}^N)} \|\rho_\varepsilon\|_{L^1(\mathbb{R}^N)} < \infty. \end{aligned}$$

Proof of the claim when ρ_ε *is compactly supported.* Let $r, R > 0$ be such that $\operatorname{supp} \zeta \subset B(0,r)$ and $\operatorname{supp} \rho_\varepsilon \subset B(0,R)$. Set $v := u \chi_{B(0,r+R)} \in L^1(\mathbb{R}^N)$. Then

$$u(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h) = v(x) \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h),\ \forall\, x, h \in \mathbb{R}^N,$$

and we then argue as in the case where $u \in L^1(\mathbb{R}^N)$.

Applying the Fubini theorem in (31), we find that

$$\begin{aligned} \int\limits_{\mathbb{R}^N} V_\varepsilon \cdot \zeta &= -N \int\limits_{\mathbb{R}^N} u(x) \left(\int\limits_{\mathbb{R}^N} \frac{h \cdot [\zeta(x+h) - \zeta(x)]}{|h|^2} \rho_\varepsilon(h)\, dh \right) dx \\ &= -N \int\limits_{\mathbb{R}^N} u(x) \left(\int\limits_{\mathbb{R}^N} \sum_{j=1}^N \frac{\zeta_j(x+h) - \zeta_j(x)}{|h|} \frac{h_j}{|h|} \rho_\varepsilon(h)\, dh \right) dx \\ &= -\sum_{j=1}^N \int\limits_{\mathbb{R}^N} u(x) \left[V_{\varepsilon, \zeta_j}(x) \cdot e_j \right] dx, \end{aligned}$$

so that (30) holds. □

Here are two quick consequences of (30), in the spirit of [7, Theorems 1.5 and 1.6].

Proposition 5. Assume (1)–(3). Let $1 < p < \infty$. Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N), \forall\, \varepsilon$.

1. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p}). \tag{32}$$

2. If there exists some $R < \infty$ such that

$$\operatorname{supp} \rho_\varepsilon \subset B(0,R),\ \forall\, 0 < \varepsilon < \varepsilon_0, \tag{33}$$

then (32) holds.

Proposition 6. Assume (1)–(3). Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N), \forall\, \varepsilon$.

1. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$, then

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathscr{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin B\dot{V}). \tag{34}$$

2. If (33) holds, then (34) holds.

Open Problem 1. Let $u \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. Is it true that (32), respectively (34), hold, *without assuming the support assumption* (33)?

Proof of Propositions 5 and 6. In view of Lemma 3 item 2 and Propositions 1 and 2, it suffices to prove the following. If $\ell := \liminf_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} < \infty$, then $u \in \dot{W}^{1,p}$ if $1 < p < \infty$, respectively $u \in B\dot{V}$ if $p = 1$. Clearly, this holds provided

$$\int_{\mathbb{R}^N} u \, \operatorname{div} \zeta \le \ell \, \|\zeta\|_{L^q(\mathbb{R}^N)}, \ \forall \, \zeta \in C^\infty_c(\mathbb{R}^N; \mathbb{R}^N), \tag{35}$$

where q is the conjugate exponent of p. In turn, (35) holds provided

$$-\int_{\mathbb{R}^N} u \, \operatorname{div} \zeta = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} V_\varepsilon \cdot \zeta, \ \forall \, \zeta \in C^\infty_c(\mathbb{R}^N; \mathbb{R}^N). \tag{36}$$

In order to complete the proof, it suffices to establish (36) under the assumptions of Proposition 5, respectively 6 (with no boundedness assumption on $\|V_\varepsilon\|_{L^p(\mathbb{R}^N)}$).

In view of (30), in order to obtain (36) it suffices to prove that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} u \, V_{\varepsilon, \zeta_j} \cdot e_j = \int_{\mathbb{R}^N} u \, \partial_j \zeta_j, \ 1 \le j \le N. \tag{37}$$

When $u \in L^\infty(\mathbb{R}^N)$, this follows from Proposition 1 applied to ζ_j with $p = 1$.

When $u \in L^1(\mathbb{R}^N)$, we note the domination

$$|u \, V_{\varepsilon, \zeta_j} \cdot e_j| \le \|\nabla \zeta_j\|_{L^\infty(\mathbb{R}^N)} \, |u| \in L^1(\mathbb{R}^N).$$

We conclude by dominated convergence, using the fact that $V_{\varepsilon, \zeta_j} \cdot e_j$ converges to $\partial_j \zeta_j$ pointwise as $\varepsilon \to 0$ (see Remark 2).

The argument for $u \in L^1_{loc}(\mathbb{R}^N)$ under the support condition (33) is similar.

The proof of Propositions 5 and 6 is complete. □

One may consider versions of Propositions 5 and 6 for families of functions instead of a fixed function. Here are, for example, two versions of [7, Theorem 3.7].

Proposition 7. Assume (1)–(3). Let $1 < p < \infty$. Let, for every ε, $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon, u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon, u_\varepsilon}) \text{ is bounded in } L^p(\mathbb{R}^N). \tag{38}$$

1. If (u_ε) is bounded in $(L^1 + L^\infty)(\mathbb{R}^N)$, then there exists some $u \in \dot{W}^{1,p}$ such that, up to a subsequence $\varepsilon_k \to 0$,

$$u_\varepsilon \rightharpoonup u \ *\text{-weakly in } \mathscr{M}_{loc}(\mathbb{R}^N), \tag{39}$$

$$\|\nabla u\|_{L^p(\Omega)} \le \liminf_{\varepsilon \to 0} \|V_{\varepsilon, u_\varepsilon}\|_{L^p(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \tag{40}$$

2. If (u_ε) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (33) holds, then (39)–(40) hold.

Proposition 8. Assume (1)–(3). Let, for every ε, $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon,u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon,u_\varepsilon}) \text{ is bounded in } L^1(\mathbb{R}^N). \tag{41}$$

1. If (u_ε) is bounded in $(L^1 + L^\infty)(\mathbb{R}^N)$, then there exists some $u \in \dot{BV}$ such that, up to a subsequence $\varepsilon_k \to 0$,

$$u_\varepsilon \rightharpoonup u \text{ *-weakly in } \mathscr{M}_{loc}(\mathbb{R}^N), \tag{42}$$

$$\|Du\|_{\mathscr{M}(\Omega)} \le \liminf_{\varepsilon\to 0} \|V_{\varepsilon,u_\varepsilon}\|_{L^1(\Omega)}, \text{ for every open set } \Omega \subset \mathbb{R}^N. \tag{43}$$

2. If (u_ε) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (33) holds, then (42)–(43) hold.

Remark 3. Note that, by Lemma 2, (38) (respectively, (41)) holds if $I_{\varepsilon,p,u_\varepsilon} \le C < \infty, \forall\, 0 < \varepsilon < \varepsilon_0$ (respectively, $I_{\varepsilon,1,u_\varepsilon} \le C < \infty, \forall\, 0 < \varepsilon < \varepsilon_0$).

Proofs of Propositions 7 and 8. We present the argument for Proposition 8; the proof of Proposition 7 is similar. Consider a (signed) Radon measure μ on $\mathbb{R}^N$ such that, up to a subsequence, $u_{\varepsilon_k} \rightharpoonup \mu$ *-weakly in $\mathscr{M}_{loc}(\mathbb{R}^N)$. Fix some $\zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$. We then have

$$\begin{aligned}\int_{\mathbb{R}^N} \operatorname{div}\zeta\, d\mu &= \lim_k \int_{\mathbb{R}^N} \operatorname{div}\zeta\, u_{\varepsilon_k} \\ &= \lim_k \int_{\mathbb{R}^N} \sum_j [V_{\varepsilon_k,\zeta_j}\cdot e_j]\, u_{\varepsilon_k} + \lim_k \int_{\mathbb{R}^N} \left(\operatorname{div}\zeta - \sum_j [V_{\varepsilon_k,\zeta_j}\cdot e_j]\right) u_{\varepsilon_k} \\ &:= \lim_k A_k + \lim_k B_k.\end{aligned} \tag{44}$$

Step 1. We have $B_k \to 0$. We have to treat three cases: (i) (u_{ε_k}) is bounded in $L^1(\mathbb{R}^N)$; (ii) (u_{ε_k}) is bounded in $L^\infty(\mathbb{R}^N)$; (iii) (u_{ε_k}) is bounded in $L^1_{loc}(\mathbb{R}^N)$ and the support condition (33) holds.

Step 1.1. Proof in case (i). We use the fact that, by Remark 2, $\operatorname{div}\zeta - \sum_j [V_{\varepsilon_k,\zeta_j}\cdot e_j] \to 0$ uniformly in $\mathbb{R}^N$, together with the boundedness of (u_{ε_k}) in $L^1(\mathbb{R}^N)$.

Step 1.2. Proof in case (ii). By Proposition 1, we have $\operatorname{div}\zeta - \sum_j [V_{\varepsilon_k,\zeta_j}\cdot e_j] \to 0$ in $L^1(\mathbb{R}^N)$. We combine this fact with the boundedness of (u_{ε_k}) in $L^\infty(\mathbb{R}^N)$.

Step 1.3. Proof in case (iii). By Remark 2, we have $\operatorname{div}\zeta - \sum_j [V_{\varepsilon_k,\zeta_j}\cdot e_j] \to 0$ uniformly in $\mathbb{R}^N$. By the support condition (33), there exists some $r > 0$ such that, for each ε, $\operatorname{div}\zeta - \sum_j [V_{\varepsilon_k,\zeta_j}\cdot e_j] = 0$ in $\mathbb{R}^N \setminus B(0,r)$. We find that

$$|B_k| \le \left\| \operatorname{div}\zeta - \sum_j [V_{\varepsilon_k,\zeta_j}\cdot e_j] \right\|_{L^\infty(\mathbb{R}^N)} \|u_{\varepsilon_k}\|_{L^1(B(0,r))} \to 0 \text{ as } k\to\infty.$$

Step 2. Conclusion. By Lemma 8, we have

$$A_k = -\int_{\mathbb{R}^N} V_{\varepsilon_k,u_{\varepsilon_k}}\cdot\zeta. \tag{45}$$

Let $\Omega \subset \mathbb{R}^N$ be an open set. Combining (44), Step 1, (45), and the assumption (41), we find that, when $\zeta \in C^\infty_c(\Omega;\mathbb{R}^N)$,

$$\int_\Omega \operatorname{div}\zeta\, d\mu \le \|\zeta\|_{L^\infty(\Omega)} \liminf_\varepsilon \|V_{\varepsilon,u_\varepsilon}\|_{L^1(\Omega)}.$$

It follows that $\mu \in \dot{BV}$ and that (43) holds. □

Remark 4. In view of [1, Theorem 4] and Ponce [8, Theorem 1.2], it is likely, but not known, that, in Propositions 7 and 8, the boundedness assumptions on u_ε can be removed, and that the $*$-weak convergence in $\mathscr{M}_{loc}$ can be improved to strong L^p_{loc} convergence. In this direction, we formulate below two open questions.

Open Problem 2. Let $1 \leq p < \infty$. Let, for every ε, $u_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ be such that $W_{\varepsilon,u_\varepsilon} \in L^1_{loc}(\mathbb{R}^N)$. Assume that

$$(V_{\varepsilon,u_\varepsilon}) \text{ is bounded in } L^p(\mathbb{R}^N), \tag{46}$$

$$\int_{B(0,1)} u_\varepsilon = 0, \ \forall\, \varepsilon. \tag{47}$$

1. Is it true that (u_ε) is bounded in $L^p_{loc}(\mathbb{R}^N)$, or at least in $L^1_{loc}(\mathbb{R}^N)$?
2. Is it true that (u_ε) is relatively compact in $L^p_{loc}(\mathbb{R}^N)$, or at least in $L^1_{loc}(\mathbb{R}^N)$?

Note the natural condition (47). Such a "normalization" condition is needed since V_ε "does not see constants"; therefore, in order to have *a priori* estimates, one has to "kill the constants".

3 A distributional approach

A natural generalization of the approach in the previous section (based on the identity (30)), consistent with the spirit of the theory of distributions, was initiated in [7]. It consists of taking the *identity* (30) as a *definition* of V_ε. More precisely, instead of assuming that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$, we assume that (30) holds for every $\zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$ and some function $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ *that is not, a priori, given by* (4). This is a *distributional version* of V_ε given by (4) and, by the proof of (30), it coincides with V_ε provided that $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$. One could even go one step beyond and define the *distribution* V_ε through the formula $V_\varepsilon(\zeta)$=the right-hand side of (30). (See also, for similar approaches in different but related settings, Shieh and Spector [9], Comi and Stefani [4], Bruè, Calzi, Comi, and Stefani [3].)

Repeating the end of the proof of the Propositions 5 and 6, we obtain, e.g., the following

Proposition 9. Assume (1)–(3). Let $1 < p < \infty$. If $u \in (L^1 + L^\infty)(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (30) holds, $\forall\, \varepsilon, \forall\, \zeta \in C^\infty_c(\mathbb{R}^N;\mathbb{R}^N)$, then

$$\lim_{\varepsilon\to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin \dot{W}^{1,p}). \tag{48}$$

And the usual variants for $p = 1$ or $u \in L^1_{loc}(\mathbb{R}^N)$, under the support condition (33).

4 Heavy tails kernels

The results in this section are in the spirit of [4].

Let ρ_ε satisfy (1) and the following variants of (2)–(3):

$$\lim_{\varepsilon\to 0} \int\limits_{|h|<1} \rho_\varepsilon(h)\, dh = 1, \tag{49}$$

$$\lim_{\varepsilon\to 0} \int\limits_{|h|>\delta} \frac{\rho_\varepsilon(h)}{|h|}\, dh = 0, \ \forall\, \delta > 0. \tag{50}$$

Note that these assumptions are weaker than (2)–(3) and that they allow *heavy tails* kernels, which are not integrable at infinity. Here is a special case, considered, e.g., in [4], of kernels satisfying (49)–(50):

$$\rho_\varepsilon(h) := \frac{2^{1-\varepsilon}}{N\,\pi^{N/2}} \frac{\Gamma\left(\dfrac{N-\varepsilon}{2}+1\right)}{\Gamma\left(\dfrac{\varepsilon}{2}\right)} \frac{1}{|h|^{N-\varepsilon}},\ 0<\varepsilon<N+2. \tag{51}$$

Indeed, the validity of (50) is straightforward, while (49) follows from the fact that

$$\frac{2^{1-\varepsilon}}{N\,\pi^{N/2}} \frac{\Gamma\left(\dfrac{N-\varepsilon}{2}+1\right)}{\Gamma\left(\dfrac{\varepsilon}{2}\right)} \sim_{\varepsilon\to 0} \frac{\varepsilon}{N\,\pi^{N/2}} \Gamma\left(\frac{N}{2}+1\right) = \frac{\varepsilon}{|\mathbb{S}^{N-1}|},$$

combined with the identity

$$\frac{\varepsilon}{|\mathbb{S}^{N-1}|} \int\limits_{|h|<1} \frac{1}{|h|^{N-\varepsilon}}\,dh = 1.$$

The results in the previous sections can be easily adapted to kernels satisfying (49)–(50). The price to pay is that the natural function setting is $W^{1,p}(\mathbb{R}^N)$, respectively $BV(\mathbb{R}^N)$, rather than $\dot{W}^{1,p}$, respectively $B\dot{V}$. Here are some results from the previous sections adapted to the assumptions (1) and (49)–(50).

Proposition 10. Assume (1) and (49)–(50). Let $1 \le p < \infty$. Let $u \in W^{1,p}(\mathbb{R}^N)$. Then

$$V_\varepsilon \to \nabla u \text{ in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \to 0.$$

Proposition 11. Assume (1) and (49)–(50). Let $u \in BV(\mathbb{R}^N)$. Then

$$V_\varepsilon \rightharpoonup Du \text{ *-weakly in } \mathscr{M}(\mathbb{R}^N) \text{ as } \varepsilon \to 0 \tag{52}$$

and

$$\lim_{\varepsilon\to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)}. \tag{53}$$

In the next two results, we assume that, for some $1 < q \le \infty$, we have

$$\lim_{\varepsilon\to 0} \|\rho_\varepsilon(h)/|h|\|_{L^q(|h|>1)} = 0. \tag{54}$$

Proposition 12. Assume (1) and (49)–(50). Let $1 \le p < \infty$. Assume that (54) holds when q is the conjugate exponent of p. Let $u \in W^{1,p}(\mathbb{R}^N)$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \to \nabla u(x)$ as $\varepsilon \to 0$.

Proposition 13. Assume (1) and (49)–(50). Assume that (54) holds when $q = \infty$. Let $u \in BV(\mathbb{R}^N)$. Then, for a.e. $x \in \mathbb{R}^N$, we have $V_\varepsilon(x) \to \nabla^{ac} u(x)$ as $\varepsilon \to 0$.

Remark 5. Given any $q > 1$, the kernel in (51) satisfies (54). Therefore, Propositions 12 and 13 apply to these kernels.

Remark 6. Propositions 12 and 13 have straightforward versions, in which the assumption on u is $u \in L^r(\mathbb{R}^N) \cap \dot{W}^{1,p}$, respectively $u \in L^r(\mathbb{R}^N) \cap B\dot{V}$ for some $r \in [1,\infty)$ (and then, in (54), q is the conjugate exponent of r).

Proposition 14. Assume (1) and (49)–(50). Let $1 < p < \infty$. If $u \in L^p(\mathbb{R}^N)$ and $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ for every ε, then

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}(\mathbb{R}^N)). \tag{55}$$

Proposition 15. Assume (1) and (49)–(50). If $u \in L^1(\mathbb{R}^N)$ and $W_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ for every ε, then

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathscr{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N)). \tag{56}$$

Proposition 16. Assume (1) and (49)–(50). Let $1 < p < \infty$. If $u \in L^p(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (30) holds, $\forall\, \varepsilon, \forall\, \zeta \in C^\infty_c(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^p(\mathbb{R}^N)} = \|\nabla u\|_{L^p(\mathbb{R}^N)} \text{ (with the convention } \|\nabla u\|_{L^p(\mathbb{R}^N)} = \infty \text{ if } u \notin W^{1,p}(\mathbb{R}^N)). \tag{57}$$

Proposition 17. Assume (1) and (49)–(50). If $u \in L^1(\mathbb{R}^N)$ and there exists some $V_\varepsilon \in L^1_{loc}(\mathbb{R}^N)$ such that (30) holds, $\forall\, \varepsilon, \forall\, \zeta \in C^\infty_c(\mathbb{R}^N; \mathbb{R}^N)$, then

$$\lim_{\varepsilon \to 0} \|V_\varepsilon\|_{L^1(\mathbb{R}^N)} = \|Du\|_{\mathscr{M}(\mathbb{R}^N)} \text{ (with the convention } \|Du\|_{\mathscr{M}(\mathbb{R}^N)} = \infty \text{ if } u \notin BV(\mathbb{R}^N)). \tag{58}$$

We prove only Propositions 10 and 12; the other results are obtained from the corresponding ones in the previous sections using similar arguments.

Proof of Proposition 10. Set

$$\rho^1_\varepsilon := \rho_\varepsilon\, \chi_{B(0,1)},\ \rho^2_\varepsilon := \rho_\varepsilon - \rho^1_\varepsilon,\ V^1_\varepsilon := V_{\varepsilon,u,\rho^1_\varepsilon},\ V^2_\varepsilon := V_{\varepsilon,u,\rho^2_\varepsilon},\ \kappa_\varepsilon(h) := \frac{\rho^2_\varepsilon(h)}{|h|}.$$

By Proposition 1 and the assumptions (1) and (49)–(50), we have $V^1_\varepsilon \to \nabla u$ in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$. On the other hand, we have the straightforward inequality

$$|V^2_\varepsilon(x)| \le |u| * \kappa_\varepsilon(x) + |u(x)|\, \|\kappa_\varepsilon\|_{L^1(\mathbb{R}^N)},\ \forall\, x \in \mathbb{R}^N. \tag{59}$$

Combining (50), (59) with $\delta = 1$, and the Young inequality, we find that $V^2_\varepsilon \to 0$ in $L^p(\mathbb{R}^N)$ as $\varepsilon \to 0$. □

Proof of Proposition 12. It suffices to note that (by (59) and (54)) we have $V^2_\varepsilon \to 0$ pointwise as $\varepsilon \to 0$. □

Remark 7. The fact that ρ is radial when $|h| > 1$ is not relevant for the above results.

5 Γ-convergence

One can associate Γ-convergence results with the above convergence statements. We present one result of this type, in the spirit of [7, Theorem 1.7].

Let $1 \le p < \infty$. Set, for $0 < \varepsilon < \varepsilon_0$ and $u \in L^1_{loc}(\mathbb{R}^N)$,

$$J_{\varepsilon,p}(u) := \begin{cases} \|V_{\varepsilon,u}\|_{L^p(\mathbb{R}^N)}, & \text{if } W_{\varepsilon,u} \in L^1_{loc}(\mathbb{R}^N) \\ \infty, & \text{otherwise} \end{cases},$$

$$J_{0,p}(u) := \begin{cases} \|\nabla u\|_{L^p(\mathbb{R}^N)}, & \text{if } u \in \dot{W}^{1,p} \\ \infty, & \text{otherwise} \end{cases}, \ \forall\, 1 < p < \infty,$$

$$J_{0,1}(u) := \begin{cases} \|Du\|_{\mathscr{M}(\mathbb{R}^N)}, & \text{if } u \in B\dot{V} \\ \infty, & \text{otherwise} \end{cases}.$$

Proposition 18. Assume (1)–(3). Let $1 \le p < \infty$.

1. For $1 \le q < \infty$, $J_{\varepsilon,p}$ Γ-converges to $J_{0,p}$ in $L^q(\mathbb{R}^N)$.
2. Under the support assumption (33), $J_{\varepsilon,p}$ Γ-converges to $J_{0,p}$ in $L^1_{loc}(\mathbb{R}^N)$.

Proof. Proof of item 2. Let $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subset L^1_{loc}(\mathbb{R}^N)$ be a family such that $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}^N)$ and $\liminf_{\varepsilon\to 0} J_{\varepsilon,p}(u_\varepsilon) < \infty$. By Proposition 7 and the proof of Propositions 5 and 6 (see, more specifically, (35) and (36)), we find that $u \in \dot{W}^{1,p}$ if $1 < p < \infty$ (respectively $u \in B\dot{V}$ if $p = 1$) and $J_{0,p}(u) \le \liminf_{\varepsilon\to 0} J_{\varepsilon,p}(u_\varepsilon)$.

In the opposite direction, we do not need the support assumption (33). Let $u \in \dot{W}^{1,p}$ if $1 < p < \infty$, respectively $u \in B\dot{V}$ if $p = 1$. Let η be a normalized bump function. By Proposition 1 applied to $u * \eta_{1/j}$, where $j \ge 1$ is an integer, there exists a sequence $(\varepsilon_j)_{j\ge 1}$ such that

$$J_{\varepsilon,p}(u * \eta_{1/j}) \le J_{0,p}(u * \eta_{1/j}) + \frac{1}{j}, \ \forall\, j \ge 1, \ \forall\, 0 < \varepsilon < \varepsilon_j.$$

With no loss of generality, we may assume that $\varepsilon_j \to 0$ and $\varepsilon_{j+1} < \varepsilon_j$. If we set

$$u_\varepsilon := u * \eta_{1/j}, \ \forall\, \varepsilon_{j+1} \le \varepsilon < \varepsilon_j,$$

then $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}^N)$ when $\varepsilon \to 0$ and

$$\limsup_{\varepsilon\to 0} J_{\varepsilon,p}(u_\varepsilon) \le \limsup_{j\to\infty} J_{0,p}(u * \eta_{1/j}) = J_{0,p}(u).$$

Proof of item 1. The proof is similar to the one of item 2. It suffices to note that Propositions 5, 6, and 7 still hold if we replace $(L^1 + L^\infty)(\mathbb{R}^N)$ with $L^q(\mathbb{R}^N)$. □

References

[1] J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces. In J. L. Menaldi, E. Rofman, and A. Sulem, editors, *Optimal Control and Partial Differential Equations – Innovations and Applications*, pages 439–455, Amsterdam, 2001. IOS Press.

[2] H. Brezis and H.-M. Nguyen. The BBM formula revisited. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 27(4):515–533, 2016.

[3] E. Bruè, M. Calzi, G. E. Comi, and G. Stefani. A distributional approach to fractional Sobolev spaces and fractional variation: asymptotics II. *C. R. Math. Acad. Sci. Paris*, 360:589–626, 2022.

[4] G. E. Comi and G. Stefani. A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up. *J. Funct. Anal.*, 277(10):3373–3435, 2019.

[5] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou. A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws. *Math. Models Methods Appl. Sci.*, 23(3):493–540, 2013.

[6] G. Gilboa and S. Osher. Nonlocal operators with applications to image processing. *Multiscale Model. Simul.*, 7(3):1005–1028, 2008.

[7] T. Mengesha and D. Spector. Localization of nonlocal gradients in various topologies. *Calc. Var.*, 52:253–279, 2015.

[8] A. C. Ponce. An estimate in the spirit of Poincaré's inequality. *J. Eur. Math. Soc. (JEMS)*, 6(1):1–15, 2004.

[9] T.-T. Shieh and D. Spector. On a new class of fractional partial differential equations. *Adv. Calc. Var.*, 8(4):321–336, 2015.

[10] D. Spector. L^p-Taylor approximations characterize the Sobolev space $W^{1,p}$. *C. R. Math. Acad. Sci. Paris*, 353:327–332, 2015.

[11] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[12] M. Šilhavý. Fractional vector analysis based on invariance requirements (critique of coordinate approaches). *Contin. Mech. Thermodyn.*, 32(1):207–228, 2020.

(1) Rutgers University, Department of Mathematics, Hill Center, Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA
brezis@math.rutgers.edu
hbrezis@gmail.com

(2) Université Claude Bernard Lyon 1; École Centrale de Lyon; INSA Lyon; Université Jean Monnet Saint-Étienne; CNRS UMR 5208 Institut Camille Jordan; 43, boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France
mironescu@math.univ-lyon1.fr