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CARLESON MEASURES AND OVERSAMPLING IN MODEL SPACE

ANTON BARANOV, PHILIPPE JAMING, KARIM KELLAY, AND MICHAEL SPECKBACHER

ABSTRACT. The aim of this paper is to extend two results from the Paley–Wiener setting to more general model spaces. The first one is an analogue of the oversampling Shannon sampling formula. The second one is a version of Donoho–Logan’s Large Sieve Theorem which is a quantitative estimate of the embedding of the Paley–Wiener space into an $L^2(\mathbb{R}, \mu)$ space.

1. INTRODUCTION

The aim of this paper is to extend two classical results on the Paley–Wiener space to more general model spaces. The first result is the sampling theorem, or more precisely the oversampling formula that improves the convergence in Shannon sampling. The second result is Donoho–Logan’s Large Sieve Principle which can be seen as one of the first results on sparsity in signal processing. In the context of complex analysis, this result is a result on Carleson measures for the Paley–Wiener space.

Let us now be more precise. Recall that the Paley–Wiener space is the subspace of L^2 signals that are band-limited to $[-c, c]$. It is a very common space used to model signal encountered in natural sciences. If we normalize the Fourier transform through

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-it\xi} dt,$$

the Paley–Wiener space $PW_c^p(\mathbb{R})$, $c > 0$, $1 \leq p < \infty$, is defined as

$$PW_c^p(\mathbb{R}) = \left\{ f \in L^p(\mathbb{R}) : \text{supp } \widehat{f} \subset [-c, c] \right\},$$

where, for $p > 2$, \widehat{f} is understood in the distributional sense. Two well known properties of the Paley–Wiener spaces is that they consist of entire functions and that that every function $f \in PW_c^2(\mathbb{R})$ can be reconstructed from samples $\{f(hk)\}_{k \in \mathbb{Z}}$ via the Kotelnikov–Nyquist–Shannon Formula

$$f(x) = \sum_{k \in \mathbb{Z}} f\left(k\frac{\pi}{b}\right) \frac{1}{2b} \widehat{\gamma}\left(x - k\frac{\pi}{b}\right), \tag{1.1}$$

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where $b \geq c$ and γ is any even function supported in $[-d, d]$ with $\gamma(\xi) = 1$ for $\xi \in [-c, c]$. Taking $b \geq c$ and $\gamma = \mathbf{1}_{[-b, b]}$ we get the classical Shannon Formula

$$f(x) = \sum_{k \in \mathbb{Z}} f\left(k \frac{\pi}{b}\right) \operatorname{sinc}\left(b\left(x - k \frac{\pi}{b}\right)\right), \quad (1.2)$$

where $\operatorname{sinc} t = \sin t/t$. However, when $b > c$ one can do better by taking γ smooth in which case $\widehat{\gamma}$ decreases faster than the sinc function. Most authors take γ to be \mathcal{C}^∞ so that, for every N , $|\widehat{\gamma}(x - k\frac{\pi}{b})| = O(k^{-N})$ when $k \rightarrow \pm\infty$ and this estimate is even uniform when x stays in a compact set. As a consequence (1.1) converges much better than (1.2), a fact that is well known in signal processing. The drawback of this is that, to the best of our knowledge, there is no example of a function γ as above for which $\widehat{\gamma}$ is explicitly known.

One way to overcome this is to give up on arbitrary fast decay and to fix N . One then fixes a parameter $a > 0$, and takes $\psi^{(1)} = \psi = \frac{1}{2a} \mathbf{1}_{[-a, a]}$ and $\psi^{(k+1)} = \psi^{(k)} * \psi$. Then $\psi^{(N)}$ is supported in $[-Na, Na]$ and $\int_{\mathbb{R}} \psi^{(N)} = 1$. It follows that $\gamma = \psi^{(N)} * \mathbf{1}_{[-c-Na, c+Na]}$ is even, supported in $[-b, b]$ with $b = c + 2Na$ and that $\gamma(\xi) = \int_{\mathbb{R}} \psi^{(N)} = 1$ for $\xi \in [-c, c]$. Computing $\widehat{\gamma}$, we get

$$f(x) = 2(c + Na) \sum_{k \in \mathbb{Z}} f\left(k \frac{\pi}{b}\right) \left[\operatorname{sinc}\left(a\left(x - k \frac{\pi}{b}\right)\right)\right]^N \operatorname{sinc}\left((c + Na)\left(x - k \frac{\pi}{b}\right)\right). \quad (1.3)$$

The second result we are dealing with in this paper is Donoho–Logan’s Large Sieve theorem. The analytic large sieve principle is a classical inequality for trigonometric polynomials which finds many applications in analytic number theory (see, *e.g.*, [19] and references therein). It has been extended from trigonometric polynomials to their near cousins the band-limited functions by Donoho and Logan [12] in the early 90s (after earlier work by Boas [8]), applying it to reconstruction of missing data in signal processing.

For I an interval, we denote by $|I|$ the length of I . When μ is a positive σ -finite measure and $\delta > 0$, we write

$$D_\mu(\delta) = \sup \left\{ \frac{\mu(I)}{|I|} : I \text{ closed interval, } |I| = \delta \right\} = \sup_{x \in \mathbb{R}} \frac{\mu([x, x + \delta])}{\delta}. \quad (1.4)$$

Donoho–Logan proved the following:

Theorem 1.1 (Donoho–Logan). *Let $c, \delta > 0$ and μ be a positive σ -finite measure. Then for every $f \in PW_c^2(\mathbb{R})$,*

$$\int_{\mathbb{R}} |f(x)|^2 d\mu(x) \leq \left(1 + \frac{c\delta}{\pi}\right) D_\mu(\delta) \int_{\mathbb{R}} |f(x)|^2 dx. \quad (1.5)$$

Moreover, for every $f \in PW_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} |f(x)| d\mu(x) \leq \frac{D_\mu(\delta)}{\operatorname{sinc}\left(\frac{c\delta}{2}\right)} \int_{\mathbb{R}} |f(x)| dx. \quad (1.6)$$

This theorem is one of the founding results of the use of sparsity constraints in signal recovery problems and has led to considerable research in the decades following it. However, until recently, it seems that this result did not get the attention it deserves in other mathematical communities. For instance, in complex analysis, Donoho–Logan’s Large Sieve theorem provides a quantitative estimate of the fact that a positive measure is a Carleson measure of the Paley–Wiener space, that is, a bound on the norm of the injection $PW_c^2(\mathbb{R}) \hookrightarrow L^2(\mu)$. Note that a first estimate of this type was given by Lin [18]. Recently, Husain and Littmann extended the result for $p = 1$ to higher dimensions [15].

This inadvertence may be due to the fact that the strategy of proof has some rigidities that are difficult to overcome. Indeed, the proof relies heavily on the interplay between convolution and the Fourier transform, and on a construction by Selberg (based on a previous construction by Beurling) of an entire function majorizing the sign function, see [19, 24]. Despite its rigidity, this strategy of proof has been extended to more general settings, see, *e.g.*, [1, 2, 17]. Our first aim in this paper is to provide two new proof strategies, one based on the oversampling formula, the second on Bernstein’s inequality. In particular, those proofs rely on real variable techniques only. The price to be paid is that the numerical constants we obtain are slightly worse. The proofs however offer more flexibility so that they apply in larger settings.

To illustrate this, we will extend the oversampling formula and the large sieve inequality to the setting of model spaces on the upper half-plane, an important family of spaces of entire functions which contains the Paley–Wiener space $PW_c^2(\mathbb{R})$. Let us now describe this family. First, let H^2 be the Hardy space on the upper half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$.

$$H^2 := \left\{ f \in \text{Hol}(\mathbb{C}^+) : \sup_{y>0} \int_{\mathbb{R}} |f(x + iy)|^2 dx < \infty \right\}.$$

Note that we may identify $H^2 = \{f \in L^2(\mathbb{R}) : \text{supp } \widehat{f} \subset [0, +\infty)\}$. Let Θ be an inner function in \mathbb{C}^+ , that is, a bounded analytic function on \mathbb{C}^+ with unimodular boundary values almost everywhere on \mathbb{R} . The corresponding *model space* is defined by

$$K_{\Theta}^2 := H^2 \cap (\Theta H^2)^{\perp}.$$

As a fundamental example, if $\Theta_c(z) = \exp(icz)$ for some $c > 0$, then

$$K_{\Theta_{2c}}^2 = \{f \in L^2(\mathbb{R}) : \text{supp } \widehat{f} \subset [0, 2c]\} = \{\Theta_c f : f \in PW_c^2(\mathbb{R})\}$$

or, equivalently,

$$PW_c^2(\mathbb{R}) = \{e^{-icz} f : f \in K_{\Theta_{2c}}^2\}.$$

Further, one can also define K_{Θ}^p model spaces in H^p , $1 \leq p \leq +\infty$, so as to also cover the PW_c^p spaces, see Section 5. In the context of model spaces, an analogue of Shannon’s sampling formula was established by de Branges [9] in the case of meromorphic inner functions and later by Clark [10] in the general case, see (3.5). However, the oversampling formula for model spaces seems to be unknown. Previously, oversampling results were obtained in [22, 23] for specific model spaces (or closely related de Branges spaces) associated to certain differential operators.

We start by establishing an oversampling formula for a class of model spaces. Once this is done, we propose a first proof of Donoho–Logan’s theorem for model spaces associated to inner functions with bounded derivative, an interesting class of model spaces that share many properties with the Paley–Wiener spaces (see, *e.g.*, [13, 21]).

Theorem 1.2. *Let Θ be a meromorphic inner function such that $\Theta' \in L^\infty(\mathbb{R})$ and μ be a σ -finite Borel measure on \mathbb{R} . For $\delta > 0$, $D_\mu(\delta)$ defined in (1.4) and $1 \leq p < +\infty$ there exists $C_p > 0$ such that for every $f \in K_\Theta^p$*

$$\int_{\mathbb{R}} |f(x)|^p d\mu(x) \leq C_p (1 + \|\Theta'\|_\infty \delta)^p D_\mu(\delta) \int_{\mathbb{R}} |f(x)|^p dx. \quad (1.7)$$

The proof relying on oversampling gives this result for $p = 2$ and a rather bad constant C but gives the right behavior with respect to δ and $\|\Theta'\|$. We thus give a second proof, based on Bernstein’s inequality which allows to obtain the constant $C_p = 1$, falling only short of Donoho–Logan’s result in the case of the Paley–Wiener space. The particular constants are discussed in more detail in Remark 5.4. We think that both proofs are interesting due to the fact that they only rely on real analytic arguments and are rather flexible. Finally, we end this article with a version of Theorem 1.2 for a larger class of inner functions, the so-called one-component inner functions but requires to modify the definition of $D_\mu(\delta)$, see Theorem 6.1. This provides a quantitative converse version of the Logvinenko–Sereda Theorem for model spaces established in [16].

The remaining of this article is organized as follows: Section 2 is devoted to a technical lemma. In Section 3, we introduce the necessary background on model spaces and prove the oversampling formula. In Section 4 we give an extension of the large sieve inequality to model spaces using the oversampling techniques. The approach based on Bernstein type inequalities is considered in Section 5, while in Section 6 a certain analog of the large sieve is given for a class of model spaces generated by the so-called one-component inner functions.

2. PRELIMINARY LEMMA

Let Ξ be a continuous function such that

$$|\Xi(x)| \leq \min(1, |x|^{-1}). \quad (2.1)$$

As an example, one can take $\Xi(x) = \operatorname{sinc} x := \frac{\sin x}{x}$.

Lemma 2.1. *Let Ξ be a function satisfying the bound (2.1). Let $a, b \in \mathbb{R}$, $\alpha, \delta > 0$ and μ be a σ -finite Borel measure on \mathbb{R} . Then we have*

$$\begin{aligned} (1) \quad & \int_{\mathbb{R}} \Xi(x-a)^2 \Xi(x-b)^2 dx \leq \frac{8\pi}{4 + (a-b)^2}, \\ (2) \quad & \int_{\mathbb{R}} \Xi(x-a)^2 \Xi(x-b)^2 d\mu(x) \leq 8\pi \frac{C_\delta^2 \sup_{x \in \mathbb{R}} \mu([x, x+\delta])}{\delta (4 + (b-a)^2)}, \text{ with } C_\delta = \max(4, 1+9\delta^2). \end{aligned}$$

Proof. The first estimate follows immediately from the simple estimate

$$|\Xi(x)|^2 \leq \min(1, |x|^{-2}) \leq \frac{2}{1+x^2},$$

and the computation of the integral

$$\int_{\mathbb{R}} \frac{dt}{(1+(t-a)^2)(1+(t-b)^2)} = \frac{2\pi}{4+(a-b)^2},$$

via the residue formula.

Next, for $\ell \in \mathbb{Z}$, let $I_\ell = [\ell\delta, (\ell+1)\delta]$, we have

$$\begin{aligned} \int_{\mathbb{R}} \Xi(x-a)^2 \Xi(x-b)^2 d\mu(x) &= \sum_{\ell \in \mathbb{Z}} \int_{I_\ell} \Xi(x-a)^2 \Xi(x-b)^2 d\mu(x) \\ &\leq \sup_{\ell \in \mathbb{Z}} \mu(I_\ell) \cdot \sum_{\ell \in \mathbb{Z}} \frac{4}{(1+d(a, I_\ell)^2)(1+d(b, I_\ell)^2)} \\ &= \frac{4}{\delta} \sup_{\ell \in \mathbb{Z}} \mu(I_\ell) \cdot \sum_{\ell \in \mathbb{Z}} \int_{I_\ell} \frac{dx}{(1+d(a, I_\ell)^2)(1+d(b, I_\ell)^2)}. \end{aligned}$$

Now, let $x \in I_\ell$. If $a \leq (\ell-2)\delta$ then $x-a \geq 2\delta$ so that

$$d(a, I_\ell) = \ell\delta - a = (\ell+1)\delta - a - \delta \geq x - a - \delta \geq \frac{x-a}{2}$$

thus

$$\frac{1}{1+d(a, I_\ell)^2} \leq \frac{4}{1+(x-a)^2}.$$

The same holds if $a \geq (\ell+3)\delta$. Finally, if $(\ell-2)\delta \leq a \leq (\ell+3)\delta$, $x-a \leq 3\delta$ thus

$$\frac{1}{1+d(a, I_\ell)^2} \leq 1 \leq \frac{1+9\delta^2}{1+(x-a)^2}.$$

It follows that

$$\begin{aligned} \int_{\mathbb{R}} \Xi(x-a)^2 \Xi(x-b)^2 d\mu(x) &\leq \frac{4}{\delta} C_\delta^2 \sup_{\ell \in \mathbb{Z}} \mu(I_\ell) \cdot \int_{\mathbb{R}} \frac{dx}{(1+(x-a)^2)(1+(x-b)^2)} \\ &\leq 8\pi \frac{C_\delta^2}{\delta} \frac{\sup_{x \in \mathbb{R}} \mu([x, x+\delta])}{4+(b-a)^2}, \end{aligned}$$

with $C_\delta = \max(4, 1+9\delta^2)$. □

Applying Lemma 2.1(2) to the dilated measure $\mu_\alpha(A) = \mu(A/\alpha)$, $\alpha > 0$ we get

$$\int_{\mathbb{R}} \Xi(\alpha x - a)^2 \Xi(\alpha x - b)^2 d\mu(x) \leq 8\pi \frac{C_\delta^2}{\delta} \frac{\sup_{x \in \mathbb{R}} \mu([x, x+\delta/\alpha])}{4+(b-a)^2}, \quad (2.2)$$

with $C_\delta = \max(4, 1+9\delta^2)$.

A slightly more evolved version of the lemma is easily available: for $m \geq 2$ and $\delta > 0$ there is a constant $C_{m,\delta}$ such that

$$\int_{\mathbb{R}} \Xi(x-a)^{2m} \Xi(x-b)^{2m} d\mu(x) \leq \frac{C_{m,\delta}}{(1+(b-a)^2)^m} \sup_{x \in \mathbb{R}} \frac{\mu([x, x+\delta])}{\delta}. \quad (2.3)$$

This follows from the easily established inequality

$$\int_{\mathbb{R}} |\Xi(x-a)|^{2m} |\Xi(x-b)|^{2m} dx \leq \sqrt{\pi} 2^{2m+1} \frac{\Gamma(m-1/2)}{\Gamma(m)} \frac{1}{(1+(b-a)^2)^m}.$$

3. OVERSAMPLING AND THE LARGE SIEVE IN MODEL SPACES

3.1. Background on model spaces. The Hardy space on the upper-half plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$, $H^2 = H^2(\mathbb{C}^+)$ contains all holomorphic functions on \mathbb{C}^+ for which

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx < \infty.$$

Every function $f \in H^2$ has an almost everywhere defined “vertical” boundary function $f(x) := \lim_{y \rightarrow 0} f(x+iy)$, and $f \in L^2(\mathbb{R})$ which may be used to define an inner product on H^2

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

We say that an analytic function Θ on \mathbb{C}^+ is *inner* if $|\Theta| \leq 1$ on \mathbb{C}^+ and if the almost everywhere defined boundary function $\Theta(x)$, $x \in \mathbb{R}$, has modulus one. If Θ is an inner function, then the corresponding *model space* is defined by

$$K_{\Theta}^2 := H^2 \ominus \Theta H^2 = (\Theta H^2)^{\perp}. \quad (3.1)$$

Recall that the reproducing kernel for functions K_{Θ}^2 is defined by

$$k_z^{\Theta}(w) = \frac{i}{2\pi} \frac{1 - \overline{\Theta(z)}\Theta(w)}{w - \bar{z}}, \quad z, w \in \mathbb{C}^+. \quad (3.2)$$

For every $f \in K_{\Theta}^2$,

$$f(z) = \langle f, k_z^{\Theta} \rangle,$$

and

$$\|k_z^{\Theta}\|^2 = \frac{1 - |\Theta(z)|^2}{4\pi \text{Im } z}.$$

Each inner function can be factored as

$$\Theta(z) = e^{i\tau} \Theta_c(z) B_{\lambda}(z) S_{\mu}(z), \quad z \in \mathbb{C}^+,$$

where τ is a real constant, $\Theta_c(z) = e^{icz}$, $c \geq 0$,

$$B_{\Lambda}(z) = \prod_{\lambda \in \Lambda} e^{i\alpha_{\lambda}} \frac{z - \lambda}{z - \bar{\lambda}},$$

is the *Blaschke product* with zeros $\lambda \in \Lambda \subset \mathbb{C}^+$, repeated according to multiplicity, satisfying the Blaschke condition

$$\sum_{\lambda \in \Lambda} \frac{\operatorname{Im} \lambda}{1 + |\lambda|^2} < \infty, \quad (3.3)$$

and $\alpha_\lambda \in \mathbb{R}$,

$$S_\mu(z) = \exp \left(i \int_{\mathbb{R}} \left(\frac{1}{x-z} - \frac{t}{x^2+1} \right) d\mu(x) \right),$$

where μ is a singular measure with respect to the Lebesgue measure such that

$$\int_{\mathbb{R}} \frac{d\mu(x)}{1+x^2} < \infty.$$

The spectrum of Θ is the closed set

$$\rho(\Theta) := \left\{ \zeta \in \overline{\mathbb{C}^+} \cup \infty : \liminf_{\substack{z \rightarrow \zeta \\ z \in \mathbb{C}^+}} |\Theta(z)| = 0 \right\}. \quad (3.4)$$

Note that Θ , along with every function in K_Θ^2 , has an analytic extension across any interval of $\mathbb{R} \setminus \rho(\Theta)$.

By the Ahern–Clark theorem [3], $k_x^\Theta \in K_\Theta^2$, $x \in \mathbb{R}$, if and only if the modulus of the angular derivative of Θ is finite. This means that

$$|\Theta'(x)| = a + 2 \sum_{\lambda \in \Lambda} \frac{\operatorname{Im} \lambda}{|x - \lambda|^2} + \int_{\mathbb{R}} \frac{d\mu(t)}{(t-x)^2} < \infty.$$

For any $\alpha \in \mathbb{C}$, $|\alpha| = 1$ the function $(\alpha + \Theta)/(\alpha - \Theta)$ has a positive real part in the upper half plane, and hence, by the Herglotz–Riesz representation theorem, there exist $c_\alpha > 0$ and a non-negative measure σ_Θ^α , called the Clark measure, such that

$$\operatorname{Re} \frac{\alpha + \Theta(z)}{\alpha - \Theta(z)} = c_\alpha \operatorname{Im} z + \frac{\operatorname{Im} z}{\pi} \int_{\mathbb{R}} \frac{d\sigma_\Theta^\alpha(x)}{|x-z|^2}, \quad z \in \mathbb{C}^+.$$

The Clark measure σ_Θ^α is carried by the set $\{x \in \mathbb{R} : \lim_{y \rightarrow 0^+} \Theta(x + iy) = \alpha\}$.

Clark [10] showed that if $c_\alpha = 0$ and σ_Θ^α is purely atomic, that is

$$\sigma_\Theta^\alpha = \sum a_n \delta_{x_n},$$

where δ_x denotes the Dirac measure at the point x , then the system $\{k_{x_n}^\Theta\}$ is an orthogonal basis in K_Θ^2 . In particular, one has

$$f(z) = \sum_{n \in \mathbb{Z}} f(x_n) \frac{k_{x_n}^\Theta(z)}{\|k_{x_n}^\Theta\|^2}, \quad f \in K_\Theta^2, \quad (3.5)$$

and

$$\|f\|^2 = \sum_{n \in \mathbb{Z}} \frac{|f(x_n)|^2}{\|k_{x_n}^\Theta\|^2}. \quad (3.6)$$

We end this section with a discussion of a special class of inner functions which will be considered throughout the paper (with exception of Section 6). A *meromorphic inner*

function on \mathbb{C}^+ is an inner function on \mathbb{C}^+ with a meromorphic continuation to \mathbb{C} . Any meromorphic inner function Θ , can be represented as

$$\Theta = \Theta_c B_\Lambda$$

where $c \geq 0$ and B_Λ is the Blaschke product associated to the zeroes $\Lambda = \{\lambda_n\}$ of Θ and the satisfy $|\lambda_n| \rightarrow \infty$ as well as the Blaschke condition (3.3). All elements of the corresponding model space K_Θ^2 are also meromorphic in \mathbb{C} , and there is a canonical isomorphism of such model spaces with de Branges' Hilbert spaces of entire functions [9].

By the Riesz–Smirnov factorization there exists an increasing, real analytic function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\Theta(x) = \exp(i\varphi(x)).$$

In that case $|\Theta'| = \varphi'$ and

$$\|k_x^\Theta\|^2 = \frac{\varphi'(x)}{2\pi}. \quad (3.7)$$

Then, by the Cauchy–Schwarz inequality one has

$$|k_x^\Theta(t)| = |\langle k_x^\Theta, k_t^\Theta \rangle| \leq \frac{\sqrt{\varphi'(x)\varphi'(t)}}{2\pi}, \quad x, t \in \mathbb{R}. \quad (3.8)$$

In the case of meromorphic inner functions the Clark measure construction becomes much more transparent. In this setting such measures were introduced by de Branges (see, *e.g.*, [9]) long before the work of Clark. For model spaces associated with meromorphic inner functions orthogonal bases of reproducing kernels can be constructed as follows. For $\gamma \in [0, 2\pi)$ define the set of points $\{x_n\}_{n \in \mathbb{Z}}$ by

$$\varphi(x_n) = \gamma + 2\pi n, \quad n \in \mathbb{Z}. \quad (3.9)$$

(Note that the points x_n may not exist for all $n \in \mathbb{Z}$.) Then the family of normalized reproducing kernels $\{k_{x_n}^\Theta / \|k_{x_n}^\Theta\|\}_{n \in \mathbb{Z}}$, with the points $\{x_n\}_{n \in \mathbb{Z}} \in \mathbb{R}$ given by (3.9), forms an orthonormal basis for K_Θ^2 for each $\gamma \in [0, 2\pi)$, except, maybe, one (in the case that $\Theta - e^{i\gamma} \in L^2(\mathbb{R})$).

In what follows we will consider the class of inner functions such that $\Theta' \in H^\infty(\mathbb{C}^+)$. This condition implies that Θ is meromorphic and is equivalent to $\Theta' = \varphi' \in L^\infty(\mathbb{R})$. It was noticed already in [13] that the model spaces K_Θ^2 with $\Theta' \in L^\infty(\mathbb{R})$ have many properties analogous to the properties of the Paley–Wiener spaces $PW_c^2(\mathbb{R})$.

3.2. Enlarging model spaces and oversampling. We need the following classical result [21]. However, for the sake of completeness, we give the complete proof here.

Lemma 3.1. *Let Θ be an inner function, let $\Theta_1 = e^{i\tau_1} \Theta_{c_1} B_{\Lambda_1}$, and $\Theta_2 = e^{i\tau_2} \Theta_{c_2} B_{\Lambda_2}$ where $\Lambda_1, \Lambda_2 \subset \mathbb{C}^+$, $i = 1, 2$, are two Blaschke sequences. Then*

- (1) $K_{\Theta_1}^2 \subseteq K_{\Theta_2}^2$ if and only if $c_1 \leq c_2$ and $\Lambda_1 \subseteq \Lambda_2$.
- (2) $K_\Theta^2 \subseteq K_{\Theta_c B_\Lambda}^2$, where $c \geq 0$ and $\Lambda \subset \mathbb{C}^+$ a Blaschke sequence.
- (3) If $0 \leq c_1 \leq c_2$, $\Lambda_1 \subseteq \Lambda_2$, and $f \in K_\Theta^2$, then $\Theta_{c_1} B_{\Lambda_1} f \in K_{\Theta_{c_2} B_{\Lambda_2}}^2$.

Proof. Let Θ_1, Θ_2 be inner functions. It is known that $K_{\Theta_1}^2 \subseteq K_{\Theta_2}^2$ if and only if Θ_2/Θ_1 is an inner function. Hence, Θ_2/Θ_1 is inner if and only if

$$\frac{\Theta_2(z)}{\Theta_1(z)} = e^{i(\tau_2 - \tau_1)} e^{i(c_2 - c_1)z} \frac{B_{\Lambda_2}(z)}{B_{\Lambda_1}(z)} = \Theta(z),$$

for some Θ inner. If $c_2 \geq c_1$ and $\Lambda_1 \subset \Lambda_2$, then $B_{\Lambda_2}/B_{\Lambda_1} = B_{\Lambda_2 \setminus \Lambda_1}$ is a Blaschke product and Θ is therefore inner.

Now assume that Θ_2/Θ_1 is inner and that there exists $\lambda^* \in \Lambda_1$ that is not contained in Λ_2 (in the sense that for $\Lambda_i = \{\lambda_n^i\}_{n \in \mathbb{N}}$, $i \in \{1, 2\}$, there is no injection $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ that satisfies $\lambda_n^1 = \lambda_{\varphi(n)}^2$ for every $n \in \mathbb{N}$), then $\#\{n \in \mathbb{N} : \lambda^* = \lambda_n^2\} < \#\{n \in \mathbb{N} : \lambda^* = \lambda_n^1\}$, which creates a pole at λ^* for Θ_2/Θ_1 , as the singular inner part is always nonzero, a contradiction to Θ_2/Θ_1 being inner. Then, as $e^{i(c_2 - c_1)z}$ is bounded on \mathbb{C}^+ if and only if $c_2 \geq c_1$, the first assertion follows. It only remains to show the last statement. By definition $f \in K_{\Theta}^2$ if and only if $f \perp \Theta g$ for every $g \in H^2$. Hence, as $\Theta_{c_1} B_{\Lambda_1}$ is an inner function we have

$$\begin{aligned} \langle \Theta_{c_1} B_{\Lambda_1} f, S_{c_1} B_{\Lambda_1} \Theta g \rangle &= \int_{\mathbb{R}} \Theta_{c_1}(x) B_{\Lambda_1}(x) f(x) \overline{\Theta_{c_1}(x) B_{\Lambda_1}(x) \Theta(x) g(x)} dx \\ &= \int_{\mathbb{R}} f(x) \overline{\Theta(x) g(x)} dx = 0, \end{aligned}$$

which shows $\Theta_{c_1} B_{\Lambda_1} f \in K_{\Theta_{c_1} B_{\Lambda_1} \Theta}^2$. The result then follows from the first part of the corollary as $\Theta_{c_2} B_{\Lambda_2} \Theta = (\Theta_{c_2 - c_1} B_{\Lambda_2 \setminus \Lambda_1})(\Theta_{c_1} B_{\Lambda_1} \Theta)$. \square

We are now ready to prove an oversampling theorem for model spaces.

Theorem 3.2. *Let $c > 0$, and Θ be an inner function such that some Clark measure for $\Theta_c \Theta$ is purely atomic and let $\{k_{x_n}^{\Theta_c \Theta}\}_{n \in \mathbb{Z}}$ be the corresponding basis of $K_{\Theta_c \Theta}^2$. Then for every integer $m \geq 1$ and for every $f \in K_{\Theta}^2$, the following sampling formula holds*

$$f(x) = \sum_{n \in \mathbb{Z}} f(x_n) e^{-\frac{ic(x-x_n)}{2}} \operatorname{sinc} \left(\frac{c(x-x_n)}{2m} \right)^m \frac{k_{x_n}^{\Theta_c \Theta}(x)}{\|k_{x_n}^{\Theta_c \Theta}\|^2}, \quad x \in \mathbb{R}. \quad (3.10)$$

In particular, one can construct sampling functions for K_{Θ}^2 of arbitrary polynomial decay.

Further,

$$\|f\|^2 = \sum_{n \in \mathbb{Z}} \frac{|f(x_n)|^2}{\|k_{x_n}^{\Theta_c \Theta}\|^2}, \quad f \in K_{\Theta}^2. \quad (3.11)$$

Proof. If $0 \leq c_k \leq c/m$, $k = 1, \dots, m$, then $0 \leq \sum_k c_k \leq c$, and $\Theta_{\sum_k c_k} f \in K_{\Theta_c \Theta}^2$ by Lemma 3.1. Hence, using the sampling formula (3.5) for $K_{\Theta_c \Theta}^2$, one has

$$e^{i \sum_k c_k z} f(z) = \sum_{n \in \mathbb{Z}} f(x_n) e^{i \sum_k c_k x_n} \frac{k_{x_n}^{\Theta_c \Theta}(z)}{\|k_{x_n}^{\Theta_c \Theta}\|^2}, \quad z \in \mathbb{C}^+,$$

and (3.11) follows directly from (3.6). Further

$$\begin{aligned}
f(z) &= \left(\frac{m}{c}\right)^m \int_0^{c/m} \cdots \int_0^{c/m} f(z) \, dc_1 \cdots dc_m \\
&= \left(\frac{m}{c}\right)^m \int_0^{c/m} \cdots \int_0^{c/m} \sum_{n \in \mathbb{Z}} f(x_n) e^{-i \sum_k c_k (z - x_n)} \frac{k_{x_n}^{\Theta_c \Theta}(z)}{\|k_{x_n}^{\Theta_c \Theta}\|^2} \, dc_1 \cdots dc_m \\
&= \sum_{n \in \mathbb{Z}} f(x_n) \left(\frac{m}{ic(z - x_n)}\right)^m \left(1 - e^{-\frac{ic(z - x_n)}{m}}\right)^m \frac{k_{x_n}^{\Theta_c \Theta}(z)}{\|k_{x_n}^{\Theta_c \Theta}\|^2}.
\end{aligned}$$

Moreover, we find that for almost every $x \in \mathbb{R}$,

$$\begin{aligned}
f(x) &= \sum_{n \in \mathbb{Z}} f(x_n) e^{-\frac{ic(x - x_n)}{2}} \left(\frac{m}{ic(x - x_n)} \left(e^{\frac{ic(x - x_n)}{2m}} - e^{-\frac{ic(x - x_n)}{2m}}\right)\right)^m \frac{k_{x_n}^{\Theta_c \Theta}(x)}{\|k_{x_n}^{\Theta_c \Theta}\|^2} \\
&= \sum_{n \in \mathbb{Z}} f(x_n) e^{-\frac{ic(x - x_n)}{2}} \operatorname{sinc}\left(\frac{c(x - x_n)}{2m}\right)^m \frac{k_{x_n}^{\Theta_c \Theta}(x)}{\|k_{x_n}^{\Theta_c \Theta}\|^2}.
\end{aligned}$$

The oversampling formula is thus established. \square

4. PROOF OF THEOREM 1.2 BASED ON OVERSAMPLING

We now have everything in place to prove Theorem 1.2.

Let $\Theta' \in L^\infty(\mathbb{R})$. Then (3.7) and (3.8) imply

$$\frac{|k_x^\Theta(t)|}{\|k_x^\Theta\|} \leq \sqrt{\frac{\|\Theta'\|_\infty}{2\pi}}, \quad x, t \in \mathbb{R}. \quad (4.1)$$

Let $k_z^{\Theta_c \Theta} \in K_{\Theta_c \Theta}^2$ be the reproducing kernel functions in $K_{\Theta_c \Theta}^2$. Denote by $\{x_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ the sequence associated in (3.9) to $\Theta_c \Theta$, that is,

$$cx_n + \varphi(x_n) = \gamma + 2\pi n, \quad \gamma \in [0, 2\pi), \quad n \in \mathbb{Z}.$$

For $c > 0$ to be chosen later, set $\alpha = \frac{c}{4}$ and $\Xi(x) = \operatorname{sinc}(x)$. First, from Theorem 3.2 (applied to $m = 2$) Fubini's theorem, and the bound (4.1) we get

$$\begin{aligned}
\int_{\mathbb{R}} |f(x)|^2 \, d\mu(x) &= \int_{\mathbb{R}} \left| \sum_{n \in \mathbb{Z}} f(x_n) e^{-\frac{ic(x - x_n)}{2}} \operatorname{sinc}\left(\frac{c(x - x_n)}{4}\right) \frac{k_{x_n}^{\Theta_c \Theta}(x)}{\|k_{x_n}^{\Theta_c \Theta}\|^2} \right|^2 \, d\mu(x) \\
&\leq \int_{\mathbb{R}} \sum_{n, k \in \mathbb{Z}} \frac{|f(x_n) f(x_k)|}{\|k_{x_n}^{\Theta_c \Theta}\| \|k_{x_k}^{\Theta_c \Theta}\|} |\Xi(\alpha(x - x_n)) \Xi(\alpha(x - x_k))|^2 \frac{|k_{x_n}^{\Theta_c \Theta}(x) k_{x_k}^{\Theta_c \Theta}(x)|}{\|k_{x_n}^{\Theta_c \Theta}\| \|k_{x_k}^{\Theta_c \Theta}\|} \, d\mu(x) \\
&\leq \frac{c + \|\Theta'\|_\infty}{2\pi} \sum_{n, k \in \mathbb{Z}} \frac{|f(x_n) f(x_k)|}{\|k_{x_n}^{\Theta_c \Theta}\| \|k_{x_k}^{\Theta_c \Theta}\|} \int_{\mathbb{R}} \Xi(\alpha(x - x_n))^2 \Xi(\alpha(x - x_k))^2 \, d\mu(x).
\end{aligned}$$

Applying the estimate (2.2) with $\eta\alpha$ (instead of δ), η to be chosen, we get

$$\int_{\mathbb{R}} |f(x)|^2 d\mu(x) \leq 4 \frac{C_{\eta\alpha}^2}{\eta\alpha} (c + \|\Theta'\|_{\infty}) \sum_{n,k \in \mathbb{Z}} \frac{|f(x_n)f(x_k)| \sup_{x \in \mathbb{R}} \mu([x, x + \eta])}{\|k_{x_n}^{\Theta_c \Theta}\| \|k_{x_k}^{\Theta_c \Theta}\| 4 + \alpha^2(x_n - x_k)^2} \quad (4.2)$$

with $C_{\eta\alpha} = \max(4, 1 + 9\eta^2\alpha^2)$. We will chose $\eta \leq \delta$ with $\eta\alpha \leq 1/\sqrt{3}$, i.e., $\eta \leq \frac{4}{\sqrt{3}c}$, so that $C_{\eta\alpha} = 4$. Now, by (4.1),

$$\begin{aligned} 2\pi|n - k| &= |cx_n + \varphi(x_n) - cx_k - \varphi(x_k)| \\ &\leq (c + \|\varphi'\|_{\infty})|x_n - x_k| = (c + \|\Theta'\|_{\infty})|x_n - x_k|. \end{aligned}$$

Set $u = (u_k)_k$ with $u_k = \frac{|f(x_k)|}{\|k_{x_k}^{\Theta_c \Theta}\|}$ so that $\|u\|_{\ell^2(\mathbb{Z})}^2 = \|f\|^2$ with (3.11). Moreover, we set

$$\lambda = \frac{\pi^2\alpha^2}{(c + \|\Theta'\|_{\infty})^2} = \left(\frac{\pi c}{4(c + \|\Theta'\|_{\infty})} \right)^2,$$

$v = (v_k)_k$ with $v_k = \frac{1}{1 + \lambda k^2}$ and notice that

$$\|v\|_{\ell^1(\mathbb{Z})} = 1 + 2 \sum_{k \geq 1} \frac{1}{1 + \lambda k^2} \leq 1 + 2 \int_0^{\infty} \frac{dx}{1 + \lambda x^2} = 1 + \frac{\pi}{\sqrt{\lambda}} = 1 + \frac{4(c + \|\Theta'\|_{\infty})}{c}.$$

Then

$$\begin{aligned} \sum_{n,k \in \mathbb{Z}} \frac{|f(x_n)f(x_k)|}{\|k_{x_n}^{\Theta_c \Theta}\| \|k_{x_k}^{\Theta_c \Theta}\| 4 + \alpha^2(x_n - x_k)^2} &\leq \frac{1}{4} \sum_{n \in \mathbb{Z}} u_n \sum_{k \in \mathbb{Z}} \frac{u_k}{1 + \lambda(n - k)^2} \\ &= \frac{1}{4} \langle u, u * v \rangle \leq \frac{1}{4} \|u\|_{\ell^2(\mathbb{Z})}^2 \|v\|_{\ell^1(\mathbb{Z})}. \end{aligned}$$

All in one, we get

$$\int_{\mathbb{R}} |f(x)|^2 d\mu(x) \leq \frac{16}{\eta\alpha} M(\eta) (c + \|\Theta'\|_{\infty}) \left(1 + 4 \frac{c + \|\Theta'\|_{\infty}}{c} \right) \|f\|^2,$$

where $M(\eta) := \sup_{x \in \mathbb{R}} \mu([x, x + \eta])$. It remains to chose c, η . To do so, we distinguish two cases.

Case 1. If $\|\Theta'\|_{\infty} \geq \frac{4}{\sqrt{3}\delta}$, we take $c = \|\Theta'\|_{\infty}$ and $\eta = \frac{4}{\sqrt{3}c} \leq \delta$ thus $\eta\alpha = \frac{1}{\sqrt{3}}$. Then

$$\int_{\mathbb{R}} |f(x)|^2 d\mu(x) \leq 288\sqrt{3} M(\eta) \|\Theta'\|_{\infty} \|f\|^2 \leq 500 \left(\frac{1}{\delta} + \|\Theta'\|_{\infty} \right) M(\delta) \|f\|^2.$$

Case 2. If $\|\Theta'\|_{\infty} \leq \frac{4}{\sqrt{3}\delta}$, we take $c = \frac{4}{\sqrt{3}\delta}$ and $\eta = \frac{4}{\sqrt{3}c} = \delta$ so that

$$\int_{\mathbb{R}} |f(x)|^2 d\mu(x) \leq 576 \left(\frac{1}{\delta} + \|\Theta'\|_{\infty} \right) M(\delta) \|f\|^2,$$

as desired. \square

Remark 4.1. In the proof, we have used the oversampling formula with $m = 2$. One may of course increase m and this would lead to slightly better numerical constants, at the price of much higher technicality. This requires in particular to replace Lemma 2.1 with (2.3). As the ultimate constants would anyway be worse than those we find in the next section, we did not follow this path further.

5. AN L^p -VERSION OF THEOREM 1.2 AND BERNSTEIN-TYPE INEQUALITIES

The previous proof gives a rather large constant C in the inequality (1.7). We present another proof that gives a better estimate; in particular, we have the constant 1 in front of $\frac{1}{\delta}$, as in the classical Donoho–Logan theorem. Also, this proof applies to L^p analogs of the model spaces, the subspaces K_Θ^p of the Hardy space H^p , where $1 \leq p < \infty$.

Recall that for $1 \leq p \leq \infty$ the subspace K_Θ^p is defined as

$$K_\Theta^p = H^p \cap \Theta \overline{H^p},$$

where H^p is understood as a closed subspace of $L^p(\mathbb{R})$. This definition agrees with the one given for K_Θ^2 earlier. The properties of the spaces K_Θ^p are very much similar to the properties of K_Θ^2 . In particular, if Θ is a meromorphic inner function, then all elements of K_Θ^p are meromorphic functions in \mathbb{C} , while for $\Theta(z) = e^{2icz}$ we have $K_\Theta^p = e^{icz} PW_c^p(\mathbb{R})$, where $PW_c^p(\mathbb{R})$ is the space of all entire functions of exponential type at most c whose restriction to \mathbb{R} belongs to $L^p(\mathbb{R})$.

Theorem 5.1. *Let $1 \leq p < \infty$, $\Theta' \in L^\infty(\mathbb{R})$, μ be a Borel measure on \mathbb{R} , and let $\delta > 0$ and $D_\mu(\delta)$ defined in (1.4). Then for any $f \in K_\Theta^p$ we have*

$$\|f\|_{L^p(\mu)}^p \leq (1 + \delta \|\Theta'\|_\infty)^p D_\mu(\delta) \|f\|_p^p. \quad (5.1)$$

The proof is based on a Bernstein-type inequality for model spaces K_Θ^p with $\Theta' \in L^\infty(\mathbb{R})$ which is due to Dyakonov [13, 14].

Theorem 5.2 (Dyakonov). *If $\Theta' \in L^\infty(\mathbb{R})$, and $1 \leq p \leq \infty$, and $f \in K_\Theta^p$, then*

$$\|f'\|_p \leq \|\Theta'\|_\infty \|f\|_p.$$

Dyakonov proved this inequality up to some constant $C(p)$ in the right-hand side. Here we present a very simple proof with the constant 1 which follows the method from [6] and is based on the formula

$$f'(x) = 2\pi i \int_{\mathbb{R}} f(t) (\overline{k_t^\Theta(x)})^2 dt, \quad f \in K_\Theta^p, \quad x \in \mathbb{R}. \quad (5.2)$$

Proof of Dyakonov's theorem. For $1 < p < \infty$, it follows from (5.2), (3.7) and the Hölder inequality that

$$\begin{aligned} |f'(x)|^p &\leq 2\pi \int_{\mathbb{R}} |f(t)|^p |k_x^\Theta(t)|^2 dt \left(2\pi \int_{\mathbb{R}} |k_x^\Theta(t)|^2 dt \right)^{p/q} \\ &= 2\pi |\Theta'(x)|^{p/q} \int_{\mathbb{R}} |f(t)|^p |k_x^\Theta(t)|^2 dt \leq 2\pi \|\Theta'\|_\infty^{p/q} \int_{\mathbb{R}} |f(t)|^p |k_x^\Theta(t)|^2 dt. \end{aligned}$$

Since $\int_{\mathbb{R}} |k_x^\Theta(t)|^2 dx = |\Theta'(t)|/(2\pi)$, we get the result integrating over x . The cases $p = 1$ or $p = \infty$ follow trivially from (5.2). \square

We will also need the following elementary lemma.

Lemma 5.3. *Let $f, f' \in L^p(\mathbb{R})$, $1 \leq p < \infty$. Then for any $t_k \in [k\delta, (k+1)\delta] =: I_k$, $k \in \mathbb{Z}$, one has*

$$\left(\sum_{k \in \mathbb{Z}} |f(t_k)|^p \right)^{1/p} \leq \delta^{-1/p} \|f\|_p + \delta^{1-1/p} \|f'\|_p. \quad (5.3)$$

Proof. From the triangle inequality,

$$\begin{aligned} \left(\sum_{k \in \mathbb{Z}} |f(t_k)|^p \right)^{1/p} &= \left(\frac{1}{\delta} \sum_{k \in \mathbb{Z}} \int_{I_k} |f(t_k)|^p dt \right)^{1/p} \\ &\leq \left(\frac{1}{\delta} \sum_{k \in \mathbb{Z}} \int_{I_k} |f(t)|^p dt \right)^{1/p} + \left(\frac{1}{\delta} \sum_{k \in \mathbb{Z}} \int_{I_k} |f(t) - f(t_k)|^p dt \right)^{1/p}. \end{aligned}$$

Note that by Hölder's inequality

$$\int_{I_k} |f(t_k) - f(t)|^p dt = \int_{I_k} \left| \int_t^{t_k} f'(s) ds \right|^p dt \leq \delta^{p/q} \int_{I_k} \int_t^{t_k} |f'(s)|^p ds dt \leq \delta^p \int_{I_k} |f'(s)|^p ds.$$

Summing these estimates we obtain (5.3). \square

Proof of Theorem 5.1. Put $M(\delta) = \sup_{x \in \mathbb{R}} \mu([x, x + \delta]) = \delta D_\mu(\delta)$. Let $t_k \in I_k$ be such that $|f(t_k)| = \sup_{I_k} |f|$. Then

$$\int_{\mathbb{R}} |f(x)|^p d\mu(x) \leq \sum_{k \in \mathbb{Z}} \mu([k\delta, (k+1)\delta]) \cdot |f(t_k)|^p \leq M(\delta) (\delta^{-1/p} \|f\|_p + \delta^{1-1/p} \|f'\|_p)^p.$$

It follows from Dyakonov's theorem that

$$\int_{\mathbb{R}} |f(x)|^p d\mu(x) \leq \frac{M(\delta)}{\delta} (1 + \delta \|\Theta'\|_\infty)^p \|f\|_p^p$$

as claimed. \square

Remark 5.4. The dependence on the parameter δ and on $\|\Theta'\|_\infty$ in (5.1) is sharp up to the numerical constant. Indeed, for the classical Paley–Wiener space $PW_c^2(\mathbb{R})$, our method (applied to $PW_c^2(\mathbb{R})$ in place of K_Θ^2 with $\Theta(z) = e^{2icz}$) gives only a factor $(1 + 2\delta c)$, while the Donoho–Logan bound is $1 + \frac{c\delta}{\pi}$.

If μ is given by $d\mu(x) = \mathbf{1}_T(x) dx$ and one considers the Paley–Wiener spaces, it is common to choose δ to be the reciprocal of the bandwidth $2c$ and write the estimates in terms of the *maximum Nyquist density*

$$D_{\max}(T, c) := \sup_{x \in \mathbb{R}} 2c \cdot |T \cap [x, x + 1/(2c)]|.$$

Donoho and Logan's results then read

$$\|f \cdot \mathbf{1}_T\|_p^p \leq C_p \cdot D_{\max}(T, c) \cdot \|f\|_p^p,$$

for $p = 1, 2$, with constants $C_1 = \text{sinc}(1/4)^{-1} \approx 1.0105$ and $C_2 = (1 + 1/2\pi) \approx 1.1592$, while our estimate (5.1) gives $C_1 = 2$ and $C_2 = 4$.

6. A DONOHO–LOGAN THEOREM FOR ONE COMPONENT INNER FUNCTIONS

For general inner functions, even if we confine ourselves with meromorphic ones, one cannot expect to obtain estimates of Donoho–Logan type considering intervals of a fixed length. The reason is that the behaviour of inner functions on \mathbb{R} can be very irregular. On the other hand, for meromorphic inner functions with a sublinear growth of the argument the assumption that the measure is uniformly bounded on intervals of a fixed length can be too strong. It looks more natural then to consider the intervals where the change of the argument of Θ is fixed, *i.e.*, consider the intervals $I = [a, b]$ such that $\varphi(b) - \varphi(a) = \delta$, where φ is an increasing continuous argument for Θ on \mathbb{R} . There exists a class of inner functions for which such a generalization is possible. These are the so-called *one-component inner functions*, that is, those inner functions for which the sublevel set

$$\Omega(\Theta, \varepsilon) = \{z \in \mathbb{C}^+ : |\Theta(z)| < \varepsilon\}$$

is connected for some $\varepsilon \in (0, 1)$. One-component inner functions were introduced by W.S. Cohn [11] in connection to Carleson-type embeddings of model spaces and were subsequently studied by many authors (see, *e.g.*, [4, 5, 6, 7, 20]).

Several important properties of one-component inner functions were obtained by A.B. Aleksandrov [4]. Recall that the spectrum $\rho(\Theta)$ of Θ (see (3.4)) is a closed set, and if we write $\mathbb{R} \setminus \rho(\Theta) = \cup_n J_n$, where J_n are disjoint open intervals, then Θ admits an analytic continuation through any J_n . Hence, Θ has an increasing C^∞ branch of the argument φ on each J_n . It is shown in [4] that $\rho(\Theta)$ has zero Lebesgue measure and, if we additionally assume that $\infty \in \rho(\Theta)$, the function $\frac{1}{\varphi'} = \frac{1}{|\Theta'|}$ (defined as 0 on $\rho(\Theta)$) is a Lipschitz function on \mathbb{R} . Moreover, it is shown in [6] that in this case for any $\varepsilon \in (0, 1)$ there exist positive constants c_1, c_2 depending on ε only such that

$$c_1 |\Theta'(x)|^{-1} \leq \text{dist}(x, \Omega(\Theta, \varepsilon)) \leq c_2 |\Theta'(x)|^{-1}, \quad x \in \mathbb{R}. \quad (6.1)$$

Therefore, there exist $A, B > 0$ such that

$$A \leq \frac{\varphi'(s)}{\varphi'(t)} \leq B, \quad |\varphi(s) - \varphi(t)| \leq 1. \quad (6.2)$$

These properties essentially characterize the class of one-component inner functions.

Also in [6] the following Bernstein type inequality was proved: if Θ is a one-component inner function and $1 < p < \infty$, then

$$\|f'/\Theta'\|_p \leq C(\Theta, p) \|f\|_p, \quad f \in K_\Theta^p. \quad (6.3)$$

The next theorem applies to general one-component (not necessarily meromorphic) functions.

Theorem 6.1. *Let Θ be a one-component inner function such that $\infty \in \rho(\Theta)$, $1 < p < \infty$. Let φ denote the branch of the argument of Θ which is an increasing C^∞ function on each subinterval of $\mathbb{R} \setminus \rho(\Theta)$. For a Borel measure μ on \mathbb{R} and $\delta > 0$ put*

$$D_\mu^\Theta(\delta) = \sup \left\{ \frac{\mu(I)}{|I|} : I = [a, b], \varphi(b) - \varphi(a) = \delta \right\}.$$

Then for any $f \in K_\Theta^p$ we have

$$\int_{\mathbb{R}} |f|^p d\mu \leq D_\mu^\Theta(\delta)(1 + C\delta)^p \|f\|_p^p, \quad (6.4)$$

where the constant $C > 0$ depends on Θ and p only.

Proof. The proof is analogous to the proof of Theorem 5.1. It follows from (6.1) that φ is unbounded on any connected component of $\mathbb{R} \setminus \rho(\Theta)$. Therefore, we can write $\mathbb{R} \setminus \rho(\Theta) = \cup_k I_k$, where $I_k = [a_k, b_k]$ are intervals with disjoint interiors such that $\varphi(b_k) - \varphi(a_k) = \delta$. Let $D = D_\mu^\Theta(\delta)$. Any function $f \in K_\Theta^p$ admits an analytic continuation through $\mathbb{R} \setminus \rho(\Theta)$. Choose $t_k \in I_k$ such that $|f(t_k)| = \max_{I_k} |f|$. Then we have

$$\begin{aligned} \|f\|_{L^p(\mu)} &\leq \left(\sum_k |f(t_k)|^p \mu(I_k) \right)^{1/p} \leq D^{1/p} \left(\sum_k \int_{I_k} |f(t_k)|^p dt \right)^{1/p} \\ &\leq M^{1/p} \left(\sum_k \int_{I_k} |f(t)|^p dt \right)^{1/p} + D^{1/p} \left(\sum_k \int_{I_k} |f(t) - f(t_k)|^p dt \right)^{1/p}. \end{aligned}$$

Furthermore, if p' is the dual index to p , $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\begin{aligned} \sum_k \int_{I_k} |f(t) - f(t_k)|^p dt &= \sum_k \int_{I_k} \left| \int_{t_k}^t f'(s) ds \right|^p dt \\ &\leq \sum_k \int_{I_k} \int_{t_k}^t \frac{|f'(s)|^p}{|\Theta'(s)|^p} ds \left(\int_{t_k}^t |\Theta'(s)|^{p'} ds \right)^{p/p'} dt. \end{aligned}$$

For any k there exists $s_k \in I_k$ such that $\delta = \varphi(b_k) - \varphi(a_k) = \varphi'(s_k) \cdot |I_k|$. In view of (6.2) we have $|\Theta'(s)| \cdot |I_k| \leq C_1 \delta$, $s \in I_k$, whence

$$\left(\int_{t_k}^t |\Theta'(s)|^q ds \right)^{p/p'} \leq C_2 \left(\frac{\delta^{p'}}{|I_k|^{p'-1}} \right)^{p/p'} = C_2 \frac{\delta^p}{|I_k|}.$$

for some constants $C_1, C_2 > 0$. Thus,

$$\sum_k \int_{I_k} \int_{t_k}^t \frac{|f'(s)|^p}{|\Theta'(s)|^p} ds \left(\int_{t_k}^t |\Theta'(s)|^{p'} ds \right)^{p/p'} dt \leq C_2 \frac{\delta^p}{|I_k|} \sum_k \int_{I_k} \int_{I_k} \frac{|f'(s)|^p}{|\Theta'(s)|^p} ds dt \leq C_3 \delta^p \|f\|_p^p$$

by (6.3). We have shown that

$$\|f\|_{L^p(\mu)} \leq D^{1/p}(1 + C_4\delta)\|f\|_p$$

which proves the theorem. \square

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