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# Lie's derivative in fluid mechanics

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## Abstract

The invariance theorems obtained in analytical mechanics and derived from Noether's theorems can be adapted to fluid mechanics. For this purpose, it is useful to give a functional representation of the fluid motion and to interpret the invariance group with respect to time in the quadri-dimensional reference space of Lagrangian variables. A powerful method of calculation uses Lie's derivative, and many invariance theorems and conservation laws can be obtained in fluid mechanics.

**Keywords:** Fluid mechanics; Lie's derivative; Invariance properties; Conservation laws

## 1 Introduction

In analytical mechanics of systems with a finite number of degrees of freedom, Noether's theorem is expressed as follows [1]:

We consider a differentiable curve ( $C$ ) of  $\mathbb{R}^n$  and a form field in the dual  $R^{n*}$  of  $\mathbb{R}^n$ ,  $\mathbf{M} \in \mathbb{R}^n \longrightarrow \boldsymbol{\Xi}^T(\mathbf{M}) \in \mathbb{R}^{n*}$ . Then

$$a = \int_{(C)} \boldsymbol{\Xi}^T(\mathbf{M}) d\mathbf{M}$$

is a functional of ( $C$ ) denoted  $a = \mathcal{G}(C)$ . We can look for paths such that  $a$  is extremal.

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We assume that  $\{\mathcal{T}_u\}$ ,  $u \in I$ , where  $I$  is an open set containing 0, is a Lie group generating the solutions of autonomous differential equation

$$\frac{d\mathbf{M}}{du} = G(\mathbf{M}),$$

where the vector field  $G(\mathbf{M})$  is the infinitesimal operator of the Lie group. We call  $\mathcal{T}_u(C)$  the transformed curve of  $(C)$  by  $\mathcal{T}_u$ . If  $a$  is invariant by  $\mathcal{T}_u$  (i.e.  $\forall u \in I, \mathcal{G}(T_u(C)) = \mathcal{G}(C)$ ), along extremal of  $a$  we obtain the relation

$$\Xi^T(\mathbf{M}) G(\mathbf{M}) = c, \quad (1)$$

where  $c$  is constant. Relation (1) is the analytical form of Noether's theorem.

We show that for fluid media, the velocity field in space-time plays a role analogous to the vector field  $G(\mathbf{M})$  and Hamilton's action states theorems of invariance allowing the search of first integrals of motion [2]; this is Noether's theorem generalized to fluids [3]. Since continuous media cannot be represented by a finite number of parameters, the forms of the first integrals are expressed as conservation laws [4]. The natural method related to the definition of motions of a fluid uses a group with one-time parameter corresponding to a diffeomorphism representing the fluid motion at each instant. The velocity field in space-time is the infinitesimal generator of this group [5]. Its main advantage is to use the variance of specific medium quantities which have the structure of scalars, vectors, forms and tensors covariant order  $p$  and contravariant order  $q$ . When they do not depend on the trajectories, the quantities can be defined on the reference space independently of the motions [6]. They are related to the Lie derivative associated with the velocity field in the time-space of the fluid motion [7]. Their variance allows us to use the linear tangent application associated with the motion.

Conservative laws are well studied in the literature [8, 9, 10, 11, 12]. In our article, we obtain conservation laws in a geometrical form, the simplest forms are associated with scalar fields allowing to write conservation laws in a time-space divergence form. We analyse the different tensors classically obtained in the literature and we can obtain first integrals of motions.

*Notations:* For any vectors  $\mathbf{a}, \mathbf{b}$ , term  $\mathbf{a}^T \mathbf{b}$  denotes the scalar product (line vector  $\mathbf{a}^T$  is multiplied by column vector  $\mathbf{b}$ ) and tensor  $\mathbf{a} \mathbf{b}^T$  (or  $\mathbf{a} \otimes \mathbf{b}$ ) denotes the product of column vector  $\mathbf{a}$  by line vector  $\mathbf{b}^T$ , where superscript

$T$  denotes the transposition. Tensor  $\mathbf{I}$  denotes the identity transformation. The gradient of scalar function  $f(\mathbf{x})$  is the transposition  $\left(\frac{\partial f}{\partial \mathbf{x}}\right)^T$  associated with linear form  $df = \left(\frac{\partial f}{\partial \mathbf{x}}\right) d\mathbf{x}$  and  $\frac{\partial \mathbf{v}(\mathbf{x})}{\partial \mathbf{x}}$  denotes the linear application defined by the relation  $d\mathbf{v} = \left(\frac{\partial \mathbf{v}(\mathbf{x})}{\partial \mathbf{x}}\right) d\mathbf{x}$ .

## 2 Lie's derivative associated with a fluid motion

### 2.1 Generalities

The fluid motion is represented by a  $t$ -dependent  $C^2$ -diffeomorphism  $\varphi_t$  [13]:

$$\mathbf{X} \in \mathcal{D}_0 \longrightarrow \mathbf{x} \in \mathcal{D}_t,$$

such that  $\mathbf{x} = \varphi_t(\mathbf{X})$  is a mapping from a reference space  $\mathcal{D}_0$  of Lagrange variables  $\mathbf{X} = (X_1, X_2, X_3)^T$  into the space  $\mathcal{D}_t$  of Euler variables  $\mathbf{x} = (x_1, x_2, x_3)^T$  occupied by the fluid at time  $t$ .

The linear tangent mapping of  $\varphi_t$  is defined by:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \equiv \frac{\partial \varphi_t(\mathbf{X})}{\partial \mathbf{X}}.$$

By derivation with respect to time, we get:

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{F} \quad \text{and} \quad \frac{d\mathbf{F}^{-1}}{dt} = -\mathbf{F}^{-1} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \quad (2)$$

where  $\mathbf{u}$  is the velocity in  $\mathcal{D}_t$ .

The motion of the fluid can be also represented by the invertible differentiable mapping  $\Phi$  such that [5, 13]:

$$\mathbf{z} = \Phi(\mathbf{Z}) \quad \text{with} \quad \mathbf{z} = \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix} \equiv \begin{bmatrix} t \\ \varphi_t(\mathbf{X}) \end{bmatrix} \in \mathcal{W} \quad \text{and} \quad \mathbf{Z} = \begin{bmatrix} t \\ \mathbf{X} \end{bmatrix} \in \mathcal{W}_0,$$

where  $\mathcal{W}$  is the 4- $D$  physical space-time and  $\mathcal{W}_0$  is the 4- $D$  reference space.

In differential manifold  $\mathcal{W}$ , the space-time velocity is  $\mathcal{U} = \begin{bmatrix} 1 \\ \mathbf{u} \end{bmatrix}$ .

In each point of  $\mathcal{W}$ , the 4- $D$  vector  $\mathcal{U}$  generates a local one-parameter group defined by the mapping

$$(h, \mathbf{z}) \in ] - \epsilon, \epsilon[ \times \mathcal{O} \longrightarrow \Theta(h, \mathbf{z}) \in \mathcal{W},$$

such that

$$\Theta(h, \mathbf{z}) = \left[ \begin{array}{c} h + t \\ \varphi_{h+t}(\varphi_t^{-1}(\mathbf{x})) \end{array} \right] \equiv \left[ \begin{array}{c} h + t \\ \varphi_{h+t}(\mathbf{X}) \end{array} \right],$$

where  $\mathcal{O}$  is an open set of  $\mathcal{W}$  and  $\epsilon$  a positive real number. The velocity field  $\mathcal{U}$  is the infinitesimal displacement of the one-parameter group associated with transformation  $\Phi$ .

We consider tensor fields with covariant order  $p$  and contravariant order  $q$  applied to the vectors of the tangent space to  $\mathcal{D}_t$  and the forms of its dual (or covector of cotangent space) at  $\mathbf{x}$ , respectively; we said tensors in vector space  $T_{\mathbf{x}}^{p,q^*}(\mathcal{D}_t)$ . At each point  $\mathbf{x}$  of  $\mathcal{D}_t$  we consider the fiber-space (fiber-bundle) which is tangent order  $p$  and cotangent order  $q$  [14].

To each tensor field  $\mathbf{v} \in T_{\varphi_t(\mathbf{X})}^{p,q^*}(\mathcal{D}_t)$  when  $\mathbf{z} \in \mathcal{O}$ , we associate its image in the tensor space  $T_{\varphi_{h+t}(\mathbf{X})}^{p,q^*}(\mathcal{D}_{t+h})$  when  $\Theta(h, \mathbf{z}) \in \mathcal{O}$ . We denote the image:

$$\mathbf{v}_h = \theta_h(\mathbf{v}).$$

**Definition 1** *A tensor field is moving with the fluid if and only if its image in the reference space represented in Lagrange variables is independent of the time.*

The definition is an extension of set  $\mathcal{E}_0$  of  $\mathcal{D}_0$  and its image  $\mathcal{E}$ :

$$\mathcal{E} = \{ \mathbf{x} \text{ such that } \mathbf{x} = \varphi(\mathbf{X}), \mathbf{X} \in \mathcal{E}_0 \},$$

when time  $t$  varies. The set  $\mathcal{E}$  is always made of the same particles. The set  $\mathcal{E}$  is material or moving with the fluid. This is in particular the case of curves, surfaces and volumes of  $\mathcal{D}_0$ .

To this definition, we associate a special derivative of the tensor fields.

**Definition 2** *The Lie derivative of  $\mathbf{v}$  is  $d_L \mathbf{v} = \lim_{h \rightarrow 0} \left( \frac{\theta_h(\mathbf{v}(\mathbf{z})) - \mathbf{v}(\mathbf{z})}{h} \right)$ .*

To interpret  $d_L \mathbf{v}$  in fluid mechanics,  $\mathcal{W}_0$  is a convenient space. Let  $\mathbf{v}_0(\mathbf{Z})$  the image of  $\mathbf{v}(\mathbf{z})$  by  $\tilde{\Psi}$  the mapping for the tensor field corresponding to  $\Psi = \Phi^{-1}$  and we denote  $\tilde{\Phi}$  the mapping corresponding to  $\Phi$ . We obtain

$$\begin{aligned} d_L \mathbf{v} &= \lim_{h \rightarrow 0} \left( \frac{\tilde{\Phi} \left( \tilde{\Psi} (\theta_h(\mathbf{v}(\mathbf{z}))) - \tilde{\Psi}(\mathbf{v}(\mathbf{z})) \right)}{h} \right) \\ &= \lim_{h \rightarrow 0} \tilde{\Phi} \left( \frac{\mathbf{v}_0(t+h, \mathbf{X}) - \mathbf{v}_0(t, \mathbf{X})}{h} \right). \end{aligned}$$

Then,

$$d_L \mathbf{v} = \tilde{\Phi} \left( \frac{d\mathbf{v}_0(t, \mathbf{X})}{dt} \right)$$

corresponds to the commutative graph:

$$\begin{array}{ccc} \mathbf{v} \in T_{\mathbf{x}}^{p,q^*}(\mathcal{D}_t) & \xrightarrow{\tilde{\Psi}} & \mathbf{v}_0 \in T_{\mathbf{X}}^{p,q^*}(\mathcal{D}_0) \\ d_L \downarrow & & \downarrow \frac{d}{dt} \\ d_L \mathbf{v} \in T_{\mathbf{x}}^{p,q^*}(\mathcal{D}_t) & \xleftarrow{\tilde{\Phi}} & \frac{d\mathbf{v}_0(t, \mathbf{X})}{dt} \in T_{\mathbf{X}}^{p,q^*}(\mathcal{D}_0) \end{array}$$

To consider tensor fields, we can also consider restrictions of tensors defined on  $\mathcal{W}$  at  $t$  given (the Lie derivative does not commute with the restriction). To consider tensor fields, we can represent their images in  $\mathcal{W}_0$  and we assume that the images depends only on  $\mathbf{X}$  and do not depend on the trajectory parameter (i.e. time). Such quantities are called *moving with the fluid* and its definition domain is  $\mathcal{D}_0$ . From the previous results, we can write:

**Theorem 3** *A tensor is moving with the fluid if and only if its Lie derivative associated with the vector field  $\mathcal{U}$  is zero.*

## 2.2 Lie's derivative of tensor fields

### 2.2.1 Scalar field

A scalar field is moving with the fluid if and only its Lie's derivative is null (which corresponds to its material derivative)

$$\exists \{ \mathbf{X} \in \mathcal{D}_0 \longrightarrow s_0(\mathbf{X}) \} \quad \text{such that} \quad s(t, \mathbf{x}) = s_0(\mathbf{X}).$$

For example, in isentropic (conservative) motions, the specific entropy is moving with the fluid:

$$\frac{ds}{dt} \equiv \frac{\partial s}{\partial t} + \frac{\partial s}{\partial \mathbf{x}} \mathbf{u} = 0,$$

where the  $d/dt$  means the material derivative.

### 2.2.2 Vector field

The tangent linear mapping  $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$  transforms vectors of the tangent vector space  $T_{\mathbf{X}}^1(\mathcal{D}_0)$  at  $\mathbf{X} \in \mathcal{D}_0$  into vectors of the tangent vector space  $T_{\mathbf{x}}^1(\mathcal{D}_t)$  at  $\mathbf{x} \in \mathcal{D}_t$ .

Let us define a vector field  $\mathbf{J}$  of  $\mathcal{D}_t$  by the mapping:

$$\mathbf{z} \in W \longrightarrow \mathbf{J}(\mathbf{x}, t) \in T_{\mathbf{x}}^1(\mathcal{D}_t).$$

The Lie derivative  $d_L$  of the vector field  $\mathbf{J}$  is deduced from the commutative diagram:

$$\begin{array}{ccc} \mathbf{J} \in T_{\mathbf{x}}^1(\mathcal{D}_t) & \xrightarrow{\tilde{\Psi}} & \mathbf{F}^{-1} \mathbf{J} \in T_{\mathbf{X}}^1(\mathcal{D}_0) \\ d_L \downarrow & & \downarrow \frac{d}{dt} \\ \frac{d\mathbf{J}}{dt} - \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \mathbf{J} \in T_{\mathbf{x}}^1(\mathcal{D}_t) & \xleftarrow{\tilde{\Phi}} & \frac{d\mathbf{F}^{-1}}{dt} \mathbf{J} + \mathbf{F}^{-1} \frac{d\mathbf{J}}{dt} \in T_{\mathbf{X}}^1(\mathcal{D}_0) \end{array}$$

Consequently, the Lie derivative of  $\mathbf{J}$  is:

$$d_L \mathbf{J} = \frac{d\mathbf{J}}{dt} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{J} \equiv \frac{\partial \mathbf{J}}{\partial t} + \frac{\partial \mathbf{J}}{\partial \mathbf{x}} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{J}$$

Let us note that  $\frac{d\mathbf{J}}{dt} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{J}$  is the convective derivative of  $\mathbf{J}$  with respect to the velocity field  $\mathcal{U}$ .

**Theorem 4**  $d_L \mathbf{J} = \mathbf{0}$  if and only if

$$\exists \{ \mathbf{X} \in \mathcal{D}_0 \longrightarrow \mathbf{J}_0(\mathbf{X}) \in T_{\mathbf{X}}^1(\mathcal{D}_0) \} \quad \text{such that} \quad \mathbf{J}(t, \mathbf{x}) = \mathbf{F} \mathbf{J}_0(\mathbf{X})$$

### 2.2.3 Form field

Let us denote  $T_{\mathbf{x}}^{1*}(\mathcal{D}_t)$  the cotangent linear space of  $\mathcal{D}_t$  at  $\mathbf{x}$  and  $T_{\mathbf{X}}^{1*}(\mathcal{D}_o)$  the cotangent linear space of  $\mathcal{D}_o$  at  $\mathbf{X}$ , and consider a field  $\mathbf{C}$  of forms (covectors) of  $\mathcal{D}_t$ :

$$\mathbf{z} \in \mathcal{W} \longrightarrow \mathbf{C}(\mathbf{x}, t) \in T_{\mathbf{x}}^{1*}(\mathcal{D}_t).$$

The Lie derivative  $d_L$  of the form field  $\mathbf{C}$  is deduced from the commutative diagram:

$$\begin{array}{ccc} \mathbf{C} \in T_{\mathbf{x}}^{1*}(\mathcal{D}_t) & \xrightarrow{\tilde{\Psi}} & \mathbf{C} \mathbf{F} \in T_{\mathbf{X}}^{1*}(\mathcal{D}_o) \\ d_L \downarrow & & \downarrow \frac{d}{dt} \\ \frac{d\mathbf{C}}{dt} + \mathbf{C} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \in T_{\mathbf{x}}^{1*}(\mathcal{D}_t) & \xleftarrow{\tilde{\Phi}} & \frac{d\mathbf{C}}{dt} \mathbf{F} + \mathbf{C} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{F} \in T_{\mathbf{X}}^{1*}(\mathcal{D}_o) \end{array}$$

Consequently, the Lie derivative of  $\mathbf{C}$  is:

$$d_L \mathbf{C} = \frac{d\mathbf{C}}{dt} + \mathbf{C} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \equiv \frac{\partial \mathbf{C}}{\partial t} + \mathbf{u}^T \left( \frac{\partial \mathbf{C}^T}{\partial \mathbf{x}} \right)^T + \mathbf{C} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}.$$

**Theorem 5**  $d_L \mathbf{C} = \mathbf{0}$  if and only if

$$\exists \{ \mathbf{X} \in \mathcal{D}_o \longrightarrow \mathbf{C}_0(\mathbf{X}) \in T_{\mathbf{X}}^{1*}(\mathcal{D}_o) \} \quad \text{such that} \quad \mathbf{C}(t, \mathbf{x}) = \mathbf{C}_0(\mathbf{X}) \mathbf{F}^{-1}.$$

For all form field  $\mathbf{z} \in \mathcal{W} \longrightarrow \mathbf{C}(t, \mathbf{x}) \in T_{\mathbf{x}}^{1*}(\mathcal{D}_t)$  and for all curve  $(\gamma)$  in  $\mathcal{D}_t$ , we associate the curvilinear integral  $I = \int_{\gamma} \mathbf{C}(t, \mathbf{x}) d\mathbf{x}$ .

**Corollary 6** *A form field is moving with the fluid if and only is, for all transported vector field, the associated scalar product is a transported scalar field.*

**Theorem 7** *Form  $\mathbf{C}$  being a transported form field, its integral along any fluid curve is constant. If we denote by  $\gamma_0$  the image of  $\gamma$  in  $\mathcal{D}_o$ , we obtain*

$$\int_{\gamma} \mathbf{C}(t, \mathbf{x}) d\mathbf{x} = \int_{\gamma_0} \mathbf{C}_0(\mathbf{X}) d\mathbf{X}.$$

This result leads to Kelvin's theorems.



### 2.2.4 2-form field

We consider a 2-form field  $\mathbf{z} \in \mathcal{W} \longrightarrow \omega(t, \mathbf{x}) \in T_{\mathbf{x}}^{2*}(\mathcal{D}_t)$ . There exists an isomorphism  $\omega(t, \mathbf{x}) \in T_{\mathbf{x}}^{2*}(\mathcal{D}_t) \longrightarrow \mathbf{W} \in T_{\mathbf{x}}^1(\mathcal{D}_t)$  defined as

$$\forall \mathbf{v}_1 \text{ and } \mathbf{v}_2 \in T_{\mathbf{x}}^1(\mathcal{D}_t), \quad \omega(\mathbf{v}_1, \mathbf{v}_2) = \det(\mathbf{W}, \mathbf{v}_1, \mathbf{v}_2), \quad (3)$$

where "det" is the application determinant. We are back to the tensorial structure of  $\mathbf{W}$  that does not have the structure of a vector field. Vector fields  $\mathbf{v}_1$  and  $\mathbf{v}_2$  defined in  $T_{\mathbf{x}}^1(\mathcal{D}_t)$  have images  $\mathbf{v}_{10}$  and  $\mathbf{v}_{20}$  in  $T_{\mathbf{X}}^1(\mathcal{D}_0)$ . From (3) we can write:

$$\det(\mathbf{W}, \mathbf{v}_1, \mathbf{v}_2) = \det(\mathbf{W}_0, \mathbf{v}_{10}, \mathbf{v}_{20}).$$

Then,

$$\mathbf{W}(t, \mathbf{x}) = \frac{\mathbf{F} \mathbf{W}_0(t, \mathbf{X})}{\det \mathbf{F}}$$

and we deduce immediately:

$$d_L(\det(\mathbf{W}, \dots)) = \det\left(\frac{d\mathbf{W}}{dt} + \mathbf{W} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{W}, \dots\right),$$

where  $\operatorname{div}$  is the application divergence in  $\mathcal{D}_t$ .

**Corollary 8** *A 2-form field is moving with the fluid if and only if, applied to any two transported vector fields, we get a transported scalar field.*

From these results we define the integral of a 2-form field on a surface  $S$  of  $\mathcal{D}_t$ . We denote  $S_0$  the image of  $S$  in  $\mathcal{D}_0$ ; we obtain

$$\iint_S \det(\mathbf{W}, d_1 \mathbf{x}, d_2 \mathbf{x}) = \iint_{S_0} \det(\mathbf{W}_0, d_1 \mathbf{X}, d_2 \mathbf{X})$$

This integral corresponds to the flux of  $\mathbf{W}$  through  $S$ . We obtain the following properties :

**Theorem 9** *The five propositions are equivalent :*

- *The flux of  $\mathbf{W}$  through any fluid surface is constant.*
- *The 2-form field  $\det(\mathbf{W}, \dots)$  is moving with the fluid.*

- The vector field  $\mathbf{W} \det \mathbf{F}$  is moving with the fluid.
- $\exists \mathbf{X} \in \mathcal{D}_0 \longrightarrow \mathbf{W}_0(\mathbf{X}) \in T_{\mathbf{X}}^1(\mathcal{D}_0)$  such that  $\mathbf{W}(t, \mathbf{x}) = \frac{\mathbf{F}}{\det \mathbf{F}} \mathbf{W}_0(\mathbf{X})$ .
- $\frac{d\mathbf{W}}{dt} + \mathbf{W} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{W} \equiv \frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{W}}{\partial \mathbf{x}} \mathbf{u} + \mathbf{W} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{W} = \mathbf{0}$ .

In the case of barotropic flow, we denote the vorticity of the fluid by

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$$

we obtain the Helmholtz equation [6]:

$$\frac{D}{Dt} \left( \frac{\boldsymbol{\omega}}{\rho} \right) = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \frac{\boldsymbol{\omega}}{\rho} \quad (4)$$

In particular, it implies the well-known Kelvin theorem. The developed form is the Helmholtz equation :

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \boldsymbol{\omega}}{\partial \mathbf{x}} \mathbf{u} + \boldsymbol{\omega} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \boldsymbol{\omega} = 0 \quad (5)$$

and  $\boldsymbol{\omega}$  is associated to a 2-form moving with the fluid.

### 2.2.5 3-form field

A 3-form of  $T_{\mathbf{x}}^{3*}(\mathcal{D}_t)$  is an element of a real vector space of one dimension corresponding to the value of an isomorphism between 3-forms and scalar fields. If the 3-forms is written  $v \det$ , for any vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  of  $T_{\mathbf{x}}^1(\mathcal{D}_t)$  with images  $\mathbf{v}_{01}, \mathbf{v}_{02}, \mathbf{v}_{03}$  of  $T_{\mathbf{X}}^1(\mathcal{D}_0)$ , it exists  $v_0$  scalar field of  $\mathcal{W}_0$  such that

$$v \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = v_0 \det(\mathbf{v}_{01}, \mathbf{v}_{02}, \mathbf{v}_{03})$$

and for any volume  $\Omega_0$  in  $\mathcal{D}_0$  of image  $\Omega$  in  $\mathcal{D}_t$

$$\iiint_{\Omega} v \det(\mathbf{d}\mathbf{x}_1, \mathbf{d}\mathbf{x}_2, \mathbf{d}\mathbf{x}_3) = \iiint_{\Omega_0} v_0 \det(\mathbf{d}\mathbf{X}_1, \mathbf{d}\mathbf{X}_2, \mathbf{d}\mathbf{X}_3).$$

Then,  $v(\mathbf{x}, t) \det \mathbf{F} = v_0(\mathbf{X}, t)$  and the Lie derivative of  $v \det$  is:

$$d_L(v \det) = \left( \frac{dv}{dt} + v \operatorname{div} \mathbf{u} \right) \det .$$

**Corollary 10** *The 3-form field  $v \det$  is moving with the fluid if and only if, applied to any three transported vector fields, we get a transported scalar field.*

*If  $v \det$  is moving with the fluid, the volume integral is constant in the convected fluid volume.*

This property is equivalent to

$$\frac{dv}{dt} + v \operatorname{div} \mathbf{u} \equiv \frac{\partial v}{\partial t} + \operatorname{div}(v \mathbf{u}) \equiv \operatorname{Div}(v \mathcal{U}) = 0,$$

where  $\operatorname{Div}$  the space-time divergence in  $\mathcal{W}$ , corresponds to the conservation of density  $v$ .

To define the specific mass, we assume a given mass distribution  $\rho_0(\mathbf{X})$  in  $\mathcal{D}_0$ . The mass density  $\rho$  is given by the relationship

$$\rho \det \mathbf{F} = \rho_0(\mathbf{X})$$

and from Jacobi's derivation of  $\det \mathbf{F}$  and relation (2)

$$\frac{d(\det \mathbf{F})}{dt} = (\det \mathbf{F}) \operatorname{Tr} \left( \mathbf{F}^{-1} \frac{d\mathbf{F}}{dt} \right) = (\det \mathbf{F}) \operatorname{Tr} \left( \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right),$$

where "Tr" is the trace operator. We deduce:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{u} \equiv \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) \equiv \operatorname{Div}(\rho \mathcal{U}) = 0.$$

### 2.2.6 A general tensor field

**Definition 11** *A tensor field with covariant order  $p$  and contravariant order  $q$  is moving with the fluid if and only if, applied to any  $p$  vectors and  $q$  forms moving with the fluid, the associated scalar is moving with the fluid.*

This property is equivalent to a zero Lie derivative of the tensor field.

An example is matrix field  $\mathbf{M}$ . We obtain the immediate property

$$\mathbf{M}(\mathbf{x}, t) = \mathbf{F} \mathbf{M}_0(\mathbf{X}) \mathbf{F}^{-1} \iff d_L \mathbf{M} = \frac{d\mathbf{M}}{dt} + \mathbf{M} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{M},$$

where  $\mathbf{M}_0(\mathbf{X})$  is the image matrix defined on  $\mathcal{D}_0$ .

We can summarize the results of previous paragraphs in form of a table:

Tensors	Lie's derivatives	Tensors moving with the fluid
Scalar $s$	$\frac{ds}{dt}$	$s(\mathbf{x}, t) = s_0(\mathbf{X})$
Vector $\mathbf{J}$	$\frac{d\mathbf{J}}{dt} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{J}$	$\mathbf{J}(\mathbf{x}, t) = \mathbf{F} \mathbf{J}(\mathbf{X})$
Form $\mathbf{C}$	$\frac{d\mathbf{C}}{dt} + \mathbf{C} \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$	$\mathbf{C}(\mathbf{x}, t) \mathbf{F} = \mathbf{C}(\mathbf{X})$
2-form $\det(\mathbf{W}, \dots)$	$\det \left( \frac{d\mathbf{W}}{dt} + \mathbf{W} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{W}, \dots \right)$	$\mathbf{W}(\mathbf{x}, t) = \frac{\mathbf{F}}{\det \mathbf{F}} \mathbf{W}_0(\mathbf{X})$
3-form $v \det$	$\left( \frac{dv}{dt} + v \operatorname{div} \mathbf{u} \right) \det$	$v(\mathbf{x}, t) \det \mathbf{F} = v_0(\mathbf{X})$
Matrix $\mathbf{M}$	$\frac{d\mathbf{M}}{dt} + \mathbf{M} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{M}$	$\mathbf{M} = \mathbf{F} \mathbf{M}_0(\mathbf{X}) \mathbf{F}^{-1}$

### 2.3 Lie's derivative and exterior derivative

It is possible to summarize the properties of integrals by noting the integral  $\int_{\Omega} \pi d\omega$  of a  $p$ -form  $\pi$  in a fluid domain  $\Omega$  convected by the flow:

**Theorem 12** *a  $p$ -form  $\pi$  in a fluid domain  $\Omega$  convected by the flow verifies*

$$\frac{d}{dt} \left( \int_{\Omega} \pi d\omega \right) = \int_{\Omega} d_L \pi d\omega.$$

This property immediately derives of the writing of the integral into  $\mathcal{D}_0$ . From the Stokes formula we deduce:

**Corollary 13** *The Lie derivative commutes with the exterior derivative of differential forms.*

**Theorem 14** *We have the properties:*

- If  $s(\mathbf{x}, t)$  is a scalar field moving with the fluid, then, the form field  $\frac{\partial s}{\partial \mathbf{x}}$  is moving with the fluid.
- If  $\mathbf{C}(\mathbf{x}, t)$  is a form field moving with the fluid, then the vector field  $\frac{1}{\rho} \text{curl } \mathbf{C}^T$  is moving with the fluid.
- If  $\mathbf{J}(\mathbf{x}, t)$  is a vector field moving with the fluid, then the 3-form field isomorphic to the scalar field  $\text{div}(\rho \mathbf{J})$  is moving with the fluid.
- If  $\alpha(\mathbf{x}, t)$  and  $\beta(\mathbf{x}, t)$  are two scalar fields moving with the fluid, the 2-form field  $\partial\alpha/\partial\mathbf{x} \wedge \partial\beta/\partial\mathbf{x}$  is moving with the fluid.

## 2.4 Complements

**Theorem 15**  $\rho$  being a scalar field isomorphic to a 3-form,  $s$  a scalar field and  $\mathbf{J}$  a vector field, all fields moving with the fluid (i.e. with a zero Lie's derivative with respect to the velocity field  $\mathcal{U}$ ), and such that

$$\frac{\partial s}{\partial \mathbf{x}} \mathbf{J} = 0 \quad \text{and} \quad \text{div}(\rho \mathbf{J}) = 0,$$

then, there locally exists a scalar field  $\eta$  moving with the fluid and a mapping  $(s, \eta) \longrightarrow f(s, \eta)$  such that

$$\rho \mathbf{J} = f(s, \eta) \text{grad } s \wedge \text{grad } \eta \quad \text{with} \quad \frac{d\eta}{dt} = 0.$$

**Proof.**  $\text{div}(\rho \mathbf{J}) = 0$  implies:

Due to Clebsch's representation, there locally exists 2 scalar fields  $\tau$  and  $\eta$  such that  $\rho \mathbf{J} = \text{grad } \tau \wedge \text{grad } \eta$  [15]. The relation  $\frac{\partial s}{\partial \mathbf{x}} \mathbf{J} = 0$  proves that there locally exists an application  $F$  such that  $F(s, \eta) = \tau$ , and we obtain  $\rho \mathbf{J} = \frac{\partial F}{\partial s}(s, \eta) \text{grad } s \wedge \text{grad } \eta$ . Vector field  $\mathbf{J}$  moving with the fluid,  $s$  and  $\eta$  can be chosen moving with the fluid (i.e. have a zero Lie's derivative with respect to the velocity field  $\mathcal{U}$ ).

All the previous properties are associated with conservation laws. The most classical ones are those of conservation of mass, momentum and energy

for conservative fluids which are represented by divergence forms in space-time  $\mathcal{W}$ , i.e. of the form [16]:

$$\frac{\partial \mathbf{a}(x, t)}{\partial t} + \operatorname{div}(\mathbf{A}(x, t)) = \mathbf{0},$$

where  $\mathbf{a}(x, t)$  is a scalar field and  $\mathbf{A}(x, t)$  a matrix field of  $\mathcal{D}_t$ , respectively. To these conservation forms, it is natural to add non-divergence representations associated with tensors with covariant order  $p$ , contravariant order  $q$  and zero Lie derivatives. To these tensors one can associate scalars or 3-forms which allow to write conservation laws still in divergence forms. It is a way to obtain universal relations in continuum mechanics [17]. As example, we obtain:

- If  $\mathbf{J}(\mathbf{x}, t)$  and  $\mathbf{C}(\mathbf{x}, t)$  have zero Lie derivatives,  $\mathbf{C}(\mathbf{x}, t) \mathbf{J}(\mathbf{x}, t)$  is a scalar field moving with the fluid.

- If  $\rho(\mathbf{x}, t)$  is a 3-form field moving with the fluid,  $\rho(\mathbf{x}, t) \mathbf{C}(\mathbf{x}, t) \mathbf{J}(\mathbf{x}, t)$  is a 3-form field moving with the fluid.

- It is the same for the field of mixed tensors  $\mathbf{J}(\mathbf{x}, t) \mathbf{C}(\mathbf{x}, t)$  and we deduce that  $\det \{\mathbf{J}(\mathbf{x}, t) \mathbf{C}(\mathbf{x}, t)\}$  is a scalar field moving with the fluid.

- It is similar for the vector field  $\rho(\mathbf{x}, t) \mathbf{W}(\mathbf{x}, t)$  and the scalar field,  $\rho(\mathbf{x}, t) \mathbf{C}(\mathbf{x}, t) \mathbf{W}(\mathbf{x}, t)$ , etc.

- For all scalar field  $\beta$  moving with the fluid, we get a conservation law in divergence form:

$$\frac{\partial \rho \beta}{\partial t} + \operatorname{div}(\rho \beta \mathbf{u}) \equiv \operatorname{Div}(\rho \beta \mathcal{U}) = 0$$

and from all these conservation laws we can deduce first integrals along the flow, and Kelvin's type theorems.

## 3 Other applications

### 3.1 Electrodynamics of fluids

In the case of perfect non-conducting fluid motion (infinite electrical resistance), the equations of the electric field  $\mathbf{D}$  and electric induction  $\mathbf{E}$  are

[18, 19]:

$$\operatorname{div} \mathbf{D} = q \quad \text{and} \quad \operatorname{curl} \mathbf{E} = \mathbf{0},$$

where  $q$  is the volumetric electric charge.  $\mathbf{D}$  and  $\mathbf{E}$  are related by the relation  $\mathbf{D} = \varepsilon \mathbf{E}$ , where  $\varepsilon$  is the electrical permeability of the medium. It depends on the density and specific entropy. Our medium being supposed isotropic, the permeability tensor of the dielectric is scalar. The charge is conserved in the motion, i.e. the field  $q$  is an isomorphic 3-form field moving with the fluid:

$$\exists \{ \mathbf{X} \in \mathcal{D}_0 \longrightarrow q_0(\mathbf{X}) \} \quad \text{such that} \quad q \det \mathbf{F} = q_0(\mathbf{X}).$$

Consequently  $\operatorname{Div}(q\mathcal{U}) = 0$  and due to the fact that  $\operatorname{div} \mathbf{D}$  is a 3-form moving with the fluid, relation  $(\det \mathbf{F}) \operatorname{div} \mathbf{D} = \operatorname{div}_0 (\det(\mathbf{F}) \mathbf{F}^{-1} \mathbf{D})$  yields:

$$\mathbf{D} = \frac{\mathbf{F}}{\det \mathbf{F}} \mathbf{D}_0(t, \mathbf{x}).$$

The field  $\mathbf{D}_0$  is not necessarily a 2-form moving with the fluid. This is nevertheless the case when the fluid is in adiabatic motion.

### 3.2 Magnetodynamics of fluids

In the case of the adiabatic motion of a perfect fluid with zero electrical resistance, the equations verified by the magnetic field  $\mathbf{H}(t, \mathbf{x})$  are [18]:

$$\operatorname{div} \mathbf{H} = 0 \quad \text{and} \quad \frac{\partial \mathbf{H}}{\partial t} - \operatorname{curl}(\mathbf{x} \wedge \mathbf{H}) = \mathbf{0}.$$

These relationships involve:

$$\frac{d\mathbf{H}}{dt} + \mathbf{H} \operatorname{div} \mathbf{u} - \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{H} = \mathbf{0}.$$

The magnetic field is such that the 2-form  $\det(\mathbf{H}, \cdot, \cdot)$  is convected by the flow and it exists  $\mathbf{H}_0(\mathbf{X})$  such that:

$$\mathbf{H}(z) = \frac{\mathbf{F}}{\det \mathbf{F}} \mathbf{H}_0(\mathbf{X}).$$

## 4 Conclusion

Various concepts, such as the particle derivative, the convective derivative, the Jaumann derivative, etc., allow to express conservative fields frozen in a moving fluid medium and whose properties are given by means of scalars and simple or multiple integrals. A more natural method, linked to the definition of the motion, is that of the Lie derivative associated with a one-parameter group, corresponding to the diffeomorphism representing the motion of a fluid in space–time. Its essential advantage is that it can express the variance of specific quantities defined on the reference space in Lagrangian variables. These quantities will have the structure of covariant order and contravariant order associated with scalars, vectors, shapes or tensors. Quantities depending only on the trajectory are moving with the fluid. They allow to express, as by Noether’s theorem, some first integrals and conservation laws.

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