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MEAN SQUARE VALUE OF L -FUNCTIONS AT $s = 1$ FOR NON-PRIMITIVE CHARACTERS, DEDEKIND SUMS AND BOUNDS ON RELATIVE CLASS NUMBERS

STÉPHANE R. LOUBOUTIN

Abstract: Let d_0 be a given square-free integer. We give an explicit formula $M_{d_0}(p) = A(d_0)(1 + B_{d_0}(p)/p)$ for the quadratic mean value at $s = 1$ of the Dirichlet L -functions associated with the non-primitive odd Dirichlet characters modulo d_0p induced by the odd Dirichlet characters modulo an odd prime p . Here $d_0 \mapsto A(d_0)$ is an explicit multiplicative arithmetic function and $B_{d_0}(f)$ is a twisted sum over the divisors d of d_0 of Dedekind sums $s(h, df)$. To prove that $f \mapsto B_{d_0}(f)$ is d_0 -periodic, we find a new and closed formula for the Dedekind sums $f \mapsto s(a + bf, c + df)$, for fixed integers a, b, c and d . We deduce explicit upper bounds for relative class numbers of cyclotomic number fields.

Keywords: Dirichlet character, L -function, mean square value, relative class number, Dedekind sums, cyclotomic field.

1. Introduction

Let $f > 2$ be an integer. Let X_f be the multiplicative group of the $\phi(f)$ Dirichlet characters modulo f . Let $X_f^- = \{\chi \in X_f; \chi(-1) = -1\}$ be the set of the $\phi(f)/2$ odd Dirichlet characters modulo f . Let $L(s, \chi)$ be the Dirichlet L -functions associated with $\chi \in X_f$. We have (see [6, Proposition 1]):

$$L(1, \chi) = \frac{\pi}{2f} \sum_{a=1}^{f-1} \chi(a) \cot\left(\frac{\pi a}{f}\right) \quad (\chi \in X_f^-). \quad (1)$$

According to [11] and [6], the mean square value $M(f)$ of $L(1, \chi)$ as χ ranges in X_f^- is defined by and given by

$$M(f) := \frac{1}{\#X_f^-} \sum_{\chi \in X_f^-} |L(1, \chi)|^2 = \frac{\pi^2 \phi(f)}{6f} \left\{ \prod_{q \substack{\text{prime} \\ q|f}} \left(1 + \frac{1}{q}\right) - \frac{3}{f} \right\}.$$

For $f = p \geq 3$ an odd prime, we recover Walum's formula in [14]:

$$M(p) := \frac{2}{p-1} \sum_{\chi \in X_p^-} |L(1, \chi)|^2 = \frac{\pi^2}{6} \left\{ \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p}\right) \right\} \quad (p \geq 3).$$

Let h_p^- denote the relative class number of the cyclotomic number field of prime conductor $p \geq 3$ (see [15] for the definition). Using the arithmetic-geometric mean inequality, we obtain (see also [6], [9]):

$$h_p^- = 2p \left(\frac{p}{4\pi^2}\right)^{\frac{p-1}{4}} \prod_{\chi \in X_p^-} L(1, \chi) \leq 2p \left(\frac{pM(p)}{4\pi^2}\right)^{\frac{p-1}{4}} \leq 2p \left(\frac{p}{24}\right)^{\frac{p-1}{4}}. \quad (2)$$

Now, let $d_0 > 1$ be a given square-free integer. Assume that $p \nmid d_0$. For $\chi \in X_p$, let χ' be the non-primitive character modulo d_0p induced by χ . Then,

$$L(1, \chi) = \left\{ \prod_{\substack{q \text{ prime} \\ q|d_0}} \frac{1}{1 - \frac{\chi(q)}{q}} \right\} L(1, \chi').$$

Therefore, we obtain

$$h_p^- = \frac{2p}{\Pi_p^-(d_0)} \left(\frac{p}{4\pi^2}\right)^{\frac{p-1}{4}} \prod_{\chi \in X_p^-} L(1, \chi') \leq \frac{2p}{\Pi_p^-(d_0)} \left(\frac{pM_{d_0}(p)}{4\pi^2}\right)^{\frac{p-1}{4}}, \quad (3)$$

where

$$\Pi_p^-(d_0) := \prod_{\substack{q \text{ prime} \\ q|d_0}} \prod_{\chi \in X_p^-} \left(1 - \frac{\chi(q)}{q}\right)$$

and

$$M_{d_0}(p) := \frac{2}{p-1} \sum_{\chi \in X_p^-} |L(1, \chi')|^2. \quad (4)$$

In [6] and [7], we explained how using (3) one could get explicit bounds better than (2) of the type $h_p^- \leq 2p(p/C)^{(p-1)/4}$ for any given constant $C < 4\pi^2$ and p large enough. In the present paper we greatly simplify the approach developed in these previous articles by giving a neat formula for these $M_{d_0}(p)$, see Theorems 3.1 and 4.3. For explicit formulas for $d_0 \in \{2, 6, 30, 210\}$, see (9), (10), (22) and (23). As a consequence of (3) and this formula, we readily obtain improvements on (2) in Theorem 5.2, by using the explicit lower bounds on $\Pi_p^-(d_0)$ given in Lemma 5.1. We invite the reader to start his reading with a quick look at our Theorems 3.1, 4.3 and 5.2.

2. Dedekind sums

The Dedekind sums $s(c, d)$ which depend only on $c \bmod d$ are defined by

$$s(c, d) = \frac{1}{4d} \sum_{n=1}^{d-1} \cot\left(\frac{\pi n}{d}\right) \cot\left(\frac{\pi nc}{d}\right) \quad (c \in \mathbb{Z}, d \geq 1, \gcd(c, d) = 1),$$

with the convention $s(c, 1) = 0$ for $c \in \mathbb{Z}$. Since $c \mapsto s(c, d)$ is d -periodic, we may consider $s(h, d)$ for $h \in (\mathbb{Z}/d\mathbb{Z})^*$. Recall the reciprocity law for Dedekind sums

$$s(c, d) + s(d, c) = \frac{c^2 + d^2 - 3cd + 1}{12cd}, \quad (c \geq 1, d \geq 1, \gcd(c, d) = 1) \quad (5)$$

(see [2], [12] or [8]). In particular,

$$s(1, d) = \frac{d^2 - 3d + 2}{12d} \quad (d \geq 1). \quad (6)$$

3. A mean square value formula for non-primitive L -functions

Theorem 3.1. *Let $d_0 > 1$ be a square-free integer. Set*

$$\phi_k(d_0) := \prod_{\substack{q \text{ prime} \\ q|d_0}} (q^k - 1).$$

Whenever $\gcd(f, d_0) = 1$, let $H_0 = H_{d_0}(f)$ be the kernel of order $\phi(d_0)$ of the surjective canonical morphism $(\mathbb{Z}/d_0 f\mathbb{Z})^ \rightarrow (\mathbb{Z}/f\mathbb{Z})^*$ and set*

$$S_{d_0}(f) := \mu(d_0) \sum_{1 \neq h \in H_0} \sum_{d|d_0} d\mu(d) s(h, df) \quad (\text{Dedekind sums}). \quad (7)$$

Let $p \geq 3$ be a prime that does not divide d_0 . For $\chi \in X_p^-$, let χ' be the non-primitive Dirichlet character modulo $d_0 p$ induced by χ . Then,

$$M_{d_0}(p) := \frac{2}{p-1} \sum_{\chi \in X_p^-} |L(1, \chi')|^2 = \frac{\pi^2}{6} \frac{\phi_2(d_0)}{d_0^2} \left(1 + \frac{12S_{d_0}(p) - 3\phi_1(d_0)}{\phi_2(d_0)p} \right).$$

Proof. The surjectivity of the canonical morphism $(\mathbb{Z}/d_0 f\mathbb{Z})^* \rightarrow (\mathbb{Z}/f\mathbb{Z})^*$ readily follows from Dirichlet theorem that asserts that there are infinitely many prime integers in the arithmetic progressions $a + f\mathbb{Z}$ for $\gcd(a, f) = 1$. The canonical morphism $(\mathbb{Z}/d_0 f\mathbb{Z})^* \rightarrow (\mathbb{Z}/d_0\mathbb{Z})^*$ is also surjective. Since $\gcd(d_0, f) = 1$, its restriction to H_0 is injective, hence bijective. It follows that whenever d divides d_0 we have

$$S := \sum_{h \in H_0} s(h, d) = \sum_{h \in (\mathbb{Z}/d_0\mathbb{Z})^*} s(h, d) = 0. \quad (8)$$

Indeed, whenever d divides d_0 , we have

$$S := \sum_{h \in (\mathbb{Z}/d_0\mathbb{Z})^*} s(h, d) = \sum_{\substack{h=1 \\ \gcd(h, d_0)=1}}^{d_0-1} s(h, d) = \sum_{\substack{h=1 \\ \gcd(h, d_0)=1}}^{d_0-1} s(d_0 - h, d) = -S.$$

Now, for $\gcd(a, d_0p) = \gcd(b, d_0p) = 1$, we have the orthogonality relations

$$\varepsilon(a, b) := \frac{2}{p-1} \sum_{\chi \in X_p^-} \chi'(a) \overline{\chi'(b)} = \begin{cases} +1 & \text{if } b \equiv +a \pmod{p}, \\ -1 & \text{if } b \equiv -a \pmod{p}, \\ 0 & \text{otherwise,} \end{cases}$$

and using (1) we have

$$M_{d_0}(p) = \frac{\pi^2}{4d_0^2 p^2} \sum_{\substack{a=1 \\ \gcd(a, d_0p)=1}}^{d_0p-1} \sum_{\substack{b=1 \\ \gcd(b, d_0p)=1}}^{d_0p-1} \varepsilon(a, b) \cot\left(\frac{\pi a}{d_0p}\right) \cot\left(\frac{\pi b}{d_0p}\right).$$

The imparity of the cotangent function and the change of variables $b \mapsto ah$ give

$$\begin{aligned} M_{d_0}(p) &= \frac{\pi^2}{2d_0^2 p^2} \sum_{\substack{a=1 \\ \gcd(a, d_0p)=1}}^{d_0p-1} \sum_{h \in H_0} \cot\left(\frac{\pi a}{d_0p}\right) \cot\left(\frac{\pi ah}{d_0p}\right) \\ &= \frac{2\pi^2}{d_0p} \sum_{h \in H_0} \tilde{s}(h, d_0p), \end{aligned}$$

where

$$\begin{aligned} \tilde{s}(c, d) &:= \frac{1}{4d} \sum_{\substack{n=1 \\ \gcd(n, d)=1}}^{d-1} \cot\left(\frac{\pi n}{d}\right) \cot\left(\frac{\pi nc}{d}\right) \\ &= \sum_{\delta|d} \frac{\mu(\delta)}{\delta} s(c, d/\delta) = \frac{\mu(d)}{d} \sum_{\delta|d} \delta \mu(\delta) s(c, \delta). \end{aligned}$$

Since $\delta \mid d_0p$ if and only if either $\delta = d$ or $\delta = dp$ where $d \mid d_0$, we have

$$\begin{aligned} M_{d_0}(p) &= \frac{2\pi^2}{d_0p} \sum_{h \in H_0} \frac{\mu(d_0p)}{d_0p} \sum_{d|d_0p} d\mu(d) s(h, d) \\ &= \frac{2\pi^2}{d_0p} \frac{\mu(d_0p)}{d_0p} \sum_{h \in H_0} \left(\sum_{d|d_0} d\mu(d) s(h, d) - p \sum_{d|d_0} d\mu(d) s(h, dp) \right) \\ &= \frac{2\pi^2}{d_0p} \frac{\mu(d_0)}{d_0} \sum_{h \in H_0} \sum_{d|d_0} d\mu(d) s(h, dp), \quad (\text{by (8)}) \\ &= \frac{2\pi^2}{d_0p} \frac{\mu(d_0)}{d_0} \left(\mu(d_0) S_{d_0}(p) + \sum_{d|d_0} d\mu(d) s(1, dp) \right). \end{aligned}$$

Finally, by (6), for $d_0 > 1$ square-free we have

$$\sum_{d|d_0} d\mu(d)s(1, df) = \sum_{d|d_0} \mu(d) \frac{f^2 d^2 - 3df + 2}{12f} = \mu(d_0) \frac{f\phi_2(d_0) - 3\phi_1(d_0)}{12}.$$

The desired formula follows. ■

Example 1. Take $d_0 = 2$ and $f \geq 3$ odd. Then $H_0 = \{1\}$ and $S_2(f) = 0$. Therefore, in accordance with [6, Théorème 3], for $p \geq 3$ we have

$$M_2(p) = \frac{\pi^2}{8} \left(1 - \frac{1}{p}\right). \quad (9)$$

Example 2. Take $d_0 = 2 \cdot 3 = 6$ and $f \geq 5$ coprime with 6. Then $H_0 = \{1, h_0\}$, where $h_0 = 1 + 4f$ if $f \equiv 1 \pmod{6}$ and $h_0 = 1 + 2f$ if $f \equiv 5 \pmod{6}$, and

$$M_6(p) = \frac{\pi^2}{9} \left(1 + \frac{2S_6(p) - 1}{4p}\right) \quad (10)$$

Moreover, using (5), (6) and a tad tedious but easy computation, we obtain:

$$\begin{aligned} S_6(f) &= 6s(h_0, 6f) - 3s(h_0, 3f) - 2s(h_0, 2f) + s(h_0, f) \\ &= \begin{cases} \frac{2f^2-13f+1}{6f} - \frac{f^2-5f+2}{12f} - \frac{2f^2-3f+1}{6f} + \frac{f^2-3f+2}{12f} = -\frac{3}{2} & \text{if } f \equiv 1 \pmod{6}, \\ \frac{2f^2-5f+1}{6f} - \frac{f^2-13f+2}{12f} - \frac{2f^2-3f+1}{6f} + \frac{f^2-3f+2}{12f} = \frac{1}{2} & \text{if } f \equiv 5 \pmod{6}. \end{cases} \end{aligned}$$

For example, for $f \equiv 1 \pmod{6}$ we have

$$\begin{aligned} s(1 + 4f, 6f) &= -\frac{10f^2 + 5f - 1}{36f(4f + 1)} - s(6f, 4f + 1) \\ &= -\frac{10f^2 + 5f - 1}{36f(4f + 1)} - s(2f - 1, 4f + 1) \\ &= -\frac{2f^2 + 7f + 1}{36f(2f - 1)} + s(4f + 1, 2f - 1) \\ &= -\frac{2f^2 + 7f + 1}{36f(2f - 1)} + s(3, 2f - 1) \\ &= \frac{2f^2 - 11f + 1}{36f} - s(2f - 1, 3) \\ &= \frac{2f^2 - 11f + 1}{36f} - s(1, 3) \\ &= \frac{2f^2 - 11f + 1}{36f} - \frac{1}{18} \\ &= \frac{2f^2 - 13f + 1}{36f}. \end{aligned}$$

Therefore, $S_6(f)$ depends only on f modulo 6, and for $p \geq 5$ we have

$$M_6(p) = \frac{\pi^2}{9} \times \begin{cases} 1 - \frac{1}{p} & \text{if } p \equiv 1 \pmod{6} \\ 1 & \text{if } p \equiv 5 \pmod{6} \end{cases} \text{ and } M_6(p) \leq \frac{\pi^2}{9}, \quad (11)$$

in accordance with [6, Théorème 4 (revised in the Addendum)].

4. On the Dedekind sums $f \mapsto s(a + bf, c + df)$

The aim of this section is to prove in Theorem 4.3 that $f \mapsto S_{d_0}(f)$ is d_0 -periodic, which makes our mean square formula in Theorem 3.1 even more interesting and useful. Indeed, we will use it to obtain explicit improvements on (2), see Theorem 5.2.

Let $d_0 > 1$ be a given square-free integer. For $f \geq 1$ we can rewrite (7) as

$$S_{d_0}(f) := \mu(d_0) \sum_{\substack{k=1 \\ \gcd(1+kf, d_0f)=1}}^{d_0-1} S_{k, d_0}(f), \quad (12)$$

where

$$S_{k, d_0}(f) := \sum_{d|d_0} d\mu(d)s(1+kf, df). \quad (13)$$

We prove in Theorem 4.3 that for f large enough $f \mapsto S_{k, d_0}(f)$ is d_0 -periodic. Therefore, $S_{d_0}(f)$ depends only on f modulo d_0 for f explicitly large enough.

To begin with, we extend the definition of Dedekind sums by setting

$$s(c, d) = \text{sign}(cd)s(|c|, |d|) = \text{sign}(d)s(c, |d|) \quad (c, d \in \mathbb{Z} \setminus \{0\}).$$

Hence $c \mapsto s(c, d)$ is still d -periodic and (5) gives the following generalized reciprocity law that holds true whatever the signs of $c, d \in \mathbb{Z} \setminus \{0\}$:

$$s(c, d) = \frac{c^2 + d^2 + 1}{12cd} - \frac{\text{sign}(cd)}{4} - s(d, c). \quad (14)$$

Let us now set some notations. Let $P_0(X) = a_0 + b_0X$ and $P_1(X) = a_1 + b_1X$ in $\mathbb{Z}[X]$ be given, with $b_0 > b_1 \geq 1$ and

$$D_0 = \det \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix} \neq 0.$$

Let $b_0 > b_1 > \dots > b_l > b_{l+1} = 0$ denote the sequence of remainders in the Euclidean algorithm for calculating $\gcd(b_0, b_1)$. For $0 \leq i \leq l-1$, define $P_i(X) = a_i + b_iX \in \mathbb{Z}[X]$ by $P_{i+2}(X) = P_i(X) - q_i P_{i+1}(X)$, where $q_i = \lfloor b_i/b_{i+1} \rfloor \geq 1$. Clearly, $\gcd(a_{i+1}, a_i)$, $\gcd(b_{i+1}, b_i)$ and $\gcd(P_{i+1}(f), P_i(f))$ do not depend on the index $i \in \{0, \dots, l\}$, where $f \in \mathbb{Z}$. In particular,

$$b_l = \gcd(0, b_l) = \gcd(b_{l+1}, b_l) = \gcd(b_1, b_0). \quad (15)$$

Moreover,

$$D_i = \det \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix} = (-1)^i D_0 \neq 0.$$

Set

$$r_{b_0, b_1} := \sum_{i=0}^{l-1} (-1)^i \left(\frac{b_i}{b_{i+1}} + \frac{b_{i+1}}{b_i} \right),$$

(which is related to the Dedekind sum $s(b'_1, b'_0)$ and $\sum_{i=0}^{l-1} \frac{(-1)^i}{b'_i b'_{i+1}}$, where $b'_i = b_i / \gcd(b_1, b_0)$, by [2, Exercise 20, page 73]),

$$T(X) = T_{a_0, b_0, a_1, b_1}(X) := P_l(X) = a_l + b_l X = a_l + \gcd(b_1, b_0) X \quad (16)$$

and

$$R(X) = R_{a_0, b_0, a_1, b_1}(X) := r_{b_0, b_1} + \frac{\frac{D_0}{b_0} + \frac{b_0}{D_0}}{P_0(X)} + \frac{\gcd(b_1, b_0)}{D_0} T(X). \quad (17)$$

We are now in a position to state our new result on Dedekind sums:

Theorem 4.1. *Let $P_0(X) = a_0 + b_0 X$ and $P_1(X) = a_1 + b_1 X$ in $\mathbb{Z}[X]$ be given, with $b_0 > b_1 \geq 1$ and*

$$D_0 = \det \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix} \neq 0.$$

For $f \geq 1$ and $\gcd(P_1(f), P_0(f)) = 1$, we have the Dedekind sums formula

$$s(P_1(f), P_0(f)) = \frac{1}{12} R(f) - \frac{1 - (-1)^l}{8} - \frac{\text{sign}(D_0)}{4} - s(T(f), D'_0), \quad (18)$$

where $D'_0 := D_0 / \gcd(b_0, b_1)$ is a rational integer not equal to 0 and the arithmetic function $f \mapsto s(T(f), D'_0)$ is D'_0 -periodic.

For example, take $P_0(X) = 6X$ and $P_1(X) = 1 + 4X$. We have $D_0 = 6$, $l = 2$, $P_2(X) = 2X - 1$ and $P_3(X) = 3$, $r_{6,4} = -1/3$, $R(X) = \frac{2X^2 - 2X + 1}{3X}$. Therefore, for $f \equiv 1 \pmod{6}$ we have $s(2f - 1, 3) = s(1, 3) = 1/18$ and we obtain

$$s(1 + 4f, 6f) = \frac{2f^2 - 11f + 1}{36f} - s(2f - 1, 3) = \frac{2f^2 - 13f + 1}{36f},$$

as in Example 2 of the previous Section.

Proof. We will use some technical results proved in the next Lemma 4.2.

Set $Q(X, Y) = \frac{X^2 + Y^2 + 1}{XY}$, and $Q_i(X) = Q(P_{i+1}(X), P_i(X))$. We have the easy to check identity

$$Q_i(X) = \frac{b_i}{b_{i+1}} + \frac{b_{i+1}}{b_i} - \frac{\frac{D_i}{b_{i+1}} + \frac{b_{i+1}}{D_i}}{P_{i+1}(X)} + \frac{\frac{D_i}{b_i} + \frac{b_i}{D_i}}{P_i(X)} \quad (0 \leq i \leq l-1),$$

and

$$Q_l(X) = \frac{\frac{D_l}{b_l} + \frac{b_l}{D_l}}{P_l(X)} + \frac{b_l}{D_l} P_l(X).$$

Noticing that $D_i = (-1)^i D_0$, we obtain

$$\sum_{i=0}^l (-1)^i Q_i(X) = r_{b_0, b_1} + \frac{\frac{D_0}{b_0} + \frac{b_0}{D_0}}{P_0(X)} + \frac{b_l}{D_0} P_l(X) = R(X). \quad (19)$$

Now, let $f \geq 1$ be such that $\gcd(P_1(f), P_0(f)) = 1$.

Then $\gcd(P_{i+1}(f), P_i(f)) = 1$, $P_i(f) \neq 0$ for $0 \leq i \leq l$, by Lemma 4.2, and $P_{l+1}(f) = a_{l+1} \neq 0$. Therefore, we can define $\varepsilon_i(f) = \text{sign}(P_i(f)) \in \{\pm 1\}$ for $0 \leq i \leq l+1$. Using (14), $P_i(f) = P_{i+2}(f) + q_i P_{i+1}(f)$ and the d -periodicity of $c \mapsto s(c, d)$, we obtain $s(P_i(f), P_{i+1}(f)) = s(P_{i+2}(f), P_{i+1}(f))$ and

$$s(P_{i+1}(f), P_i(f)) = \frac{1}{12} Q_i(f) - \frac{\varepsilon_{i+1}(f) \varepsilon_i(f)}{4} - s(P_{i+2}(f), P_{i+1}(f))$$

for $0 \leq i \leq l-1$ and

$$s(P_{l+1}(f), P_l(f)) = \frac{1}{12} Q_l(f) - \frac{\varepsilon_{l+1}(f) \varepsilon_l(f)}{4} - s(P_l(f), P_{l+1}(f)).$$

Taking the alternating sums of these $l+1$ equalities and using (19) we obtain

$$s(P_1(f), P_0(f)) = \frac{1}{12} R(f) - \frac{1}{4} \varepsilon(f) - (-1)^l s(P_l(f), P_{l+1}(f)),$$

where

$$\varepsilon(f) = \sum_{i=0}^l (-1)^i \varepsilon_{i+1}(f) \varepsilon_i(f). \quad (20)$$

By Lemma 4.2, we have

$$\varepsilon(f) = \frac{1 - (-1)^l}{2} + \text{sign}(D_0).$$

Finally, $P_{l+1}(f) = a_{l+1} = \frac{a_{l+1} b_l}{\gcd(b_0, b_1)} = \frac{D_l}{\gcd(b_0, b_1)} = (-1)^l D_0 / \gcd(b_0, b_1)$, by (15). Therefore,

$$(-1)^l s(P_l(f), P_{l+1}(f)) = s(P_l(f), D_0 / \gcd(b_0, b_1)),$$

and the desired result follows. ■

Let us now prove what we used for proving Theorem 4.1, namely that the $P_i(f)$'s are never equal to 0, which enabled us to deal with their signs, their associated Dedekind sums and the reciprocity law for Dedekind sums:

Lemma 4.2. *Let $P_0(X) = a_0 + b_0X$ and $P_1(X) = a_1 + b_1X$ in $\mathbb{Z}[X]$ be given, with $b_0 > b_1 \geq 1$, $a_0, a_1 \geq 0$ and*

$$D_0 = \det \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix} \neq 0.$$

Let $f \geq 1$ be such that $\gcd(P_1(f), P_0(f)) = 1$. Then $P_i(f) \neq 0$ for $0 \leq i \leq l + 1$ and $\varepsilon(f)$ defined in (20) does not depend on f and is given by

$$\varepsilon(f) = \frac{1 - (-1)^l}{2} + \text{sign}(D_0).$$

Proof. Recall that $q_i \geq 1$ for $0 \leq i \leq l$. We claim that $a_i \geq 0$ for $0 \leq i \leq i_0$ and $a_i \neq 0$ with $\text{sign}(a_i) = (-1)^{i-i_0}$ for $i_0 < i \leq l + 1$, where

$$1 \leq i_0 := \max\{i; 0 \leq i \leq l + 1 \text{ and } a_i \geq 0\}.$$

Indeed, if $i_0 \leq l$, then $a_{i_0} \geq 0$ and $a_{i_0+1} \leq -1$. Hence if $i_0 \leq l - 1$ then $a_{i_0+2} = a_{i_0} - q_{i_0}a_{i_0+1} \geq 1$, i.e. $a_{i_0+1} \leq -1$ and $a_{i_0+2} \geq 1$. Hence if $i_0 \leq l - 2$ then $a_{i_0+3} = a_{i_0+1} - q_{i_0+1}a_{i_0+2} \leq -1$, i.e. $a_{i_0+2} \geq 1$ and $a_{i_0+3} \leq -1$. We then proceed inductively. Consequently, (i) if $a_{i+1} \leq -1$ for some index $i \leq l$ then $a_i \geq 0$ for the previous index and (ii) the indices in

$$A_- := \{i; 0 \leq i \leq l + 1 \text{ and } a_i \leq -1\}$$

have all the same parity.

Now assume that $P_{i+1}(f) = 0$ for some $f \geq 1$ and some index $i \leq l$. Since $P_{l+1}(X) = a_{l+1} \neq 0$, we have $i \leq l - 1$ and $b_i > b_{i+1} \geq b_l > b_{l+1} = 0$. Hence, $b_i \geq 2$ and $b_{i+1} \geq 1$. Therefore, $0 = P_{i+1}(f) = a_{i+1} + b_{i+1}f \geq a_{i+1} + 1$ gives $a_{i+1} \leq -1$. Hence, $a_i \geq 0$, by the assertion above, and $P_i(f) = a_i + b_i f \geq a_i + 2 \geq 2$. However, $P_{i+1}(f) = 0$ and $\gcd(P_{i+1}(f), P_i(f)) = \gcd(P_1(f), P_0(f)) = 1$ give $P_i(f) = \pm 1$ and we obtain a contradiction.

Hence, $P_i(f) \neq 0$ for $f \geq 1$ and $0 \leq i \leq l + 1$ and the $\varepsilon_i(f)$ are well defined. We claim that $\varepsilon_i(f) = +1$ for $0 \leq i \leq i_0(f)$ and $\varepsilon_i(f) = (-1)^{i-i_0(f)}$ for $i_0(f) < i \leq l + 1$, where

$$i_0(f) := \max\{i; 0 \leq i \leq l + 1 \text{ and } P_i(f) \geq 1\}.$$

Indeed, if $i_0(f) \leq l$, then $P_{i_0}(f) \geq 1$ and $P_{i_0+1}(f) \leq -1$. Hence if $i_0 \leq l - 1$ then $P_{i_0+2}(f) = P_{i_0}(f) - q_{i_0}P_{i_0+1}(f) \geq 1$, i.e. $P_{i_0+1}(f) \leq -1$ and $P_{i_0+2}(f) \geq 1$. Hence if $i_0(f) \leq l - 2$ then $P_{i_0+3}(f) = P_{i_0+1}(f) - q_{i_0+1}P_{i_0+2}(f) \leq -1$, i.e. $P_{i_0+2}(f) \geq 1$ and $P_{i_0+3}(f) \leq -1$. We then proceed inductively.

Hence, $\varepsilon_{i+1}(f)\varepsilon_i(f) = +1$ for $0 \leq i \leq i_0(f) - 1$ and $\varepsilon_{i+1}(f)\varepsilon_i(f) = -1$ for $i_0(f) \leq i \leq l$. By (20), it follows that $\varepsilon(f) := \frac{1 - (-1)^l}{2} - (-1)^{i_0(f)}$.

Finally, $I(f) := \{i; 0 \leq i \leq l + 1 \text{ and } P_i(f) \leq -1\}$ is included in A_- . Since the i 's in A_- have all the same parity, the parity of $i_0(f) = \min I(f) - 1$ does not depend on $f \geq 1$. Since $b_i \geq 1$ for $0 \leq i \leq l$, for $f >$ large enough we have $\varepsilon_i(f) = +1$ for $0 \leq i \leq l$, $i_0(f) = l + \frac{1 + \text{sign}(a_{l+1})}{2}$ and $\varepsilon(f) = \frac{1 - (-1)^l}{2} + \text{sign}(D_0)$. ■

Theorem 4.3. *Let $0 \leq k < d$ be given. There exist $r_{k,d} \in \mathbb{Q}$ and $T_{k,d}(X) = a + bX \in \mathbb{Z}[X]$ such that for $f \geq 1$ and $\gcd(1 + kf, df) = 1$, we have*

$$s(1 + kf, df) = \frac{\gcd(k, d)^2 f^2 + 2}{12df} + r_{k,d} - s(T_{k,d}(f), d/\gcd(k, d)). \quad (21)$$

Consequently, the rational numbers $S_{d_0}(f)$ defined in (7) depend only on $f \geq 1$ modulo d_0 .

Proof. For $(a_0, b_0, a_1, b_1) = (0, d, 1, k)$ we have $\gcd(b_1, b_0) = \gcd(k, d)$, $D_0 = d$, $D_0/b_0 + b_0/D_0 = 2$ and $P_0(X) = dX$. By (16), (17) and (18), we do have the first assertion (21). For the second assertion, we use (21) and notice that $\sum_{d|d_0} \mu(d) = 0$ for $d_0 > 1$ and that the function $d_0 \mapsto G(d_0, k) := \sum_{d|d_0} \mu(d) \gcd(k, d)^2$ is multiplicative. Therefore,

$$G(d_0, k) = \prod_{\substack{q|d_0, \\ q \text{ prime}}} (1 - \gcd(k, q)^2).$$

Hence, for d_0 square-free, $G(d_0, k) = 0$ if and only if $d_0 \nmid k$.

It follows that for $d_0 > 1$ square-free and k not divisible by d_0 there exists $r'_{k,d_0} \in \mathbb{Q}$ such that for $f \geq 1$ and $\gcd(1 + kf, df) = 1$, we have

$$S_{k,d_0}(f) := \sum_{d|d_0} d\mu(d)s(1 + kf, df) = r'_{k,d_0} - \sum_{d|d_0} d\mu(d)s(T_{k,d}(f), d/\gcd(k, d)).$$

Hence, the rational numbers $S_{k,d_0}(f)$ defined in (13) depends only on f modulo d_0 and the last assertion follows from (12). \blacksquare

We refer the reader to [5, Theorem 4.4 and Proposition 4.6] for a different proof of the last assertion of Theorem 4.3.

5. Explicit upper bounds on relative class numbers

Lemma 5.1. *Let $q \geq 2$ be a given prime integer. Let $p \geq 3$ be a prime integer, with $p \neq q$. Let f be the order of q in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$. Then*

$$\Pi_p^-(q) := \prod_{\chi \in X_p^-} \left(1 - \frac{\chi(q)}{q}\right) = \begin{cases} \left(1 - \frac{1}{q^f}\right)^{\frac{p-1}{2f}} & \text{if } f \text{ is odd} \\ \left(1 + \frac{1}{q^{f/2}}\right)^{\frac{p-1}{f}} & \text{if } f \text{ is even.} \end{cases}$$

Moreover, we have

$$\Pi_p^-(d_0) := \prod_{\substack{q \text{ prime} \\ q|d_0}} \Pi_p^-(q) \geq \begin{cases} 6/7 & \text{for } d_0 = 2 \text{ and } p \geq 3, \\ 2/3 & \text{for } d_0 = 6 \text{ and } p \geq 5, \\ 1/2 & \text{for } d_0 = 30 \text{ and } p \geq 7, \\ 5/13 & \text{for } d_0 = 210 \text{ and } p \geq 11. \end{cases}$$

Proof. By [13, Lemma 6 page 72], we have $\Pi_p(q) := \prod_{\chi \in X_p} (1 - \chi(q)T) = (1 - T^f)^{\frac{p-1}{2f}}$ and $\Pi_p^+(q) := \prod_{\chi \in X_p^+} (1 - \chi(q)T) = (1 - T^{f_+})^{\frac{p-1}{2f_+}}$, where f_+ is the order of q in the multiplicative group $G_+ := (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$ of order $(p-1)/2$ and $X_p^+ = \{\chi \in X_p; \chi(-1) = +1\}$ is the group of characters of G_+ . Now, clearly, $f_+ = f$ if f is odd and $f_+ = f/2$ if f is even. Since $\Pi_p^-(q) = \Pi_p(q)/\Pi_p^+(q)$, the desired first result follows. Now, we have $q^f \geq p+1$. Hence,

$$\Pi_p^-(q) \geq \left(1 - \frac{1}{p+1}\right)^{\frac{p-1}{2f}} \geq \left(1 - \frac{1}{p+1}\right)^{\frac{(p-1)\log q}{2\log(p+1)}} = \exp\left(\frac{\log q}{2} F(p+1)\right)$$

where for $x > 1$ we set $F(x) := \frac{(x-2)\log(1-\frac{1}{x})}{\log x}$. Therefore,

$$\Pi_p^-(d_0) \geq \exp\left(\frac{\log d_0}{2} F(p+1)\right).$$

The desired lower bounds easily follow. ■

Theorem 5.2. *By taking successively $d_0 = 2, 6, 30$ and 210 in (3) we obtain*

$$h_p^- \leq \begin{cases} \frac{7p}{3} \left(\frac{p}{32}\right)^{(p-1)/4} & \text{for } p \geq 3 \quad (\text{compare with [3, (1)]}), \\ 3p \left(\frac{p}{36}\right)^{(p-1)/4} & \text{for } p \geq 5, \\ 4p \left(\frac{2p+1}{75}\right)^{(p-1)/4} & \text{for } p \geq 7, \\ \frac{26p}{5} \left(\frac{32p+649}{1225}\right)^{(p-1)/4} & \text{for } p \geq 11. \end{cases}$$

Proof. For $d_0 = 2$ we use (3), (9) and Lemma 5.1 to get the first bound.

For $d_0 = 6$ we use (3), (11) and Lemma 5.1 to get the second.

Take $d_0 = 30$ and $p \geq 7$. Then

$$M_{30}(p) = \frac{8\pi^2}{75} \left(1 + \frac{S_{30}(p) - 2}{48p}\right). \quad (22)$$

Now, $S_{30}(f)$ depends only on f modulo 30 for $f \geq 1$ and $\gcd(30, f) = 1$, and easy numerical computations that, with :

f modulo 30	1	7	11	13	17	19	23	29
$S_{30}(f)$	-46	-42	26	2	-30	-54	14	18

In particular, $\frac{S_{30}(f)-2}{48} \leq \frac{1}{2}$. Using (3), (22) and Lemma 5.1, the desired third bound follows.

Finally, take $d_0 = 210$ and $p \geq 11$. Then

$$M_{210}(p) = \frac{128\pi^2}{1225} \left(1 + \frac{S_{210}(p) - 12}{2304p}\right). \quad (23)$$

Now, $S_{210}(f)$ depends only on f modulo 30 for $f \geq 1$ and $\gcd(210, f) = 1$, and easy numerical computations that $\frac{S_{210}(f)-12}{2304} \leq \frac{649}{576}$. Using (3), (23) and Lemma 5.1, the desired fourth bound follows. ■

By [1], we have $h_p^- = 2p^{1+o(1)} \left(\frac{p}{4\pi^2}\right)^{\frac{p-1}{4}}$. According to [4], Kummer's conjecture on the asymptotic behavior $h_p^- \sim 2p \left(\frac{p}{4\pi^2}\right)^{(p-1)/4}$ is unlikely to be true. Notice also the explicit bound $h_p^- \leq p^{31/4} \left(\frac{p}{4\pi^2}\right)^{p/4}$ (see [10]).

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