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Stéphane R Louboutin

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## MEAN SQUARE VALUE OF $L$ -FUNCTIONS AT $s = 1$ FOR NON-PRIMITIVE CHARACTERS, DEDEKIND SUMS AND BOUNDS ON RELATIVE CLASS NUMBERS

STÉPHANE R. LOUBOUTIN

**Abstract:** Let  $d_0$  be a given square-free integer. We give an explicit formula  $M_{d_0}(p) = A(d_0)(1 + B_{d_0}(p)/p)$  for the quadratic mean value at  $s = 1$  of the Dirichlet  $L$ -functions associated with the non-primitive odd Dirichlet characters modulo  $d_0p$  induced by the odd Dirichlet characters modulo an odd prime  $p$ . Here  $d_0 \mapsto A(d_0)$  is an explicit multiplicative arithmetic function and  $B_{d_0}(f)$  is a twisted sum over the divisors  $d$  of  $d_0$  of Dedekind sums  $s(h, df)$ . To prove that  $f \mapsto B_{d_0}(f)$  is  $d_0$ -periodic, we find a new and closed formula for the Dedekind sums  $f \mapsto s(a+bf, c+df)$ , for fixed integers  $a, b, c$  and  $d$ . We deduce explicit upper bounds for relative class numbers of cyclotomic number fields.

**Keywords:** Dirichlet character,  $L$ -function, mean square value, relative class number, Dedekind sums, cyclotomic field.

### 1. Introduction

Let  $f > 2$  be an integer. Let  $X_f$  be the multiplicative group of the  $\phi(f)$  Dirichlet characters modulo  $f$ . Let  $X_f^- = \{\chi \in X_f; \chi(-1) = -1\}$  be the set of the  $\phi(f)/2$  odd Dirichlet characters modulo  $f$ . Let  $L(s, \chi)$  be the Dirichlet  $L$ -functions associated with  $\chi \in X_f$ . We have (see [6, Proposition 1]):

$$L(1, \chi) = \frac{\pi}{2f} \sum_{a=1}^{f-1} \chi(a) \cot\left(\frac{\pi a}{f}\right) \quad (\chi \in X_f^-). \quad (1)$$

According to [11] and [6], the mean square value  $M(f)$  of  $L(1, \chi)$  as  $\chi$  ranges in  $X_f^-$  is defined by and given by

$$M(f) := \frac{1}{\#X_f^-} \sum_{\chi \in X_f^-} |L(1, \chi)|^2 = \frac{\pi^2 \phi(f)}{6f} \left\{ \prod_{q \substack{\text{prime} \\ q|f}} \left(1 + \frac{1}{q}\right) - \frac{3}{f} \right\}.$$

For  $f = p \geq 3$  an odd prime, we recover Walum's formula in [14]:

$$M(p) := \frac{2}{p-1} \sum_{\chi \in X_p^-} |L(1, \chi)|^2 = \frac{\pi^2}{6} \left\{ \left(1 - \frac{1}{p}\right) \left(1 - \frac{2}{p}\right) \right\} \quad (p \geq 3).$$

Let  $h_p^-$  denote the relative class number of the cyclotomic number field of prime conductor  $p \geq 3$  (see [15] for the definition). Using the arithmetic-geometric mean inequality, we obtain (see also [6], [9]):

$$h_p^- = 2p \left(\frac{p}{4\pi^2}\right)^{\frac{p-1}{4}} \prod_{\chi \in X_p^-} L(1, \chi) \leq 2p \left(\frac{pM(p)}{4\pi^2}\right)^{\frac{p-1}{4}} \leq 2p \left(\frac{p}{24}\right)^{\frac{p-1}{4}}. \quad (2)$$

Now, let  $d_0 > 1$  be a given square-free integer. Assume that  $p \nmid d_0$ . For  $\chi \in X_p$ , let  $\chi'$  be the non-primitive character modulo  $d_0p$  induced by  $\chi$ . Then,

$$L(1, \chi) = \left\{ \prod_{\substack{q \text{ prime} \\ q|d_0}} \frac{1}{1 - \frac{\chi(q)}{q}} \right\} L(1, \chi').$$

Therefore, we obtain

$$h_p^- = \frac{2p}{\Pi_p^-(d_0)} \left(\frac{p}{4\pi^2}\right)^{\frac{p-1}{4}} \prod_{\chi \in X_p^-} L(1, \chi') \leq \frac{2p}{\Pi_p^-(d_0)} \left(\frac{pM_{d_0}(p)}{4\pi^2}\right)^{\frac{p-1}{4}}, \quad (3)$$

where

$$\Pi_p^-(d_0) := \prod_{\substack{q \text{ prime} \\ q|d_0}} \prod_{\chi \in X_p^-} \left(1 - \frac{\chi(q)}{q}\right)$$

and

$$M_{d_0}(p) := \frac{2}{p-1} \sum_{\chi \in X_p^-} |L(1, \chi')|^2. \quad (4)$$

In [6] and [7], we explained how using (3) one could get explicit bounds better than (2) of the type  $h_p^- \leq 2p(p/C)^{(p-1)/4}$  for any given constant  $C < 4\pi^2$  and  $p$  large enough. In the present paper we greatly simplify the approach developed in these previous articles by giving a neat formula for these  $M_{d_0}(p)$ , see Theorems 3.1 and 4.3. For explicit formulas for  $d_0 \in \{2, 6, 30, 210\}$ , see (9), (10), (22) and (23). As a consequence of (3) and this formula, we readily obtain improvements on (2) in Theorem 5.2, by using the explicit lower bounds on  $\Pi_p^-(d_0)$  given in Lemma 5.1. We invite the reader to start his reading with a quick look at our Theorems 3.1, 4.3 and 5.2.

## 2. Dedekind sums

The Dedekind sums  $s(c, d)$  which depend only on  $c \pmod d$  are defined by

$$s(c, d) = \frac{1}{4d} \sum_{n=1}^{d-1} \cot\left(\frac{\pi n}{d}\right) \cot\left(\frac{\pi nc}{d}\right) \quad (c \in \mathbb{Z}, d \geq 1, \gcd(c, d) = 1),$$

with the convention  $s(c, 1) = 0$  for  $c \in \mathbb{Z}$ . Since  $c \mapsto s(c, d)$  is  $d$ -periodic, we may consider  $s(h, d)$  for  $h \in (\mathbb{Z}/d\mathbb{Z})^*$ . Recall the reciprocity law for Dedekind sums

$$s(c, d) + s(d, c) = \frac{c^2 + d^2 - 3cd + 1}{12cd}, \quad (c \geq 1, d \geq 1, \gcd(c, d) = 1) \quad (5)$$

(see [2], [12] or [8]). In particular,

$$s(1, d) = \frac{d^2 - 3d + 2}{12d} \quad (d \geq 1). \quad (6)$$

## 3. A mean square value formula for non-primitive $L$ -functions

**Theorem 3.1.** *Let  $d_0 > 1$  be a square-free integer. Set*

$$\phi_k(d_0) := \prod_{\substack{q \text{ prime} \\ q|d_0}} (q^k - 1).$$

*Whenever  $\gcd(f, d_0) = 1$ , let  $H_0 = H_{d_0}(f)$  be the kernel of order  $\phi(d_0)$  of the surjective canonical morphism  $(\mathbb{Z}/d_0 f\mathbb{Z})^* \rightarrow (\mathbb{Z}/f\mathbb{Z})^*$  and set*

$$S_{d_0}(f) := \mu(d_0) \sum_{1 \neq h \in H_0} \sum_{d|d_0} d\mu(d) s(h, df) \quad (\text{Dedekind sums}). \quad (7)$$

*Let  $p \geq 3$  be a prime that does not divide  $d_0$ . For  $\chi \in X_p^-$ , let  $\chi'$  be the non-primitive Dirichlet character modulo  $d_0 p$  induced by  $\chi$ . Then,*

$$M_{d_0}(p) := \frac{2}{p-1} \sum_{\chi \in X_p^-} |L(1, \chi')|^2 = \frac{\pi^2}{6} \frac{\phi_2(d_0)}{d_0^2} \left( 1 + \frac{12S_{d_0}(p) - 3\phi_1(d_0)}{\phi_2(d_0)p} \right).$$

**Proof.** The surjectivity of the canonical morphism  $(\mathbb{Z}/d_0 f\mathbb{Z})^* \rightarrow (\mathbb{Z}/f\mathbb{Z})^*$  readily follows from Dirichlet theorem that asserts that there are infinitely many prime integers in the arithmetic progressions  $a + f\mathbb{Z}$  for  $\gcd(a, f) = 1$ . The canonical morphism  $(\mathbb{Z}/d_0 f\mathbb{Z})^* \rightarrow (\mathbb{Z}/d_0\mathbb{Z})^*$  is also surjective. Since  $\gcd(d_0, f) = 1$ , its restriction to  $H_0$  is injective, hence bijective. It follows that whenever  $d$  divides  $d_0$  we have

$$S := \sum_{h \in H_0} s(h, d) = \sum_{h \in (\mathbb{Z}/d_0\mathbb{Z})^*} s(h, d) = 0. \quad (8)$$

Indeed, whenever  $d$  divides  $d_0$ , we have

$$S := \sum_{h \in (\mathbb{Z}/d_0\mathbb{Z})^*} s(h, d) = \sum_{\substack{h=1 \\ \gcd(h, d_0)=1}}^{d_0-1} s(h, d) = \sum_{\substack{h=1 \\ \gcd(h, d_0)=1}}^{d_0-1} s(d_0 - h, d) = -S.$$

Now, for  $\gcd(a, d_0p) = \gcd(b, d_0p) = 1$ , we have the orthogonality relations

$$\varepsilon(a, b) := \frac{2}{p-1} \sum_{\chi \in X_p^-} \chi'(a) \overline{\chi'(b)} = \begin{cases} +1 & \text{if } b \equiv +a \pmod{p}, \\ -1 & \text{if } b \equiv -a \pmod{p}, \\ 0 & \text{otherwise,} \end{cases}$$

and using (1) we have

$$M_{d_0}(p) = \frac{\pi^2}{4d_0^2p^2} \sum_{\substack{a=1 \\ \gcd(a, d_0p)=1}}^{d_0p-1} \sum_{\substack{b=1 \\ \gcd(b, d_0p)=1}}^{d_0p-1} \varepsilon(a, b) \cot\left(\frac{\pi a}{d_0p}\right) \cot\left(\frac{\pi b}{d_0p}\right).$$

The imparity of the cotangent function and the change of variables  $b \mapsto ah$  give

$$\begin{aligned} M_{d_0}(p) &= \frac{\pi^2}{2d_0^2p^2} \sum_{\substack{a=1 \\ \gcd(a, d_0p)=1}}^{d_0p-1} \sum_{h \in H_0} \cot\left(\frac{\pi a}{d_0p}\right) \cot\left(\frac{\pi ah}{d_0p}\right) \\ &= \frac{2\pi^2}{d_0p} \sum_{h \in H_0} \tilde{s}(h, d_0p), \end{aligned}$$

where

$$\begin{aligned} \tilde{s}(c, d) &:= \frac{1}{4d} \sum_{\substack{n=1 \\ \gcd(n, d)=1}}^{d-1} \cot\left(\frac{\pi n}{d}\right) \cot\left(\frac{\pi nc}{d}\right) \\ &= \sum_{\delta|d} \frac{\mu(\delta)}{\delta} s(c, d/\delta) = \frac{\mu(d)}{d} \sum_{\delta|d} \delta \mu(\delta) s(c, \delta). \end{aligned}$$

Since  $\delta \mid d_0p$  if and only if either  $\delta = d$  or  $\delta = dp$  where  $d \mid d_0$ , we have

$$\begin{aligned} M_{d_0}(p) &= \frac{2\pi^2}{d_0p} \sum_{h \in H_0} \frac{\mu(d_0p)}{d_0p} \sum_{d|d_0p} d\mu(d) s(h, d) \\ &= \frac{2\pi^2}{d_0p} \frac{\mu(d_0p)}{d_0p} \sum_{h \in H_0} \left( \sum_{d|d_0} d\mu(d) s(h, d) - p \sum_{d|d_0} d\mu(d) s(h, dp) \right) \\ &= \frac{2\pi^2}{d_0p} \frac{\mu(d_0)}{d_0} \sum_{h \in H_0} \sum_{d|d_0} d\mu(d) s(h, dp), \quad (\text{by (8)}) \\ &= \frac{2\pi^2}{d_0p} \frac{\mu(d_0)}{d_0} \left( \mu(d_0) S_{d_0}(p) + \sum_{d|d_0} d\mu(d) s(1, dp) \right). \end{aligned}$$

Finally, by (6), for  $d_0 > 1$  square-free we have

$$\sum_{d|d_0} d\mu(d)s(1, df) = \sum_{d|d_0} \mu(d) \frac{f^2 d^2 - 3df + 2}{12f} = \mu(d_0) \frac{f\phi_2(d_0) - 3\phi_1(d_0)}{12}.$$

The desired formula follows. ■

**Example 1.** Take  $d_0 = 2$  and  $f \geq 3$  odd. Then  $H_0 = \{1\}$  and  $S_2(f) = 0$ . Therefore, in accordance with [6, Théorème 3], for  $p \geq 3$  we have

$$M_2(p) = \frac{\pi^2}{8} \left(1 - \frac{1}{p}\right). \quad (9)$$

**Example 2.** Take  $d_0 = 2 \cdot 3 = 6$  and  $f \geq 5$  coprime with 6. Then  $H_0 = \{1, h_0\}$ , where  $h_0 = 1 + 4f$  if  $f \equiv 1 \pmod{6}$  and  $h_0 = 1 + 2f$  if  $f \equiv 5 \pmod{6}$ , and

$$M_6(p) = \frac{\pi^2}{9} \left(1 + \frac{2S_6(p) - 1}{4p}\right) \quad (10)$$

Moreover, using (5), (6) and a tad tedious but easy computation, we obtain:

$$\begin{aligned} S_6(f) &= 6s(h_0, 6f) - 3s(h_0, 3f) - 2s(h_0, 2f) + s(h_0, f) \\ &= \begin{cases} \frac{2f^2-13f+1}{6f} - \frac{f^2-5f+2}{12f} - \frac{2f^2-3f+1}{6f} + \frac{f^2-3f+2}{12f} = -\frac{3}{2} & \text{if } f \equiv 1 \pmod{6}, \\ \frac{2f^2-5f+1}{6f} - \frac{f^2-13f+2}{12f} - \frac{2f^2-3f+1}{6f} + \frac{f^2-3f+2}{12f} = \frac{1}{2} & \text{if } f \equiv 5 \pmod{6}. \end{cases} \end{aligned}$$

For example, for  $f \equiv 1 \pmod{6}$  we have

$$\begin{aligned} s(1 + 4f, 6f) &= -\frac{10f^2 + 5f - 1}{36f(4f + 1)} - s(6f, 4f + 1) \\ &= -\frac{10f^2 + 5f - 1}{36f(4f + 1)} - s(2f - 1, 4f + 1) \\ &= -\frac{2f^2 + 7f + 1}{36f(2f - 1)} + s(4f + 1, 2f - 1) \\ &= -\frac{2f^2 + 7f + 1}{36f(2f - 1)} + s(3, 2f - 1) \\ &= \frac{2f^2 - 11f + 1}{36f} - s(2f - 1, 3) \\ &= \frac{2f^2 - 11f + 1}{36f} - s(1, 3) \\ &= \frac{2f^2 - 11f + 1}{36f} - \frac{1}{18} \\ &= \frac{2f^2 - 13f + 1}{36f}. \end{aligned}$$

Therefore,  $S_6(f)$  depends only on  $f$  modulo 6, and for  $p \geq 5$  we have

$$M_6(p) = \frac{\pi^2}{9} \times \begin{cases} 1 - \frac{1}{p} & \text{if } p \equiv 1 \pmod{6} \\ 1 & \text{if } p \equiv 5 \pmod{6} \end{cases} \text{ and } M_6(p) \leq \frac{\pi^2}{9}, \quad (11)$$

in accordance with [6, Théorème 4 (revised in the Addendum)].

#### 4. On the Dedekind sums $f \mapsto s(a + bf, c + df)$

The aim of this section is to prove in Theorem 4.3 that  $f \mapsto S_{d_0}(f)$  is  $d_0$ -periodic, which makes our mean square formula in Theorem 3.1 even more interesting and useful. Indeed, we will use it to obtain explicit improvements on (2), see Theorem 5.2.

Let  $d_0 > 1$  be a given square-free integer. For  $f \geq 1$  we can rewrite (7) as

$$S_{d_0}(f) := \mu(d_0) \sum_{\substack{k=1 \\ \gcd(1+kf, d_0f)=1}}^{d_0-1} S_{k, d_0}(f), \quad (12)$$

where

$$S_{k, d_0}(f) := \sum_{d|d_0} d\mu(d)s(1+kf, df). \quad (13)$$

We prove in Theorem 4.3 that for  $f$  large enough  $f \mapsto S_{k, d_0}(f)$  is  $d_0$ -periodic. Therefore,  $S_{d_0}(f)$  depends only on  $f$  modulo  $d_0$  for  $f$  explicitly large enough.

To begin with, we extend the definition of Dedekind sums by setting

$$s(c, d) = \text{sign}(cd)s(|c|, |d|) = \text{sign}(d)s(c, |d|) \quad (c, d \in \mathbb{Z} \setminus \{0\}).$$

Hence  $c \mapsto s(c, d)$  is still  $d$ -periodic and (5) gives the following generalized reciprocity law that holds true whatever the signs of  $c, d \in \mathbb{Z} \setminus \{0\}$ :

$$s(c, d) = \frac{c^2 + d^2 + 1}{12cd} - \frac{\text{sign}(cd)}{4} - s(d, c). \quad (14)$$

Let us now set some notations. Let  $P_0(X) = a_0 + b_0X$  and  $P_1(X) = a_1 + b_1X$  in  $\mathbb{Z}[X]$  be given, with  $b_0 > b_1 \geq 1$  and

$$D_0 = \det \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix} \neq 0.$$

Let  $b_0 > b_1 > \dots > b_l > b_{l+1} = 0$  denote the sequence of remainders in the Euclidean algorithm for calculating  $\gcd(b_0, b_1)$ . For  $0 \leq i \leq l-1$ , define  $P_i(X) = a_i + b_iX \in \mathbb{Z}[X]$  by  $P_{i+2}(X) = P_i(X) - q_i P_{i+1}(X)$ , where  $q_i = \lfloor b_i/b_{i+1} \rfloor \geq 1$ . Clearly,  $\gcd(a_{i+1}, a_i)$ ,  $\gcd(b_{i+1}, b_i)$  and  $\gcd(P_{i+1}(f), P_i(f))$  do not depend on the index  $i \in \{0, \dots, l\}$ , where  $f \in \mathbb{Z}$ . In particular,

$$b_l = \gcd(0, b_l) = \gcd(b_{l+1}, b_l) = \gcd(b_1, b_0). \quad (15)$$

Moreover,

$$D_i = \det \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix} = (-1)^i D_0 \neq 0.$$

Set

$$r_{b_0, b_1} := \sum_{i=0}^{l-1} (-1)^i \left( \frac{b_i}{b_{i+1}} + \frac{b_{i+1}}{b_i} \right),$$

(which is related to the Dedekind sum  $s(b'_1, b'_0)$  and  $\sum_{i=0}^{l-1} \frac{(-1)^i}{b'_i b'_{i+1}}$ , where  $b'_i = b_i / \gcd(b_1, b_0)$ , by [2, Exercise 20, page 73]),

$$T(X) = T_{a_0, b_0, a_1, b_1}(X) := P_l(X) = a_l + b_l X = a_l + \gcd(b_1, b_0) X \quad (16)$$

and

$$R(X) = R_{a_0, b_0, a_1, b_1}(X) := r_{b_0, b_1} + \frac{\frac{D_0}{b_0} + \frac{b_0}{D_0}}{P_0(X)} + \frac{\gcd(b_1, b_0)}{D_0} T(X). \quad (17)$$

We are now in a position to state our new result on Dedekind sums:

**Theorem 4.1.** *Let  $P_0(X) = a_0 + b_0 X$  and  $P_1(X) = a_1 + b_1 X$  in  $\mathbb{Z}[X]$  be given, with  $b_0 > b_1 \geq 1$  and*

$$D_0 = \det \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix} \neq 0.$$

*For  $f \geq 1$  and  $\gcd(P_1(f), P_0(f)) = 1$ , we have the Dedekind sums formula*

$$s(P_1(f), P_0(f)) = \frac{1}{12} R(f) - \frac{1 - (-1)^l}{8} - \frac{\text{sign}(D_0)}{4} - s(T(f), D'_0), \quad (18)$$

*where  $D'_0 := D_0 / \gcd(b_0, b_1)$  is a rational integer not equal to 0 and the arithmetic function  $f \mapsto s(T(f), D'_0)$  is  $D'_0$ -periodic.*

For example, take  $P_0(X) = 6X$  and  $P_1(X) = 1 + 4X$ . We have  $D_0 = 6$ ,  $l = 2$ ,  $P_2(X) = 2X - 1$  and  $P_3(X) = 3$ ,  $r_{6,4} = -1/3$ ,  $R(X) = \frac{2X^2 - 2X + 1}{3X}$ . Therefore, for  $f \equiv 1 \pmod{6}$  we have  $s(2f - 1, 3) = s(1, 3) = 1/18$  and we obtain

$$s(1 + 4f, 6f) = \frac{2f^2 - 11f + 1}{36f} - s(2f - 1, 3) = \frac{2f^2 - 13f + 1}{36f},$$

as in Example 2 of the previous Section.

**Proof.** We will use some technical results proved in the next Lemma 4.2.

Set  $Q(X, Y) = \frac{X^2 + Y^2 + 1}{XY}$ , and  $Q_i(X) = Q(P_{i+1}(X), P_i(X))$ . We have the easy to check identity

$$Q_i(X) = \frac{b_i}{b_{i+1}} + \frac{b_{i+1}}{b_i} - \frac{\frac{D_i}{b_{i+1}} + \frac{b_{i+1}}{D_i}}{P_{i+1}(X)} + \frac{\frac{D_i}{b_i} + \frac{b_i}{D_i}}{P_i(X)} \quad (0 \leq i \leq l-1),$$



and

$$Q_l(X) = \frac{\frac{D_l}{b_l} + \frac{b_l}{D_l}}{P_l(X)} + \frac{b_l}{D_l} P_l(X).$$

Noticing that  $D_i = (-1)^i D_0$ , we obtain

$$\sum_{i=0}^l (-1)^i Q_i(X) = r_{b_0, b_1} + \frac{\frac{D_0}{b_0} + \frac{b_0}{D_0}}{P_0(X)} + \frac{b_l}{D_0} P_l(X) = R(X). \quad (19)$$

Now, let  $f \geq 1$  be such that  $\gcd(P_1(f), P_0(f)) = 1$ .

Then  $\gcd(P_{i+1}(f), P_i(f)) = 1$ ,  $P_i(f) \neq 0$  for  $0 \leq i \leq l$ , by Lemma 4.2, and  $P_{l+1}(f) = a_{l+1} \neq 0$ . Therefore, we can define  $\varepsilon_i(f) = \text{sign}(P_i(f)) \in \{\pm 1\}$  for  $0 \leq i \leq l+1$ . Using (14),  $P_i(f) = P_{i+2}(f) + q_i P_{i+1}(f)$  and the  $d$ -periodicity of  $c \mapsto s(c, d)$ , we obtain  $s(P_i(f), P_{i+1}(f)) = s(P_{i+2}(f), P_{i+1}(f))$  and

$$s(P_{i+1}(f), P_i(f)) = \frac{1}{12} Q_i(f) - \frac{\varepsilon_{i+1}(f) \varepsilon_i(f)}{4} - s(P_{i+2}(f), P_{i+1}(f))$$

for  $0 \leq i \leq l-1$  and

$$s(P_{l+1}(f), P_l(f)) = \frac{1}{12} Q_l(f) - \frac{\varepsilon_{l+1}(f) \varepsilon_l(f)}{4} - s(P_l(f), P_{l+1}(f)).$$

Taking the alternating sums of these  $l+1$  equalities and using (19) we obtain

$$s(P_1(f), P_0(f)) = \frac{1}{12} R(f) - \frac{1}{4} \varepsilon(f) - (-1)^l s(P_l(f), P_{l+1}(f)),$$

where

$$\varepsilon(f) = \sum_{i=0}^l (-1)^i \varepsilon_{i+1}(f) \varepsilon_i(f). \quad (20)$$

By Lemma 4.2, we have

$$\varepsilon(f) = \frac{1 - (-1)^l}{2} + \text{sign}(D_0).$$

Finally,  $P_{l+1}(f) = a_{l+1} = \frac{a_{l+1} b_l}{\gcd(b_0, b_1)} = \frac{D_l}{\gcd(b_0, b_1)} = (-1)^l D_0 / \gcd(b_0, b_1)$ , by (15). Therefore,

$$(-1)^l s(P_l(f), P_{l+1}(f)) = s(P_l(f), D_0 / \gcd(b_0, b_1)),$$

and the desired result follows. ■

Let us now prove what we used for proving Theorem 4.1, namely that the  $P_i(f)$ 's are never equal to 0, which enabled us to deal with their signs, their associated Dedekind sums and the reciprocity law for Dedekind sums:

**Lemma 4.2.** *Let  $P_0(X) = a_0 + b_0X$  and  $P_1(X) = a_1 + b_1X$  in  $\mathbb{Z}[X]$  be given, with  $b_0 > b_1 \geq 1$ ,  $a_0, a_1 \geq 0$  and*

$$D_0 = \det \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \end{pmatrix} \neq 0.$$

*Let  $f \geq 1$  be such that  $\gcd(P_1(f), P_0(f)) = 1$ . Then  $P_i(f) \neq 0$  for  $0 \leq i \leq l + 1$  and  $\varepsilon(f)$  defined in (20) does not depend on  $f$  and is given by*

$$\varepsilon(f) = \frac{1 - (-1)^l}{2} + \text{sign}(D_0).$$

**Proof.** Recall that  $q_i \geq 1$  for  $0 \leq i \leq l$ . We claim that  $a_i \geq 0$  for  $0 \leq i \leq i_0$  and  $a_i \neq 0$  with  $\text{sign}(a_i) = (-1)^{i-i_0}$  for  $i_0 < i \leq l + 1$ , where

$$1 \leq i_0 := \max\{i; 0 \leq i \leq l + 1 \text{ and } a_i \geq 0\}.$$

Indeed, if  $i_0 \leq l$ , then  $a_{i_0} \geq 0$  and  $a_{i_0+1} \leq -1$ . Hence if  $i_0 \leq l - 1$  then  $a_{i_0+2} = a_{i_0} - q_{i_0}a_{i_0+1} \geq 1$ , i.e.  $a_{i_0+1} \leq -1$  and  $a_{i_0+2} \geq 1$ . Hence if  $i_0 \leq l - 2$  then  $a_{i_0+3} = a_{i_0+1} - q_{i_0+1}a_{i_0+2} \leq -1$ , i.e.  $a_{i_0+2} \geq 1$  and  $a_{i_0+3} \leq -1$ . We then proceed inductively. Consequently, (i) if  $a_{i+1} \leq -1$  for some index  $i \leq l$  then  $a_i \geq 0$  for the previous index and (ii) the indices in

$$A_- := \{i; 0 \leq i \leq l + 1 \text{ and } a_i \leq -1\}$$

have all the same parity.

Now assume that  $P_{i+1}(f) = 0$  for some  $f \geq 1$  and some index  $i \leq l$ . Since  $P_{l+1}(X) = a_{l+1} \neq 0$ , we have  $i \leq l - 1$  and  $b_i > b_{i+1} \geq b_l > b_{l+1} = 0$ . Hence,  $b_i \geq 2$  and  $b_{i+1} \geq 1$ . Therefore,  $0 = P_{i+1}(f) = a_{i+1} + b_{i+1}f \geq a_{i+1} + 1$  gives  $a_{i+1} \leq -1$ . Hence,  $a_i \geq 0$ , by the assertion above, and  $P_i(f) = a_i + b_i f \geq a_i + 2 \geq 2$ . However,  $P_{i+1}(f) = 0$  and  $\gcd(P_{i+1}(f), P_i(f)) = \gcd(P_1(f), P_0(f)) = 1$  give  $P_i(f) = \pm 1$  and we obtain a contradiction.

Hence,  $P_i(f) \neq 0$  for  $f \geq 1$  and  $0 \leq i \leq l + 1$  and the  $\varepsilon_i(f)$  are well defined. We claim that  $\varepsilon_i(f) = +1$  for  $0 \leq i \leq i_0(f)$  and  $\varepsilon_i(f) = (-1)^{i-i_0(f)}$  for  $i_0(f) < i \leq l + 1$ , where

$$i_0(f) := \max\{i; 0 \leq i \leq l + 1 \text{ and } P_i(f) \geq 1\}.$$

Indeed, if  $i_0(f) \leq l$ , then  $P_{i_0}(f) \geq 1$  and  $P_{i_0+1}(f) \leq -1$ . Hence if  $i_0 \leq l - 1$  then  $P_{i_0+2}(f) = P_{i_0}(f) - q_{i_0}P_{i_0+1}(f) \geq 1$ , i.e.  $P_{i_0+1}(f) \leq -1$  and  $P_{i_0+2}(f) \geq 1$ . Hence if  $i_0(f) \leq l - 2$  then  $P_{i_0+3}(f) = P_{i_0+1}(f) - q_{i_0+1}P_{i_0+2}(f) \leq -1$ , i.e.  $P_{i_0+2}(f) \geq 1$  and  $P_{i_0+3}(f) \leq -1$ . We then proceed inductively.

Hence,  $\varepsilon_{i+1}(f)\varepsilon_i(f) = +1$  for  $0 \leq i \leq i_0(f) - 1$  and  $\varepsilon_{i+1}(f)\varepsilon_i(f) = -1$  for  $i_0(f) \leq i \leq l$ . By (20), it follows that  $\varepsilon(f) := \frac{1 - (-1)^l}{2} - (-1)^{i_0(f)}$ .

Finally,  $I(f) := \{i; 0 \leq i \leq l + 1 \text{ and } P_i(f) \leq -1\}$  is included in  $A_-$ . Since the  $i$ 's in  $A_-$  have all the same parity, the parity of  $i_0(f) = \min I(f) - 1$  does not depend on  $f \geq 1$ . Since  $b_i \geq 1$  for  $0 \leq i \leq l$ , for  $f >$  large enough we have  $\varepsilon_i(f) = +1$  for  $0 \leq i \leq l$ ,  $i_0(f) = l + \frac{1 + \text{sign}(a_{l+1})}{2}$  and  $\varepsilon(f) = \frac{1 - (-1)^l}{2} + \text{sign}(D_0)$ . ■

**Theorem 4.3.** *Let  $0 \leq k < d$  be given. There exist  $r_{k,d} \in \mathbb{Q}$  and  $T_{k,d}(X) = a + bX \in \mathbb{Z}[X]$  such that for  $f \geq 1$  and  $\gcd(1 + kf, df) = 1$ , we have*

$$s(1 + kf, df) = \frac{\gcd(k, d)^2 f^2 + 2}{12df} + r_{k,d} - s(T_{k,d}(f), d/\gcd(k, d)). \quad (21)$$

Consequently, the rational numbers  $S_{d_0}(f)$  defined in (7) depend only on  $f \geq 1$  modulo  $d_0$ .

**Proof.** For  $(a_0, b_0, a_1, b_1) = (0, d, 1, k)$  we have  $\gcd(b_1, b_0) = \gcd(k, d)$ ,  $D_0 = d$ ,  $D_0/b_0 + b_0/D_0 = 2$  and  $P_0(X) = dX$ . By (16), (17) and (18), we do have the first assertion (21). For the second assertion, we use (21) and notice that  $\sum_{d|d_0} \mu(d) = 0$  for  $d_0 > 1$  and that the function  $d_0 \mapsto G(d_0, k) := \sum_{d|d_0} \mu(d) \gcd(k, d)^2$  is multiplicative. Therefore,

$$G(d_0, k) = \prod_{\substack{q|d_0, \\ q \text{ prime}}} (1 - \gcd(k, q)^2).$$

Hence, for  $d_0$  square-free,  $G(d_0, k) = 0$  if and only if  $d_0 \nmid k$ .

It follows that for  $d_0 > 1$  square-free and  $k$  not divisible by  $d_0$  there exists  $r'_{k,d_0} \in \mathbb{Q}$  such that for  $f \geq 1$  and  $\gcd(1 + kf, df) = 1$ , we have

$$S_{k,d_0}(f) := \sum_{d|d_0} d\mu(d)s(1 + kf, df) = r'_{k,d_0} - \sum_{d|d_0} d\mu(d)s(T_{k,d}(f), d/\gcd(k, d)).$$

Hence, the rational numbers  $S_{k,d_0}(f)$  defined in (13) depends only on  $f$  modulo  $d_0$  and the last assertion follows from (12).  $\blacksquare$

We refer the reader to [5, Theorem 4.4 and Proposition 4.6] for a different proof of the last assertion of Theorem 4.3.

## 5. Explicit upper bounds on relative class numbers

**Lemma 5.1.** *Let  $q \geq 2$  be a given prime integer. Let  $p \geq 3$  be a prime integer, with  $p \neq q$ . Let  $f$  be the order of  $q$  in the multiplicative group  $(\mathbb{Z}/p\mathbb{Z})^*$ . Then*

$$\Pi_p^-(q) := \prod_{\chi \in X_p^-} \left(1 - \frac{\chi(q)}{q}\right) = \begin{cases} \left(1 - \frac{1}{q^f}\right)^{\frac{p-1}{2f}} & \text{if } f \text{ is odd} \\ \left(1 + \frac{1}{q^{f/2}}\right)^{\frac{p-1}{f}} & \text{if } f \text{ is even.} \end{cases}$$

Moreover, we have

$$\Pi_p^-(d_0) := \prod_{\substack{q \text{ prime} \\ q|d_0}} \Pi_p^-(q) \geq \begin{cases} 6/7 & \text{for } d_0 = 2 \text{ and } p \geq 3, \\ 2/3 & \text{for } d_0 = 6 \text{ and } p \geq 5, \\ 1/2 & \text{for } d_0 = 30 \text{ and } p \geq 7, \\ 5/13 & \text{for } d_0 = 210 \text{ and } p \geq 11. \end{cases}$$

**Proof.** By [13, Lemma 6 page 72], we have  $\Pi_p(q) := \prod_{\chi \in X_p} (1 - \chi(q)T) = (1 - T^f)^{\frac{p-1}{2f}}$  and  $\Pi_p^+(q) := \prod_{\chi \in X_p^+} (1 - \chi(q)T) = (1 - T^{f_+})^{\frac{p-1}{2f_+}}$ , where  $f_+$  is the order of  $q$  in the multiplicative group  $G_+ := (\mathbb{Z}/p\mathbb{Z})^*/\{\pm 1\}$  of order  $(p-1)/2$  and  $X_p^+ = \{\chi \in X_p; \chi(-1) = +1\}$  is the group of characters of  $G_+$ . Now, clearly,  $f_+ = f$  if  $f$  is odd and  $f_+ = f/2$  if  $f$  is even. Since  $\Pi_p^-(q) = \Pi_p(q)/\Pi_p^+(q)$ , the desired first result follows. Now, we have  $q^f \geq p+1$ . Hence,

$$\Pi_p^-(q) \geq \left(1 - \frac{1}{p+1}\right)^{\frac{p-1}{2f}} \geq \left(1 - \frac{1}{p+1}\right)^{\frac{(p-1)\log q}{2\log(p+1)}} = \exp\left(\frac{\log q}{2} F(p+1)\right)$$

where for  $x > 1$  we set  $F(x) := \frac{(x-2)\log(1-\frac{1}{x})}{\log x}$ . Therefore,

$$\Pi_p^-(d_0) \geq \exp\left(\frac{\log d_0}{2} F(p+1)\right).$$

The desired lower bounds easily follow. ■

**Theorem 5.2.** *By taking successively  $d_0 = 2, 6, 30$  and  $210$  in (3) we obtain*

$$h_p^- \leq \begin{cases} \frac{7p}{3} \left(\frac{p}{32}\right)^{(p-1)/4} & \text{for } p \geq 3 \quad (\text{compare with [3, (1)]}), \\ 3p \left(\frac{p}{36}\right)^{(p-1)/4} & \text{for } p \geq 5, \\ 4p \left(\frac{2p+1}{75}\right)^{(p-1)/4} & \text{for } p \geq 7, \\ \frac{26p}{5} \left(\frac{32p+649}{1225}\right)^{(p-1)/4} & \text{for } p \geq 11. \end{cases}$$

**Proof.** For  $d_0 = 2$  we use (3), (9) and Lemma 5.1 to get the first bound.

For  $d_0 = 6$  we use (3), (11) and Lemma 5.1 to get the second.

Take  $d_0 = 30$  and  $p \geq 7$ . Then

$$M_{30}(p) = \frac{8\pi^2}{75} \left(1 + \frac{S_{30}(p) - 2}{48p}\right). \quad (22)$$

Now,  $S_{30}(f)$  depends only on  $f$  modulo 30 for  $f \geq 1$  and  $\gcd(30, f) = 1$ , and easy numerical computations that, with :

$f$ modulo 30	1	7	11	13	17	19	23	29
$S_{30}(f)$	-46	-42	26	2	-30	-54	14	18

In particular,  $\frac{S_{30}(f)-2}{48} \leq \frac{1}{2}$ . Using (3), (22) and Lemma 5.1, the desired third bound follows.

Finally, take  $d_0 = 210$  and  $p \geq 11$ . Then

$$M_{210}(p) = \frac{128\pi^2}{1225} \left(1 + \frac{S_{210}(p) - 12}{2304p}\right). \quad (23)$$

Now,  $S_{210}(f)$  depends only on  $f$  modulo 30 for  $f \geq 1$  and  $\gcd(210, f) = 1$ , and easy numerical computations that  $\frac{S_{210}(f)-12}{2304} \leq \frac{649}{576}$ . Using (3), (23) and Lemma 5.1, the desired fourth bound follows. ■

By [1], we have  $h_p^- = 2p^{1+o(1)} \left(\frac{p}{4\pi^2}\right)^{\frac{p-1}{4}}$ . According to [4], Kummer's conjecture on the asymptotic behavior  $h_p^- \sim 2p \left(\frac{p}{4\pi^2}\right)^{(p-1)/4}$  is unlikely to be true. Notice also the explicit bound  $h_p^- \leq p^{31/4} \left(\frac{p}{4\pi^2}\right)^{p/4}$  (see [10]).

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**Address:** Stéphane R. Louboutin: Institut de Mathématiques de Marseille, Aix Marseille Université, CNRS, 163 Avenue de Luminy, Case 907, 13288 Marseille Cedex 9, France.

**E-mail:** stephane.louboutin@univ-amu.fr

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