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► **To cite this version:**

Antoine Soulas. Logical implications between fundamental properties of relativistic quantum theories. 2023. hal-04056859v2

HAL Id: hal-04056859

<https://hal.science/hal-04056859v2>

Preprint submitted on 7 Sep 2023 (v2), last revised 18 Jun 2024 (v3)

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Logical implications between fundamental properties of relativistic quantum theories

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August 31, 2023

Abstract

A mathematical consistency condition constraining any relativistic quantum theory is formulated. It turns out to be equivalent to the locality of physics as well as, in the context of quantum field theory, microcausality, thereby revealing that these are actually two redundant hypotheses. It also promotes an epistemic interpretation of the wavefunction collapse, helps address unsolved problems related to nonlocal measurements and provides a new proof of the non-measurability of fermionic fields.

Keywords: relativistic quantum mechanics, relativistic measurements, locality, microcausality, axiomatic QFT.

Acknowledgements

I would like to gratefully thank my PhD supervisor Dimitri Petritis for the great freedom he grants me in my research, while being nevertheless always present to guide me. I also thank my friends Dmitry Chernyak and Matthieu Dolbeault for illuminating discussions.

Statements and Declarations

Funding: no funding was received for conducting this study.

Competing Interests: the author has no competing interests to declare that are relevant to the content of this article.

Ethics approval, Consent, Data, Materials and code availability, Authors' contribution statements: not applicable.

Introduction

The aim of this paper is to clarify and present rigorously a lot of well-known, but often badly presented, facts about relativistic quantum theories. In doing so, interesting new questions and results will emerge, as well as a better understanding of the interrelationships between the fundamental properties of these theories.

A key aspect of this investigation is the notion of measurement. Standard quantum field theory (QFT) textbooks usually focus on constructing the quantum fields and the dynamics in the Fock space, but usually don't tackle the question of the measurement as though it were a purely non-relativistic topic. More generally, most of them don't check the compatibility of quantum mechanics (QM) with the disappearance of the notion of instantaneity in special relativity, and there have been surprisingly few works addressing this problem.

Although this paper is not directly concerned by the measurement problem, we will sometimes get close to interpretational issues. This is why we begin with some preliminaries on the status of the wavefunction collapse in QM (§1). Two possible sources of instantaneities in QM are then exhibited but, in the light of the previous section, only one of them is *a priori* problematic. To deal with the latter, a consistency condition called (C), constraining the statistics of measurements outcomes in any relativistic quantum theory, is formulated and is shown to be necessary to prevent faster-than-light communication and non-covariance (§2.1). This condition takes a precise mathematical form, known as the no-communication theorem, which can be derived under some (generally accepted) assumption on the unitary evolution operator of two isolated systems. (C) being satisfied allows, and even fosters, an epistemic interpretation of the collapse (§2.2). We compare our treatment with the existing literature in §2.3.

At this point, we try to go further than the previous works by embracing a more axiomatic point of view. Rather than considering it as a theorem, we argue that (C) is the truly fundamental postulate because it is required by special relativity. We then look for the whole logical structure relating (C), and the previously introduced properties, with the locality of physics on one hand (§3) and with microcausality and the spin-statistics theorem in the context of QFT on the other hand (§4). Our study will reveal rather unexpectedly that locality and microcausality are actually two redundant hypotheses of QFT; it will also provide us with an original proof that the Dirac field can in no sense be measured. The particular unsolved issue of non-local measurements is addressed in §3.4.

1 Preliminaries

It is not the aim of this paper to address thoroughly the interpretational issues of QM, but some basic considerations will be needed at some point. The material for this section is based on an ongoing article [25] where we propose a detailed investigation of the measurement problem.

1.1 Selective measurement, non-selective measurement

When discussing the implementation of measurements in QM, it is crucial to distinguish clearly between selective and non-selective measurement. The former is what is usually referred to as the 'wave function collapse'. When measuring an observable \hat{A} , of spectral decomposition $\hat{A} = \sum_{x \in \text{spec} \hat{A}} x \Pi_x$, relative to a system in a pure state (resp. mixed state) $|\Psi\rangle$ (resp. ρ), if the outcome is the eigenvalue x_0 , then the system's state is standardly postulated to evolve as:

$$|\Psi\rangle \longrightarrow \frac{\Pi_{x_0} |\Psi\rangle}{\|\Pi_{x_0} |\Psi\rangle\|} \quad \text{resp.} \quad \rho \longrightarrow \frac{\Pi_{x_0} \rho \Pi_{x_0}}{\text{tr}(\rho \Pi_{x_0})}. \quad (1)$$

Note that the implementation of such a selective measurement requires to know, or to assume, the outcome. Alternatively, a non-selective measurement describes the update of the state without distinguishing the actual outcome: it is only concerned about the statistics that would be obtained if the experiment were repeated. Except in specific cases, the result is always a mixed state:

$$\rho \longrightarrow \sum_{x \in \text{spec} \hat{A}} \Pi_x \rho \Pi_x. \quad (2)$$

Selective measurement is a non-linear probabilistic operation on ρ , whereas non-selective measurement is a linear deterministic one, since it merely consists of extracting the diagonal part of the density matrix in the eigenbasis of the measured observable. The theory of decoherence explains how non-selective measurements (2) arise in QM, due to the entanglement between the system and its measurement apparatus [31] [16]. What about the selective measurement (1)?

1.2 The status of the collapse

In any probabilistic model, an update of the probabilities has to be performed when one obtains new empirical information about the system considered.¹ Since QM is a probabilistic theory, it is very natural that the wavefunction needs to be updated after a gain of information on the system, namely after a selective measurement. It is therefore tempting to say that the physical process affecting the wavefunction during a measurement (ontic side) is the one entailed by decoherence (2), whereas the collapse (1) is not a physical process (epistemic side). Further arguments in favor of the non-physicality of the collapse is that objective collapse models still suffer from issues in their compatibility with special relativity despite numerous attempts [11] [26] [3], and that large portions of the set of possible parameters have already been ruled out experimentally [8]. They are also at odds with the phenomenon of coherence revival [9]. Besides, what can be the status of decoherence in a collapse model, apart from a curious redundant phenomenon that also happens to destroy very efficiently the quantum interferences?

For these reasons, and also because it will appear particularly consistent in the relativistic context (see §2.3), we will adopt in this paper the stance that the collapse is a purely epistemic operation. This view, so natural in classical probabilities, obviously comes with its own lot of interpretational issues. This is in particular because quantum probabilities display interferences between all the possible histories before a measurement, so that every potential outcome seems to have had an influence on the result of the experiment, hence a sort of ‘reality’, whereas a probabilistic update is simply supposed to suppress the weights of all the truly unreal histories (again, we refer to [25] for a complete discussion). Anyway, one direct consequence of an epistemic collapse is that there are as many wavefunctions of a system as there are observers having different knowledge about it.

2 The conditions (C) and (MC)

2.1 Literal and mathematical formulation of the conditions

There are two different sources of instantaneity in QM that could cause troubles when trying to build a relativistic quantum theory. When two subsystems are entangled, they must be considered as a whole², therefore:

- any physical evolution (for instance a non-selective measurement (2)) on the first instantaneously affects the whole state no matter how far the other part may be,
- any measurement performed on the first allows the observer to apply a collapse (1) on the whole state, which is generally presented as an instantaneous update.

However, according to the understanding of the collapse defended in §1.2, the second item is not problematic at all, because the epistemic update of one particular observer has no reason to be constrained by the speed of light. As Bell wrote [5]: ‘When the Queen dies in London (may it long be delayed) the Prince of Wales, lecturing on modern architecture in Australia, becomes instantaneously King’, and he could have added: from the Queen and her entourage’s point of view. On the other hand, what has to be constrained by special relativity are the physically predictable effects, so that no experiment conducted on the King can determine faster than light whether the Queen is alive. For this to be prevented in

¹As a trivial example, suppose for instance that you ask your computer to choose uniformly a number x_1 in $\{0; 1\}$, and then to choose uniformly a second number x_2 in $\{0; 1\}$ if $x_1 = 0$, or to choose uniformly x_2 in $\{2; 3\}$ if $x_1 = 1$. If you don’t look at the result for x_1 , you predict x_2 to follow a uniform law in $\{0; 1; 2; 3\}$, but if you know the outcome for x_1 , you obviously update your probabilities.

²It has been experimentally confirmed that non-local correlations are satisfied (almost) immediately: in [29], the authors present an incredibly precise experimental test which has allowed them to ‘set a lower bound on the speed on quantum information to 10^7c , *i.e.* seven orders of magnitude larger than the speed of light.’

QM despite the non-locality of the entanglement phenomenon, the following consistency condition has to hold:

(C) *For all quantum systems composed of two entangled subsystems, any physical evolution of the first must leave invariant **the statistical results** of any measurement on the second, if the two are spacelike separated.*

In the following, we will also consider the following more specific consistency condition (dealing only with measurements), that will subsequently be referred to as the condition (MC), because it is easier to manipulate mathematically and will turn out to be equivalent to (C):

(MC) *For all quantum systems composed of two entangled subsystems, any ideal projective measurement of the first must leave invariant **the statistical results** of any measurement on the second, if the two measurements are spacelike separated.*

Indeed, if (MC) were not satisfied (*a fortiori* (C)), the theory would face two types of inconsistencies:

- *Non covariance* — Consider two entangled quantum systems that violate the condition (MC). Then there exists an experimental protocol concerning the second system that yields different statistical results depending on whether a certain measurement has been performed on the first system or not, such that the two measurements are spacelike separated. Thus one can find a reference frame in which the measurement on the first system happens *before* the other measurement, and another reference frame in which it happens *after*. Consequently, the statistical predictions of the theory depend on the reference frame.
- *Faster-than-light communication* — Consider a reference frame in which Alice and Bob are apart, motionless, and share N entangled pairs that violate the condition (MC) as well as two synchronized clocks. At $t = 0$, Alice performs the suitable measurement on each of the N subsystems if she wants to communicate the bit 0, or do nothing if she wants to communicate the bit 1. At $t = 0^+$, Bob performs the corresponding measurement on each of the subsystems in his possession: the statistical distribution he obtains allows him to distinguish if Alice has sent the bit 0 or 1, with an error margin arbitrarily small when N goes larger.

The condition (MC) can be given a precise mathematical formulation. Let $\mathcal{S}_1 + \mathcal{S}_2$ be two entangled systems described by a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, prepared in a state ρ . Let also \hat{A} (resp. \hat{B}) be a hermitian operator of \mathcal{H}_1 (resp. \mathcal{H}_2) of spectral decomposition $\hat{A} = \sum_{x \in \text{spec} \hat{A}} x \Pi_x^{(1)}$ (resp. $\hat{B} = \sum_{y \in \text{spec} \hat{B}} y \Pi_y^{(2)}$) where the $\Pi_x^{(1)}$ (resp. $\Pi_y^{(2)}$) are the spectral projectors of the observable. If an ideal projective measurement of A is performed on \mathcal{S}_1 , the whole state evolves to $\sum_{x \in \text{spec} \hat{A}} (\Pi_x^{(1)} \otimes \mathbb{1}_2) \rho (\Pi_x^{(1)} \otimes \mathbb{1}_2)$ (non-selective operation (2)). After this, in a given reference frame, the system evolves according to a unitary operator U (see Fig. 1). Since the state of \mathcal{S}_2 , obtained by tracing over \mathcal{S}_1 , fully characterizes the probabilities of any measurement on \mathcal{S}_2 , the invariance of the statistics of \mathcal{S}_2 under the measurement of \mathcal{S}_1 is expressed by:

Mathematical formulation of (MC)

$$\text{(MC)} \quad \forall \rho, \forall U, \forall \hat{A}, \quad \text{tr}_1 \left(\sum_{x \in \text{spec} \hat{A}} U (\Pi_x^{(1)} \otimes \mathbb{1}_2) \rho (\Pi_x^{(1)} \otimes \mathbb{1}_2) U^\dagger \right) = \text{tr}_1 (U \rho U^\dagger).$$

Or, equivalently, due to the universal property of the partial trace:

$$\forall \rho, \forall U, \forall \hat{A}, \forall \hat{B}, \forall y_0 \in \text{spec} \hat{B},$$

$$\sum_{x \in \text{spec} \hat{A}} \text{tr} \left(U (\Pi_x^{(1)} \otimes \mathbb{1}_2) \rho (\Pi_x^{(1)} \otimes \mathbb{1}_2) U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right) = \text{tr} (U \rho U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})). \quad (3)$$

Concerning (C), the most general physical evolution that \mathcal{S}_1 may undergo is a unitary V in $\mathcal{H}_1 \otimes \mathcal{H}_\mathcal{E}$ (possibly non-trivial only on \mathcal{H}_1), where \mathcal{E} stands for any external third system. It is not restrictive to consider them initially non-entangled, hence in an initial state $\rho \otimes \rho_\mathcal{E} \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{S}(\mathcal{H}_\mathcal{E})$, up

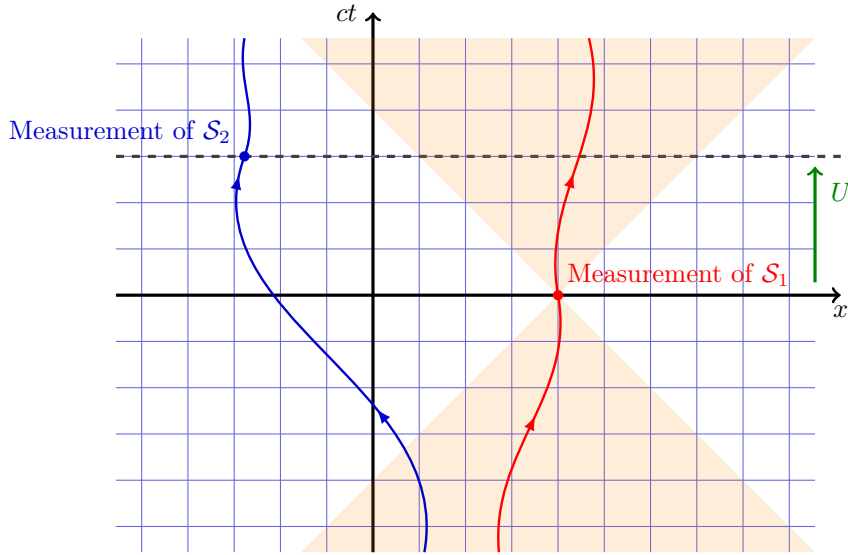


Figure 1: Spacelike measurements

to integrating some initial entanglement in the unitary V . Afterwards, $\mathcal{S}_1 + \mathcal{S}_2$ evolves, as previously, according to a unitary U in $\mathcal{H}_1 \otimes \mathcal{H}_2$. For the theory to be consistent, whether V has been applied or not must not modify the statistics of \mathcal{S}_2 , hence the following criterion:

Mathematical formulation of (C)

$$(C) \quad \forall \rho, \forall \rho_{\mathcal{E}}, \forall U, \forall V, \quad \text{tr}_1 \left(U \text{tr}_{\mathcal{E}} \left[(V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] U^\dagger \right) = \text{tr}_1 (U \rho U^\dagger).$$

Or, equivalently, due to the universal property of the partial trace:

$$\forall \rho, \forall \rho_{\mathcal{E}}, \forall U, \forall V, \forall \hat{B}, \forall y_0 \in \text{spec} \hat{B},$$

$$\text{tr} \left(U \text{tr}_{\mathcal{E}} \left[(V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right) = \text{tr} \left(U \rho U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right).$$

Clearly, when \mathcal{E} is a measurement apparatus that causes perfect and immediate decoherence of \mathcal{S}_1 in the eigenbasis of \hat{A} , then $\text{tr}_{\mathcal{E}} \left[(V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] = \sum_{x \in \text{spec} \hat{A}} (\Pi_x^{(1)} \otimes \mathbb{1}_2) \rho (\Pi_x^{(1)} \otimes \mathbb{1}_2)$ (recall that perfect decoherence in a given basis amounts, by definition, to extracting the diagonal part of the density matrix in this basis) and we recover the previous condition (MC). Therefore **(C) implies (MC)**.

Remark 2.1. on the unavoidable probabilistic nature of QM. In the literal formulations of (C) and (MC) given above, we have highlighted the fact that these conditions only constrain the statistical results of measurements. This is because special relativity only imposes the covariance of what is predictable by a given theory; and QM, being a probabilistic theory, only predicts the statistics. Similarly, the ability to transmit information depends on the best theory available to the communicators. If they can't do better than probabilistic predictions, Alice has to be able to modify the statistical results of a repeated experiment on Bob's side in order to send him a bit.

Alternatively, if they had a deterministic (hidden-variable) theory superseding QM, only one run could suffice. Such a theory should then be constrained by a stronger version of (C) where the word 'statistical' is removed. But this is precisely forbidden by Bell's theorem [4] which implies that such a theory would necessarily display non-local features. More precisely, either the outcome or parameter independence assumption would have to be violated, allowing in any case for superluminal signalling and covariance issues; see [19] for an excellent introduction to everything about Bell's theorem. See also [13] for a simple but illuminating analytic example of non-locality in Bohmian mechanics compared to QM,

where it is also shown that no probabilistic knowledge other than the one given by the wave function is compatible with special relativity. Quantum entanglement seemed spooky to Einstein because the collapse was perceived as physical and the wave function as observer-independent, but it is actually the contrary: QM is the only possible non-spooky theory! This remark proves that we will never be able to build a deterministic theory supplanting QM (at least not without a radical change in our physicists' paradigms). This is also why the terms added to the Schrödinger equation in collapse models will always be stochastic, 'because otherwise [they] would allow for faster-than-light communication' [2].

2.2 (F) implies (C)

Let's now check that the condition (C) (*a fortiori* (MC)) is indeed satisfied in any relativistic quantum theory provided it satisfies the following factorization property (F), generally assumed in QM:

(F) For all pairs of isolated systems³ \mathcal{S}_1 and \mathcal{S}_2 , the unitary evolution operator of $\mathcal{S}_1 + \mathcal{S}_2$ takes the factorized form $U = U_1 \otimes U_2$.

Let's keep the notations introduced in §2.1. It is clear that the systems are isolated between the two measurements, since they are spacelike separated; therefore, assuming (F), we may write $U = U_1 \otimes U_2$ and:

$$\begin{aligned} \mathbb{P}(B = y_0) &= \text{tr} \left(U \text{tr}_{\mathcal{E}} \left[(V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right) \\ &= \text{tr} \left(\text{tr}_{\mathcal{E}} \left[(V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2) \right] (\mathbb{1}_1 \otimes U_2^\dagger \Pi_{y_0}^{(2)} U_2) \right) \\ &= \text{tr} \left((V \otimes \mathbb{1}_2)(\rho_{\mathcal{E}} \otimes \rho)(V^\dagger \otimes \mathbb{1}_2)(\mathbb{1}_{\mathcal{E}} \otimes \mathbb{1}_1 \otimes U_2^\dagger \Pi_{y_0}^{(2)} U_2) \right) \\ &= \text{tr} \left(\rho_{\mathcal{E}} \otimes (\rho U_2^\dagger \Pi_{y_0}^{(2)} U_2) \right) \\ &= \text{tr} \left(U \rho U^\dagger (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}) \right) \end{aligned}$$

where we have used the fact that $\text{tr}(\rho_{\mathcal{E}}) = 1$ in the last step, and the property of the partial trace $\text{tr}(\text{tr}_1(A_{12})B_2) = \text{tr}(A_{12}(\mathbb{1}_1 \otimes B_2))$ in the third step. [To prove this equality, it is enough by linearity to check it for pure tensors $A_{12} = A_1 \otimes A_2$; it then simply reads: $\text{tr}(\text{tr}_1(A_1 \otimes A_2)B_2) = \text{tr}(\text{tr}(A_1)A_2 B_2) = \text{tr}(A_1) \text{tr}(A_2 B_2) = \text{tr}(A_1 \otimes A_2 B_2) = \text{tr}((A_1 \otimes A_2)(\mathbb{1}_1 \otimes B_2))$.] Thus **(F) implies (C)**.

We have therefore established that any relativistic quantum theory which satisfies (F) is consistent with an instantaneous effect of physical interactions on an entangled state, even though the notion of instantaneity is frame dependent. Different observers may write different states for the entangled pair, but they agree on the statistics. Accepting that there is one wave function per observer (for a probability distribution always depends on what is known, recall especially note 1 in §1.2 and Bell's example in §2.1) solves a lot of spooky problems, because then the wave function doesn't have to be covariant, only the physics have to. Also, what precedes constitutes the proof that an entangled pair can not convey information in QM, known as the no-communication theorem.

Remark 2.2. In §2.1, we have motivated the importance of (C) by the fact that two kinds of inconsistencies appear if it were not satisfied. The above proof suffices to exclude the second kind (faster-than-light communication), but lacks an additional argument to get rid of the first kind (non covariance). Indeed, it has assumed the choice of a fixed reference frame, in which the time evolutions of the systems between the instants t_{1i} , t_{1f} and t_2 (corresponding to the beginning and the end of the evolution undergone by \mathcal{S}_1 and the measurement performed on \mathcal{S}_2 respectively) are given by unitary operators $V_{t_{1i}, t_{1f}}$ and U_{t_{1f}, t_2} that we have simply denoted V and U . In another reference frame, however, neither the temporal axis nor the unitary operators are conserved. Even the possible outcomes of the measurements may undergo

³We say that two systems are isolated if no particles of \mathcal{S}_1 and \mathcal{S}_2 can meet each other. The possibility to write U as $U_1 \otimes U_2$ is sometimes taken as a definition for being isolated, or we could have replaced 'isolated' by 'spacelike separated' in all this section §2, but our choice will become clear when discussing locality in §3. Of course, the main difficulty with our definition is that the concept of particle is ill-defined. In particular, Unruh [27] have showed that the very presence of a particle is frame-dependent. Also, should we consider virtual particles? How to include gravitational interactions in our treatment if the graviton actually doesn't exist?

a Lorenz transformation, if they correspond to the position observable for example. The invariance of the statistics between two reference frames \mathcal{R} and \mathcal{R}' may be written more precisely as follows (one assumes for simplicity $t_{1i} = t_{1f} = t_1 > t_2$, with $t_2 < t_1$ in \mathcal{R} and denote t_0 an earlier time at which the two entangled systems were separated):

$$\left\{ \begin{array}{ll} \bullet \operatorname{tr} (U_{t_0, t_2}^{(\mathcal{R})} \rho U_{t_0, t_2}^{(\mathcal{R})\dagger} (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})) = \operatorname{tr} (U_{t'_0, t'_2}^{(\mathcal{R}')} \rho U_{t'_0, t'_2}^{(\mathcal{R}')\dagger} (\mathbb{1}_1 \otimes \Pi_{y'_0}^{(2)})) & \text{if } t'_2 < t'_1. \\ \bullet \operatorname{tr} (U_{t_0, t_2}^{(\mathcal{R})} \rho U_{t_0, t_2}^{(\mathcal{R})\dagger} (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})) \\ = \operatorname{tr} (U_{t'_1, t'_2}^{(\mathcal{R}')} \operatorname{tr}_{\mathcal{E}} [(V_{t'_1}^{(\mathcal{R}')} \otimes \mathbb{1}_2) (\rho_{\mathcal{E}} \otimes U_{t'_0, t'_1}^{(\mathcal{R}')} \rho U_{t'_0, t'_1}^{(\mathcal{R}')\dagger}) (V_{t'_1}^{(\mathcal{R}')} \otimes \mathbb{1}_2)] U_{t'_1, t'_2}^{(\mathcal{R}')\dagger} (\mathbb{1}_1 \otimes \Pi_{y'_0}^{(2)})) & \text{if } t'_2 > t'_1 \end{array} \right.$$

The first line is simply the covariance of the theory in the absence of measurements: we suppose it already established. The only case to examine involves the changes of reference frames that reverse the temporal order of t_1 and t_2 . In particular, since the only potential discontinuity occur when the sign of $t_2 - t_1$ flips, it suffices to write the second line above in the limit $\varepsilon, \varepsilon' \rightarrow 0$ of an infinitesimal change of frame with $t_2 = t_1 - \varepsilon$ and $t'_2 = t'_1 + \varepsilon'$. In that case, $U_{t'_a, t'_b}^{(\mathcal{R}')} \rightarrow U_{t_a, t_b}^{(\mathcal{R})}$ for all instants t_a and t_b , and $y'_0 \rightarrow y_0$ so that the condition to verify reads:

$$\operatorname{tr} (U_{t_0, t_2}^{(\mathcal{R})} \rho U_{t_0, t_2}^{(\mathcal{R})\dagger} (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)})) = \operatorname{tr} (U_{t_1, t_2}^{(\mathcal{R})} \operatorname{tr}_{\mathcal{E}} [(V_{t_1}^{(\mathcal{R})} \otimes \mathbb{1}_2) (\rho_{\mathcal{E}} \otimes U_{t_0, t_1}^{(\mathcal{R})} \rho U_{t_0, t_1}^{(\mathcal{R})\dagger}) (V_{t_1}^{(\mathcal{R})} \otimes \mathbb{1}_2)] U_{t_1, t_2}^{(\mathcal{R})\dagger} (\mathbb{1}_1 \otimes \Pi_{y_0}^{(2)}))$$

but since $U_{t_0, t_2}^{(\mathcal{R})} = U_{t_1, t_2}^{(\mathcal{R})} U_{t_0, t_1}^{(\mathcal{R})}$, this is nothing but the condition (C) that has been proved applied to $U_{t_0, t_1}^{(\mathcal{R})} \rho U_{t_0, t_1}^{(\mathcal{R})\dagger}$ instead of ρ .

2.3 Comparison with previous works

The above verification of (C) is simple but essential to establish the compatibility between QM and special relativity. Of course, some previous works have already investigated these topics, although remarkably few. Bloch [6] pointed out some apparent inconsistencies and new intuitions, notably: ‘it appears that either causality or Lorentz covariance of wave functions must be sacrificed’. In reply, Hellwig and Kraus [14] published a paper in which they clarified Bloch’s ideas, checked a simple version (MC) (equation (6)) based on an assumption called locality (commutation of projectors associated to spacelike measurements), and then proposed that the effect of measurements (selective and non-selective) should be implemented along the past light cone so that their description becomes covariant. In a long footnote, Malament [17] actually proved the logical equivalence between a simple version of (MC) and the fact that spacelike measurements projectors commute. A general proof of (MC), presented as a no-communication theorem, was detailed by Ghirardi, Rimini and Weber in [12]. In a very inspiring work, Aharonov and Albert [1] explained how non-local measurements can actually be implemented based on local interactions only. As a consequence, they claim that ‘the proposal that the reduction be taken to occur covariantly along the backward light cone (...) or along *any* hypersurface other than $t = 0$ will fail’ (although we must confess that we didn’t understand their argument), and that ‘the covariance of relativistic quantum theories (...) resides *exclusively* in the experimental probabilities, and not in the underlying quantum states. The states themselves make sense only within a given frame’. Surprisingly enough, all these works were concerned with (MC) (even the Wikipedia entry ‘No-communication theorem’ only deals with (MC)), which is after all a very artificial statement compared to (C).

Much more recently, in [22], the authors exactly proved the condition (C) (equation (29)) in the context of QFT based on the microcausality hypothesis, used to derive a relation not very different from (F) (equation (28)). They also argued that selective measurements can not be compatible with special relativity unless one accepts to define as many wave functions as there are observers and that the update occurs along the future light cone. Contrary to the latter and in accordance to [1], we consider that, as soon as the wave function is understood to be observer dependent (hence, in particular, frame dependent), it is pointless and even awkward to ask for frame independent updates. Why would a given observer in her reference frame wait some time before using all the information she has (in the case of a selective measurement), or before taking into account the evolution she knows has happened on \mathcal{S}_1 to write down the whole state of $\mathcal{S}_1 + \mathcal{S}_2$? Finally, as a concluding remark for this section, we note that none of these approaches have found out how fruitful the precise formulation of (C) and (MC) may be, as we shall now see in the following.

3 The locality hypothesis

3.1 (L') implies (L) and (F)

In §2.2, we have seen that (F) ensures a theory to be consistent with special relativity. The property (F) is directly linked to a fundamental principle of physics, namely the locality hypothesis (L) which states the absence of interactions at a distance:

(L) *Two localized particles can interact⁴ at a given time only if they are located at the same point in space.*

In QFT, this hypothesis has a more specific formulation:

(L') *The interaction Hamiltonian of any system can be written in the form $H_{int}(t) = \int_{\mathbb{R}^3} H_{int}(\vec{x}, t) d^3x$ where H_{int} is a field of operators defined on the whole spacetime.*

Why is (L') called a locality hypothesis? Suppose one wants to compute the coefficient $S_{\alpha\beta}$ of the S -matrix, where $|\alpha\rangle$ is the state composed of two particles localized in \vec{x}_1 and \vec{x}_2 , *i.e.*

$$|\alpha\rangle = \int \frac{d^3\vec{p}_1}{(2\pi)^{3/2}} e^{-i\vec{p}_1 \cdot \vec{x}_1} a^\dagger(\vec{p}_1) \int \frac{d^3\vec{p}_2}{(2\pi)^{3/2}} e^{-i\vec{p}_2 \cdot \vec{x}_2} b^\dagger(\vec{p}_2) |0\rangle.$$

Under the hypothesis (L'), the Born approximation reads: $S_{\alpha\beta} = -i \int d^4x \langle \alpha | H_{int}(x) | \beta \rangle$. Furthermore, even without specifying $H_{int}(x)$, one knows by covariance that the latter can only be built from the different quantum fields of the theory, and that if an interaction is possible it necessarily contains two quantum fields of the type of the particles of the state $|\alpha\rangle$, and so includes at least an expression of the form $\int \frac{d^3\vec{p}}{(2\pi)^{3/2}} e^{-ipx} u(\vec{p}) a^\dagger(\vec{p}) \int \frac{d^3\vec{p}'}{(2\pi)^{3/2}} e^{-ip'x} v(\vec{p}') b^\dagger(\vec{p}')$ where u and v are objects that depend on the nature of the quantum fields. Using the (anti-)commutation relation for the operators a and b , one finds:

$$\langle \alpha | H_{int}(x) | \beta \rangle = \iint d^3\vec{p}_1 d^3\vec{p}_2 [\dots] e^{i\vec{p}_1 \cdot \vec{x}_1} e^{i\vec{p}_2 \cdot \vec{x}_2} e^{-i(p_1+p_2)x} \propto \delta^{(3)}(\vec{x} - \vec{x}_1) \delta^{(3)}(\vec{x} - \vec{x}_2) \propto \delta^{(3)}(\vec{x}_1 - \vec{x}_2).$$

This means that two localized particles can interact at a given time only if they are located at the same point in space, namely (L). The electromagnetic force between two charged particles, for example, is an interaction 'at a distance' only because (virtual) photons are exchanged between them. Furthermore, under the hypothesis (L), two isolated⁵ systems \mathcal{S}_1 and \mathcal{S}_2 can not interact, therefore the total Hamiltonian reads $H = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$ with H_1 and H_2 the internal Hamiltonians of \mathcal{S}_1 and \mathcal{S}_2 . Therefore, the unitary evolution operator of $\mathcal{S}_1 + \mathcal{S}_2$ is $U = e^{itH} = e^{itH_1} \otimes e^{itH_2} = U_1 \otimes U_2$. We have just shown that **(L') implies (L) which implies (F)**.

3.2 (F) implies (L) and (almost) (L')

Conversely, suppose that (F) is true. Then, if \mathcal{S}_1 and \mathcal{S}_2 are two isolated systems, the total Hamiltonian of $\mathcal{S}_1 + \mathcal{S}_2$ reads:

$$H = \frac{1}{i} \frac{dU}{dt} = \frac{1}{i} \frac{dU_1 \otimes U_2}{dt} \Big|_{t=0} = \frac{1}{i} \frac{dU_1}{dt} \Big|_{t=0} \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes \frac{1}{i} \frac{dU_2}{dt} \Big|_{t=0} = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2 \Rightarrow H_{int} = 0,$$

hence **(F) implies (L)**. Now, it is natural to imagine (this is not a proof!) that in a theory satisfying (L), the interaction Hamiltonian of an arbitrary system should be composed of a combination of local operators of the form $\int_{\mathcal{D}} H_{int}(\vec{x}, t)$ where \mathcal{D} is an integration domain *a priori* unknown. However, to preserve the translation invariance of the laws of physics, \mathcal{D} must be the whole space \mathbb{R}^3 . Thus **(L) justifies the locality hypothesis in QFT (L')**.

⁴Again, as in note 3, this term can be confusing. One can think of possible interactions as allowed vertices in Feynman diagrams, but what about the gravitational interaction? If the latter is to be background independent [18], then there is no pre-existing spacetime in which to define a notion of locality...

⁵Recall that we defined 'isolated' as the fact that no particles of \mathcal{S}_1 and \mathcal{S}_2 meet each other.

3.3 (MC) implies (F)

We need our theories to satisfy (C). In Section 2.2, we have shown that (F) \Rightarrow (C) and remarked that (F) is generally postulated in QM. But, unlike (F), (C) is a direct consistency condition, so that it seems after all more natural to postulate the latter. One can then try to determine the set of unitaries compatible with (C), yielding a constraint on any unitary evolution operator associated to a system of the form $\mathcal{S}_1 + \mathcal{S}_2$ with \mathcal{S}_1 and \mathcal{S}_2 spacelike separated. It happens that the more restrictive condition (MC) suffices to deduce (F), as shown in the following theorem.

Theorem 3.1 ((MC) \Rightarrow (F)). *Let $U(t)$ be the unitary evolution operator, expressed in a fixed reference frame \mathcal{R} , of a quantum system $\mathcal{S}_1 + \mathcal{S}_2$ composed of two isolated subsystems \mathcal{S}_1 and \mathcal{S}_2 , described in a relativistic quantum theory. Then there exist two unitary operators $U_1(t)$ and $U_2(t)$ such that:*

$$U(t) = U_1(t) \otimes U_2(t)$$

Proof. The two subsystems being isolated, it is possible to divide the temporal axis of \mathcal{R} into small time intervals such that \mathcal{S}_1 and \mathcal{S}_2 are spacelike separated⁶ during any of these time intervals. On each interval, the theory satisfies (in particular) the condition (MC) by consistency, and it suffices to show there the factorization result. From now, for the sake of clarity, we will not write the parameter t which plays no role in the proof anymore. Let's denote \mathcal{H}_1 and \mathcal{H}_2 the Hilbert spaces associated with \mathcal{S}_1 and \mathcal{S}_2 , and assume first that they are finite dimensional with $n_1 = \dim(\mathcal{H}_1)$ and $n_2 = \dim(\mathcal{H}_2)$. One can write U in the following generic form:

$$U = \sum_{\substack{i \\ 1 \leq k, l \leq n_2}} \alpha_{ikl} T_i \otimes |k\rangle \langle l|$$

with $(|k\rangle \langle l|)_{1 \leq k, l \leq n_2}$ the canonical basis of $\mathcal{L}(\mathcal{H}_2)$ associated with an orthonormal basis $(|k\rangle)_{1 \leq k \leq n_2}$, and $(T_i)_i$ a collection of operators of $\mathcal{L}(\mathcal{H}_1)$. Up to a reorganization of the sum, one can assume that the $(T_i)_i$ are linearly independent in $\mathcal{L}(\mathcal{H}_1)$.

When replacing U in the expression (3) for the condition (MC), one gets for all hermitian operators \hat{A} and \hat{B} of \mathcal{H}_1 and \mathcal{H}_2 and for all $y_0 \in \text{spec}\hat{B}$:

$$\begin{aligned} & \forall \rho, \quad \text{tr} \left(\rho \sum_{\substack{i, k, l \\ j, k', l'}} \left[\sum_{x \in \text{spec}\hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes \alpha_{ikl} \overline{\alpha_{jk'l'}} |l'\rangle \langle k'| \Pi_{y_0}^{(2)} |k\rangle \langle l| \right) = 0 \\ \Rightarrow & \sum_{\substack{i, k, l \\ j, k', l'}} \left[\sum_{x \in \text{spec}\hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes \alpha_{ikl} \overline{\alpha_{jk'l'}} |l'\rangle \langle k'| \Pi_{y_0}^{(2)} |k\rangle \langle l| = 0. \end{aligned}$$

Note that \hat{B} may be chosen arbitrarily, as well as $\Pi_{y_0}^{(2)}$. In particular, for any pair $\{k_1, k_2\} \subset \llbracket 1, n_2 \rrbracket$ and $\mu, \nu \in \mathbb{C}$ such that $|\mu|^2 + |\nu|^2 = 1$, one can define $\Pi_{y_0}^{(2)}$ to be the projector on the vector $\mu |k_1\rangle + \nu |k_2\rangle$, that is $|\mu|^2 |k_1\rangle \langle k_1| + \mu \bar{\nu} |k_1\rangle \langle k_2| + \bar{\mu} \nu |k_2\rangle \langle k_1| + |\nu|^2 |k_2\rangle \langle k_2|$. Inserting into the previous equation divides it in four sums:

$$\begin{aligned} & \forall k_1, k_2, \quad \sum_{\substack{i, l \\ j, l'}} \left[\sum_{x \in \text{spec}\hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \\ & \quad \otimes \left[|\mu|^2 \alpha_{ik_1 l} \overline{\alpha_{jk_1 l'}} + \mu \bar{\nu} \alpha_{ik_2 l} \overline{\alpha_{jk_1 l'}} + \bar{\mu} \nu \alpha_{ik_1 l} \overline{\alpha_{jk_2 l'}} + |\nu|^2 \alpha_{ik_2 l} \overline{\alpha_{jk_2 l'}} \right] |l'\rangle \langle l| = 0. \end{aligned}$$

The particular cases $\mu = 1, \nu = 0$ or $\mu = 0, \nu = 1$ imply that the first and fourth terms actually always

⁶Meaning that every point of the spacetime region spanned by \mathcal{S}_1 is spacelike separated to every point of the spacetime region spanned by \mathcal{S}_2 .

vanish. Setting $\mu = \nu = \frac{1}{\sqrt{2}}$ or $\mu = \frac{1}{\sqrt{2}}, \nu = \frac{i}{\sqrt{2}}$ leads to:

$$\begin{cases} \sum_{\substack{i,l \\ j,l'}} \left[\sum_{x \in \text{spec } \hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes \left[\frac{1}{2} \alpha_{ik_2} \overline{\alpha_{jk_1 l'}} + \frac{1}{2} \alpha_{ik_1} \overline{\alpha_{jk_2 l'}} \right] |l'\rangle \langle l| = 0 \\ \sum_{\substack{i,l \\ j,l'}} \left[\sum_{x \in \text{spec } \hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes \left[-\frac{i}{2} \alpha_{ik_2} \overline{\alpha_{jk_1 l'}} + \frac{i}{2} \alpha_{ik_1} \overline{\alpha_{jk_2 l'}} \right] |l'\rangle \langle l| = 0 \end{cases}$$

and taking appropriate linear combinations of these shows that the second and third terms vanish as well. Therefore:

$$\begin{aligned} \forall k, k', \sum_{\substack{i,l \\ j,l'}} \alpha_{ikl} \overline{\alpha_{jk'l'}} \left[\sum_{x \in \text{spec } \hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] \otimes |l'\rangle \langle l| &= 0 \\ \Rightarrow \forall k, k', l, l', \sum_{i,j} \alpha_{ikl} \overline{\alpha_{jk'l'}} \left[\sum_{x \in \text{spec } \hat{A}} \Pi_x^{(1)} T_j^\dagger T_i \Pi_x^{(1)} - T_j^\dagger T_i \right] &= 0 \end{aligned}$$

because $(|l'\rangle \langle l|)_{1 \leq l, l' \leq n_2}$ is a basis of $\mathcal{L}(\mathcal{H}_2)$. This being true for all hermitian operators \hat{A} , the operator $\sum_{i,j} \alpha_{ikl} \overline{\alpha_{jk'l'}} T_j^\dagger T_i = (\sum_i \alpha_{ik'l'} T_i)^\dagger (\sum_i \alpha_{ikl} T_i)$ is diagonal in every orthonormal basis of \mathcal{H}_1 , so it is a dilation:

$$\forall k, k', l, l', \exists \lambda_{kk'l'l'} \in \mathbb{C} : \left(\sum_i \alpha_{ik'l'} T_i \right)^\dagger \left(\sum_i \alpha_{ikl} T_i \right) = \lambda_{kk'l'l'} \mathbb{1}.$$

Obviously, $\lambda_{kk'l'l'} \neq 0$, otherwise $\sum_i \alpha_{ikl} T_i = 0$ (for $\mathcal{L}(\mathcal{H}_2)$ is a C^* -algebra) which would contradict the linear independence of the $(T_i)_i$. Moreover, by unicity of the inverse, we have for all k, l , $\frac{1}{\lambda_{k1l1}} \sum_i \alpha_{ikl} T_i = \frac{1}{\lambda_{1111}} \sum_i \alpha_{i11} T_i$ and since the $(T_i)_i$ are linearly independent:

$$\forall k, l, \exists \beta_{kl} \in \mathbb{C} : \forall i, \alpha_{ikl} = \beta_{kl} \alpha_{i11}.$$

It is now possible to factorize:

$$U = \sum_{1 \leq k, l \leq n_2} \beta_{kl} \alpha_{i11} T_i \otimes |k\rangle \langle l| = \left(\sum_i \alpha_{i11} T_i \right) \otimes \left(\sum_{1 \leq k, l \leq n_2} \beta_{kl} |k\rangle \langle l| \right) = U_1 \otimes U_2$$

where one can identify U_1 and U_2 to the evolution operators of \mathcal{S}_1 and \mathcal{S}_2 , which are necessarily unitary. Finally, this proof is purely algebraic and can be transposed without difficulty in the infinite dimensional case by replacing the sums by integrals. \square

A consequence of this theorem is that (F), (C) and (MC) imply themselves circularly, therefore **(F), (C) and (MC) are logically equivalent.**

3.4 Nonlocal measurements

We are now in position to address a thorny problem. Let's reproduce briefly how the aforementioned [1] proceed to implement what they call 'nonlocal measurements'. Consider a system \mathcal{S} composed of two spin $\frac{1}{2}$ particles \mathcal{S}_1 and \mathcal{S}_2 to be measured, and an apparatus \mathcal{A} composed of two measuring particles \mathcal{A}_1 and \mathcal{A}_2 . Let the interaction be given by $\hat{H}_{int} = g_1(t) \hat{X}_1 \hat{\sigma}_1^z + g_2(t) \hat{X}_2 \hat{\sigma}_2^z$, where the g_i are two couplings which are non zero only during a short interval of time centered around t_0 , \hat{X}_i is \mathcal{A}_i 's position operator and $\hat{\sigma}_i^z$ is \mathcal{S}_i 's spin operator along z . This Hamiltonian is simply the composition of two separate local measurements processes for \mathcal{S}_i 's spin, recorded in \mathcal{A}_i 's momentum P_i , since we have in the Heisenberg picture $\Delta \hat{P}_i = \hat{P}_i(t > t_0) - \hat{P}_i(t < t_0) \propto \hat{\sigma}_i^z$. If \mathcal{A} is initially prepared in an entangled state such that $(\hat{P}_1 + \hat{P}_2) |\mathcal{A}\rangle = (\hat{X}_1 - \hat{X}_2) |\mathcal{A}\rangle = 0$ (check that this is indeed possible, in particular because $[\hat{P}_1 + \hat{P}_2, \hat{X}_1 - \hat{X}_2] = 0$), then the nonlocal quantity J^z corresponding to the sum of the two spins is now measured by \mathcal{A} since $\hat{J}^z = \hat{\sigma}_1^z + \hat{\sigma}_2^z \propto \hat{P}_1(t > t_0) + \hat{P}_2(t > t_0)$. Moreover, it is possible to verify whether \mathcal{S} is in the state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = |J^z = 0, J^z = 0\rangle$ (a nonlocal statement), by measuring J^x ,

J^y and J^z simultaneously (using 3 pairs of measuring particles) since this state is uniquely defined by the requirements $J^x = J^y = J^z = 0$.

Clearly, no relativistic inconsistencies can occur with such ‘nonlocal measurements’, because they arise from local Hamiltonians generating a unitary evolution of the form $U_1 \otimes U_2$ (where U_i acts on $\mathcal{H}_{\mathcal{S}_i} \otimes \mathcal{H}_{\mathcal{A}_i}$), therefore (C) is satisfied and the statistics of $\mathcal{S}_1 + \mathcal{A}_1$ (*a fortiori* those of \mathcal{S}_1) are unaltered by whatever may happen on $\mathcal{S}_2 + \mathcal{A}_2$. Quoting Aharonov and Albert: ‘It is a requirement of relativistic causality, then, that although we may measure nonlocal properties of various systems, we must always carry out such measurements by means of local observations on the measuring apparatus; all measurements must ultimately be local ones’. On the other hand, they present a simple experimental setup in which the ability to perform a nonlocal measurement of J^2 would allow for faster-than-light communication. Similarly, Sorkin [24] points out that being able to measure the observable Π of orthogonal projection onto $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$ would also bypass the speed of light. Certainly, these two measurements can not be realized by means of local Hamiltonians (otherwise they would satisfy (C)). But is there a simple criterion to determine whether a given nonlocal observable can be measured (like J^x) or not (like J^2 or Π)? As Sorkin deplors: ‘the need to resort to a case-by-case analysis would still leave us without any clear formal criterion for which “observables” can be ideally-measured, and which cannot’.

The answer depends on the definition of the (non-selective) measurement used. Let’s use the terminology introduced in [25]:

- We say that the observable A of a system \mathcal{S} is **recorded** by an environment \mathcal{E} , when \mathcal{S} and \mathcal{E} are entangled with perfect decoherence in the eigenbasis \mathcal{B} of \hat{A} . Mathematically, this means that $\mathcal{S} + \mathcal{E}$ ’s state takes the form $\sum_i c_i |A_i\rangle |\mathcal{E}_i\rangle$, where $\mathcal{B} = (|A_i\rangle)_i$ are the eigenstates of A and $\langle \mathcal{E}_i | \mathcal{E}_j \rangle = 0$ for $i \neq j$. Note that any entanglement constitutes a recording in the basis where the density matrix of \mathcal{S} is diagonal after interaction (this basis may depend on the initial state of the system and/or of the environment).
- A **non-selective measurement process** for A is a unitary evolution that entangles \mathcal{S} with \mathcal{E} such that, *for any initial state* of \mathcal{S} , A is recorded by \mathcal{E} after interaction. If moreover \mathcal{B} is a conserved basis during the process, we call it a **projective** (or non-demolition) measurement.

Going back to the above setups, J^2 and Π had to be *projectively* measured to lead to relativistic inconsistencies. If one rather uses our broader definition, then J^2 and Π *can* be measured. Indeed, a few calculations show that a nonlocal measurement of J^x as above followed by one of J^z is an interaction that records \mathcal{S} in the basis $(|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle))$ regardless of its initial state. These are nothing but the eigenbasis of J^2 (a simple flip of spin also yields an eigenbasis of Π), although the interaction admits no conserved basis. So the question becomes: which observables can be (i) recorded, (ii) measured, (iii) projectively measured, by means of local Hamiltonians? The question (i) could be reformulated this way: what are the possible eigenbases of $\text{tr}_{\mathcal{A}}(U_1 \otimes U_2(\rho_{\mathcal{S}} \otimes \rho_{\mathcal{A}})U_1^\dagger \otimes U_2^\dagger)$ for arbitrary unitaries U_1 and U_2 ? For (ii), one demands moreover that the eigenbasis be independent of $\rho_{\mathcal{S}}$, and for (iii) one adds the hypothesis that there exists a conserved basis in the measurement process. We keep the resolution of these questions for a future work, although we conjecture that the answers are (i) & (ii) any observable (iii) any observable diagonal in a product basis $\mathcal{B} \otimes \mathcal{B}'$.

4 The microcausality hypothesis

In addition of the locality hypothesis, another deep postulate of QFT is the microcausality hypothesis (M).

(M) *For all quantum fields Φ and spacelike intervals $x - y$, $[\Phi(x), \Phi^\dagger(y)]_\pm = 0$ where $[\ ,]_\pm$ stands for an anti-commutator or a commutator depending on the fermionic or bosonic nature of Φ .*

Usually [21, p.28] [15, p.106] [30, p.121], standard QFT textbooks justify this hypothesis by invoking — for once! — the concept of measurement, but they generally make do with the affirmation that two spacelike measurements must be independent... without more explanations. Not only is the argument too vague, but it is hard to see how could the relation differ according to the fermionic or bosonic nature of the field. Weinberg, on the other hand, makes an interesting remark:

‘The condition [(M)] is often described as a causality condition, because if $x - y$ is spacelike then no signal can reach y from x , so that a measurement of Φ at point x should not be able to interfere with a measurement of Φ or Φ^\dagger at point y . Such considerations of causality are plausible for the electromagnetic field, any one of whose components may be measured at a given spacetime point, as shown in a classic paper of Bohr and Rosenfeld [7]. However, we will be dealing here with fields like the Dirac field of the electron that do not seem in any sense measurable. The point of view taken here is that [(M)] is needed for the Lorentz invariance of the S-matrix, without any ancillary assumptions about measurability or causality.’ [28, p.198]

Nevertheless, as Weinberg notes himself, microcausality is only a sufficient condition for the invariance of the S-matrix:

‘Theories of this class [satisfying (M)] are not the only ones that are Lorentz invariant, but the most general Lorentz invariant theories are not very different. In particular, there is always a commutation condition something like [(M)] that needs to be satisfied. This condition has no counterpart for non-relativistic systems, for which time-ordering is always Galilean-invariant. *It is this condition that makes the combination of Lorentz invariance and QM so restrictive.*’ [28, p.145]

However, this argument is valid only when one works with normal-ordered fields, but this writing is only a computation convenience and doesn’t have, as it seems, any physical meaning. The main effect of the normal ordering is to get rid of the infinite constants that appear in certain computations, by making finite all the matrix elements of the operators manipulated. This operation is justified when the divergences have no influence in the considered context. Alternatively, the prediction of the Casimir effect or the Lamb shift by the vacuum energy is only possible *without* normal-ordering (one substantially uses the fact that $\langle 0|\Phi^2|0\rangle \neq 0$, see [15, p.111]). Presumably, a stronger argument is needed to justify the microcausality hypothesis (M). Note that (M) is especially crucial to prove the famous spin-statistics theorem (SS):

(SS) *Scalar and vector fields correspond to bosons, while Dirac fields correspond to fermions.*

Here is how most QFT textbooks proceed to establish (SS). Since we don’t know the result yet, we have to compute both commutators and anti-commutators. A short calculation first shows that when Φ is a Dirac field, we have:

$$[\Phi(x), \Phi^\dagger(y)]_\pm = \Delta_+(x - y) \mp \Delta_+(y - x) \quad (4)$$

depending on Φ describing fermions or bosons, *i.e.* $[a(\vec{p}), a^\dagger(\vec{p}')]_\pm = \delta^{(3)}(\vec{p} - \vec{p}')$, and where $\Delta_+(x) = \int \frac{d^3p}{(2\pi)^3 2p_0} e^{ip_\mu x^\mu}$ is shown to be a Lorentz invariant quantity. Likewise, when Φ is a scalar or a vector field:

$$[\Phi(x), \Phi^\dagger(y)]_\pm = \Delta_+(x - y) \pm \Delta_+(y - x) \quad (5)$$

depending on Φ describing fermions or bosons. It is quite easy to see that when $x - y$ is spacelike, $\Delta_+(x - y) - \Delta_+(y - x) = 0$, while $\Delta_+(x - y) + \Delta_+(y - x)$ is not identically zero [28, p.202]. All these remarks, in addition to (M), imply (SS). Not the reverse: these computations generally don’t justify (M) but only the fermionic or bosonic nature of a field, whereas one can intuitively feel that they contain much more information.

Let’s present now a simultaneous proof of the microcausality hypothesis and the spin-statistics theorem that relies only on the consistency of the theory and on the above (anti-)commutators computations.

Theorem 4.1 ((MC) \Rightarrow (M) and (SS)). *Let Φ be a quantum field. Then, for all spacelike intervals $x - y$, $[\Phi(x), \Phi^\dagger(y)]_\pm = 0$ where $[\ ,]_\pm$ stands for an anti-commutator or a commutator depending on Φ being a Dirac field (and in that case it describes a fermion) or a scalar or vector field (and in that case it describes a boson).*

Proof. Let’s first suppose that Φ is effectively a measurable field, in other words $\Phi(x)$ is an observable for all x , *i.e.* a hermitian operator. As stated by Weinberg in the previous quote, it is for example the case for the electromagnetic field that describes the photon. One can write its spectral decomposition $\Phi(x) = \sum_{\lambda \in \text{spec } \Phi(x)} \lambda \Pi_\lambda^{(x)}$ (rigorously speaking, one should work with smeared fields, and the sum could be an integral in the case of a continuous spectrum). Contrary to the previous sections, there is now only one Hilbert space, the Fock space \mathcal{H}_{Fock} , and the system’s state is given by a density matrix

$\rho \in \mathcal{S}(\mathcal{H}_{Fock})$. For all spacelike intervals $x - y$, a measurement of $\Phi(x)$ doesn't affect the statistics of a measurement of $\Phi(y)$ if and only if the following condition, variant of (MC), is satisfied:

$$\forall \rho, \forall \mu \in \text{spec } \Phi(y), \quad \text{tr} \left(\left(\sum_{\lambda \in \text{spec } \Phi(x)} \Pi_{\lambda}^{(x)} \rho \Pi_{\lambda}^{(x)} \right) \Pi_{\mu}^{(y)} \right) = \text{tr}(\rho \Pi_{\mu}^{(y)})$$

where we implicitly moved to a reference frame \mathcal{R} in which $x^0 = y^0$, so as to avoid to introduce the (non covariant) unitary evolution operators⁷. It yields:

$$\begin{aligned} & \forall \rho, \forall \mu \in \text{spec } \Phi(y), \quad \text{tr} \left(\rho \left(\sum_{\lambda \in \text{spec } \Phi(x)} \Pi_{\lambda}^{(x)} \Pi_{\mu}^{(y)} \Pi_{\lambda}^{(x)} - \Pi_{\mu}^{(y)} \right) \right) = 0 \\ \Rightarrow & \forall \mu \in \text{spec } \Phi(y), \quad \sum_{\lambda \in \text{spec } \Phi(x)} \Pi_{\lambda}^{(x)} \Pi_{\mu}^{(y)} \Pi_{\lambda}^{(x)} = \Pi_{\mu}^{(y)} \\ \Rightarrow & \sum_{\lambda \in \text{spec } \Phi(x)} \Pi_{\lambda}^{(x)} \Phi(y) \Pi_{\lambda}^{(x)} = \Phi(y). \end{aligned}$$

Thus, $\Phi(y)$ is (block) diagonal in the eigenbasis of $\Phi(x)$, so they are codiagonalisable and $[\Phi(x), \Phi(y)] = 0$. This relation *a priori* holds in the frame \mathcal{R} , but when applying the appropriate representation of the Lorenz group under which Φ transforms, one sees that $\Phi(x)$ and $\Phi(y)$ commute in all reference frames.

If now Φ is not supposed hermitian anymore, we still know that $\Phi \Phi^\dagger$ is. Applying what precedes to $\Phi \Phi^\dagger$ instead of Φ , we obtain that for all $x - y$ spacelike, $[\Phi(x) \Phi^\dagger(x), \Phi(y) \Phi^\dagger(y)] = 0$. Moreover, as Φ is not hermitian, one shows as usual that for all x and y , $[\Phi(x), \Phi(y)] = [\Phi^\dagger(x), \Phi^\dagger(y)] = 0$ ⁸. But on the other hand, it is also possible to compute the commutator $[\Phi(x) \Phi^\dagger(x), \Phi(y) \Phi^\dagger(y)]$ using the commutation relations (4) and (5). Suppose for instance that Φ is a Dirac field. Since we don't know yet if it is a boson or a fermion, let's distinguish the possible cases:

- if the particle described by Φ is a fermion, we have (4) with the upper signs. Then:

$$\begin{aligned} \Phi(x) \underbrace{\Phi^\dagger(x) \Phi(y)}_{\substack{= -\Phi(y) \Phi^\dagger(x) \\ -\Delta_+(x-y) \\ +\Delta_+(y-x)}} \Phi^\dagger(y) &= -\Phi(x) \Phi(y) \Phi^\dagger(x) \Phi^\dagger(y) + \left(\Delta_+(y-x) - \Delta_+(x-y) \right) \Phi(x) \Phi^\dagger(y) \\ &= -\Phi(y) \underbrace{\Phi(x) \Phi^\dagger(y)}_{\substack{= -\Phi^\dagger(y) \Phi(x) \\ +\Delta_+(x-y) \\ -\Delta_+(y-x)}} \Phi^\dagger(x) + \left(\Delta_+(y-x) - \Delta_+(x-y) \right) \Phi(x) \Phi^\dagger(y) \\ &= \Phi(y) \Phi^\dagger(y) \Phi(x) \Phi^\dagger(x) + \underbrace{\left(\Delta_+(y-x) - \Delta_+(x-y) \right)}_{=0 \text{ if } x-y \text{ spacelike}} (\Phi(y) \Phi^\dagger(x) + \Phi(x) \Phi^\dagger(y)), \end{aligned}$$

⁷We are aware that there still doesn't exist a proper and consensual mathematical framework for the implementation of measurements in QFT [10]). This is probably why the same computation leads to oddities if one doesn't move to the special reference frame \mathcal{R} beforehand. Still, it seems to us that this simple reasoning actually captures something of physical interest.

⁸This is because the only commutators that can appear between a creation and an annihilation operator are $[a(\vec{p}), a^{c\dagger}(\vec{p}')]]$ or $[a^c(\vec{p}), a^\dagger(\vec{p}')]]$ (with the label c standing for the antiparticle) which are zero if $a(\vec{p}) \neq a(\vec{p})^c$.

- if the particle described by Φ is a boson, we have (4) with the lower signs. Then

$$\begin{aligned}
\Phi(x) \underbrace{\Phi^\dagger(x)\Phi(y)}_{\substack{=\Phi(y)\Phi^\dagger(x) \\ -\Delta_+(x-y) \\ -\Delta_+(y-x)}} \Phi^\dagger(y) &= \Phi(x)\Phi(y)\Phi^\dagger(x)\Phi^\dagger(y) - \left(\Delta_+(x-y) + \Delta_+(y-x)\right)\Phi(x)\Phi^\dagger(y) \\
&= \Phi(y) \underbrace{\Phi(x)\Phi^\dagger(y)}_{\substack{=\Phi^\dagger(y)\Phi(x) \\ +\Delta_+(x-y) \\ +\Delta_+(y-x)}} \Phi^\dagger(x) - \left(\Delta_+(x-y) + \Delta_+(y-x)\right)\Phi(x)\Phi^\dagger(y) \\
&= \Phi(y)\Phi^\dagger(y)\Phi(x)\Phi^\dagger(x) + \underbrace{\left(\Delta_+(x-y) + \Delta_+(y-x)\right)}_{\substack{\text{non identically zero} \\ \text{if } x-y \text{ spacelike}}}\left(\Phi(y)\Phi^\dagger(x) - \Phi(x)\Phi^\dagger(y)\right).
\end{aligned}$$

As a consequence, one recovers the commutation relation $[\Phi(x)\Phi^\dagger(x), \Phi(y)\Phi^\dagger(y)] = 0$ imposed by the condition (MC) if, and only if, Φ is a fermionic field and in this case, the relation (4) implies that $\{\Phi(x), \Phi^\dagger(y)\} = 0$ for all spacelike intervals $x - y$. These are indeed the statements (M) and (SS). When Φ is a scalar or vector field, the proof is similar. \square

Remark 4.2. Note that this proof makes use, as usually in physics, of the identification between the notion of observable in the mathematical sense (hermitian operator) and in the physical sense (quantity measurable by a concrete experimental protocol). However, the second if far more restrictive, since in practice we can only measure a few very specific observables. Rigorously speaking, only the physical notion of observable is constrained by (MC), since the latter must ensure the absence of inconsistencies between *actual* measurements. Therefore, in the above proof, although $\Phi\Phi^\dagger$ is hermitian, one could question the legitimacy of imposing it (MC). Of course, any mathematical observable could be in principle measured by applying a suitable unitary evolution to the system that would map its eigenbasis to the eigenbasis of a physically measurable observable. But is it satisfactory to rely on the idea that all unitary evolutions are a priori feasible, even though we will never be able to implement them? Nonetheless, it is still possible to adapt the proof by replacing $\Phi\Phi^\dagger$ by an undoubtedly physical observable, such that a function of the components of $\hat{T}_{\mu\nu}$ or even the charge \hat{Q} , that one can express in terms of Φ . For example, for a Dirac field, $\hat{T}_\mu^\mu = m\Phi^\dagger\gamma^0\Phi$, would allow a quite similar proof.

What precedes has an unexpected consequence, expressed in the corollary below.

Corollary 4.3. *A fermionic field is not measurable.*

Proof. In the previous proof, we have seen that if a field is an observable (*i.e.* $\Phi = \Phi^\dagger$), then the condition (MC) implies that for all $x - y$ spacelike, $[\Phi(x), \Phi(y)] = 0$. But if Φ is a fermionic field, it also satisfies $\{\Phi(x), \Phi(y)\} = 0$. Adding these two relations yields for all $x - y$ spacelike, $\Phi(x)\Phi(y) = 0$, which is too strong a constraint for a quantum field. In particular, $\Phi(x)\Phi(y)|0\rangle$ is the state containing two localized particles at x and y ; in any case it is a non-zero vector of the Fock space⁹. \square

5 Conclusion

The aim of this paper was to investigate the logical interrelationships between fundamental properties in relativistic QM. As we have seen, such a work is not only useful to found quantum theory on solid bases, but it also delivered fruitful and unexpected results, and lead to various philosophical remarks.

Our starting point was the apparent incompatibility between special relativity and two kinds of instantaneities that seem to appear in QM. By the time it was historically developed, quantum theory was not built to integrate special relativity, so that it is always surprising to contemplate their ‘peaceful coexistence’ (expression coined by Shimony in [23]). We formulated a mathematical consistency condition

⁹This result is not new, and can also be derived, for instance, by the fact that the equation $\Phi = \Phi^\dagger$ is neither covariant nor independent of the representation of the gamma matrices [20], so it can’t have a physical meaning for the Dirac field. But it is interesting to see that it can be directly derived from considerations about measurements. Again, our argument relies on the widespread identification between the notions of mathematical and physical observable. Here, Φ may indeed be hermitian, but is it a physically measurable quantity subject to the condition (MC)?

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