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Modeling and scoring dynamic probabilistic forecasts

Thibault Modeste * Clément Dombry † Anne-Laure Fougères *

Abstract

Probabilistic forecast plays a major role in many applications where forecast is needed together with an assessment of its uncertainty. Verification of probabilistic forecast has become increasingly important and mostly relies on two sets of tools: scoring rules and calibration diagnostics. Proper scoring rules assign forecasts numerical scores such that the correct forecast achieves a minimal expected score. Calibration theory aims at verifying that the observations and the forecasts are consistent.

In practice, using a probabilistic forecast commonly involves a sequential decision making process where the environment evolves over time. In this article, we propose a mathematical framework for dynamic probabilistic forecast. The forecasts take therein the form of stochastic processes adapted to a filtration that encodes the available information. Under minimal assumptions, we show that proper scoring rules can still be used in this dynamic framework to discriminate the ideal forecast - more precisely, we prove that the long term average score is close to minimum if and only if the forecasts are close to ideal. Some connections are also done in terms of Wasserstein distance, and links are given between scoring rules and reproducing kernels.

Keywords: dynamic probabilistic forecast; scoring rules.

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1 Introduction

In a wide range of applications, probabilistic forecasts (Dawid, 1984; Gneiting et al., 2007) have become an essential tool, as recently illustrated for example in hydrology (Tiberi-Wadier et al., 2021), health (Henzi et al., 2021), demography (Raftery and Ševčíková, 2021), or meteorology (Vannitsem et al., 2021). In such contexts, verification is of particular importance, and is based on two sets of tools: calibration diagnostics, and scoring rules. Calibration theory aims at verifying that the observations and the forecasts are consistent. See e.g. Tsyplakov (2011), Strähl and Ziegel (2017), or Taillardat et al. (2022, Appendix A) for formal definitions. Scoring rules are used for evaluating the quality of a forecast and to compare different forecasts, and proper scoring rules assign forecasts numerical scores such that the correct forecast achieves a minimal expected score.

Most of the phenomena considered in applications have a dynamical nature, see among others Holzmann and Eulert (2014) in a risk management scoring framework, or Bröcker and Ben Bouallègue (2020) for verification of ensemble weather forecasts. To meet these needs in terms of assumptions, the stationary setting is the most popular one, as is required in the papers cited above. Such hypotheses happen however to be too restrictive in most situations, and there is a real practical interest to get general models for probabilistic forecasts in a dynamical context. From this perspective, Strähl and Ziegel (2017) proposed a framework allowing for quite general serial dependence as well as a definition of calibration dedicated to this setting. Our work is in the same vein, and consists of proposing to consider as dynamic probabilistic forecasts some stochastic processes adapted to a filtration that encodes the available information. We show in such a general framework that proper scoring rules can still be used to discriminate the ideal forecast.

More precisely, we introduce in Section 2 a general model – called Model 1 – which only requires assumptions of measurability with respect to a σ -field gathering the available information. Various examples are also given to illustrate the defined structure. In Section 3, Theorem 1 shows that even under weak assumptions including Model 1, the long term averaged score is still almost surely minimized by the ideal forecast; it states additionally that the long term averaged score of a dynamic forecast is asymptotically equivalent to the long term averaged score of the ideal forecast if and only if the average divergence between the two forecasts tends to 0. Similar results can be easily obtained with stationarity assumptions. This theorem therefore justifies the common use of averaging under more general assumptions than the stationary one. In some particular cases of energy scores, it is also stated that the average divergence between two forecasts tends to 0 if and only if the average Wasserstein distance between two forecasts tends to 0. Finally, Section 4 focuses on the family of kernel scores, for which sufficient conditions on the kernel are discussed that guarantee a proper scoring rule, and the links with the theory of reproducing kernels is discussed. Section 5 contains all the proofs, and an appendix gives some results on the regularity of scoring rules.

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2 Dynamic probabilistic forecast

2.1 Mathematical models

In this section, we propose a simple mathematical framework for dynamic probabilistic forecasts. Let \mathcal{U} be a Polish space considered as the *universe*. Let \mathcal{Y} be a second Polish space and $f : \mathcal{U} \rightarrow \mathcal{Y}$ be a measurable application considered as the *observable*, that is to say the quantity we are interested in and we want to forecast. The space of Borel probability measures on \mathcal{Y} is denoted by $\mathcal{M}_1(\mathcal{Y})$ and seen as the *space of predictive distributions*. It is equipped with the σ -algebra generated by the applications $\pi \in \mathcal{M}_1(\mathcal{Y}) \mapsto \pi(B) \in \mathbb{R}$, $B \subset \mathcal{Y}$ Borel.

Model 1. On an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider:

- a sequence $(U_n)_{n \in \mathbb{N}}$ of \mathcal{U} -valued random variables;
- the sequence $Y_n = f(U_n)$, $n \in \mathbb{N}$;
- a sub-filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of the natural filtration $(\mathcal{G}_n)_{n \in \mathbb{N}}$ associated with $(U_n)_{n \in \mathbb{N}}$, i.e. $\mathcal{F}_n \subset \mathcal{G}_n = \sigma(U_k; k \leq n)$ for all $n \in \mathbb{N}$;
- a sequence $(F_n)_{n \in \mathbb{N}}$ of $\mathcal{M}_1(\mathcal{Y})$ -valued random variables adapted to $(\mathcal{F}_n)_{n \in \mathbb{N}}$, i.e. F_n is \mathcal{F}_n -measurable for all $n \in \mathbb{N}$.

The sequence $(U_n)_{n \in \mathbb{N}}$ represents the evolution of the environment over time and $(Y_n)_{n \in \mathbb{N}}$ the quantity of interest. The sub-filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ represents the *information* and we call the sequence $(F_n)_{n \in \mathbb{N}}$ a *dynamic probabilistic forecast*. The forecast is understood relatively to a *lead time* $T \geq 1$, meaning that the forecaster produces at time n a forecast F_n for the future value Y_{n+T} and this predictive distribution F_n is built in view of the limited information encoded in \mathcal{F}_n only.

In a context of meteorological forecast, the space \mathcal{U} may represent e.g. the different possible states of the atmosphere, and U_n its state at time n . If the quantity of interest is the temperature at some location, we may take $\mathcal{Y} = \mathbb{R}$ and Y_n is the temperature at time n . The available information \mathcal{F}_n may be a record of temperature, pressure, precipitation at several locations up to time n . Using this information, the forecast for the future temperature Y_{n+T} is given by the predictive distribution F_n with lead time T .

For the forecaster, the holy graal is the so-called *ideal forecast* that we define below.

Definition 1. *The ideal forecast with respect to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and with lead time $T \geq 1$ is the random sequence $(F_{n,T}^*)_{n \in \mathbb{N}}$ defined by*

$$F_{n,T}^* = \mathcal{L}(Y_{n+T} \mid \mathcal{F}_n) \quad a.s., \quad n \in \mathbb{N}.$$

Clearly, the ideal forecast $(F_{n,T}^*)_{n \in \mathbb{N}}$ is adapted to the filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and is an admissible dynamic probabilistic forecast. The conditional distribution $F_{n,T}^*$ is considered as the best possible predictive distribution for Y_{n+T} given the information \mathcal{F}_n . We refer to [Gneiting and Ranjan \(2013\)](#) and [Tsyplakov \(2013\)](#) for a general discussion on ideal forecast.

For the purpose of asymptotics, we also consider a stronger model assuming stationarity. The model is very similar to Model 1, but we assume additionally the strict stationarity and that the time index is $n \in \mathbb{Z}$. We also assume that the information is stemming from auxiliary observations of the form $X_n = h(U_n)$, with $h : \mathcal{U} \rightarrow \mathcal{X}$ measurable mapping between Polish spaces.

Model 2. On an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider:

- a strictly stationary sequence $(U_n)_{n \in \mathbb{Z}}$ of \mathcal{U} -valued random variables;
- the sequences $Y_n = f(U_n)$ and $X_n = h(U_n)$, $n \in \mathbb{Z}$;
- the sub-filtration $(\mathcal{F}_n)_{n \in \mathbb{Z}}$ associated to $(X_n)_{n \in \mathbb{Z}}$, i.e. $\mathcal{F}_n = \sigma(X_k; k \leq n)$;
- a sequence $(F_n)_{n \in \mathbb{Z}}$ of $\mathcal{M}_1(\mathcal{Y})$ -valued random variables adapted to $(\mathcal{F}_n)_{n \in \mathbb{Z}}$.

The sequence $(Y_n)_{n \in \mathbb{Z}}$ again corresponds to the quantity of interest that we want to forecast, whereas $(X_n)_{n \in \mathbb{Z}}$ gathers the observations that are available, generating the information encoded by the filtration $(\mathcal{F}_n)_{n \in \mathbb{Z}}$. Clearly, both sequences $(X_n)_{n \in \mathbb{Z}}$ and $(Y_n)_{n \in \mathbb{Z}}$ are strictly stationary, and we will mostly consider stationary forecast as defined below.

Definition 2. *The dynamic probabilistic forecast $(F_n)_{n \in \mathbb{Z}}$ is said stationary if the sequence $(U_n, F_n)_{n \in \mathbb{Z}}$ is strictly stationary.*

Under Model 2, a stationary forecast takes the form

$$F_n = \Phi(X_n, X_{n-1}, X_{n-2}, \dots), \quad n \in \mathbb{Z},$$

for some measurable mapping $\Phi : \mathcal{X}^{\mathbb{N}} \rightarrow \mathcal{M}_1(\mathcal{Y})$. The mapping Φ is seen as the forecast algorithm that produces the predictive distribution given the past observations. See Lemma 1 in Section 5.1.

One can also show that the ideal forecast in Model 2 is stationary and takes the form

$$F_{n,T}^* = \mathcal{L}(Y_{n+T} \mid X_n, X_{n-1}, X_{n-2}, \dots) = \Phi_T^*(X_n, X_{n-1}, X_{n-2}, \dots) \quad a.s.,$$

with $\Phi_T^* : \mathcal{X}^{\mathbb{N}} \rightarrow \mathcal{M}_1(\mathcal{Y})$ and $n \in \mathbb{Z}$. See Lemma 2 in Section 5.1.

2.2 Examples

Several examples are discussed in this section to illustrate the framework of dynamic probabilistic forecasts introduced previously.

Example 1. As a simple example of Model 1, consider the Gaussian autoregressive model of order 1 (Brockwell and Davis (1991), Chapter 3) defined by the initial value $U_0 = 0$ and the recursive relation

$$U_{n+1} = \alpha U_n + \varepsilon_{n+1}, \quad n \in \mathbb{N},$$

where $\alpha \in \mathbb{R}$ and $(\varepsilon_n)_{n \geq 1}$ is an i.i.d. centered gaussian sequence with variance $\sigma^2 > 0$. We assume that $\mathcal{U} = \mathcal{Y} = \mathbb{R}$ and that the quantity of interest is $Y_n = U_n$.

Consider first the trivial case where no information is available, i.e. $\mathcal{F}_n = \{\emptyset, \mathcal{U}\}$ is the trivial σ -field for all $n \in \mathbb{N}$. Then the ideal forecast with lead time $T \geq 1$ is

$$F_{n,T}^* = \mathcal{L}(U_{n+T} | \mathcal{F}_n) = \mathcal{L}(U_{n+T}), \quad n \in \mathbb{N},$$

because conditional distribution with respect to the trivial σ -field reduces to marginal distribution. Simple computations show that

$$U_{n+T} = \sum_{i=1}^{n+T} \alpha^{n+T-i} \varepsilon_i$$

whence we deduce

$$F_{n,T}^* = \mathcal{N}\left(0, \frac{1 - \alpha^{2(n+T)}}{1 - \alpha^2} \sigma^2\right), \quad n \in \mathbb{N}.$$

We next discuss the opposite case of complete information where $\mathcal{F}_n = \sigma(U_k, k \leq n)$ for all $n \in \mathbb{N}$. The ideal forecast with lead time $T = 1$ is then

$$F_{n,1}^* = \mathcal{L}(U_{n+1} | U_0, \dots, U_n) = \mathcal{N}(\alpha U_n, \sigma^2), \quad n \in \mathbb{N}.$$

For a general lead time $T \geq 1$, the relation

$$U_{n+T} = \alpha^T U_n + \sum_{i=1}^T \alpha^{T-i} \varepsilon_{n+i}$$

implies that the ideal forecast is given by

$$F_{n,T}^* = \mathcal{N}\left(\alpha^T U_n, \frac{1 - \alpha^{2T}}{1 - \alpha^2} \sigma^2\right), \quad n \in \mathbb{N}. \quad (1)$$

Note that the variances are smaller than in the case with no information which corresponds to the general fact that exploiting information reduces forecast uncertainty and leads to sharper predictive distributions.

If the parameters α, σ^2 are unknown, the ideal forecast is not accessible but the forecaster may naturally provide sequential parameter estimates based on the observation record. Maximum likelihood estimation ([Brockwell and Davis \(1991\)](#), Chapter 8.7) yields

$$\hat{\alpha}_n = \frac{\sum_{i=1}^n U_{i-1} U_i}{\sum_{i=1}^n U_{i-1}^2} \quad \text{and} \quad \hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^n (U_i - \hat{\alpha}_n U_{i-1})^2, \quad n \geq 1.$$

In view of Equation (1), the plug-in method suggests the dynamic probabilistic forecast with lead time T ,

$$F_{n,T} = \mathcal{N}\left(\hat{\alpha}_n^T U_n, \frac{1 - \hat{\alpha}_n^{2T}}{1 - \hat{\alpha}_n^2} \hat{\sigma}_n^2\right), \quad n \geq 1, \quad (2)$$

which is adapted with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$, i.e. accessible in view of the available observation record.

A simple illustration of Model 2 can be obtained when $\alpha \in (-1, 1)$. Let $(\varepsilon_n)_{n \in \mathbb{Z}}$ be an i.i.d. sequence with distribution $\mathcal{N}(0, \sigma^2)$, and consider the infinite moving average

$$U_n = \sum_{i \geq 0} \alpha^i \varepsilon_{n-i}, \quad n \in \mathbb{Z}.$$

The sequence $(U_n)_{n \in \mathbb{Z}}$ is strictly stationary and satisfies the auto-regressive property. The marginal distribution $\mathcal{N}(0, \sigma^2/(1 - \alpha^2))$ corresponds to the ideal forecast in absence of information. To produce a stationary forecast, one may consider maximum likelihood estimation based on the last p observations, where p is an integer in $[1, n]$, i.e.

$$\hat{\alpha}_n = \frac{\sum_{i=0}^{p-1} U_{n-i-1} U_{n-i}}{\sum_{i=0}^{p-1} U_{n-i}^2} \quad \text{and} \quad \hat{\sigma}_n^2 = \frac{1}{p} \sum_{i=0}^{p-1} (U_{n-i} - \hat{\alpha}_n U_{n-i-1})^2.$$

Clearly, the sequence of parameter estimates $(\hat{\alpha}_n, \hat{\sigma}_n^2)_{n \in \mathbb{Z}}$ is strictly stationary and $(F_{n,T})_{n \in \mathbb{Z}}$ defined by Equation (2) is a stationary forecast.

Example 2. An extension of the previous model can be built by incorporating an additive error of measure. To do this, consider the sequence

$$\begin{cases} U_{n+1}^{(1)} &= \alpha U_n^{(1)} + \varepsilon_{n+1}, \\ U_{n+1}^{(2)} &= U_{n+1}^{(1)} + \delta_{n+1}, \end{cases}$$

where the two sequences of innovation $(\varepsilon_n)_{n \geq 1}$ and noise $(\delta_n)_{n \geq 1}$ are assumed i.i.d. and with respective distribution $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(0, \tau^2)$. Assume that the quantity of interest is the first component $Y_n = U_n^{(1)}$ and is observed with a measurement error δ_n , so that the observation available is the second component $X_n = U_n^{(2)}$. Here, one thus has $\mathcal{U} = \mathbb{R}^2$ and $\mathcal{Y} = \mathbb{R}$, and the information is given by the natural filtration associated with (X_n) , i.e. $\mathcal{F}_n = \sigma(X_k, k \leq n)$.

The sequence $(U_n)_{n \in \mathbb{N}} = (U_n^{(1)}, U_n^{(2)})_{n \in \mathbb{N}}$ is a bivariate gaussian vector as soon as the initial value U_0 is assumed to be bivariate gaussian. When $\alpha \in (-1, 1)$, it is strictly stationary for a suitable choice of the initial distribution.

Example 3. An example in a spatio-temporal setting can be constructed starting from a vectorial AR(1). Let $A \in \mathcal{M}_d(\mathbb{R})$, U_0 be a d -variate random vector and $(\varepsilon_n)_{n \in \mathbb{N}}$ an i.i.d. sequence of d -variate Gaussian vectors with distribution $\mathcal{N}_d(0, \Sigma)$. Consider for $n \in \mathbb{N}$

$$U_{n+1} = AU_n + \varepsilon_n.$$

We write

$$U_n = (X_{1,n}, \dots, X_{d,n})^T.$$

The quantity of interest is the first coordinate, ie $Y_n = X_{1,n}$. For $I \subset \{1, \dots, d\}$, we note

$$\mathcal{F}_{I,n} = \sigma(X_{i,k}, i \in I, k \leq n).$$

Example 4. As discussed above, the three examples considered can be made stationary by choosing a specific initialisation U_0 . Let finally illustrate a case of Model 1 that can not be plug into Model 2; let $(t_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ be two sequences, and assume that $(s_n)_{n \in \mathbb{N}}$ is periodic. It typically represents seasonal variability. The sequence $(t_n)_{n \in \mathbb{N}}$ is a general trend that can represent a global warming in the context of climate change. For a given r.v. U_0 , one can define for all $n \in \mathbb{N}$ the sequence

$$U_{n+1} = t_n + s_n + \alpha U_n + \varepsilon_{n+1} ,$$

which fulfills Model 1's general assumptions but not Model 2's ones.

3 Score theory for dynamic probabilistic forecast

3.1 Background on score theory

Scoring rules (Gneiting and Raftery, 2007) provide a major tool for forecast validation. A scoring rule compares forecasts and realizations and assigns a numerical score assessing the forecast quality. Proper scoring rules have the property that the correct forecast minimizes the expected score. They are commonly used to compare different forecast methods and the forecast with the lowest score is preferred.

First we recall some basic definitions. Let (\mathcal{L}, d) be a subset of $\mathcal{M}_1(\mathcal{Y})$ and d a distance on this space, as for example the Wasserstein distance, see Section 3.3 for a definition. A score on \mathcal{L} is a measurable real valued function $S: \mathcal{L} \times \mathcal{Y} \rightarrow \mathbb{R}$, and $S(F, y)$ is the score assigned to the probabilistic forecast F when the outcome y occurs.

Remark 1. Let specify here that we add the assumption of measurability in the definition of the score S , as done in Holzmann and Eulert (2014), in order to lighten the presentation. Some sufficient conditions to get this measurability are provided in Appendix A.

Definition 3. The score S is said proper on \mathcal{L} if for all $F, G \in \mathcal{L}$, the integral

$$\bar{S}(F, G) = \int_{\mathcal{Y}} S(F, y) \, dG(y)$$

is well-defined and if the following inequality holds

$$\bar{S}(G, G) \leq \bar{S}(F, G), \quad \text{for all } F, G \in \mathcal{L}.$$

The score is said strictly proper if the equality holds above if and only if $F = G$.

The quantity $\bar{S}(F, G)$ is the average score when the forecast is F and the observations have distribution G . For a proper scoring rule, the minimum of $F \mapsto \bar{S}(F, G)$ is achieved when $F = G$. The divergence associated with a proper score S is the non-negative function

$$\mathbf{div}_S(F, G) = \bar{S}(F, G) - \bar{S}(G, G).$$

For a strictly proper scoring rule, the divergence vanishes if and only if $F = G$, so that the divergence can be seen as a pseudo-distance between F and G (the symmetry or triangle inequality may not be satisfied).

Example 5. For real observations, the most important and widely used score is the Continuous Ranked Probability Score, shortly noted CRPS (Epstein (1969); Hersbach (2000); Bröcker (2012)). The CRPS is a strictly proper scoring rule on the class $\mathcal{P}_1(\mathbb{R})$ of probability measures on \mathbb{R} having a finite first moment. It is defined for $F \in \mathcal{P}_1(\mathbb{R})$ and $y \in \mathbb{R}$ by

$$\text{CRPS}(F, y) = \int_{\mathbb{R}} (F(x) - \mathbf{1}_{\{y \leq x\}})^2 dx,$$

where the probability F is identified with its cumulative distribution function. An alternative representation is

$$\text{CRPS}(F, y) = \mathbb{E}[|X - y|] - \frac{1}{2} \mathbb{E}[|X - X'|] \quad (3)$$

where X, X' are independent random variables with distribution F . Several other decompositions are available, see e.g. Taillardat et al. (2022) and references therein.

Example 6. A generalization of the CRPS can be obtained from (3) by introducing the so-called energy kernel, defined for $\alpha > 0$ and $\beta \in (0, \infty]$ by

$$\rho_{\alpha, \beta}(x, y) = \|x - y\|_{\beta}^{\alpha}, \quad x, y \in \mathbb{R}^d, \quad (4)$$

with $\|x\|_{\beta} = (\sum_{i=1}^d |x_i|^{\beta})^{1/\beta}$ and $\|x\|_{\infty} = \max_{1 \leq i \leq d} |x_i|$. More precisely, let consider the energy score, defined as

$$S_{\rho_{\alpha, \beta}}(F, y) = \int_{\mathbb{R}^d} \rho_{\alpha, \beta}(x, y) dF(x) - \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho_{\alpha, \beta}(x, x') F(dx) F(dx').$$

The case $d = 1$ and $\alpha = 1$ boils down to Equation (3) and hence to the CRPS.

It can be shown that the energy score is a proper scoring rule in the following cases:

- $d = 1$ and $\alpha \in (0, 2]$;
- $d \geq 2$, $\beta \in (0, 2]$ and $\alpha \in (0, \beta]$;
- $d = 2$, $\beta \in (2, +\infty]$ and $\alpha \in (0, 2]$.

See e.g. Schoenberg (1938) (case $\beta = 2$), Koldobsky (1992) and Zastavnyi (1993), as well as the discussion in Section 4. Additionally, some interesting results can be established on the divergence of the energy score when $\beta = 2$ and $\alpha \in (0, 2)$, see Section 3.3.

Example 7. Another score that is widely used in practice is the logarithm score, introduced in Good (1952). It is defined for measures dominated by the Lebesgue measure with non vanishing density functions. Consider such a measure F , denote by f its density, and define for $y \in \mathbb{R}$ the score as

$$S(F, y) = \log f(y).$$

This score is strictly proper, and its divergence is the well-known Kullback–Leibler divergence,

$$\mathbf{div}_S(F, G) = \int_{\mathbb{R}} \log \left(\frac{f(y)}{g(y)} \right) g(y) dy.$$

3.2 Asymptotic results for dynamic forecast scoring

The aim of this section is to discuss the score theory for dynamic probabilistic forecasts as defined in Model 1. It is established in particular that the ideal forecast minimizes the averaged score in different ways. Note that Model 1 provides a sequential model and, for simplicity, we first look at a single step.

Consider a random forecast F on $(\Omega, \mathcal{F}, \mathbb{P})$, which is measurable with respect to a sub- σ -algebra $\mathcal{F}_0 \subset \mathcal{F}$, and consider an observation Y . The ideal forecast is thus defined as $F^* = \mathcal{L}(Y \mid \mathcal{F}_0)$. A natural consequence of the definition of proper scoring rule is that the ideal forecast minimizes the expected score. Here, the expectation is taken with respect to both the observation and the forecast randomness. The following proposition is an important result in the non dynamic framework, and can be found in [Holzmann and Eulert \(2014\)](#) (Theorem 3).

Proposition 1. *Let S be a proper scoring rule. Then a.s.*

$$E[S(F, Y) - S(F^*, Y) \mid \mathcal{F}_0] = \mathbf{div}_S(F, F^*) \geq 0,$$

and this implies $\mathbb{E}[S(F, Y)] \geq \mathbb{E}[S(F^*, Y)]$. Moreover, if the score is strictly proper, then the equality holds if and only if $F = F^*$ a.s..

As a consequence, in the sequential framework defined by Model 1, the forecaster has to predict at each time n the ideal forecast $F_{n,T}^*$ in order to minimize its expected score. The main result of this section is a stronger optimality property in the sense of almost sure convergence. It states that the ideal forecast minimizes the long-term score *almost surely*. In some sense, expectation is replaced by a temporal average but it should be stressed that this is not straightforward because we do not assume any stationary condition.

Theorem 1. *Let $(F_n)_{n \in \mathbb{N}}$ be a probabilistic dynamic forecast as defined in Model 1, measurable with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$. Let $(F_{n,T}^*)_{n \in \mathbb{N}}$ be the ideal forecast with lead time $T \geq 1$. Let S be a proper scoring rule with associated divergence \mathbf{div}_S . We assume that, for $k = 1, \dots, T$,*

$$\sum_{i=1}^n \mathbb{E}[(\delta_i^k)^2 \mid \mathcal{F}_{i+T-k}] = O(n), \quad n \in \mathbb{N}. \quad (5)$$

where $\delta_i^k = \mathbb{E}[\Delta_i \mid \mathcal{F}_{i+T+1-k}] - \mathbb{E}[\Delta_i \mid \mathcal{F}_{i+T-k}]$ and $\Delta_i = S(F_i, Y_{i+T}) - S(F_{i,T}^*, Y_{i+T})$. Then a.s., the following inequality holds

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n S(F_i, Y_{i+T}) - S(F_{i,T}^*, Y_{i+T}) \geq 0. \quad (6)$$

Moreover, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n S(F_i, Y_{i+T}) - S(F_{i,T}^*, Y_{i+T}) = 0 \quad a.s. \quad (7)$$

if and only if

$$\frac{1}{n} \sum_{i=1}^n \mathbf{div}_S(F_i, F_{i,T}^*) \rightarrow 0 \quad a.s. .$$

Equation (6) states that the ideal forecast minimizes the long term averaged score almost surely – here the average is temporal and for a fixed realization, in opposition to the expected score considered in Proposition 1. The vanishing limit (7) means that the long term averaged score of the dynamic forecast $(F_n)_{n \in \mathbb{N}}$ is equal to the one of the ideal forecast $(F_{n,T}^*)_{n \in \mathbb{N}}$, stating that both predictions are equally good in this sense. This is characterized by an asymptotically negligible average divergence between the two sequences.

The proof of Theorem 1 is based on the strong law of large numbers for square integrable martingales and do not assume any stationarity condition. The technical condition (5) is required for the martingale convergence theorem. See Section 5.2 for more details.

Remark 2. In the case of the energy score with $\alpha \leq 1$ and $1 \leq \beta$ (see Example 6), a simple sufficient condition for condition (5) to hold is

$$\sup_i m(F_i) + m(F_{i,T}^*) < +\infty ,$$

where $m(F)$ is the first moment of a probability F . In other words, the probabilistic forecast and the ideal forecast have uniformly bounded moments. See Proposition 3 in the Appendix A for more details.

Example 8. Theorem 1 has an interesting application dealing with partial information. Assume that two experts have access to different information and that the first expert is better informed. This is formalized by two filtrations $(\mathcal{F}_n^1)_{n \in \mathbb{N}}$ and $(\mathcal{F}_n^2)_{n \in \mathbb{N}}$ with $\mathcal{F}_n^2 \subset \mathcal{F}_n^1$ for all $n \in \mathbb{N}$. The best possible forecast for each expert is the ideal forecast with respect to the available information, noted $F_{n,T}^{*,j} = \mathcal{L}(Y_{n+T} | \mathcal{F}_n^j)$, $j = 1, 2$. Theorem 1 yields

$$\liminf \frac{1}{n} \sum_{i=1}^n S(F_{i,T}^{*,2}, Y_{i+T}) - S(F_{i,T}^{*,1}, Y_{i+T}) = 0 \quad a.s.,$$

meaning that the extra information possessed by the first expert allows him to reach a lower averaged score in the long term.

Example 9. Elaborating on the previous example is the notion of cross-calibration (Strähl and Ziegel (2017)). Assume J different experts produce dynamic forecasts $(F_n^j)_{n \in \mathbb{N}}$ with respect to different filtrations $(\mathcal{F}_n^j)_{n \in \mathbb{N}}$, $1 \leq j \leq J$. Assume that the information $(\mathcal{F}_n^j)_{n \in \mathbb{N}}$ is private but the forecasts $(F_n^j)_{n \in \mathbb{N}}$ are public. Then the information encoded in the filtration

$$\mathcal{F}_n = \sigma(F_i^j; i \leq n, 1 \leq j \leq J), \quad n \in \mathbb{N},$$

is publicly accessible. Note that \mathcal{F}_n does not necessarily contain \mathcal{F}_n^j but nevertheless F_n^j is measurable with respect to \mathcal{F}_n . Considering the ideal forecast $F_{n,T}^* = \mathcal{L}(Y_{n+T} | \mathcal{F}_n)$, Theorem 1 yields, for all $1 \leq j \leq J$,

$$\liminf \frac{1}{n} \sum_{i=1}^n S(F_i^j, Y_{i+T}) - S(F_{i,T}^*, Y_{i+T}) = 0 \quad a.s. .$$

This means that the public forecasts can be used to produce a new forecast that outperforms the J experts in terms of averaged score.

We briefly comment the stationary case and state simple results for Model 2. Note that stationarity combined with ergodicity allows to use the ergodic theorem and greatly simplifies the proof.

Corollary 1. *In the framework of Model 2 with the ergodicity condition, let $(F_n)_{n \in \mathbb{Z}}$ be a stationary dynamic forecast measurable with respect to $(\mathcal{F}_n)_{n \in \mathbb{Z}}$ and $(F_{n,T}^*)_{n \in \mathbb{Z}}$ be the ideal forecast with lead time $T \geq 1$. Let S be a proper scoring rule. Then,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n S(F_i, Y_{i+T}) = \mathbb{E}[S(F_0, Y_T)] \geq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n S(F_{i,T}^*, Y_{i+T}) = \mathbb{E}[S(F_{0,T}^*, Y_T)],$$

where the limit are meant almost surely. Moreover, if S is strictly proper, equality holds if and only if $(F_n)_{n \in \mathbb{Z}} = (F_{n,T}^*)_{n \in \mathbb{Z}}$ a.s.

3.3 Links between energy scores and Wasserstein distance

The purpose of this section is to provide a more explicit interpretation of the divergence condition (7) in the case of the energy scores defined via (4). We assume $\beta = 2$ so that $\|\cdot\|_2$ denotes the Euclidean norm on \mathbb{R}^d . The p -Wasserstein space on \mathbb{R}^d consists in the set $\mathcal{P}_p(\mathbb{R}^d)$ of Borel probability measures F on \mathbb{R}^d with finite p -moment, i.e.

$$\mathcal{P}_p(\mathbb{R}^d) = \{F \in \mathcal{M}_1(\mathbb{R}^d) : \int \|x\|_2^p dF(x) < \infty\}.$$

It follows from Hölder's inequality that $\mathcal{P}_p(\mathbb{R}^d) \subset \mathcal{P}_1(\mathbb{R}^d)$. In the case of $p = 1$, it is endowed with the Kantorovich-Rubinstein distance

$$W_1(F_1, F_2) = \sup_{\text{Lip}(\phi) \leq 1} \left| \int \phi(x) dF_1(x) - \int \phi(x) dF_2(x) \right|,$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a Lipschitz function and $\text{Lip}(\phi) = \sup_{x \neq y} |\phi(x) - \phi(y)| / \|x - y\|_2$. For more details on Wasserstein spaces, we refer to Villani (2008) Chapter 6.

Theorem 2. *Let $\alpha \in (0, 2)$ and $(F_n)_{n \in \mathbb{N}}$ and $(G_n)_{n \in \mathbb{N}}$ be sequences in $\mathcal{P}_{\max(1, \alpha)}(\mathbb{R}^d)$ and assume the sequences are uniformly integrables, i.e.*

$$\forall \varepsilon > 0, \exists K \subset \mathbb{R}^d \text{ compact}, \forall n \in \mathbb{N}, \int_{K^c} \|x\| d(F_n + G_n)(x) < \varepsilon.$$

Let \mathbf{div}_S be the divergence of the energy score with $\alpha \in (0, 2)$ and $\beta = 2$. Then

$$\frac{1}{n} \sum_{i=1}^n \mathbf{div}_S(F_i, G_i) \rightarrow 0 \quad \text{if and only if} \quad \frac{1}{n} \sum_{i=1}^n W_1(F_i, G_i) \rightarrow 0.$$

Consequently, Theorem 1 can be rewritten as follows: assuming condition (5) together with uniform integrability, we have the equivalence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n S(F_i, Y_{i+T}) - S(F_{i,T}^*, Y_{i+T}) = 0 \iff \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n W_1(F_i, F_{i,T}^*) = 0.$$

This means that the dynamic forecast $(F_n)_{n \in \mathbb{N}}$ achieves asymptotically the minimal averaged score if and only if it is closed to the ideal forecast $(F_{n,T}^*)_{n \in \mathbb{N}}$ in Wasserstein distance.

4 Kernel Score

4.1 Presentation

The energy score defined in Example 6 is a specific case of the larger family of the so-called kernel scores. On the observation space \mathcal{Y} , consider a measurable kernel, $\rho: \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$ and assume that

$$\forall y \in \mathcal{Y}, \rho(y, y) = 0. \quad (8)$$

Such a condition is useful to characterize the definition set of the associated score. Recall that ρ is said *conditionally negative definite* if it is symmetric and if for all $n \geq 2$, $(y_1, \dots, y_n) \in \mathcal{Y}^n$ and $(a_1, \dots, a_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n a_i = 0$, it holds

$$\sum_{1 \leq i, j \leq n} a_i a_j \rho(y_i, y_j) \leq 0. \quad (9)$$

Note that a kernel ρ satisfying Assumptions (8) and (9) is necessarily non negative. Indeed, for $x, y \in \mathcal{Y}$, these conditions yield, when $a_1 = 1$ and $a_2 = -1$,

$$0 \geq \rho(x, x) + \rho(y, y) - 2\rho(x, y) = -2\rho(x, y).$$

The following subset of probability measures is then introduced for such kernels

$$\mathcal{L}_\rho := \left\{ F \in \mathcal{M}_1(\mathcal{Y}) \mid \exists y_0 \in \mathcal{Y}, \int_{\mathcal{Y}} \rho(y, y_0) dF(y) < +\infty \right\}, \quad (10)$$

and the score S_ρ associated with the kernel ρ is defined on $\mathcal{L}_\rho \times \mathcal{Y}$ by

$$S_\rho: (F, y) \mapsto \int_{\mathcal{Y}} \rho(x, y) dF(x) - \frac{1}{2} \int_{\mathcal{Y} \times \mathcal{Y}} \rho(x, x') F(dx) F(dx').$$

Remark 3. The well-defined character of these integrals is justified in Remark 21 of [Sejdinovic et al. \(2013\)](#). Note that conditions (8) and (9) are essential. We will propose a quick justification similar to the previous article but using the terminology from geostatistics. If the kernel ρ verifies conditions (8) and (9) then we can consider a Gaussian process $(B_y)_{y \in \mathcal{Y}}$ verifying

$$\forall x, y \in \mathcal{Y}, \rho(x, y) = \text{Var}(B_x - B_y).$$

The kernel ρ is then called the variogram of the process $(B_y)_{y \in \mathcal{Y}}$. Let $F \in \mathcal{L}_\rho$ and $y_0 \in \mathcal{Y}$ from the definition (10),

$$\begin{aligned} \rho(x, y) &= \text{Var}(B_x - B_y) = \text{Var}(B_x - B_{y_0} + B_{y_0} - B_y) \\ &= \text{Var}(B_x - B_{y_0}) + \text{Var}(B_{y_0} - B_y) + 2\text{Cov}(B_x - B_{y_0}, B_{y_0} - B_y) \\ &\leq \rho(x, y_0) + \rho(y, y_0) + 2\sqrt{\rho(x, y_0)}\sqrt{\rho(y, y_0)}. \end{aligned}$$

This concludes because $L^1(\mathcal{Y}, dF) \subset L^{1/2}(\mathcal{Y}, dF)$ as F is a finite measure. This also shows that $\bar{S}(F, G)$ is well defined for $F, G \in \mathcal{L}_\rho$.

The score S_ρ is a proper scoring rule as soon as the kernel ρ is continuous. This comes from Hoeffding's inequality, see [Berg et al. \(1984\)](#), section 7, Theorem 2.1. Note that it is not always simple to show that a kernel score is strictly proper. Sufficient conditions are discussed in [Steinwart and Ziegel \(2021\)](#), especially the case where \mathcal{Y} is compact.

4.2 Scoring rules and Reproducing Kernel

The scores from a kernel can be compared to the Maximum Mean Discrepancy (MMD) of the Reproducing Kernel Hilbert Space (RKHS) theory. This comparison is the key to the proof of Theorem 2. We will not give details of this theory but only the essence. We invite the curious reader to refer to [Berlinet and Thomas-Agnan \(2004\)](#), [Smola et al. \(2007\)](#) or [Steinwart and Christmann \(2008, Section 4\)](#). The idea of this theory is to embed any topological space \mathcal{Y} into a Hilbert space \mathcal{H} , i.e. each point $y \in \mathcal{Y}$ is represented by a vector $K(y) \in \mathcal{H}$. To do this embedding, we define the scalar product between each point of the space \mathcal{Y} . This scalar product is represented by a kernel k , which is this time positive definite, i.e.

$$\forall n \geq 2, (y_1, \dots, y_n) \in \mathcal{Y}^n, (a_1, \dots, a_n) \in \mathbb{R}^n, \sum_{1 \leq i, j \leq n} a_i a_j k(y_i, y_j) \geq 0$$

and not conditionally definite negative as previously. In the following, k will represent a positive definite kernel and ρ a conditionally negative definite kernel. We will see in Theorem 3 the links between these two properties. Returning to the RKHS, after embedding the space \mathcal{Y} into a Hilbert space \mathcal{H} , one can embed a set $\mathcal{L} \subset \mathcal{M}_1(\mathcal{Y})$ into this same Hilbert. Thus each measure $F \in \mathcal{L}$ is represented by a vector $K(F)$. This idea allows then to compare two measures $F, G \in \mathcal{L}$ through the Hilbert space, i.e.

$$\gamma_k(F, G) = \|K(F) - K(G)\|_{\mathcal{H}}.$$

The function γ_k comparing these two measures is called the MMD. It has another more explicit form in terms of the kernel k ,

$$\gamma_k(F, G)^2 = \int_{\mathcal{Y}^2} k(x, y) d(F - G) \otimes (F - G)(x, y).$$

Moreover, this integral is well defined for F, G in the set

$$\mathcal{L}_k := \left\{ F \in \mathcal{M}_1(\mathcal{Y}) \mid \int \sqrt{k(y, y)} dF(y) < +\infty \right\}. \quad (11)$$

The links between kernel scores and this theory have already been explained in several articles (see ([Sejdinovic et al., 2013](#), Section 4 and 5), [Steinwart and Ziegel \(2021\)](#)).

Theorem 3. [*(Berg et al., 1984, Section 3 Lemma 2.1.)*, (*Sejdinovic et al., 2013, Proposition 20, Theorem 22 and Remark 23*)] *Let $y_0 \in \mathcal{Y}$, let ρ be a kernel verifying the Assumption (8) and (9), then the kernel defined by*

$$\forall x, y \in \mathcal{Y}, k(x, y) = \rho(y_0, x) + \rho(y_0, y) - \rho(x, y) \quad (12)$$

is positive definite. Moreover, this kernel is defined on a larger set of probability measures, i.e. $\mathcal{L}_\rho \subset \mathcal{L}_k$. The divergence \mathbf{div}_{S_ρ} and the MMD γ_k satisfy

$$\forall F, G \in \mathcal{L}_\rho, \mathbf{div}_{S_\rho}(F, G) = 2\gamma_k^2(F, G).$$

Example 10. The CRPS corresponds to the conditionally negative kernel $\rho(x, y) = |x - y|$ and is defined on $\mathcal{P}_1(\mathbb{R})$, the set of probability measures with a first absolute moment. An associated positive definite kernel is

$$k(x, y) = |x| + |y| - |x - y|.$$

Then $k(x, x) = 2|x|$ so that the MMD is defined on the space $\mathcal{P}_{1/2}(\mathbb{R})$ of probability measures with a half absolute moment. So we notice that the MMD γ_k is defined for *strictly* more probability measures.

Remark 4. Following Dawid (2007), kernel scores can also be defined for positive definite kernel by

$$S_k(F, y) = \int_{\mathcal{Y}^2} k(x, x') d(F - \delta_y) \otimes (F - \delta_y)(x, x'),$$

for $F \in \mathcal{L}_k$ defined in (11) and $y \in \mathbb{R}^d$. If the kernel k is associated with the conditionally negative kernel ρ by Equation (12), then the scoring rules S_ρ and S_k are defined on different distribution sets $\mathcal{L}_\rho \subset \mathcal{L}_k$ and

$$\forall F, G \in \mathcal{L}_\rho, \mathbf{div}_{S_\rho}(F, G) = \mathbf{div}_{S_k}(F, G).$$

The construction with positive definite kernels is more general since it is defined on a larger space.

5 Proofs

5.1 Proofs of Section 2

Lemma 1. Under Model 2, if $(F_n)_{n \in \mathbb{Z}}$ is stationary, there exists a map $\Phi: \mathcal{X}^{\mathbb{N}} \rightarrow \mathcal{M}_1(\mathcal{Y})$ such that

$$F_n \stackrel{d}{=} \Phi(X_n, X_{n-1}, \dots).$$

Proof. As $\mathcal{M}_1(\mathcal{Y})$ is a Polish Space, there exists for all $n \in \mathbb{Z}$, a map $\Phi_n: \mathcal{X}^{\mathbb{N}} \rightarrow \mathcal{M}_1(\mathcal{Y})$ such that

$$F_n = \Phi_n(X_n, X_{n-1}, \dots),$$

because F_n is $\sigma(X_n, X_{n-1}, \dots)$ -measurable. Moreover, $(F_n)_{n \in \mathbb{Z}}$ is stationary, so that

$$\Phi_n(X_n, X_{n-1}, \dots) = F_n \stackrel{d}{=} F_0 = \Phi_0(X_0, X_{-1}, \dots).$$

But $(X_n)_{n \in \mathbb{Z}}$ is also stationary, and thus $\Phi_0(X_0, X_{-1}, \dots) \stackrel{d}{=} \Phi_0(X_n, X_{n-1}, \dots)$. Then we note $\Phi := \Phi_0$ and get the proof. \square

Lemma 2. Under Model 2, the ideal forecast with respect to the filtration $(\mathcal{F}_n)_{n \geq \mathbb{Z}}$ and with lead time $T \geq 1$ can be written as

$$F_{n,T}^* = \Phi_T^*(X_n, X_{n-1}, \dots) \text{ a.s.,}$$

where $\Phi_T^*: \mathcal{X}^{\mathbb{N}} \rightarrow \mathcal{M}_1(\mathcal{Y})$ is a measurable map.

Proof. By definition, we have $F_{n,T}^* = \mathcal{L}(Y_{n+T} | \mathcal{F}_n) = F_{Y_{n+T}}^{\mathcal{F}_n}$ for all $n \in \mathbb{Z}$. Standard properties of conditional distributions then give that

$$F_{n,T}^* = F_{Y_{n+T}}^{(X_n, X_{n-1}, \dots)}(X_n, X_{n-1}, \dots).$$

See e.g. [Kallenberg \(1997, Chapter 5\)](#), or [Modeste et al. \(2023, Appendix A\)](#) for more details. Now, $(X_n)_{n \in \mathbb{Z}}$ and $(Y_n)_{n \in \mathbb{Z}}$ are strictly stationary (since $(U_n)_{n \in \mathbb{Z}}$ is so), and one thus gets for all $k \in \mathbb{Z}$,

$$\mathcal{L}(Y_{n+T} | X_n, X_{n-1}, \dots) = \mathcal{L}(Y_{n+T+k} | X_{n+k}, X_{n+k-1}, \dots).$$

This yields

$$\Phi_T^* := F_{Y_{n+T}}^{(X_n, X_{n-1}, \dots)} = F_{Y_{n+T+k}}^{(X_{n+k}, X_{n+k-1}, \dots)},$$

implying that $F_{n,T}^* = \Phi_T^*(X_n, X_{n-1}, \dots)$. □

5.2 Proofs for Section 3

Proof of Theorem 1. Defining $\Delta_i = S(F_i, Y_{i+T}) - S(F_{i,T}^*, Y_{i+T})$, our goal is to prove that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \Delta_i \geq 0.$$

Applying Proposition 1 for all $i \geq 1$ yields $\mathbb{E}[\Delta_i | \mathcal{F}_i] = \mathbf{div}_S(F_i, F_{i,T}^*) \geq 0$. Now, since Δ_i is \mathcal{F}_{i+T} -measurable, let introduce the following decomposition as a telescopic sum

$$\Delta_i - \mathbb{E}[\Delta_i | \mathcal{F}_i] = \sum_{k=1}^T (\mathbb{E}[\Delta_i | \mathcal{F}_{i+T+1-k}] - \mathbb{E}[\Delta_i | \mathcal{F}_{i+T-k}]).$$

Defining $\delta_i^k = \mathbb{E}[\Delta_i | \mathcal{F}_{i+T+1-k}] - \mathbb{E}[\Delta_i | \mathcal{F}_{i+T-k}]$ and $M_n^k = \sum_{i=1}^n \delta_i^k$ implies

$$\frac{1}{n} \sum_{i=1}^n \Delta_i = \frac{1}{n} \sum_{i=1}^n \mathbf{div}_S(F_i, F_{i,T}^*) + \frac{1}{n} \sum_{k=1}^T M_n^k.$$

The announced results therefore follow as soon as the second term of the right hand side in the equality above is shown to converge a.s. to 0. To see this, notice that for $1 \leq k \leq T$, the sequence $(M_n^k)_{n \in \mathbb{N}}$ is a martingale with respect to the filtration $(\mathcal{F}_{n+T+1-k})_{n \in \mathbb{N}}$ and its quadratic variation is defined by

$$\langle M^k \rangle_n = \sum_{i=1}^n \mathbb{E}[(\delta_i^k)^2 | \mathcal{F}_{i+T-k}], \quad n \in \mathbb{N}.$$

It is a nondecreasing process and we denote by $\langle M^k \rangle_\infty$ its almost sure limit in $[0, +\infty]$. The strong law of large numbers for square-integrable martingales, see e.g. [\(Hall and Heyde, 1980, Section 2.6\)](#), implies that:

- i) on the event $\langle M^k \rangle_\infty < +\infty$, the martingale M_n^k converges to a finite limit as $n \rightarrow \infty$;
- ii) on the event $\langle M^k \rangle_\infty = +\infty$, the ratio $M_n^k / \langle M^k \rangle_n$ converges to 0 as $n \rightarrow \infty$.

The first case clearly implies that $M_n^k/n \rightarrow 0$ as $n \rightarrow \infty$. This also holds in the second case thanks to the Assumption (5) cause $\langle M^k \rangle_n = O(n)$. As a conclusion, one gets also in ii) that $M_n^k/n \rightarrow 0$ as $n \rightarrow \infty$. □

Our proof of Theorem 2 rely on the inequality stated in Modeste and Dombry (2022, Proposition 3.9). This results is based on the following representation of the Energy Score divergence in the case $\alpha \in (0, 2)$ and $\beta = 2$ due to Szekely (2003). For $F, G \in \mathcal{P}_\alpha(\mathbb{R}^d)$,

$$\mathbf{div}_S(F, G) = \frac{1}{C(d, \alpha)} \int_{\mathbb{R}^d} \frac{|\hat{F}(t) - \hat{G}(t)|^2}{\|t\|_2^{d+\alpha}} dt, \quad (13)$$

with $C(d, \alpha) > 0$ and \hat{F} (resp. \hat{G}) the characteristic function of F (resp. G) defined by $\hat{F}(t) = \int_{\mathbb{R}^d} e^{it \cdot x} dF(x)$. Note that the subject of Modeste and Dombry (2022) is not conditionally negative kernels, but their Formula (14) shows that we consider the same object. The link between these two articles is detailed in Subsection 4.2.

Proof of the Theorem 2. The direct implication is a consequence of this inequality present in Modeste and Dombry (2022, Proposition 3.9 and Formula (14))

$$\forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists C > 0, W_1(F_n, G_n) \leq C \mathbf{div}_S(F_n, G_n) + \varepsilon,$$

because the sequences are uniformly integrables. Then for $\varepsilon > 0$, let $C > 0$ of the previous inequality. We deduce that

$$\limsup \frac{1}{n} \sum_{j=1}^n W_1(F_j, G_j) \leq C \limsup \frac{1}{n} \sum_{j=1}^n \mathbf{div}_S(F_j, G_j) + \varepsilon = \varepsilon.$$

We now prove the converse implication and assume that the sequences $(F_n)_n, (G_n)_n$ are uniformly integrables and that $n^{-1} \sum_{i=1}^n W_1(F_i, G_i) \rightarrow 0$. Because, for all $t \in \mathbb{R}^d$, the functions $x \mapsto \cos(t \cdot x)$ and $x \mapsto \sin(t \cdot x)$ are $\|t\|$ -Lipschitz continuous, we have

$$\begin{aligned} & \frac{1}{n} \sum_{j=1}^n |\hat{F}_j(t) - \hat{G}_j(t)| \\ & \leq \frac{1}{n} \sum_{j=1}^n \left| \int_{\mathbb{R}^d} \cos(t \cdot x) (F_j - G_j)(dx) \right| + \left| \int_{\mathbb{R}^d} \sin(t \cdot x) (F_j - G_j)(dx) \right| \\ & \leq \frac{2\|t\|}{n} \sum_{j=1}^n W_1(F_j, G_j) \rightarrow 0. \end{aligned}$$

By the Cauchy-Schwarz inequality, we also have

$$\frac{1}{n} \sum_{j=1}^n |\hat{F}_j(t) - \hat{G}_j(t)|^2 \rightarrow 0, \quad t \in \mathbb{R}^d.$$

Thanks to Equation (13)

$$\frac{1}{n} \sum_{j=1}^n \mathbf{div}_S(F_j, G_j) = \int_{\mathbb{R}^d} \frac{1}{nC(d, \alpha)} \sum_{j=1}^n \frac{|\hat{F}_j(t) - \hat{G}_j(t)|^2}{\|t\|^{d+\alpha-2}} dt.$$

This is shown to converge to 0 by dominated convergence. Indeed,

$$h_n(t) = \frac{1}{n} \sum_{j=1}^n \frac{|\hat{F}_j(t) - \hat{G}_j(t)|^2}{\|t\|^{d+\alpha-2}} \rightarrow 0, \quad \text{for all } t \in \mathbb{R}^d \setminus \{0\}.$$

Furthermore, the uniform integrability of $(F_n)_{n \in \mathbb{N}}$, $(G_n)_{n \in \mathbb{N}}$ implies a first moment uniformly bounded by some constant $M > 0$ so that the characteristic functions are M -Lipschitz continuous and

$$|h_n(t)| \leq \frac{M^2}{\|t\|^{d+\alpha-2}} \mathbf{1}_{\|t\| \leq 1} + \frac{2}{\|t\|^{d+\alpha}} \mathbf{1}_{\|t\| > 1} \in L^1(\mathbb{R}^d),$$

for $\alpha \in (0, 2)$. The integrability of the dominant function comes from the following lemma

Lemma 3. *Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a measurable map and $\|\cdot\|$ be a norm on \mathbb{R}^d . The map $\phi \circ \|\cdot\| \in L^1(\mathbb{R}^d)$ if and only if $t \in \mathbb{R}^+ \mapsto t^{d-1}\phi(t) \in L^1(\mathbb{R})$.*

□

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A Measurability of the score

The assumption of measurability added in the definition is there for mathematical reasons. In this small Appendix, we will give some natural properties about a score involving measurability.

Definition 4. *An application S is said continuous for a distance δ on \mathcal{L} if for each $y \in \mathcal{Y}$, the map $S(\cdot, y)$ is continuous for δ .*

Lemma 4. *Let $S: \mathcal{L} \times \mathcal{Y} \rightarrow \mathbb{R}$ be an application measurable in its second variable, if S is continuous for a metric δ such that \mathcal{L} is separable then S is a score.*

Proof. Let (F_0, F_1, F_2, \dots) be a countable dense family of \mathcal{L} . We define for $n \in \mathbb{N}^*$

$$k_n(F) = \inf\{k \in \mathbb{N} \mid F \in B_\delta(F_k, 1/n)\},$$

where $B_\delta(F, \varepsilon)$ is the closed ball centered at point F . By density, this set is not empty. The map k_n is measurable cause

$$\{k_n(F) = k\} = B_\delta(F_k, 1/n) \setminus \bigcup_{i=0}^{k-1} B_\delta(F_i, 1/n).$$

We define

$$S_n(F, y) = \sum_{k=0}^{+\infty} \mathbb{1}_{\{k_n(F)=k\}} S(F_k, y).$$

This map is measurable by the measurability of k_n . By continuity of S and construction of k_n , we have

$$S_n(F, y) \rightarrow S(F, y), \text{ for all } F, y \in \mathcal{L} \times \mathcal{Y}.$$

Then S is limit of measurable maps, then S is measurable. \square

Remark 5. If (\mathcal{Y}, d) is separable, the space of probability measure $\mathcal{M}_1(\mathcal{Y})$ is still measurable for the Levy-Prokhorov metric which metrizes the weak convergence. This is still true when the set $\mathcal{P}_1(\mathbb{R}^d)$ is fitted with the Wasserstein distance W_1 , introduced in Section 3.3. Moreover, a subset of a separable metric space remains separable.

In the Section 4, we introduced the family of kernel score. We will show the continuity of these scores for the Wasserstein distance introduced in Section 3.3. We need another writing of this distance, for $F, G \in \mathcal{P}_1(\mathbb{R}^d)$,

$$W_1(F, G) = \inf_{X \sim F, Y \sim G} \mathbb{E}[\|X - Y\|], \quad (14)$$

where $X \sim F$ means that F is the distribution of the random variable X .

Proposition 2. *Let ρ be a Lipschitz continuous kernel, ie there exists $C > 0$,*

$$\forall x_1, x_2, y_1, y_2 \in \mathbb{R}^d, |\rho(x_1, y_1) - \rho(x_2, y_2)| \leq C(\|x_1 - x_2\| + \|y_1 - y_2\|),$$

then the score S_ρ is defined on $\mathcal{P}_1(\mathbb{R}^d)$ and is continuous for the Wasserstein distance W_1 . More precisely for all $F, G \in \mathcal{P}_1(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$,

$$|S_\rho(F, y) - S_\rho(G, y)| \leq 2CW_1(F, G).$$

Proof. Let $C > 0$ be a Lipschitz constant of ρ , then S_ρ is well defined on $\mathcal{P}_1(\mathbb{R}^d)$. Let $F, G \in \mathcal{P}_1(\mathbb{R}^d)$ and X, X', Z, Z' be four random variables associated with this probabilities and X (resp. Z) and X' (resp. Z') independents. For $y \in \mathbb{R}^d$, we have

$$\left| \int_{\mathbb{R}^d} \rho(x, y) dF(x) - \int_{\mathbb{R}^d} \rho(z, y) dG(z) \right| = |\mathbb{E}[\rho(X, y) - \rho(Z, y)]| \leq C\mathbb{E}[\|X - Z\|]$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho(x, x') dF(x)dF(x') - \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho(z, z') dG(z)dG(z') \right| &= |\mathbb{E}[\rho(X, X') - \rho(Z, Z')]| \\ &\leq C\mathbb{E}[\|X - Z\| + \|X' - Z'\|]. \end{aligned}$$

As we do not do assumption on the dependence between X (resp. X') and Z (resp. Z'), we conclude with the formulation (14) of the Wasserstein distance that

$$\begin{aligned} \left| \int_{\mathbb{R}^d} k(x, y) dF(x) - \int_{\mathbb{R}^d} k(z, y) dG(z) \right| &\leq CW_1(F, G) \\ \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} k(x, x') dF(x)dF(x') - \int_{\mathbb{R}^d \times \mathbb{R}^d} k(z, z') dG(z)dG(z') \right| &\leq 2CW_1(F, G). \end{aligned}$$

Then

$$|S_\rho(F, y) - S_\rho(G, y)| \leq 2CW_1(F, G).$$

□

Proposition 3. Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a filtration and $F_{n,T}^*$ be the ideal forecast with respect to this filtration with a lead time $T \geq 1$. We consider also an admissible dynamic forecast $(F_n)_{n \in \mathbb{N}}$. If ρ is Lipschitz continuous and

$$\sum_{i=1}^n \left(\int_{\mathbb{R}^d} \|y\| d(F_i + F_{i,T}^*)(y) \right)^2 = O(n),$$

then one verifies the condition (5). This is true specially in the case where the moments are uniformly bounded

$$\sup_i \int_{\mathbb{R}^d} \|y\| d(F_i + F_{i,T}^*)(y) < +\infty.$$

Proof. Let $k \in \{1, \dots, T\}$ and $C > 0$ be a Lipschitz constant of the kernel ρ , let's remember the definition of

$$\Delta_i = S(F_i, Y_{i+T}) - S(F_{i,T}^*, Y_{i+T}) \text{ and } \delta_i^k = \mathbb{E}[\Delta_i | \mathcal{F}_{i+T+1-k}] - \mathbb{E}[\Delta_i | \mathcal{F}_{i+T-k}], \text{ for } i \in \mathbb{N}$$

For $a, b \in \mathbb{R}$, $(a + b)^2 \leq 2(a^2 + b^2)$, then with the Jensen Inequality

$$\mathbb{E}[(\delta_i^k)^2 | \mathcal{F}_{i+T-k}] \leq 4\mathbb{E}[\Delta_i^2 | \mathcal{F}_{i+T-k}].$$

The Proposition 2 rewrites this inequality in terms of Wasserstein distance

$$\mathbb{E}[(\delta_i^k)^2 \mid \mathcal{F}_{i+T-k}] \leq 16C^2 \mathbb{E}[W_1^2(F_i, F_{i,T}^*) \mid \mathcal{F}_{i+T-k}] = 16C^2 W_1^2(F_i, F_{i,T}^*)$$

Then by Triangular Inequality and Equation (14),

$$\begin{aligned} W_1^2(F_i, F_{i,T}^*) &\leq \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - x'\| \, dF_i \otimes F_{i,T}^*(x, x') \right)^2 \\ &\leq \left(\int_{\mathbb{R}^d} \|y\| \, d(F_i + F_{i,T}^*)(y) \right)^2 . \end{aligned}$$

So the condition (5) is checked. □