



HAL
open science

Chaotic trajectories in the standard map. The concept of anti-integrability

Serge Aubry, Gilles Abramovici

► **To cite this version:**

Serge Aubry, Gilles Abramovici. Chaotic trajectories in the standard map. The concept of anti-integrability. *Physica D: Nonlinear Phenomena*, 1990, 43 (2-3), pp.199-219. <10.1016/0167-2789(90)90133-A>. <hal-04055834>

HAL Id: hal-04055834

<https://hal.science/hal-04055834v1>

Submitted on 3 Apr 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Copyright - All rights reserved

CHAOTIC TRAJECTORIES IN THE STANDARD MAP. THE CONCEPT OF ANTI-INTEGRABILITY

Serge AUBRY and Gilles ABRAMOVICI

Laboratoire Léon Brillouin¹, CEN Saclay, 91191 Gif-Sur Yvette Cedex, France

Received 6 December 1989

Accepted 31 January 1990

Communicated by Y. Pomeau

A rigorous proof is given in the standard map (associated with a Frenkel–Kontorowa model) for the existence of chaotic trajectories with unbounded momenta for large enough coupling constant $k > k_0$. These chaotic trajectories (with finite entropy per site) are coded by integer sequences $\{m_i\}$ such that the sequence $b_i = |m_{i+1} + m_{i-1} - 2m_i|$ be bounded by some integer b . The bound k_0 in k depends on b and can be lowered for coding sequences $\{m_i\}$ fulfilling more restrictive conditions.

The obtained chaotic trajectories correspond to stationary configurations of the Frenkel–Kontorowa model with a finite (non-zero) phonon gap (called gap parameter in dimensionless units). This property implies that the trajectory (or the configuration $\{u_i\}$) can be uniquely continued as a uniformly continuous function of the model parameter k in some neighborhood of the initial configuration. A non-zero gap parameter implies that the Lyapunov coefficient is strictly positive (when it is defined). In addition, the existence of dilating and contracting manifolds is proven for these chaotic trajectories. “Exotic” trajectories such as ballistic trajectories are also proven to exist as a consequence of these theorems.

The concept of anti-integrability emerges from these theorems. In the anti-integrable limit which can be only defined for a discrete time dynamical system, the coordinates of the trajectory at time i do not depend on the coordinates at time $i - 1$. Thus, at this singular limit, the existence of chaotic trajectories is trivial and the dynamical system reduces to a Bernoulli shift. It is well known that the KAM tori of symplectic dynamical originates by continuity from the invariant tori which exists in the integrable limit (under certain conditions). In a similar way, it appears that the chaotic trajectories of dynamical systems originate by continuity from those which exists at the anti-integrable limits (also under certain conditions).

1. Introduction

It is well known that the trajectories of the standard map [1] correspond to the stationary configurations of the Frenkel–Kontorowa (FK) model [2–10]. Among this large set of configurations, the ground states of the FK models can be studied independently of the trajectories of the standard map. These configurations are either periodic (commensurate) or quasi-periodic (incommensurate). The incommensurate ground states undergo a transition by breaking of analyticity (TBA) as a function of the model parameter

λ (or C) which is associated with a transformation of the quasi-periodic trajectory in the standard map which represents this ground state. When the ground state is “analytic”, this trajectory is dense on a KAM torus. Beyond the TBA, this KAM torus breaks up into an invariant Cantorus and the incommensurate ground state becomes “non-analytic”. A similar variational approach for studying the standard map has been used independently in ref. [6].

The representation of the standard map by a mechanical FK model has the advantage that the existence of chaotic trajectories which looked some years ago very mysterious for a dynamical system, appears rather intuitive (and physically

¹Laboratoire Commun. CEA–CNRS.

obvious) for this structural model. These chaotic trajectories represent configurations where the atoms can be randomly distributed in the wells of the periodic potential when the elastic constant between the atoms is small enough. Despite chaotic configurations (i.e. chaotic trajectories) can be found explicitly for example in the FK model with a piecewise parabolic potential [7] (which is associated with the discontinuous sawtooth map), a general rigorous proof for their existence was still missing.

The aim of this paper is to associate the existence of chaotic trajectories with the concept of anti-integrability in a similar way, the existence of KAM tori can be associated with the concept of integrability. This approach applied to the FK model, yields a rigorous proof for the existence of unbounded chaotic trajectories in the standard map close enough to its anti-integrable limit. This approach can be generalized to a wider class of models.

This work is of interest not only for understanding of symplectic dynamical systems, but can also be extended to some non-symplectic dynamical system. It is also useful for understanding structural problems in one, two or more dimensions which cannot be associated with dynamical systems. These new ideas will be applied in a forthcoming paper to the problem of high physical interest of coupled electron-phonon systems [8]. Then the existence of chaotic bipolaronic states will be proven.

2. The standard map and the FK model

For pedagogical reasons, we shall mostly focus on the FK model in its original and simplest form, where the concept of anti-integrability appears to be very simple. It is a chain of atoms i at position u_i with energy

$$\Phi(\{u_i\}) = \sum_i [W(u_{i+1} - u_i) + V(u_i)]. \quad (1a)$$

The atoms are coupled between nearest neigh-

bors by the convex potential

$$W(x) = \frac{1}{2}Cx^2 - \mu x \quad (1b)$$

and submitted to the 2π -periodic potential $V(x)$

$$V(x) = \lambda(1 - \cos x). \quad (1c)$$

We have chosen an elastic potential with constant $C > 0$ and a sine potential with a positive amplitude λ . μ is a constant which in the absence of periodic potential ($k = 0$) fixes the atomic mean distance $l = \mu/C$ of the ground state,

$$l = \lim_{N-N' \rightarrow \infty} \frac{u_N - u_{N'}}{N - N'}, \quad (2a)$$

which can be either commensurate or incommensurate with 2π , or in other words that the incommensurability ratio

$$\zeta = \frac{l}{2\pi} \quad (2b)$$

is either a rational or an irrational number. Exact results concerning the ground states of this model have been obtained [3–6]. These ground state configurations $\{u_i\}$ fulfill the stationary equation

$$\begin{aligned} 0 &= -\frac{\partial \Phi(\{u_i\})}{\partial u_i} \\ &= C(u_{i+1} + u_{i-1} - 2u_i) - \lambda \sin u_i. \end{aligned} \quad (3a)$$

Defining

$$p_i = u_i - u_{i-1}, \quad (3b)$$

this equation generates the standard map \mathcal{S} [1]

$$\{u_{i+1}, p_{i+1}\} = \mathcal{S}(\{u_i, p_i\}) \quad (4a)$$

with parameter

$$k = \frac{\lambda}{C}, \quad (4b)$$

which maps the cylinder $\mathbb{C} \otimes \mathbb{R} = \{u \bmod 2\pi, p\}$ onto itself as

$$p_{i+1} = p_i + k \sin u_i \quad (4c)$$

$$u_{i+1} = (u_i + p_i + k \sin u_i) \bmod 2\pi. \quad (4d)$$

The trajectories $\{u_i, p_i\} = \mathcal{S}^i(\{u_0, p_0\})$ have different behaviors depending on the initial point $\{u_0, p_0\}$. Knowing a trajectory is equivalent to knowing the sequence $\{u_i\}$ because $\{p_i\}$ is given by (3b). They could be periodic cycles, Kolmogorov–Arnol’d–Moser (KAM) trajectory or chaotic trajectories, etc. The KAM trajectories which are dense on KAM tori, are perturbations of the trivial trajectories obtained for $k = 0$ ($\lambda = 0$).

Let us briefly recall some aspects of this integrable limit. This 2D map (4) is *integrable* when a single invariant can be found. When $k = 0$, eq. (4a) yields the invariant $p_i = \text{constant}$. In the integrable limit, the solutions of eq. (3) take the form

$$u_i = il + \alpha = 2\pi i\zeta + \alpha = f(il + \alpha; 0)$$

with $f(x; 0) = x$, (5a)

where α is an arbitrary phase and l is the atomic nearest distance. For $l/2\pi$ irrational, the corresponding trajectory on the cylinder $\{u \bmod 2\pi, p\}$ is dense on the invariant circle $p = l$ (1-torus). The KAM theorem asserts that this trajectory (5a) “survives” for $k < k_c(\zeta)$ where $k_c(\zeta)$ is some finite non-zero value when $\zeta = l/2\pi$ is a “good irrational number” (this set of irrational numbers has full Lebesgue measure 1). In other words, there exists a trajectory with the same rotation number ζ which is dense on a circle close to the unperturbed circle ($k = 0$). This solution $\{u_i\}$ is then described by an analytic function of x and k with the form $f(x; k) = x + g(x; k)$, where $g(x; k)$ is a 2π -periodic function of x

$$u_i = f(il + \alpha; k) = il + \alpha + g(il + \alpha; k). \quad (5b)$$

(For $k = 0$, we have $g(x; 0) = 0$.) Thus, $g(x; k)$ is a 2π -periodic analytic solution of the functional equation

$$g(x + l; k) + g(x - l; k) - 2g(x, k) - k \sin[x + g(x; k)] = 0. \quad (5c)$$

This corresponding solution $\{u_i\}$ can be obtained by a Newton method or more precisely as a fixed point of the non-linear operator $\{u_i^{(1)}\} = T(\{u_i^{(0)}\})$ defined implicitly by

$$u_{i+1}^{(1)} + u_{i-1}^{(1)} - 2u_i^{(1)} = k \sin u_i^{(0)} \quad (6a)$$

or equivalently by the functional relation

$$g^{(1)}(x + l; k) + g^{(1)}(x - l; k) - 2g^{(1)}(x; k) = k \sin[x + g^{(0)}(x; k)]. \quad (6b)$$

In general, the sequence $\{u_i^{(n)}\}$ recursively defined by $\{u_i^{(n)}\} = T(\{u_i^{(n-1)}\})$ does not necessarily converge to a fixed point of the operator T . For having a convergent sequence, one needs to restrict the space of configurations where T applies to a subspace of configurations $\{u_i\}$ which is invariant by T and where T is contracting. First, this subspace only contains configurations with a rotation number which is a sufficiently “good irrational” number (and particularly is not a Liouville number) and second they can be described as $u_i = f(2\pi i\zeta + \alpha)$ with a hull function $g(x) = f(x) - x$ which is 2π -periodic and analytic. It is not our purpose to describe here the complex machinery of this approach in details. As far as we understood, this space is a “Frechet” space with an unusual topology which is generated by a “graded metric”. Then when k is smaller than a certain bound $k_c(\zeta)$ which depends on the incommensurability ratio, the sequence of configurations $\{u_i^{(n)}\} = T^n(\{u_i^{(0)}\})$ converges to a fixed point $\{u_i\}$ in the same space. Particularly, with the initial condition $u_i^{(0)} = 2\pi i\zeta + \alpha$, $\{u_i^{(n)}\}$ does converge to an “analytic” limit. Thus, the functional equation (5c) has an analytic solution. The proof of this result which involves the “smaller denominator problem” is rather complex and has first been done by Arnol’d [9].

When a KAM solution exists, the integrability of the non-perturbed dynamical system, is “locally” preserved. In other words, new variables “action-angle” can be defined for the trajectories

which are uniformly close to the considered KAM torus. Thus, as explained in section 2 of ref. [5], new “angle” variables $\{\alpha_i\}$ can be defined instead of the old variable $\{u_i\}$ such that for small fluctuations of α_i , the energy (1) takes the same form up to a constant but with $k = 0$ and a renormalized elastic constant $C_\zeta(k)$. $\{\alpha_i\}$ describes the phase fluctuations of the incommensurate ground state which physically corresponds to the gapless phason modes. Thus, for irrational ζ and $k < k_c(\zeta)$, the periodic perturbation $V(x)$ can be eliminated and the renormalized elastic constant $C_\zeta(k)$ corresponds to the stiffness of the phase fluctuations. At $k_c(\zeta)$, numerical analysis have shown that the stiffness $C_\zeta(k)$ goes to zero (the FK chain becomes soft).

3. The anti-integrable limit

At the opposite of this integrable limit $k = 0$ ($C = +\infty$) which has been very much studied, it is useful to define the “anti-integrable” limit. We set a general definition for symplectic dynamical systems (or more generally for dynamical systems which can be defined from the extremalization of a generating functional $\sum_i L_i(u_{i+1}, u_i)$).

Definition 1. We say that a symplectic dynamical system with trajectories $\{u_i\}$ and discrete time i ($i \in \mathbb{Z}$) is *anti-integrable* when its action (or generating functional) $\sum_i L(u_{i+1}, u_i)$ can be written as a sum $\sum_i V(u_i)$.

When this property is fulfilled, the stationary equations which are fulfilled by the trajectories become

$$\frac{\partial V(u_i)}{\partial u_i} = 0. \quad (7a)$$

Thus, u_i only takes values which are extrema of potential $V(x)$. Therefore, at this anti-integrable limit, the phase space becomes singular since all

the trajectories defined by extremalizing the generating functional, “collapse” on a discrete set of points of the full space $\{u_i, p_i\} = \{u_i, \partial L(u_i, u_{i-1})/\partial u_i\}$. Let us also note that for dynamical systems with a continuous time, the concept of anti-integrable limit is not apparent unless it is represented within a Poincaré map.

When $V(x)$ has more than one extrema, these extrema can be denoted by several symbols. The trajectory at time $i + 1$ does not depend on the trajectory at time i . Then an arbitrary sequence of symbols can be associated with a unique trajectory of this anti-integrable system and vice versa. The dynamical system appears to be a trivial Bernoulli shift. The existence of chaotic trajectories with finite entropy is then trivial.

Let us also remark that for models with several parameters, the anti-integrable limit may not be unique. For example, take the same model (1) but with a second harmonics with amplitude k_2 in the 2π -periodic potential $V(x)$ (1c), the limit $k = 0$ and $k_2 = \infty$ also corresponds to an anti-integrable limit where the coding of the chaotic trajectories is different from the anti-integrable limit $k = \infty$ and $k_2 = 0$. There exist other anti-integrable limits where k_2 and k have a given ratio. The nature of this anti-integrable limit is essential for the linear decomposition theorem of the ground states into discommensurations [11]. It might be also important in view of understanding recent works concerning the universality classes for the breaking of KAM tori [12]. In any case, by analogy with the KAM theorem which applies to perturbed integrable systems, we can prove that under certain conditions the chaotic trajectories of an anti-integrable dynamical system can be preserved under small enough perturbation

Let us now return to the simplest standard map which reaches its anti-integrable limit for $C = 0$ (or $k = +\infty$). Then, *all the solutions* of eq. (3a) can also be obtained and have the form

$$u_i = m_i \pi, \quad (7b)$$

where $\{m_i\}$ are arbitrary sequences of integers. In

that case, the set of integers can be used as the set of symbols which represent the dynamical system for $C = 0$. Unlike the integrable limit, most trajectories are chaotic since the sequences of integers $\{m_i\}$ are generally random.

By analogy with the KAM theorem applied to an integrable system, we ask: What do the chaotic trajectories of a perturbed anti-integrable dynamical system become? More precisely in our case, what do the chaotic trajectories (7a) become when the coupling constant C is switched to a finite value?

We prove that under some conditions, these chaotic trajectories “survive” for small enough perturbation. Thus, this theorem yields information about trajectories of the standard map in a domain where the KAM theorem does not. This theorem concerning the perturbation of the anti-integrable limit of the standard map is

Theorem 1. Let b be an arbitrary positive integer and $\{m_i\}$ ($i \in \mathbb{Z}$) an arbitrary sequence of integers, fulfilling for all i the condition

$$|m_{i+1} + m_{i-1} - 2m_i| \leq b. \quad (8a)$$

Then, for

$$k \geq B(b) = [16 + (b + 2)^2 \pi^2]^{1/2} \quad (8b)$$

there exists a unique solution $\{u_i(k)\}$ of eq. (3a) (i.e. a trajectory of the standard map) such that for all i

$$|u_i(k) - m_i \pi| \leq \pi/2. \quad (8c)$$

In addition, this solution depends continuously on k and we have

$$\lim_{k \rightarrow \infty} u_i(k) = m_i \pi. \quad (8d)$$

Particularly, when the sequence of integers $\{\sigma_i\} = \{m_i - m_{i-1}\}$ is periodic with period s ($\sigma_{i+s} = \sigma_i$ for all i) the corresponding trajectory $\{u_i \bmod 2\pi,$

$p_i = u_i - u_{i-1}\}$ is represented in the standard map as a periodic cycle with the same period s ($p_{i+s} = p_i$ for all i).

Most (i.e. with probability 1) integer sequences which fulfill (8a) have a finite topological entropy which is readily found to be $\ln(2b + 1)$ with b an integer. Let us prove theorem 1.

Proof. As we noticed above when discussing the KAM solutions, any solution of eq. (3a) is a fixed point of the non-linear transformation T defined by eq. (6a). For large k , it is more convenient to consider the inverse S of this transformation. Then we should have

$$\{u_i^{(1)}\} = S(\{u_i^{(0)}\}) \quad (9a)$$

defined as

$$k \sin u_i^{(1)} = u_{i+1}^{(0)} + u_{i-1}^{(0)} - 2u_i^{(0)}. \quad (9b)$$

However, this equation (9b), which yields $u_i^{(1)}$ from $\{u_i^{(0)}\}$,

- (1) may have no solution and
- (2) when this solution exists, it is not unique.

Therefore, we need first to determine a domain of configurations $\{u_i\}$ where this operator S can be defined and second we have to choose a unique determination for S in this domain. For example, for a given sequence of integers $\{m_i\}$, we consider the domain $E(\{m_i\}; \pi/2)$ defined as the set of configurations $\{u_i\}$ fulfilling for all $i \in \mathbb{Z}$.

$$|u_i - \pi m_i| \leq \pi/2. \quad (10a)$$

For $\{u_i\} \in E(\{m_i\}; \pi/2)$, we have

$$\begin{aligned} |u_{i+1} + u_{i-1} - 2u_i| \\ \leq \pi |m_{i+1} + m_{i-1} - 2m_i| + 2\pi \\ \leq (b + 2)\pi. \end{aligned} \quad (10b)$$

When

$$k = \frac{\lambda}{C} > (b + 2)\pi \quad (11)$$

eq. (9b) has infinitely many solutions $\{u_i^{(1)}\}$ for $\{u_i^{(0)}\} \in \mathbf{E}(\{m_i\}; \pi/2)$, but for each interval $]m\pi - \pi/2, m\pi + \pi/2]$ with m integer, there is a value for $u_i^{(1)}$ and only one which fulfills eq. (9b). Transformation S is uniquely defined for each domain $\mathbf{E}(\{m_i\}; \pi/2)$ which is thus mapped onto itself, by requiring that for any $\{u_i^{(0)}\} \in \mathbf{E}(\{m_i\}; \pi/2)$ we have

$$\{u_i^{(1)}\} = S(\{u_i^{(0)}\}) \in \mathbf{E}(\{m_i\}; \pi/2). \quad (12)$$

This transformation S is clearly continuous and differentiable. The Jacobian operator $\mathbf{J} = \{J_{i,j}\}$ defined as

$$\frac{\partial u_i^{(1)}}{\partial u_j^{(0)}} = J_{i,j} \quad (13)$$

is tridiagonal, that is

$$J_{i,j} = 0 \quad \text{for } |i-j| > 1. \quad (14a)$$

One has

$$J_{i,i} = -\frac{2(-1)^{m_i}}{k} \times \left(1 - \frac{(u_{i+1}^{(0)} + u_{i-1}^{(0)} - 2u_i^{(0)})^2}{k^2}\right)^{-1/2} \quad (14b)$$

and

$$J_{i,i+1} = J_{i,i-1} = \frac{(-1)^{m_i}}{k} \times \left(1 - \frac{(u_{i+1}^{(0)} + u_{i-1}^{(0)} - 2u_i^{(0)})^2}{k^2}\right)^{-1/2}. \quad (14c)$$

Using (14) and (10b), the spectral norm of this matrix \mathbf{J} defined as

$$\|\mathbf{J}\| = \sup_X \frac{\|\mathbf{J}\mathbf{X}\|}{\|\mathbf{X}\|} \quad (15a)$$

is bounded as

$$\|\mathbf{J}\| \leq 4 \left[k^2 - (b+2)^2 \pi^2 \right]^{-1/2} = \kappa(k, b). \quad (15b)$$

We can now use the Banach fixed point theorem^{#1} for the transformation S restricted to the compact domain $\mathbf{E}(\{m_i\}; \pi/2)$. The application of this theorem requires that

(1) the operator S maps $\mathbf{E}(\{m_i\}; \pi/2)$ into itself or equivalently

$$\mathbf{E}(\{m_i\}; \pi/2) \supseteq S(\mathbf{E}(\{m_i\}; \pi/2)); \quad (16)$$

(2) $\mathbf{E}(\{m_i\}; \pi/2)$ is closed and non-empty for the metric topology associated with the distance $d(\{u_i\}, \{v_i\}) = \text{Sup}_i |u_i - v_i|$;

(3) transformation S is contracting in $\mathbf{E}(\{m_i\}; \pi/2)$, which means that there exists a positive constant χ strictly smaller than 1, such that for any couple of points $\{u_i\}$ and $\{v_i\}$ in the domain $\mathbf{E}(\{m_i\}; \pi/2)$, we have

$$d(S(\{u_i\}), S(\{v_i\})) < \chi d(\{u_i\}, \{v_i\}). \quad (17a)$$

Since the spectral norm of the Jacobian linear operator $\mathbf{J} = \nabla S$ is strictly smaller than $4 [k^2 - (b+2)^2 \pi^2]^{-1/2} = \kappa(k, b)$ for any point in the considered domain $\mathbf{E}(\{m_i\}; \pi/2)$, then when

$$k > [16 + (b+2)^2 \pi^2]^{1/2} > (b+2)\pi, \quad (17b)$$

condition (17a) is fulfilled for $\chi = \kappa(k, b) < 1$.

Therefore, when condition (17b) is fulfilled, this theorem yields that transformation S has a fixed point and only one in the domain $\mathbf{E}(\{m_i\}; \pi/2)$. Equivalently, eq. (3a) has a unique solution $\{u_i\}$ in this domain. In practice, the sequence of configurations $\{u_i^{(n)}\}$ which is recursively defined as $\{u_i^{(n)}\} = S(\{u_i^{(n-1)}\})$ converges uniformly to $\{u_i\}$ for any initial configuration $\{u_i^{(0)}\}$ chosen in $\mathbf{E}(\{m_i\}; \pi/2)$.

In addition, since the transformation S is continuous with respect to the parameter k , a complementary theorem (ref. [13] p. 18) allows one to prove that the fixed point $\{u_i(k)\}$ is also a continuous function of this parameter k . (Let us consider an arbitrary constant $k'_0 > k_0 = [16 +$

^{#1}For the Banach fixed point theorem, see for example ref. [13], p. 17, theorem 1.A and p. 19 proposition 1.2.

$(b + 2)^2 \pi^2]^{1/2}$, then for $k \geq k'_0$, the contraction parameter is uniformly bounded as $0 < \kappa(k) \leq \kappa(k'_0) < 1$, which is the hypothesis required for the application of this complementary theorem.) Thus, $\{u_i(k)\}$ is a continuous function of k on the open interval $]k_0, \infty[$. Since for $k = \infty$ (or $C = 0$), the solution $E(\{m_i\}; \pi/2)$ to eq. (3a) is $m_i \pi$, property (8c) is fulfilled.

When the integer sequence $\{m_{i+1} - m_i\}$ is periodic with period s and when condition (11) is fulfilled, the intersection $\mathcal{P}_s \cap E(\{m_i\}; \pi/2)$ of the space \mathcal{P}_s of configurations $\{u_i\}$ such that $\{m_{i+1} - m_i\}$ is periodic with the same period s with the domain $E(\{m_i\}; \pi/2)$ is invariant by transformation S . Then, condition (8b) for the existence of a fixed point for transformation S in this domain $\mathcal{P}_s \cap E(\{m_i\}; \pi/2)$ still hold. The fixed point configuration of S corresponds to an s -periodic cycle of the standard map \mathcal{S} and is also the unique fixed point configuration of S in the full domain $E(\{m_i\}; \pi/2)$. The associate structure of the FK model is commensurate.

QED

Theorem 1 thus asserts that for large enough k , the standard map does possess chaotic trajectories where the sequence $\{p_i = u_i - u_{i-1}\}$ is unbounded. Taking the smallest possible value for b , which is 1, we obtain a lower bound for k :

$$k > (16 + 9\pi^2)^{1/2} \cong 10.24. \tag{18a}$$

When (18a) is fulfilled, for any sequence of integers $\{m_i\}$ (which may be random or not) such that

$$|m_{i+1} + m_{i-1} - 2m_i| \leq 1, \tag{18b}$$

there exists a *unique* trajectory $\{u_i\}$ in the standard map which “follows” the sequence πm_i . This bound is found to be much larger than the bound beyond which chaotic trajectories with unbounded momenta are numerically observed ($k > \text{Sup}_\zeta k_c(\zeta) \cong 0.96$, since it is believed that the unbounded chaotic trajectories exist as soon as all KAM tori disappeared). However, if there

exist unbounded chaotic trajectories for much lower values of k ($k > 1$), they have less entropy than those predicted by theorem 1.

In fact, for a finite value of k , any trajectory of the standard map belongs to some domain $E(\{m_i\}; \pi/2)$, but because of eq. (3a), the sequence $\{m_i\}$ must fulfill for all i

$$|m_{i+1} + m_{i-1} - 2m_i| \leq k/\pi + 2. \tag{18c}$$

This remark shows that for

$$\text{Sup}_i |m_{i+1} + m_{i-1} - 2m_i| \geq 2,$$

the trajectories of the standard map which existed uniquely in domains $E(\{m_i\}; \pi/2)$ for large k cannot exist anymore when k becomes too small.

The lower bound (18a) in k for the existence of chaotic trajectories can be decreased by considering sequences of integers $\{m_i\}$ fulfilling conditions which are more restrictive than (8a). But, then we loose also uniqueness in the domain $E(\{m_i\}; \pi/2)$, which means that there could exist several (and likely infinitely many) trajectories $\{u_i\}$ fulfilling the condition $|u_i - m_i \pi| < \pi/2$ for a given sequence $\{m_i\}$.

For example, we can require for $\{m_i\}$ that the sequence $|m_{i+1} + m_{i-1} - 2m_i| = b_i$ contains many zeros and that the consecutive non-zero values of b_i are obtained for indices i_n which are distant enough, that is $|i_{n+1} - i_n| \geq s$, where s is a given number. In physical terms, this condition means that the bending of the chain cannot be too large. Thus we obtain a second theorem which partially completes the results of theorem 1.

Theorem 2. δ being an arbitrary number between 0 and $\pi/2$, for

$$k > \text{Max} \left(\frac{2\delta + (4\delta^2 + \pi^2)^{1/2}}{\sin \delta}, \frac{4}{\cos \delta} \right) \tag{19}$$

there exist unbounded chaotic trajectories in the standard map. More precisely, there exists an integer $s(k)$ such that for any arbitrary sequence

of integers $\{m_i\}$ fulfilling

$$|m_{i+1} + m_{i-1} - 2m_i| = b_i \leq 1 \quad (20a)$$

and

$$b_i b_{i+n} = 0 \quad \text{for all } i \text{ and all } |n| < s(k) \quad (20b)$$

there exists a trajectory of the standard map such that $|u_i - m_i\pi| < \delta$.

As for theorem 1, when the sequence of integers $\{\sigma_i = m_{i+1} - m_i\}$ is periodic with period s ($\sigma_{i+s} = \sigma_i$ for all i), the corresponding trajectory $\{u_i\}$ has the same period s ($u_{i+s} = u_i \bmod 2\pi$ and $p_{i+s} = p_i$ for all i).

The proof of this theorem is similar to that of theorem 1 but becomes more technical. It is given in the appendix. The smaller bound for k is obtained when δ fulfills

$$\frac{2\delta + (4\delta^2 + \pi^2)^{1/2}}{\sin \delta} = \frac{4}{\cos \delta}, \quad (21a)$$

which yields the special value

$$\delta \cong 0.948 \quad (21b)$$

and the lower bound for k ,

$$k > 6.86. \quad (21c)$$

Other conditions on the integers $\{m_i\}$ concerning the distribution of discommensurations can be found by reducing the k lower bound but require more complicated proofs with asymmetric domains. These calculations become tedious and non-essential. They are not presented here.

3.1. Ballistic trajectories in the standard map^{#2}

Theorems 1 and 2 have interesting and surprising consequences because they predict the existence of ballistic trajectories when k is large

^{#2}Some of these ballistic trajectories were already known by Chirikov [1].

enough. A ballistic trajectory in the standard map is a trajectory where the momentum p_i diverges linearly as a function of time i . For having such a ballistic trajectory, it suffices to choose a sequence of integers $\{m_i\}$ which diverges as i^2 for $i \rightarrow \infty$. For example, with condition (8a) and $b = 1$, take

$$m_{i+1} + m_{i-1} - 2m_i = 1$$

or

$$m_i = \frac{1}{2}i(i-1) + ip + q$$

(p and q are arbitrary integers). For more restrictive conditions as in theorem 2, take

$$m_i = \text{Int}[i(i-1)/2s] = ip + q$$

(Int means the integer part of). The associated momentum $p_i = u_i - u_{i-1}$ then grows linearly as i goes to infinity.

Thus for large enough k , there exist trajectories in the standard map with a uniform acceleration! Many other kinds of unbounded trajectories with a non-diffusive behavior can be built. However, it is clear that for most sequences of integers $\{m_i\}$ (in probability) fulfilling (8a), the behavior of the associated $\{u_i\}$ will be a diffusive motion.

Let us end this section with another application of the concept of anti-integrability used for theorems 1 and 2. It is the proof of existence of trajectories asymptotic to these chaotic trajectories

Theorem 3. Let us consider a trajectory $\{u_i\}$ of the standard map obtained within the conditions of theorem 1 or 2 for the integer sequence $\{m_i\}$. Then, for v an arbitrary given number in the interval $m_0\pi - \delta < v < m_0\pi + \delta$, there exists a unique trajectory $\{v_i(v)\}$ of the standard map (solution of eq. (3) for $i > 0$) which is a uniformly continuous function of v such that

$$v_0 = v, \quad (22a)$$

$$m_i\pi - \delta < v_i < m_i\pi + \delta \quad \text{for } i > 0. \quad (22b)$$

In addition, we have

$$v_i(u_0) = u_i. \quad (22c)$$

Proof. We now consider the space $E_+(\{m_i\}; \delta)$ of semi-infinite configurations $\{v_i\}$ with $v_0 = v$ and where i only varies from 0 to $+\infty$ instead of the space of infinite configurations $\{v_i\}$ with $-\infty < i < +\infty$ as for the proof of theorems 1 and 2. Then, in this semi-infinite system, transformation S is defined by (9b) for $i > 0$ only and the first value $u_0^{(1)} = u_0^{(0)} = v$ remains fixed. This transformation S_+ fulfills the conditions for the application of the Banach fixed point theorem, with the same bounds on k as in the infinite system. Thus, when either the conditions of theorem 1 or those of theorem 2 on the sequence $\{m_i\}$ are fulfilled for $i > 0$ only, together with condition (8b) or (19) respectively, the sequence of configurations $\{u_i^{(n+1)}\} = S_+(\{u_i^{(n)}\})$ converges to a fixed point configuration $\{v_i\}$ defined for $i \geq 0$. This configuration is the unique solution of eq. (3a) for $i > 0$ in the domain by (22). It can be continued for negative i by using eq. (3a). Since the operator S_+ depends continuously on the parameter v , the limit configuration $\{v_i(v)\}$ continuously depends on v . When, $v = u_0$, the solution $\{v_i\}$ which is unique, is necessarily identical to $\{u_i\}$ for $i \geq 0$.

QED

The initial point $\{v, q\}$ generates by the standard map \mathcal{S} , the trajectory associated with the configuration $\{v_i(v)\}$. As a consequence of this theorem 3, the locus of this initial point is a continuous 1D manifold called \mathcal{L}_0 . We now anticipate on proposition 3 proven in the next section. This proposition establishes that \mathcal{L}_0 is a piece of the contracting manifold $\mathcal{V}^-(\{u_0, p_0\})$ of the trajectory $\{u_i\}$ at the initial point $\{u_0, p_0\}$. Let us recall that by definition, this manifold $\mathcal{V}^-(\{u_0, p_0\})$ is the locus of the initial points $\{v, q\}$ in phase space of the standard map with a trajectory $\{v_i\}$ asymptotic to the trajectory $\{u_i\}$ for $i \rightarrow \infty$, that is such that $\lim_{i \rightarrow +\infty} |v_i - u_i| = 0$.

We also define \mathcal{L}_n as the manifold of the standard map defined by changing the origin from

site 0 to site n . (More precisely, we fix $v_n = v$ with $m_n\pi - \delta < v < m_n\pi + \delta$ and we consider the semi-infinite configurations $\{v_i\}$ for $i > n$. The locus of the points $\{v_n, q_n\}$ which is found according to theorem 3 is a manifold called \mathcal{L}_n .) For $n \geq 0$, it appears that $\mathcal{S}^{-n}(\mathcal{L}_n) \supset \mathcal{S}^{-n-1}(\mathcal{L}_{n-1}) \supset \dots \supset \mathcal{L}_0$. The definition of $\mathcal{V}^-(\{u_0, p_0\})$ implies that for any x in $\mathcal{V}^-(\{u_0, p_0\})$, there necessarily exists n such that $\mathcal{S}^n(x) \in \mathcal{L}_n$. Then, we have $\mathcal{V}^-(\{u_0, p_0\}) = \bigcup_{n \geq 0} \mathcal{S}^{-n}(\mathcal{L}_n)$.

Of course, a theorem similar to theorem 3 can be established for semi-infinite configurations with negative i . The dilating manifold $\mathcal{V}^+(\{u_0, p_0\})$ is the set of initial points $\{v, q\}$ in the phase space of the standard map such that for negative times i , the trajectory $\mathcal{S}^i(\{v, q\})$ is asymptotic to the trajectory $\{u_i, p_i\}$ for $i \rightarrow \infty$. Its existence is thus proven identically.

Particularly, when the sequence $\{m_i\}$ is periodic with period s , these manifolds $\mathcal{V}^+(\{u_0, p_0\})$ and $\mathcal{V}^-(\{u_0, p_0\})$ become the well-known dilating and contracting sheets of the fixed point $\{u_0, p_0\}$ of the standard map \mathcal{S}^s at power s .

4. Gap parameter and chaotic trajectories

In section 3, we proved the existence of unbounded chaotic trajectories close enough to the anti-integrable limit. However, the fact that there exist chaotic trajectories which cannot be found as attractive fixed points of transformation S defined in some given domain, does not imply the absence of fixed point configurations (i.e. solutions of eq. (3a)) in this domain. Our purpose in this section is to analyze by continuity for smaller values of k , the behavior of the chaotic trajectories of the standard map which originate from those which exist at the anti-integrable limit. We shall discuss whether these trajectories fill up the full chaotic area of the standard map or not.

For that purpose, it is useful to introduce the concept of gap parameter (GP) for a trajectory in the standard map. This concept is different from

the concept of the Lyapunov coefficient although it has some connection with it: A non-zero GP implies a non-zero Lyapunov coefficient when it is defined, but the reverse is not true. We prove that the trajectories with a non-vanishing GP depend continuously on the model parameter k (without any bifurcation). We show that the trajectories predicted by theorem 1 or 2 have finite non-zero GP. At the present stage, our study is not complete and we propose two conjectures concerning the GP and the chaotic region in the standard map which suggest that although the chaotic trajectories predicted by theorem 1 or 2 have zero Lebesgue measure in the phase space, their closure may have a finite measure.

Let us first explain the concept of GP. A stationary configuration $\{u_i\}$ (fulfilling eq. (3a)) is said to be configurationally stable [3–5] when the quadratic form defined by its phonon dynamical matrix $\mathbf{M}(\{u_i\})$ is positive or zero. Then, the eigenvalues of $\mathbf{M}(\{u_i\})$ which are positive or zero are physically the phonon square frequencies of the configuration (with the assumption that the atoms with coordinates $\{u_i\}$ have unit mass). Extending this concept to arbitrary sequences $\{u_i\}$ (which do not necessarily correspond either to stable configurations or to a stationary one), we rename this dynamical matrix of the configurations as quadratic expanded action matrix (QEAM) for the associated configuration $\{u_i\}$.

Definition 2. The “quadratic expanded action matrix” $\mathbf{M}(\{u_i\}) = \{M_{i,j}\}$ of a given trajectory $\{u_n\}$ is defined as the (infinite) matrix of second order derivatives of the action $\Phi(\{u_n\})$ of this symplectic map,

$$M_{i,j} = \frac{\partial^2 \Phi(\{u_n\})}{\partial u_i \partial u_j}. \quad (23a)$$

The GP Δ of this trajectory is the (dimensionless) coefficient defined as

$$\Delta = \frac{1}{\|\mathbf{M}^{-1}\| \|\mathbf{M}\|}. \quad (23b)$$

In the case where $\|\mathbf{M}^{-1}\|$ is not defined (i.e. infinite) Δ is set to be zero. These definitions extend identically for configurations $\{u_i\}$ which are not extrema of eq. (3a) and thus which do not correspond to trajectories of the standard map.

We use the standard spectral norm $\|\mathbf{M}\|$ for \mathbf{M} (which is above defined for $\|\mathbf{J}\|$ in (15a)). For the symmetric matrix \mathbf{M} , this definition is equivalent to $\Delta = \lambda_{\max}/\lambda_{\min}$, where λ_{\max} is the supremum of the spectrum modulus of $\mathbf{M}(\{u_i\})$ and λ_{\min} is the infimum of this spectrum modulus. Since it is easy to show that $\lambda_{\max} = \|\mathbf{M}\|$ has an upper bound $4C + \lambda = C(4 + k)$ independent of the choice of the trajectory $\{u_i\}$, the condition that zero belongs to the spectrum of \mathbf{M} is equivalent to $\lambda_{\min} = 1/\|\mathbf{M}^{-1}\| = 0$ and to $\Delta = 0$.

For sake of clarity, let us discuss with some details the connection between the Lyapunov coefficient and the gap parameter. For a given trajectory, the Lyapunov coefficient yields an *average* information for the given trajectory $\{u_i\}$. It is defined by the behavior of the time-ordered matrix product

$$\mathbf{L}_N = \prod_{i=0}^{N-1} \mathbf{K}_i, \quad (24a)$$

where \mathbf{K}_i is the Jacobian derivative of the considered map at the trajectory point reached at time i . In the case of the standard map \mathcal{S} , we have

$$\mathbf{K}_i = \begin{pmatrix} 1 + k \cos u_i & 1 \\ k \cos u_i & 1 \end{pmatrix}. \quad (24b)$$

The Oseledec multiplicative ergodic theorem [14] asserts that the set of trajectories for which the limit

$$\Lambda = \lim_{N \rightarrow +\infty} (\mathbf{L}_N^t \mathbf{L}_N)^{1/2N} \quad (24c)$$

is defined has full measure with respect to the preserved Lebesgue measure. (\mathbf{L}_N^t is the transposed matrix of \mathbf{L}_N .) The matrix Λ is both

symmetric and symplectic and thus has real positive eigenvalues. The logarithm $\lambda = \ln \lambda_m$ of the largest eigenvalue λ_m of \mathbf{A} , which is necessarily larger than or equal to 1, is positive or zero and is called Lyapunov coefficient.

In the case of the standard map, this Lyapunov coefficient can be related to the measure density of states of the QEAM. For any initial

$$\begin{pmatrix} \varepsilon_0 \\ \pi_0 \end{pmatrix}$$

the recursive sequence of vectors

$$\begin{pmatrix} \varepsilon_i \\ \pi_i \end{pmatrix} = \mathbf{K}_i \begin{pmatrix} \varepsilon_{i-1} \\ \pi_{i-1} \end{pmatrix}$$

yields a solution of

$$(\varepsilon_{i+1} + \varepsilon_{i-1} - 2\varepsilon_i) - k \cos u_i \varepsilon_i = y\varepsilon_i \quad \text{for } y = 0 \quad (25a)$$

or

$$\mathbf{M}\boldsymbol{\varepsilon} = 0 \quad \text{with } \boldsymbol{\varepsilon} = \{\varepsilon_i\}. \quad (25b)$$

The Thouless formula [15] relates the generalized Lyapunov coefficient $\gamma(y)$ (defined as in (24)) but with

$$\mathbf{K}_i(y) = \begin{pmatrix} 1 + k \cos u_i + y & 1 \\ k \cos u_i + y & 1 \end{pmatrix} \quad (25c)$$

to the measure “density of states” $d\mu(x)$ of the QEAM of the trajectory $\{u\}$ defined as

$$d\mu(x) = \lim_{N-N' \rightarrow \infty} \frac{1}{N-N'} \sum_{i=1}^{N-N'} \delta(x - x_i), \quad (26a)$$

where $\delta(x - x_i)$ are the Dirac measures at the eigenvalues of x_i of the finite truncated QEAM $\mathbf{M}_{N,N'}$ with the coefficients $M_{i,j}$ of \mathbf{M} for $N' < i \leq N$ and $N' < j \leq N$). If this limit is well defined for $N - N' \rightarrow \infty$, the Thouless formula yields the Lyapunov coefficient at “energy” y

$$\gamma(y) = \int_{-\infty}^{+\infty} \ln|y - x| d\mu(x), \quad (26b)$$

which is always positive (or possibly zero if y belongs to the spectrum). As we noticed in ref. [16], the Thouless formula is exact when y does not belong to the spectrum of \mathbf{M} but may be wrong for y in the spectrum. For $y = 0$, $\gamma(0)$ is the Lyapunov coefficient of the associated trajectory $\{u_i, p_i\}$ in the standard map.

When the GP is not zero, zero does not belong to the spectrum of \mathbf{M} and formula (26b) can be applied for $y = 0$. It comes out that the Lyapunov coefficient of the associated trajectories in the standard map is well defined and strictly positive when the measure “density of states” $d\mu(x)$ is well defined. It is easy to show that the momenta

$$\mathcal{M}_n = \int x^n d\mu(x)$$

of this positive measure, are just equal to

$$\lim_{N-N' \rightarrow \infty} \frac{1}{N-N'} \text{Tr } \mathbf{M}_{N,N'}^n,$$

which is the average of the diagonal terms of \mathbf{M}^n the QEAM of the trajectory of power n . According to the ergodic theorem, these momenta are well defined for all n and for most trajectories of the standard map. They determine uniquely the measure $d\mu(x)$.

However, there exist sequences $\{m_i\}$ such that theorem 1 yields a trajectory $\{u_i, p_i\}$ for which the measure $d\mu(x)$ and the Lyapunov coefficient are both undefined. For example for k large enough, we can construct a sequence $\{\sigma_i\} = \{m_{i+1} - m_i\}$ with two blocks $\{0\} = \{a\}$ and $\{01\} = \{b\}$ arranged intermittently as

$$\{abaabbbaaaabbbbbaaaaaaabbabbbbbb \\ aaaaaaaaaaaaaaaaaa \dots\}.$$

In this sequence of symbols, the length of each constant subsequence of symbols a (or b) is twice those of the previous one. Theorem 1 associates the corresponding sequence $\{m_i\}$ with a trajectory which has two accumulation orbits. The first one

is those of the fixed point determined by $\{\sigma_i\} = \{00000000\dots\}$ for all i and the second one is those of the two-periodic cycle determined by $\{01010101\dots\}$. In general, the Lyapunov coefficients (as well as the density of state $d\mu(x)$) are not the same for both accumulation orbits. Then, it is clear that the sequence of matrices (24c) which determines the Lyapunov coefficient has also two accumulation points, which then shows that the Lyapunov coefficient cannot be defined.

Unlike the Lyapunov coefficient, the gap parameter is defined for all trajectories because the GP is not defined as an average but as a uniform bound for the whole spectrum and thus for the trajectory. For having a zero GP, it is sufficient that the QEAM $\mathbf{M}(\{u_i\})$ of a trajectory $\{u_i\}$ only possesses a single localized eigenstate at zero eigenenergy while the distance of the remainder of the spectrum to the origin does not vanish. In that situation, the support of the measure $d\mu(x)$ "average density of states" is smaller than that of the spectrum of the QEAM and has a non-zero gap. Then, the Thouless formula shows that the Lyapunov coefficient cannot vanish. This situation occurs for example for the trajectories of a separatrix. When the dilating and contracting manifolds of an hyperbolic fixed point H merge into a smooth separatrix, the Lyapunov coefficients of these trajectories are strictly positive and equal those of the hyperbolic fixed point H. Otherwise, since these trajectories form a continuum, their QEAMs exhibit an isolated zero eigenvalue [4, 5] corresponding physically to the translation mode of this configuration. However, although this example might suggest that this situation has probability zero to occur, we believe that most trajectories with non-vanishing Lyapunov coefficients have zero GPs! (see conjectures 1, 2 and their explanations in the next). In summary:

The condition for a trajectory of having a non-vanishing gap parameter Δ is a much stronger condition than having a non-vanishing Lyapunov coefficient: A zero Lyapunov coefficient implies a

zero GP, but a zero GP does not imply a zero Lyapunov coefficient.

Thus, it is interesting to note that the trajectories given in theorem 1 or 2 fulfill the strongest property:

Proposition 1. The gap parameter Δ of the trajectory of the standard map obtained as an attractive fixed point of the operator S defined by eq. (9) is strictly positive.

As a result, when they are defined, the Lyapunov coefficients of these trajectories are strictly positive. (In the exceptional cases where they would not be defined, it can be proven^{#3} that

$$\liminf_{N \rightarrow \infty} \|\mathbf{L}_N\|^{1/N} > 1,$$

which confirms that the exponential instability of these trajectories always exist.) Another consequence of this proposition is that any trajectory dense on a KAM tori cannot be obtained as an attractive fixed point of transformation S since it has been shown that $\Delta = 0$ for these trajectories (e.g. see ref. [5], ch. 6).

Proof. For the FK model (1)

$$M_{i,i} = C(1 + k \cos u_i), \quad (27a)$$

$$M_{i,i+1} = M_{i,i-1} = -C, \quad (27b)$$

$$M_{i,j} = 0 \quad \text{for } |i-j| > 1. \quad (27c)$$

Since $\{u_n\}$ is an attractive fixed point for the operator S in some domain $\mathbf{E}(\{m_i\}; \{\delta_i\})$, the Jacobi operator \mathbf{J} defined by (14) is at the fixed point

$$J_{i,i} = -\frac{2}{k \cos u_i}, \quad (28a)$$

$$J_{i,i+1} = J_{i,i-1} = \frac{1}{k \cos u_i} \quad (28b)$$

^{#3}This result can also be obtained as a consequence of the more general proposition 3 in ref. [11].

and

$$J_{i,j} = 0 \quad \text{for } |i-j| > 1. \quad (28c)$$

The QEAM can be written as

$$\mathbf{M} = \mathbf{C}\mathbf{D}(\mathbf{1} - \mathbf{J}), \quad (29a)$$

where \mathbf{D} is a diagonal matrix defined as $D_{i,j} = k \cos u_i \delta_{i,j}$. Thus, we have

$$\|\mathbf{M}\| \leq Ck(1 + \|\mathbf{J}\|) \sup_i |\cos u_i|. \quad (29b)$$

Since

$$\|\mathbf{J}\| < 1$$

and

$$\inf_i |\cos u_i| \geq \left(1 - \frac{(b+2)^2 \pi^2}{k^2}\right)^{1/2} > 0,$$

$(\mathbf{1} - \mathbf{J})$ and \mathbf{D} are both invertible. So

$$\mathbf{M}^{-1} = \frac{1}{C}(\mathbf{1} - \mathbf{J})^{-1}\mathbf{D}^{-1} = \frac{1}{C} \left(\sum_{n=0}^{\infty} \mathbf{J}^n \right) \mathbf{D}^{-1}, \quad (29c)$$

which yields

$$\|\mathbf{M}^{-1}\| \leq \frac{1}{Ck} \frac{1}{\inf_i |\cos u_i|} \frac{1}{1 - \|\mathbf{J}\|}. \quad (29d)$$

Then, the GP fulfills the inequality

$$\Delta \geq \frac{1 - \|\mathbf{J}\|}{1 + \|\mathbf{J}\|} \frac{\inf_i |\cos u_i|}{\sup_i |\cos u_i|} > 0, \quad (30)$$

which proves the proposition. This property does not imply the configurational stability of configuration $\{u_i\}$ but it is easy to prove:

Corollary. The stationary configurations which are obtained as attractive fixed points of operator S , in the domains $E(\{m_i\}; \pi/2)$ determined by sequences $\{m_i\}$ of even integers m_i , are metastable configurations $\{u_i\}$ (i.e. the spectrum

of the dynamical matrix (or QEAM) is strictly positive).

For $C = 0$, it is straightforward that this stability is obtained if and only if all the integers of the sequence $\{m_i\}$ are even (the uncoupled atoms are at the bottom of the wells of the periodic potential). This property is preserved by continuity. Indeed, since the configuration $\{u_i(k)\}$ is uniformly continuous with respect to k , the spectrum of its QEAM \mathbf{M} also depends continuously on k . Since for $C = 0$, this spectrum is included in the positive part of the real axis, it will remain entirely in this positive part because, according to the above corollary, zero cannot belong to the spectrum of \mathbf{M} .

For large k , the ground states of the FK model belong to this class of metastable states. From their properties in refs. [3–5], it is straightforward to check that for large k , they are obtained from the sequences of even numbers $m_i = 2 \text{Int}[\frac{1}{2}(i\zeta + \alpha)]$, where ζ is the commensurability ratio (rational or irrational) and α is an arbitrary phase.

It is now useful to give an essential property of the trajectories with non-zero GP which is that these trajectories depend continuously in a *unique manner* on the model parameter k in some open interval.

Proposition 2. For some value k_0 of the model parameter k , let $\{u_i\}$ be a trajectory of the standard map (fulfilling eq. (3a)) with a strictly positive Δ . Then

(1) this trajectory $\{u_i\}$ cannot be the *uniform limit* of a sequence of other trajectories obtained for the same parameter k_0 ;

(2) for k in some infinite open interval containing k_0 , there exists a trajectory of the standard map $\{v_i(k)\}$ (fulfilling eq. (3a) for the parameter k) which depends continuously and uniformly on k and such that $\{v_i(k_0)\} = \{u_i\}$.

When zero belongs to the spectrum of the dynamical matrix, it might become ambiguous to

define the stationary configuration as a continuous function of the model parameter k because this configuration may bifurcate into many other configurations. In those cases, for following a stationary configuration as a continuous function of k , it is necessary to introduce an extra condition on the trajectory which is for example a fixed periodicity (for the standard bifurcations of periodic cycles) or both a fixed rotation number and a fixed phase (for the stationary configurations corresponding to a KAM tori). For non-periodic and non-quasiperiodic stationary configurations, there are no precise rules.

Thus, in physical terms, this proposition asserts that these stationary configurations are not bifurcating, which means that they can be followed unambiguously as a continuous function of the model parameter k at least on some local finite interval.

Proof. Let us assume that there exists a sequence of stationary configurations $\{v_i^{(n)}\}$ which converges uniformly to $\{u_i\}$ for $n \rightarrow \infty$. Since configurations $\{u_i\}$ and $\{v_i^{(n)}\}$ both fulfill eq. (3a), one obtains by difference

$$(u_{i+1} - v_{i+1}^{(n)}) + (u_{i-1} - v_{i-1}^{(n)}) - 2(u_i - v_i^{(n)}) - k \cos \xi_i^{(n)} (u_i - v_i^{(n)}) = 0, \quad (31)$$

where $\xi_i^{(n)}$ is some real number which belongs to the interval determined by u_i and $v_i^{(n)}$. Thus for $n \rightarrow \infty$, $\{\xi_i^{(n)}\}$ converges uniformly to $\{u_i\}$.

Because of the definition of the uniform convergence, $|u_i - v_i^{(n)}|$ is bounded by some finite positive number η_n for all i (and $\lim_{n \rightarrow \infty} \eta_n = 0$). Then, the sequence $\varepsilon_i^{(n)} = \{u_i - v_i^{(n)}\}$ is a bounded eigenvector for the matrix $\mathbf{M}(\{\xi_i^{(n)}\})$ corresponding to a zero eigenvalue. Since for $n \rightarrow \infty$, $\{\xi_i^{(n)}\}$ converges uniformly to $\{u_i\}$, $\mathbf{M}(\{\xi_i^{(n)}\})$ converges uniformly to $\mathbf{M}(\{u_i\})$. Then, since zero belongs to the spectrum of $\mathbf{M}(\{\xi_i^{(n)}\})$, zero should also be the spectrum of $\mathbf{M}(\{u_i\})$. The fact is in contradiction with the initial assumption. As a result, the stationary configuration $\{u_i\}$ cannot be the

uniform limit of stationary configurations which proves the first assertion of this proposition. We now prove the second assertion:

Let us assume that the derivative $\varepsilon = \{\varepsilon_i(k)\} = \{dv_i(k)/dk\}$ with respect to the parameter k exists, and thus that eq. (3) can be differentiated. One obtains the equation

$$\varepsilon_{i+1}(k) + \varepsilon_{i-1}(k) - 2\varepsilon_i(k) - k \cos v_i(k) \varepsilon_i(k) = \sin v_i(k), \quad (32a)$$

which can be written as

$$\mathbf{M}(\{v_i\}; k) \varepsilon = CV, \quad (32b)$$

where $\mathbf{M}(\{v_i\}; k)$ is the QEAM of trajectory $\{v_i\}$ with the parameter k defined by (32) and V is the vector with components $\{\sin v_i\}$. Since the GP $\Delta(\{u_i\}; k_0)$ is strictly positive, $\mathbf{M}(\{u_i\}; k_0)$ is invertible. Since $\mathbf{M}(\{v_i\}; k)$ is a uniformly continuous function of the vector $\{v_i\}$ and of the parameter k , the GP $\Delta(\{v_i\}; k)$ remains strictly positive for $(\{v_i\}; k)$ in some neighborhood D of the point $(\{u_i\}; k_0)$. In that domain, $\mathbf{M}(\{v_i\}; k)$ is invertible and its inverse $\mathbf{M}^{-1}(\{v_i\}; k)$ is a uniformly continuous function of $(\{v_i\}; k)$ in some open domain D' included in D . Thus, the set of differential equations

$$\frac{dv_i(k)}{dk} = \sum_j \mathbf{M}_{i,j}^{-1}(\{v_n\}; k) \sin v_j \quad (32c)$$

can be integrated in this domain D' and yields a solution $\{v_i(k)\}$. Choosing the initial condition $\{v_i(k_0)\} = \{u_i\}$ for $k = k_0$, this solution $\{v_i(k)\}$ fulfills eq. (32a) with $\varepsilon_i(k) = dv_i(k)/dk$ and by integration over k , it also fulfills eq. (3). Thus, the solution $\{v_i(k)\}$ is a trajectory for the parameter k which depends uniformly and continuously on $k \in I$, where I is some open interval $]k_0^-, k_0^+[$, which contains k_0 .

This trajectory is no longer defined or uniquely defined when parameter k reaches a value k_0^- or k_0^+ where the GP vanishes. Thus, k_0^- and k_0^+ correspond to ‘‘generalized’’ bifurcation points

for the stationary configuration $\{v_i(k)\}$ (or for the associated trajectory on the standard map).

QED

For the configurations which are predicted to exist by theorem 1 or 2, k_0^+ is equal to infinity. The commensurate ground state of the FK model considered here were found numerically to be always pinned, that is with a finite gap in their phonon spectrum and consequently a non-zero GP Δ . Thus, with this assumption, the corresponding trajectories in the standard map which are periodic cycles is a well-defined continuous function of k for k belonging to the whole positive axis ($k_0^- = 0; k_0^+ = +\infty$). However, these configurations are attractive fixed points of operator S only while they remain in the initial domain $E(\{m_i\}; \pi/2)$, where they were for $C = 0$ or $k = \infty$.

By contrast, the incommensurate ground states with incommensurability ratio ζ should "bifurcate" at $k_c(\zeta) = k_0^-$ ($k_0^+ = +\infty$), that is at the TBA since for $k \leq k_c(\zeta)$, the phonon spectrum is gapless, while it exhibits a finite gap for $k_c(\zeta) < k$. Hyperbolic periodic cycles "bifurcate" when they become elliptic etc.

For k outside of the closed interval $]k_0^-, k_0^+[$, there may exist either no stationary configuration or infinitely many stationary configurations $\{v_i(k)\}$ which continuously depend on k (for the uniform topology) and fulfills $v_i(k_0) = u_i$.

Now, we establish the proposition which completes theorem 3:

Proposition 3. Under the same conditions as for theorem 3, two stationary configurations $\{v_i\}$ and $\{w_i\}$ in $E_+(\{m_i\}; \delta)$ with $v_0 = v$ and $w_0 = w$ fulfill $\lim_{i \rightarrow +\infty} |v_i - w_i| = 0$. Particularly, we have $\lim_{i \rightarrow +\infty} |v_i - u_i| = 0$.

In addition, according to the remarks at the beginning of this section, when the Lyapunov exponent γ of trajectory $\{u_i\}$ is defined, $|v_i - w_i|$ behaves exponentially as $\exp(-\gamma i)$ for $i \rightarrow +\infty$.

Proof. Let us consider the semi-infinite QEAM, $\mathbf{M}_+ = \{M_{i,j}\}$ defined as above but for $i \geq 1$ and

$j \geq 1$ only for the configurations in $E_+(\{m_i\}; \delta)$. With similar proofs as in the infinite case (proposition 1) which essentially use the fact that transformation S is contracting, zero does not belong to the spectrum of \mathbf{M}_+ and $\|\mathbf{M}_+^{-1}\|$ has a uniform upper bound B for all configurations in $E_+(\{m_i\}; \delta)$.

Let us assume that $\{v_i(v)\}$, which is continuous with respect to $v = v_0$, is also differentiable. By differentiating (3a), we readily obtain

$$\frac{dv_i(v)}{dv} = -\mathbf{M}_{+,i,1}^{-1}(\{v_i(v)\}). \tag{33a}$$

Since $\|\mathbf{M}_+^{-1}\|$ is finite, this differential equation is integrable. It has a solution which fulfills eq. (3a) for $i > 0$ and yields $v = v_0$. Since this solution is unique in $E_+(\{m_i\}; \delta)$, this result proves that $\{v_i(v)\}$ was indeed differentiable. More precisely, we have

$$\left\| \left\{ \frac{dv_i(v)}{dv} \right\} \right\| = \left(\sum_{i \geq 1} \left| \frac{dv_i(v)}{dv} \right|^2 \right)^{1/2} \leq \|\mathbf{M}_+^{-1}\| \leq B \tag{33b}$$

and by integrating on v between u_0 and w , we have

$$\left(\sum_{i \geq 1} [v_i(w) - u_i]^2 \right)^{1/2} < B|w - u_0|. \tag{33c}$$

Since the sum in the first member (33c) is finite, we necessarily have $\lim_{i \rightarrow \infty} |v_i - u_i| = 0$.

QED

The existence of chaotic unbounded trajectories in the standard map for large enough parameter k is now rigorously proven but the configurations which are proven to exist, have necessarily a finite GP. In addition, these trajectories are uniquely defined as uniformly continuous function of k in the domain in k where their GP is non-vanishing. In addition, they possess both contracting and

dilating manifolds. Concerning the role of these chaotic trajectories in the full set of chaotic trajectories of the standard map, we suggest the two following conjectures:

Conjecture 1. The Lebesgue measure of the full set of trajectories which have a strictly non-zero GP is zero.

Equivalently, we conjecture that the set of trajectories of the standard map with a zero GP, has the full Lebesgue measure. This conjecture could suggest that the set of chaotic trajectories which we have proven to exist, have a negligible role. But, the next conjecture suggests that this set could be a dense network inside the whole chaotic region (for example as the set of rational numbers inside the whole set of real numbers) and thus would be essential for understanding it.

Let us give our arrangements for this conjecture which are partially based on numerical observations. They could perhaps give the idea for a rigorous proof of this conjecture.

First, at the singular limit, $k = \infty$, each of these trajectories collapses onto a discrete set of points $\{u_i = m_i\pi; p_i = (m_i - m_{i-1})\pi\}$ which clearly has zero Lebesgue measure.

Second, for finite k , we remark that invariant KAM tori or “KAM islands” appear between the points of this discrete set. The trajectories in these KAM islands have a zero GP because the phase mode is an explicit eigen vector of their QEAM for the zero eigenvalue (see for example ref. [4]).

Third, we observed that the closed domain K filled by chaotic trajectories looks like a Cantor set (likely with finite measure?). The complementary set of this domain contains many KAM tori and islands which are dense at the border of K . These trajectories have zero GPs. These GPs Δ remain zero by continuity, for the orbits at the border of K which are limits of KAM orbits. The numerical observations suggest that most trajectories (in probability) can approach infinitely close to the border of this domain K where the

GP Δ is zero. Since as we mentioned above, the GP of a trajectory does not depend on its average behavior but is the lower bound of its full spectrum, the regions of the trajectories which approach closer and closer to the boarder of K should force the GP of these chaotic trajectories to be zero. Therefore we expect that most trajectories on domain K also have zero gap. The minimum distance d_I of a given trajectory $\{u_i, p_i\}$ with finite GP from the given KAM island I should not vanish. (Nevertheless, we expect that the lower bound of d_I for all KAM islands is likely zero!)

This conjecture suggests that the chaotic trajectories predicted by theorem 1 or 2 have zero measure in the standard map. However, if we define \bar{K} as the closure of the whole set trajectories $\{u_i(k)\}$ with a finite gap ($\Delta \neq 0$) which can be continued up to $k = +\infty$, it should contain trajectories with smaller and smaller GPs and by continuity trajectories with zero GPs.

We can call this set \bar{K} the main chaotic component. Note that for two-dimensional maps, it does not remain connected, which is certainly true at least when k becomes smaller than $\text{Sup}_\zeta k_c(\zeta)$. In that case, there may exist other chaotic components for example in between invariant curves of KAM islands. However, for $2d$ -dimensional maps with $d > 1$, a well-known conjecture due to Arnol’d suggests that there is a single chaotic component.

We expect when the gap parameter Δ of a trajectory becomes closer and closer to zero (for example by varying the model parameter k), its distance to the KAM islands could become smaller and smaller. See for example the continuous transformation (transition by breaking of analyticity) of the Cantorus which represents an incommensurate ground state of the Frenkel–Kontorowa model, into a KAM torus at $k = k_c(\zeta)$. Thus we suggest the following conjecture for the main component:

Conjecture 2. For the standard map and $k \neq 0$, the Lebesgue measure of \bar{K} (restricted on the

cylinder $\{u; p\}$ to the annulus $0 \leq p < 2\pi$) is finite non-zero.

If, this conjecture is true, it would suggest that many chaotic trajectories of the standard map (and may be all of them) obtained for large enough k , originate by continuity from the trivial chaotic trajectories obtained at an anti-integrable limit. (This may not be unique if there is several model parameter, i.e. the amplitude of the second harmonics in (1a).)

5. Extensions of the concept of anti-integrability: concluding remarks

The theorems for the anti-integrable limit were presented for sake of simplicity on the example of the standard map. On the same footing, similar theorems can be found for any dynamical system at its anti-integrable limit, the action of which takes the form

$$\Phi_{AI}(\{u_i\}) = \sum_i V(u_i). \tag{34a}$$

However, we need to assume that the behavior of $V(x)$ be not too much pathological. More precisely, we need the following properties:

(1) $V(x)$ is twice differentiable with continuous derivatives and has a finite or countable set of extrema on the set $\{x_\nu\}$ of values of x : $\nu \in \mathcal{A}$ which contains at least two extrema.

(2) There exist two positive constants δ and K such that for any extremal value x_ν and any x in the interval $[x_\nu - \delta, x_\nu + \delta]$, we have

$$\left| \frac{d^2V(x)}{dx^2} \right| > K. \tag{34b}$$

These conditions imply that the extrema of $V(x)$ be isolated and non-degenerate ($d^2V(x)/dx^2 \neq 0$). Close to the anti-integrable limit, the action of the dynamical system can be written for

example as

$$\Phi(\{u_i\}) = \sum_i V(u_i) + C \sum_i L(u_{i+1}, u_i), \tag{34c}$$

where $L(x, y)$ is also twice differentiable with continuous derivatives and C is the control parameter of the perturbation. We define the domain $\mathcal{E}(\{\nu_i\}; \delta)$ as the set of configurations $\{u_i\}$ fulfilling

$$|u_i - x_{\nu_i}| \leq \delta \quad \text{for all } i \tag{35a}$$

and the upper bound $B(\{\nu_i\}, \delta)$ on this domain

$$\begin{aligned} \text{Sup}_{i \in \mathbb{Z}, \{u_i\} \in \mathcal{E}(\{\nu_i\}; \delta)} & |\partial_2 L(u_{i+1}, u_i) + \partial_1 L(u_i, u_{i-1})| \\ & = B(\{\nu_i\}, \delta). \end{aligned} \tag{35b}$$

We denote the derivatives of $L(x, y)$ as

$$\begin{aligned} \partial_1 L(x, y) &= \frac{\partial L(x, y)}{\partial x}, \\ \partial_2 L(x, y) &= \frac{\partial L(x, y)}{\partial y}, \\ \partial_{11} L(x, y) &= \frac{\partial^2 L(x, y)}{\partial x^2}, \text{ etc.} \end{aligned}$$

Then, using the Banach fixed point theorem, we prove in a similar way as for the standard map:

For any arbitrary sequences of extrema (or symbols in \mathcal{A}) $\{\nu_i\} = \mathcal{A}^{\mathbb{Z}}$ ($i \in \mathbb{Z}$ and $\nu_i \in \mathcal{A}$) and for

$$C \leq \frac{K\delta}{B(\{\nu_i\}, \delta)} \tag{36a}$$

there exists a unique solution to the stationary equation

$$C(\partial_2 L(u_{i+1}, u_i) + \partial_1 L(u_i, u_{i-1})) + V'(u_i) = 0 \tag{36b}$$

in the domain $\mathcal{E}(\{\nu_i\}; \delta)$.

Of course, this theorem requires that the upper bound (35b) be finite. The resulting bound (36a) for C depends of the initial sequence of symbols $\{v_i\}$. As for the standard map, it is easy to prove that the GPs of these stationary configurations $\{u_i\}$ are strictly positive. Moreover, we also prove similarly that they have dilating and contracting manifolds.

Note, however, that in this generalized case, we do not prove necessarily the existence of chaotic trajectories $\{u_i\}$ with unbounded momenta close enough to the anti-integrable limit as for the standard map, but only of bounded chaotic trajectories. This restriction is not surprising because considering for example, the FK model with $W(x) = \frac{1}{2}x^2 + \frac{1}{4}x^4$ in (1b), it can be found numerically that there always exist KAM tori at large enough momenta p , and thus there never exist chaotic trajectories with unbounded momenta. In fact, we need the extra condition that $\partial^2 L(x, y)/\partial x \partial y$ be bounded for all x and y as it is in the standard map for proving the stronger result.

The concept of anti-integrability and its associated theorems, can also be extended to structural models

(1) which are not associated with a dynamical system. This situation occurs for example when $W(x)$ in (1b) is a non-convex function. Then, the associated map is multivalued in some regions of the phase space and this is undefined;

(2) which are defined in d -dimensional space ($i \in \mathbb{Z}^d$). Then, the phase space of the associated map has infinitely many dimensions;

(3) which contain non-local interactions. In that case, the concept of anti-integrability may not appear so clearly. However, this method can still be used (with some care) for example for proving the existence of bipolaronic chaotic states in models with electron-phonon coupling [8];

(4) which are not translationally invariant, that is when the function $V_i(x)$ depends on i as well as $L_i(x, y)$ in the energy functional (1a). Then, the associated map is time dependent and non-symplectic. For example, we consider a non-transla-

tionally invariant model which is a variation of the FK model with energy functional:

$$\Phi(\{u_i\}) = \sum_i h^{-i} [W(u_{i+1} - u_i) + V(u_i)], \tag{37a}$$

where W and V have the forms given by (1b) and (1c). h is a positive parameter smaller than 1. The equation of extremalization of (39a)

$$-C(u_{i+1} - u_i) + hC(u_i - u_{i-1}) + (1 - h)\mu + \lambda \sin u_i = 0 \tag{37b}$$

can be associated to the map \mathcal{S}_h (with $p_i = u_i - u_{i-1}$)

$$p_{i+1} = hp_i + k \sin u_i + (1 - h)\frac{\mu}{C}, \tag{38a}$$

$$u_{i+1} = \left(hp_i + u_i + (1 - h)\frac{\mu}{C} + k \sin u_i \right) \text{ mod } 2\pi \tag{38b}$$

which maps the cylinder $\mathbb{C} \otimes \mathbb{R} = \{u \text{ mod } 2\pi, p\}$ onto itself and which is area contracting by the factor h . For $h = 1$, one recovers the standard map and for $h = 0$, a circle map (when $\mu \neq 0$, this circle map is $u_{i+1} = u_i + \mu/C + k \sin u_i$). This map has an anti-integrable limit obtained for $C = 0$. The Banach fixed point theorem can be applied as above close to the trajectories obtained at this anti-integrable limit. We find for example the theorem:

Theorem. For $\mu = 0$ and any given sequence of integers $\{m_i\}$ fulfilling

$$|(m_{i+1} - m_i) - h(m_i - m_{i-1})| < b \quad \text{for all } i \tag{39a}$$

and

$$k > [4(1+h)^2 + \pi^2(b+1+h)^2]^{1/2}, \quad (39b)$$

there exists a unique trajectory of the extended standard map \mathcal{S}_h inside $E(\{m_i\}; \pi/2)$.

Although this result is not sufficient in itself for proving the existence of a strange attractor [17], it strongly suggests its existence for k large enough and should be easily completed for a rigorous proof.

For $h = 0$, any trajectory of \mathcal{S}_h drops after one iteration on the attractor of \mathcal{S}_h with equation $p = k \sin u + (1-h)\mu/C$. Although this degenerate attractor is just a circle and has not a Cantor set structure, this result implies that there also exist chaotic trajectories in this circle map for $\mu = 0$ and when $k > 2(1 + \pi^2)^{1/2}$.

In summary, the purpose of the work presented in this paper was to emphasize the role of the anti-integrable limit where the dynamical system reduces to a Bernoulli shift with trivial chaotic trajectories. For proving that these chaotic trajectories are preserved under perturbations, the Banach fixed point theorem can be used in appropriately defined domains in the configuration space. We have shown the efficiency of this method on a pedagogical example, which is by far much simpler than the KAM theory. Of course, we could not explore in this relatively short paper all the possible applications and consequences of this concept. This work should be completed by numerical studies concerning the bifurcation phenomena of the chaotic trajectories with non-zero GP at smaller k and its conjectures presented in section 4.

For understanding either dynamical problems or structural problems, the concept of anti-integrability appears to be the concept opposite to the concept of integrability and seems to be as much important. Systems in the regime intermediates between integrable and anti-integrable, generally exhibits two kinds of trajectories which

are mixed in a complex fashion: smooth KAM tori originating from the integrable limit and chaotic Cantori originating from the anti-integrable limit. The dynamical behavior only becomes simple either at the integrable or at anti-integrable limit.

Acknowledgements

The authors thank Yves Pomeau for careful reading and useful criticisms concerning the manuscript of this paper.

Appendix. Proof of theorem 2

Improved bound in k for the existence of unbounded chaotic trajectories

We consider a sequence of positive numbers fulfilling

$$0 < \delta_i < \pi/2. \quad (A1.a)$$

and we define the domain $E(\{m_i\}; \{\delta_i\})$ of configurations $\{u_i\}$ which is included in the domain $E(\{m_i\}; \pi/2)$ by the condition

$$\{u_i\} \in E(\{m_i\}; \{\delta_i\}) \Leftrightarrow |u_i - m_i\pi| < \delta_i \quad \text{for all } i. \quad (A1.b)$$

The sequence $\{\delta_i\}$ will be chosen in an optimized way which depends on the sequence $\{m_i\}$ or more precisely on the sequence of integers

$$m_{i+1} + m_{i-1} - 2m_i = b_i. \quad (A2.a)$$

We are now going to determine conditions which will allow the application of the Banach fixed point theorem to the map S restricted to this smaller domain. The map S which applies $E(\{m_i\}; \{\delta_i\})$ onto itself can be defined uniquely by

eqs. (9) when

$$k \sin \delta_i \geq |b_i| \pi + \delta_{i+1} + \delta_{i-1} + 2\delta_i \quad \text{for all } i. \quad (\text{A2.b})$$

For finding a sequence $\{\delta_i\}$ which fulfills (22b), we choose for example a parameter δ and use the inequality $\sin x \geq mx$ with $m = \sin(\delta)/\delta$ which is valid for $0 < x < \delta$. Then, we choose for $\{\delta_i\}$ the positive solution of

$$km\delta_i = |b_i| \pi + \delta_{i+1} + \delta_{i-1} + 2\delta_i, \quad (\text{A3.a})$$

which only exists when $km > 4$ and requires that $0 < \delta_i \leq \delta$. This solution is

$$\delta_i = K \pi \sum_n \eta^{|n|} |b_{i+n}| \quad (\text{A3.b})$$

with

$$K = \frac{1}{[km(km - 4)]^{1/2}} \quad (\text{A3.c})$$

and

$$\eta = \frac{(km)^{1/2} - (km - 4)^{1/2}}{(km)^{1/2} + (km - 4)^{1/2}}. \quad (\text{A3.d})$$

With the condition that $\|b_i\| \leq 1$ and $b_i b_{i+n} = 0$ for all i and all $n < s$, δ_i in (22b) is bounded as

$$|\delta_i| \leq K \pi \frac{1 + \eta^s}{1 - \eta^s}. \quad (\text{A4.a})$$

Thus when

$$K \pi \frac{1 + \eta^s}{1 - \eta^s} < \delta \quad (\text{A4.b})$$

the domain $E(\{m_i\}; \{\delta_i\})$ is invariant by the transformation S . For s going to infinity, condition (A4.b) yields

$$k > \frac{2\delta + (4\delta^2 + \pi^2)^{1/2}}{\sin \delta}. \quad (\text{A4.c})$$

When this condition is fulfilled and since

$$\frac{1 + \eta^s}{1 - \eta^s} \rightarrow 1 \quad \text{for } s \rightarrow \infty,$$

there exists s_0 such that for the configurations $\{m_i\}$ which fulfills (19) with $s > s_0$, (A4.a) is also fulfilled. Then, $E(\{m_i\}; \{\delta_i\})$ is invariant by the transformation S for these particular coding sequences $\{m_i\}$.

For applying the Banach fixed point theorem for the map S in this domain $E(\{m_i\}; \{\delta_i\})$ we also require $\|J\| < 1$. Because

$$\begin{aligned} \|J\| &\leq \sup_i \frac{4}{[k^2 - (|b_i| \pi + \delta_{i+1} + \delta_{i-1} + 2\delta_i)^2]^{1/2}} \\ &= \sup_i \frac{4}{(k^2 - k^2 m^2 \delta_i^2)^{1/2}} \\ &\leq \frac{4}{(k^2 - k^2 m^2 \delta^2)^{1/2}} = \frac{4}{k \cos \delta} \end{aligned} \quad (\text{A5.a})$$

this condition is fulfilled when

$$k > \frac{4}{\cos \delta}. \quad (\text{A5.b})$$

When both conditions (A4.c) and (A5.b) are fulfilled, there exists a fixed point for S in the domain $E(\{m_i\}; \{\delta_i\})$. This fixed point is a solution for eq. (3a) and corresponds to a trajectory of the standard map. QED

References

- [1] B.V. Chirikov, Phys. Rep. 52 (1979) 263; J.M. Greene, R.S. MacKay, F. Vivaldi and M.J. Feigenbaum, Physica D 3 (1981) 468.
- [2] S. Aubry, in: Solid State Sciences, Vol. 8, eds. A.R. Bishop and T. Schneider (1978); S. Aubry and G. André, Ann. Israel Phys. Soc. 3 (1980) 133-164.
- [3] S. Aubry and P.Y. Le Daeron, Physica D 8 (1983) 381-422.
- [4] S. Aubry, Physica D 7 (1983) 240-258.
- [5] S. Aubry, in: Structures et Instabilités, ed. C. Godreche (Editions de Physique, Paris, 1986) pp. 73-194.

- [6] Percival, J. Phys. A 12 (1979) L57;
J. Mather, Topology 21 (1982) 457–468.
- [7] S. Aubry, J. Phys. C 16 (1983) 2497–2508.
- [8] S. Aubry, G. Abramovici, P. Quémérais and J.L. Raimbault, Bipolaronic chaotic states in the adiabatic Holstein model, in preparation.
- [9] V.I. Arnol'd, Russ. Math. Sur. 18 (1963) 87–191.
- [10] L.H. Tang and R.B. Griffiths, J. Stat. Phys. 53 (1988) 853–892.
- [11] S. Aubry, J.P. Gosso, G. Abramovici and P. Quémérais, Discommensurations in the incommensurate ground states of the Frenkel–Kontorowa model, submitted to Physica D.
- [12] J.M. Greene, H. Johanneson, B. Schaub and H. Suhl, Phys. Rev. A 36 (1988) 5858–5861;
H. Johanneson, B. Schaub and H. Suhl, Phys. Rev. B 37 (1988) 9625–9637.
- [13] E. Zeidler, Nonlinear Functional Analysis and its applications I. Fixed-point Theorems (Springer, Berlin, 1986).
- [14] V.I. Oseledec, Trudy Moskow, Mat. Obsc. 19 (1968) 179–210;
D. Ruelle, Adv. Math. 32 (1979) 68.
- [15] D.J. Thouless, J. Phys. C 5 (1972) 77–81.
- [16] S. Aubry and P. Quémérais, in: Low Dimensional Electronic Properties of Molybdenum Bronzes and Oxides, ed. Claire Schlenker (Kluwer Academic Publishers, Dordrecht, 1989) pp. 295–405.
- [17] S. Smale, Bull. Am. Math. Soc. 73 (1967) 747;
D. Ruelle and F. Takens, Comm. Math. Phys. 20 (1972) 167; 23 (1972) 343.
- [18] J.C. Yoccoz, private communication.
- [19] M. Henon, Commun. Math. Phys. 50 (1976) 69.