Refinement of Gál-Koksma's theorems and applications

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Abstract

In this paper, we investigate the strong laws of large numbers with general normalizing sequences for arbitrary dependent sequences whose partial sums satisfy certain conditions on the p-th moment. To do it, we adapt results due to Móricz concerning moment inequalities. We also give applications that were difficult to deal with previously. Our results extend and improve some well-known corresponding ones.

Keywords: Strong laws of large numbers, Moment inequalities

AMS Subject Classification: 60F15

1 Introduction

Kolmogorov's strong law of large numbers [10] is one of the most important results in probability limit theorems. This result has been generalized by Marcinkiewicz-Zygmund [14] to obtain rates of convergence. Specifically, given a centered sequence of independent and identically distributed (i.i.d.) random variables $\{X_k, k \geq 1\}$ and S_n, the associated partial sum of rank n: $S_n = X_1 + \cdots + X_n$, for $1 \leq p < 2$, $\mathbb{E}|X_1|^p < \infty$ if and only if $n^{-1/p} S_n \longrightarrow 0$ almost surely (a.s.). It is worth noting that the last equivalence constitutes Kolmogorov's strong law of large numbers in the case $p = 1$ and that of Marcinkiewicz-Zygmund in the case $1 < p < 2$. Many studies have been conducted concerning the rates of convergence in the Marcinkiewicz-Zygmund's strong law of large numbers, and there is a vast body of literature on this topic.

It is natural to extend the idea behind the Kolmogorov-Marcinkiewicz-Zygmund result to dependent random sequences. However, this extension imposes conditions on the p-th moment of S_n. In this context, we refer to several studies such as [4, 5, 6, 9, 13, 16, 17, 21, 22].

Let $p \geq 1$ be a constant and let $\{X_k, k \geq 1\} \subset L^p(X, \mathcal{A}, \mu)$ be a sequence of random variables. We say that $\{X_k, k \geq 1\}$ satisfies the strong law of large numbers with respect to a non-decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(x) \uparrow \infty$ as $x \to \infty$, if

$$\lim_{N\to\infty} \frac{1}{\phi(N)} \sum_{k=1}^{N} X_k = 0 \quad a.s..$$

Gál-Koksma [9] derived weak conditions of the form

$$\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^p \leq d(m,n) \quad \text{for any} \quad m,n \in \mathbb{N} \quad \text{with} \quad m < n, \tag{1}$$

for some function $d : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$, to ensure that the above limit holds.

Weber [21, 22] recently established a strong law of large numbers for random variables satisfying increment conditions of the type considered in Gál-Koksma theorems. In this paper, we aim to establish a more general strong law of large numbers by providing conditions of the type (1) that allow us to extend Gál-Koksma and Weber's results. Our approach is based on adapting results from Móricz's work, and it enables us to obtain new results in several cases. One of the strengths of our approach is that our proof is much simpler than the existing ones.

Before presenting our main result, let us first recall the following definition. A function $d(m,n)$ is said to be super-additive if, for any $j \in [m,n]$, we have $d(m,j) + d(j,n) \leq d(m,n)$. It is important to note that if $d(m,n)$ is super-additive and $\eta(m,n)$ is a non-decreasing function with respect to n and a non-increasing function with respect to m, then $d(m,n)\eta(m,n)$ is also super-additive. Throughout this paper, the symbol C denotes a positive constant which may vary from one appearance to another. With this definition in mind, we can now proceed to state our main theorem.

Theorem 1. *Let $p \geq 1$. Consider a sequence of random variables $\{X_k, k \geq 1\} \subset L^p(X, \mathcal{A}, \mu)$. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function satisfying $\Psi(2x) \leq C\Psi(x)$. Assume that, there exists a constant $\eta \geq 0$ such that the function $\frac{\Psi(x)}{x^{1+\eta}}$ is non-decreasing and for any $m,n \in \mathbb{N}$ with $m < n$, we have*

$$\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^p \leq \Psi(d(m,n)). \tag{2}$$

1. *If one of the following cases holds:*
   (a) *$\eta > 0$, $d(m,n)$ is a super-additive function and $\tau > 1$;*
   (b) *$\eta = 0$, there exists $\alpha > 1$ such that $d^{\frac{1}{\alpha}}(m,n)$ is super-additive and $\tau > 1$;*
   (c) *$\eta = 0$, $d(m,n)$ is super-additive, there exists $\sigma > 0$ such that $n^\sigma = O\left(d(0,n)\right)$ and $\tau > p+1$,*

   then we have

$$\lim_{N\to\infty} \frac{1}{\left(\Psi(d(0,N)) \log^\tau(d(0,N))\right)^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s.$$

   and $\sup_{N\geq 1} \left| \frac{\sum_{k=1}^N X_k}{\left(\Psi(d(0,N)) \log^\tau(d(0,N))\right)^{1/p}} \right| \in L^p(\mu)$.

2. *If one of the preceding cases, namely a, b, or c without the condition that there exists $\sigma > 0$ such that $n^\sigma = O(d(0,n))$, holds and if $d(0,2n) \leq Cd(0,n)$, then*

$$\lim_{N\to\infty} \frac{1}{(\Psi(d(0,N))\log^\tau(N))^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s.$$

and $\sup_{N\geq 1}\left|\frac{\sum_{k=1}^{N} X_k}{(\Psi(d(0,N))\log^\tau(N))^{1/p}}\right| \in L^p(\mu)$.

Theorem 1 extends the results of Weber [21, 22] and, as a result, extends the well-known Gál-Koksma theorems [9]. More details on this will be provided in Section 2 below. Additionally, this theorem can be used to improve upon the ideas presented in Theorem 1.1. of [17] in a one-dimensional setting (see also Theorem 1.1. in [16] and Theorem 1. in [4]). The paper by Nane et al. [17] provides strong laws of large numbers with normalizations that satisfy more conditions than ours.

The next corollary considers the scenario where only $d(m,n)$ is present. It is a direct application of the previous theorem with $\eta = 0$ and Ψ is the identity function.

Corollary 1. *Let $p \geq 1$. Consider a sequence of random variables $\{X_k, k \geq 1\} \subset L^p(X, \mathcal{A}, \mu)$ and suppose that*

$$\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^p \leq d(m,n), \quad \text{for any} \quad m,n\in\mathbb{N} \quad \text{with} \quad m<n.$$

1. *If one of the following cases holds:*
   - *(a) there exists $\alpha > 1$ such that $d^{\frac{1}{\alpha}}(m,n)$ is super-additive and $\tau > 1$;*
   - *(b) $d(m,n)$ is super-additive, there exists $\sigma > 0$ such that $n^\sigma = O(d(0,n))$ and $\tau > p+1$,*

   then we have

$$\lim_{N\to\infty} \frac{1}{(d(0,N)\log^\tau(d(0,N)))^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s..$$

2. *If one of the preceding cases, namely a or b without the condition that there exists $\sigma > 0$ such that $n^\sigma = O(d(0,n))$, holds and if $d(0,2n) \leq Cd(0,n)$, then*

$$\lim_{N\to\infty} \frac{1}{(d(0,N)\log^\tau(N))^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s..$$

As an application of Theorem 1, we are able to give an example of function ϕ such that the series $\sum_{k\geq 1}\frac{X_k}{\phi(k)}$ converges. In particular, we establish the following corollary.

Corollary 2. *Suppose that*

$$\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^p \leq (n-m)^\beta \quad \text{for any} \quad m,n\in\mathbb{N} \quad \text{with} \quad m<n.$$

If $\beta = 1$ and $\tau > p+1$, then

$$\sum_{k>1} \frac{X_k}{(k\log^\tau(k))^{1/p}} \quad \text{converges} \quad a.s..$$

If $1 < \beta < p$ *and* $\tau > 1$*, then*

$$\sum_{k>1} \frac{X_k}{(k^\beta \log^\tau(k))^{1/p}} \quad \textit{converges} \quad \textit{a.s.}.$$

This paper also provides strong laws of large numbers for sequences of non-negative random variables. Two notable strands of research in this literature are: (i) the works of Etemadi [8] and Csörgő, Tandori, and Totik [7], and (ii) the Petrov-type approach, inspired by Petrov [18], which has been studied by several authors such as Korchevsky [11], Kuczmaszewska [12], and Petrov [19]. Chen and Sung [3] established a theorem that unifies both strands. Specifically, they demonstrated that if $\{Y_k, k \geq 1\} \subset L^p(X, \mathcal{A}, \mu)$, $p \geq 1$, is a sequence of non-negative random variables and $(b_k)_{k\geq 1}$ is a non-decreasing unbounded sequence of positive numbers such that

$$\sup_{n\geq 1} \frac{\sum_{k=1}^n \mathbb{E}Y_k}{b_n} < \infty, \quad \exists\, \delta_n \geq 0 : \mathbb{E}\left|\sum_{k=1}^n (Y_k - \mathbb{E}Y_k)\right|^p \leq \sum_{k=1}^n \delta_k \quad \text{and} \quad \sum_{k\geq 1} \frac{\delta_k}{b_k^p} < \infty,$$

then we have

$$\lim_{N\to\infty} \frac{1}{b_N} \sum_{k=1}^N (Y_k - \mathbb{E}Y_k) = 0 \quad \text{a.s.}.$$

This paper aims to provide additional contributions to this topic. We can prove the following theorem.

Theorem 2. *Let* $p \geq 1$*. Let* $\{Y_k, k \geq 1\} \subset L^p(X, \mathcal{A}, \mu)$ *be a sequence of non-negative random variables. Suppose that, for any* $m < n$*,*

$$\mathbb{E}\left|\sum_{k=m+1}^n (Y_k - \mathbb{E}Y_k)\right|^p \leq d(m,n) \quad \textit{and} \quad \mathbb{E}Y_n \leq C d^{\frac{1}{p}}(0,n) d(n-1,n),$$

where $d(m,n)$ *is a super-additive function. Then, for any* $\tau > p+1$*, we have*

$$\lim_{N\to\infty} \frac{1}{(d(0,N)\log^\tau(d(0,N)))^{1/p}} \sum_{k=1}^N (Y_k - \mathbb{E}Y_k) = 0 \quad \textit{a.s.}.$$

Notice that this theorem can also be used for some sequences of non-positive random variables. For example, consider a sequence of random variables $\{X_k, k \geq 1\}$ taking values in $\{-1, 1\}$ such that $\mathbb{P}(X_n = 1) = p_n$ and $p_n \leq C d^{\frac{1}{p}}(0,n) d(n-1,n)$. Then, we have $\mathbb{E}\left|\sum_{k=m+1}^n (X_k - \mathbb{E}X_k)\right|^p = \mathbb{E}\left|\sum_{k=m+1}^n (Y_k - \mathbb{E}Y_k)\right|^p$, where $Y_k = X_k + 1$ for any $k \geq 1$. Moreover, note that the result of the theorem remains true even if the condition $\mathbb{E}Y_n \leq C d^{\frac{1}{p}}(0,n) d(n-1,n)$ is replaced by the condition $sup_{n\geq 1} \frac{\sum_{k=1}^n \mathbb{E}Y_k}{d^{1/p}(0,n)} < \infty$.

The rest of this paper is organized as follows. In Section 2, we compare our results with those of Gál-Koksma and Weber. In Section 3, we recall some known results that will be used in the proofs of our main theorems and then proceed to prove them. One of the main analytical tools used in the proofs is Móricz's work. In the last section, we provide several examples illustrating our main results.

2 Extension of Gál-Koksma's and Weber's theorems

In this section, we will show how our theorem extends the results of Gál-Koksma [9] and Weber [21, 22]. Let $\{X_k, k \geq 1\}$ be a sequence of random variables with probability space $(X, \mathcal{A}, \mu)$ and let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function such that $\Psi(2x) \leq C\Psi(x)$. Assume that there exists a constant $\eta \geq 0$ such that $\frac{\Psi(x)}{x^{1+\eta}}$ is non-decreasing. Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers, and define $A(n) = \sum_{k=1}^{n} a_k$. Now, assume that the condition (2) holds with:

$$d(m,n) = \sum_{k=m+1}^{n} a_k.$$

Then several theorems of Gál-Koksma and Weber can be seen as particular cases of our Theorem 1. More precisely, let $\epsilon > 0$, and depending on the value of a_k, we have the following:

Case $a_k > C$. Applying Theorem 1, we have :

If $\eta > 0$,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} X_k}{\left(\Psi(A(N)) \log^{1+\epsilon} A(N))\right)^{1/p}} = 0 \quad a.s. \quad \text{(Theorem 8.4.1 [22]; } \eta > 0\text{)}.$$

If $\eta = 0$,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} X_k}{\left(\Psi(A(N)) \log^{p+1+\epsilon}(A(N))\right)^{1/p}} = 0 \quad a.s. \quad \text{(Theorem 8.4.1 [22]; } \eta = 0\text{)}.$$

Case $a_k = 1$. This case covers two theorems of [9]. Indeed, when $a_k = 1$, we have A(n)=n. Then it is clear that if $\eta = 0$,

$$\lim_{N\to\infty} \frac{1}{\left(\Psi(N) \log^{p+1+\epsilon}(N)\right)^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s. \quad \text{(Theorem 3 [9])}.$$

If $\eta > 0$,

$$\lim_{N\to\infty} \frac{1}{\left(\Psi(N) \log^{1+\epsilon}(N)\right)^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s. \quad \text{(Theorem 5 [9])}.$$

Moreover, let $s > 0$ be such that $p - s > 1$, then if $\Psi(x) = x^{p-s}$, we obtain

$$\lim_{N\to\infty} \frac{1}{\left(N^{p-s} \log^{1+\epsilon}(N)\right)^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s.$$

which improves Theorem 7 in [9] in the case where $\eta(x) = \frac{1}{x^s}$.

Case $a_k = \frac{1}{k}$. This case allows us to establish a new result which apparently cannot be obtained by applying Weber's work. Since we have $A(n) \sim \log(n)$, then if $\eta > 0$,

$$\lim_{N\to\infty} \frac{1}{\left(\Psi(\log(N)) \log^{1+\epsilon}(\log(N))\right)^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s.$$

and, since $\log(2n) \leq 2\log(n)$ for $n > 1$, then if $\eta = 0$,

$$\lim_{N\to\infty} \frac{1}{\left(\Psi(\log(N))\log^{p+1+\epsilon}(N)\right)^{1/p}} \sum_{k=1}^{N} X_k = 0 \quad a.s..$$

3 Proofs of the main results

3.1 Additivity and maximal inequalities

To prove the main results, we need the following lemmas.

Lemma 1. *Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function. Suppose there exists a constant $\eta \geq 0$ such that $\frac{\Psi(x)}{x^{1+\eta}}$ is non-decreasing. Then there exists a constant $\lambda \geq 1$ such that*

$$\Psi^{\frac{1}{\lambda}}(x) + \Psi^{\frac{1}{\lambda}}(y) \leq \Psi^{\frac{1}{\lambda}}(x+y)$$

for all $x, y \in \mathbb{R}^+$. In particularly, if $\eta = 0$ and if there exists $\alpha > 1$ such that $x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}} \leq z^{\frac{1}{\alpha}}$, $z \in \mathbb{R}^+$, then

$$\Psi^{\frac{1}{\alpha}}(x) + \Psi^{\frac{1}{\alpha}}(y) \leq \Psi^{\frac{1}{\alpha}}(z).$$

Proof. For all $x, y \in \mathbb{R}^+$, we have

$$\begin{aligned}
\Psi^{\frac{1}{\lambda}}(x) + \Psi^{\frac{1}{\lambda}}(y) &\leq (x^{1+\eta})^{\frac{1}{\lambda}} \left(\frac{\Psi(x+y)}{(x+y)^{1+\eta}}\right)^{\frac{1}{\lambda}} + (y^{1+\eta})^{\frac{1}{\lambda}} \left(\frac{\Psi(x+y)}{(x+y)^{1+\eta}}\right)^{\frac{1}{\lambda}} \\
&\leq \left(x^{\frac{1+\eta}{\lambda}} + y^{\frac{1+\eta}{\lambda}}\right) \left(\frac{\Psi(x+y)}{(x+y)^{1+\eta}}\right)^{\frac{1}{\lambda}} \\
&\leq (x+y)^{\frac{1+\eta}{\lambda}} \left(\frac{\Psi(x+y)}{(x+y)^{1+\eta}}\right)^{\frac{1}{\lambda}} \\
&= \Psi^{\frac{1}{\lambda}}(x+y)
\end{aligned}$$

where the last inequality holds if we take $1 \leq \lambda \leq 1+\eta$. Moreover, if $\eta = 0$ and if there exists $\alpha > 1$ such that $x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}} \leq z^{\frac{1}{\alpha}}$, we have

$$\begin{aligned}
\Psi^{\frac{1}{\alpha}}(x) + \Psi^{\frac{1}{\alpha}}(y) &\leq x^{\frac{1}{\alpha}} \left(\frac{\Psi(z)}{z}\right)^{\frac{1}{\alpha}} + (y)^{\frac{1}{\alpha}} \left(\frac{\Psi(z)}{z}\right)^{\frac{1}{\alpha}} \\
&\leq \left(x^{\frac{1}{\alpha}} + y^{\frac{1}{\alpha}}\right) \left(\frac{\Psi(z)}{z}\right)^{\frac{1}{\alpha}} \\
&\leq z^{\frac{1}{\alpha}} \left(\frac{\Psi(z)}{z}\right)^{\frac{1}{\alpha}} \\
&= \Psi^{\frac{1}{\alpha}}(z).
\end{aligned}$$

□

The following results are provided by Móricz. A direct consequence of Theorems 1 in [15] is the following lemma.

Lemma 2. *Let $p \geq 1$ and let $\{X_k, k \geq 1\}$ be a sequence of random variables on a probability space $(X, \mathcal{A}, \mathbb{P})$. Assume that there exist $\lambda > 1$ and a non-negative function $g : \mathbb{N} \times \mathbb{N} \longmapsto \mathbb{R}^+$, such that for any $m, n \in \mathbb{N}$ with $m < n$,*

$$\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^p \leq C g^{\lambda}(m,n),$$

where $g(m,n)$ is a super-additive function. Then

$$\mathbb{E}\left(\sup_{j=m+1}^{n}\left|\sum_{k=m+1}^{j} X_k\right|^p\right) \leq C(\lambda) g^{\lambda}(m,n),$$

where $C(\lambda)$ is a constant only depending on λ.

We are now dealing with the case where $\lambda = 1$. By adapting the result of Theorem 3 in [15], we obtain the following lemma. Before presenting it, let us define the following sequences. Let $(\lambda(n))_{n\in\mathbb{N}}$ be a positive and non-decreasing sequence, and define $(\Lambda(n))_{n\in\mathbb{N}}$ as follows:

$$\Lambda(1) = 1 \quad \text{and} \quad \Lambda(n) = \lambda(m) + \Lambda(m-1), \quad n \geq 2,$$

where m denotes the integer part of $\frac{n+2}{2}$. For instance, if $\lambda \equiv 1$, then $\Lambda(n) = \log_2(2n)$ because $1 + \log_2(2(m-1)) \leq \log_2(2n)$.

Lemma 3. *Let $\{X_k, k \geq 1\}$ be a sequence of random variables on a probability space $(X, \mathcal{A}, \mathbb{P})$. Suppose that*

$$\mathbb{E}\left|X_{b+n} - X_b\right|^p \leq \lambda^p(n) g(b, b+n)$$

where $g(m,n)$ is a super-additive function. Then

$$\mathbb{E}\sup_{1\leq k\leq n}\left|X_{b+k} - X_b\right|^p \leq \Lambda^p(n) g(b, b+n).$$

Proof. Let $n > 1$. If $m \leq k \leq n$, we have

$$\begin{aligned}\left|X_{b+k} - X_b\right| &\leq \left|X_{b+k} - X_{b+m}\right| + \left|X_{b+m} - X_b\right| \\ &\leq \sup_{m\leq k\leq n}\left|X_{b+k} - X_{b+m}\right| + \left|X_{b+m} - X_b\right|.\end{aligned}$$

If $k < m$, we have

$$\left|X_{b+k} - X_b\right| \leq \sup_{1\leq k\leq m-1}\left|X_{b+k} - X_b\right|.$$

By Minkowski's inequality

$$\begin{aligned}\left(\mathbb{E}\sup_{1\leq k\leq n}\left|X_{b+k} - X_b\right|^p\right)^{1/p} &\leq \left(\mathbb{E}\left|X_{b+m} - X_b\right|^p\right)^{1/p} \\ &\quad + \left(\mathbb{E}\sup_{1\leq k\leq m-1}\left|X_{b+k} - X_b\right|^p + \mathbb{E}\sup_{m\leq k\leq n}\left|X_{b+k} - X_{b+m}\right|^p\right)^{1/p}.\end{aligned} \tag{3}$$

Assuming that the conclusion of the lemma holds for $k < n$ and for any $b \geq 0$. Then, by the choice of m, we have

$$\mathbb{E} \sup_{1 \leq k \leq m-1} |X_{b+k} - X_b|^p \leq \Lambda^p(m-1) g(b, b+m-1)$$

and thus

$$\begin{aligned}\mathbb{E} \sup_{m \leq k \leq n} |X_{b+k} - X_{b+m}|^p &= \mathbb{E} \sup_{1 \leq k \leq n-m} |X_{b+m+k} - X_{b+m}|^p \\ &\leq \Lambda^p(n-m) g(b+m, b+n).\end{aligned}$$

Since $n - m \leq m - 1$ and $d(m, n)$ is super-additive, we have that the second term of the right-hand side of Equation (3) is

$$\begin{aligned}\mathbb{E} \sup_{1 \leq k \leq m-1} |X_{b+k} - X_b|^p + \mathbb{E} \sup_{m \leq k \leq n} |X_{b+k} - X_{b+m}|^p \\ \leq \Lambda^p(m-1) \left(g(b, b+m) + g(b+m, b+n) \right) \\ \leq \Lambda^p(m-1) g(b, b+n).\end{aligned} \tag{4}$$

Now by assumption, the first term is

$$\mathbb{E} |X_{b+m} - X_b|^p \leq \lambda^p(m) g(b, b+m). \tag{5}$$

The proof follows directly by using (3), (4), and (5) and performing recurrence over n. □

3.2 Proof of Theorem 1

Firstly, let's consider that $2^{\psi(n)} \leq C d(0, n)$ and $\frac{\psi(n)}{n} \uparrow \infty$ as $n \to \infty$. Since $\mathbb{E} \left| \sum_{k=1}^N X_k \right|^p \leq \Psi(d(0, N))$, we can derive that

$$\mathbb{E} \left| \frac{\sum_{k=1}^N X_k}{(\Psi(d(0, N)) \log^\tau(d(0, N)))^{1/p}} \right|^p \leq C \frac{1}{\log^\tau(d(0, N))} \leq C \frac{1}{n^\tau}.$$

Summing over n with $\tau > 1$, by Beppo Levi's theorem, we can clearly obtain the result of the first part of the theorem.

Secondly, we consider that $d(0, n) \leq C 2^n$. For any $j \in \mathbb{N}^*$, we have

$$\{d(0, n); n \in \mathbb{N}^*\} \cap [2^j, 2^{j+1}] \neq \emptyset.$$

We denote by N_j the smallest integer N such that $d(0, N)$ belongs to $[2^j, 2^{j+1}]$. In other words, we write

$$N_j = \inf\{N \in \mathbb{N}^* : d(0, N) \in [2^j, 2^{j+1}]\}.$$

To provide proof, we write

$$\begin{aligned}\left| \frac{\sum_{k=1}^N X_k}{(\Psi(d(0, N)) \log^\tau(d(0, N)))^{1/p}} \right| &\leq \left| \frac{\sum_{k=1}^{N_j} X_k}{(\Psi(d(0, N_j)) \log^\tau(d(0, N_j)))^{1/p}} \right| \\ &+ \sup_{N_j < N \leq N_{j+1}} \left| \frac{\sum_{k=N_j+1}^N X_k}{(\Psi(d(0, N_j)) \log^\tau(d(0, N_j)))^{1/p}} \right|.\end{aligned} \tag{6}$$

The proof is divided into 6 steps. In *Step 1*, we show that the first term of the right-hand side of Equation (6) converges to zero almost surely, which is a common step for all cases. From *Step 2* to *Step 4*, we demonstrate that the second term converges to zero almost surely in all cases of the first part of the theorem. *Step 5* is dedicated to the second part. Finally, *Step 6* is performed to prove the maximal inequality.

Step 1. First, since $\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^p \leq \Psi(d(m,n))$ for $m<n$, we observe that

$$\mathbb{E}\left|\frac{\sum_{k=1}^{N_j} X_k}{\left(\Psi(d(0,N_j))\log^{\tau}(d(0,N_j))\right)^{1/p}}\right|^p \leq \frac{\Psi((d(0,N_j))}{\Psi(d(0,N_j))\log^{\tau}(d(0,N_j))} \leq C\frac{1}{j^{\tau}}.$$

With $\tau > 1$ in all cases of the theorem, this implies that the last quantity is a term of a convergent series. By Beppo Levi's theorem

$$\lim_{j\to\infty} \frac{\sum_{k=1}^{N_j} X_k}{\left(\Psi(d(0,N_j))\log^{\tau}(d(0,N_j))\right)^{1/p}} = 0 \quad a.s.. \tag{7}$$

Step 2. To control the second term on the right-hand side of (6), we can write for any $n,m \in [N_j, N_{j+1}]$ with $m<n$,

$$\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^p \leq \Psi(d(m,n)). \tag{8}$$

Moreover, if there exists $\eta > 0$ such that $\frac{\Psi(x)}{x^{1+\eta}}$ is non-decreasing, then, by Lemma 1, we have $\Psi^{\frac{1}{\lambda}}(d(m,n))$ is super-additive for some $\lambda > 1$. We can now apply Lemma 2 to obtain

$$\begin{aligned}
\mathbb{E}\sup_{N_j<N\leq N_{j+1}}\left|\frac{\sum_{k=N_j+1}^{N} X_k}{\left(\Psi(d(0,N_j))\log^{\tau}(d(0,N_j))\right)^{1/p}}\right|^p &\leq \frac{\Psi(d(N_j,N_{j+1}))}{\Psi(d(0,N_j))\log^{\tau}(d(0,N_j))}\\
&\leq \frac{\Psi(d(0,N_{j+1}))}{\Psi(d(0,N_j))\log^{\tau}(d(0,N_j))}\\
&\leq C\frac{1}{j^{\tau}}.
\end{aligned}$$

The last line follows from the fact that $d(0,N_j)$ belongs to $[2^j, 2^{j+1}]$, $d(0,N_{j+1})$ belongs to $[2^{j+1}, 2^{j+2}]$ and $\Psi(2x) \leq C\Psi(x)$ for any $x \in \mathbb{R}^+$. Now, by summing over j and using the Beppo Levi's theorem, we obtain

$$\lim_{j\to\infty}\sup_{N_j<N\leq N_{j+1}}\left|\frac{\sum_{k=N_j+1}^{N} X_k}{\left(\Psi(d(0,N_j))\log^{\tau}(d(0,N_j))\right)^{1/p}}\right| = 0 \quad a.s.. \tag{9}$$

This, together with (7), concludes the proof the first part of the theorem in case a.

Step 3. By Lemma 1, if there exists $\alpha > 1$ such that $d^{\frac{1}{\alpha}}(m,n)$ is super-additive then $\Psi^{\frac{1}{\alpha}}(d(m,n))$ is also super-additive. Therefore, if we follow the same procedure as in *Step 2*, we can conclude that

$$\lim_{j\to\infty}\sup_{N_j<N\leq N_{j+1}}\left|\frac{\sum_{k=N_j+1}^{N} X_k}{\left(\Psi(d(0,N_j))\log^{\tau}(d(0,N_j))\right)^{1/p}}\right| = 0 \quad a.s..$$

This, together with (7), concludes the proof the first part of the theorem in case b.

Step 4. We will now begin the proof in case c. Since $d(m,n)$ is assumed to be a super-additive function, then $\Psi(d(m,n))$ is also super-additive. Applying now Lemma 3 to equation (8) yields,

$$\mathbb{E} \sup_{N_j<N\leq N_{j+1}} \left| \frac{\sum_{k=N_j+1}^{N} X_k}{(\Psi(d(0,N_j))\log^\tau(d(0,N_j)))^{1/p}} \right|^p \leq \frac{\log^p(2(N_{j+1}-N_j))\,\Psi(d(N_j,N_{j+1}))}{\Psi(d(0,N_j))\log^\tau(d(0,N_j))}$$
$$\leq C\frac{\log^p(2(N_{j+1}-N_j))}{j^\tau},$$

where the last inequality follows from the fact that $d(0,N_j)$ belongs to $[2^j,2^{j+1}]$, $d(0,N_{j+1})$ belongs to $[2^{j+1},2^{j+2}]$ and $\Psi(2x)\leq C\Psi(x)$ for any $x\in\mathbb{R}^+$. We recall that by assumption, $n^\sigma \leq Cd(0,n)$ for some $\sigma>0$. Now, from this fact and the definition of N_j, we have $(N_{j+1})^\sigma \leq Cd(0,N_{j+1}) \leq C2^j$. Then

$$\log^p(2(N_{j+1}-N_j)) \leq Cj^p.$$

And, therefore

$$\mathbb{E} \sup_{N_j<N\leq N_{j+1}} \left| \frac{\sum_{k=N_j+1}^{N} X_k}{(\Psi(d(0,N_j))\log^\tau(d(0,N_j)))^{1/p}} \right|^p \leq C\frac{1}{j^{\tau-p}}.$$

Summing over j and using Beppo Levi's theorem, this gives

$$\lim_{j\to\infty} \sup_{N_j<N\leq N_{j+1}} \left| \frac{\sum_{k=N_j+1}^{N} X_k}{(\Psi(d(0,N_j))\log^\tau(d(0,N_j)))^{1/p}} \right| = 0 \quad a.s..$$

This together with (7) concludes the proof of the first part of the theorem in case c.

Step 5. To prove the second part of the theorem, we follow the previous steps but with some modifications. Specifically, we set $N_j = 2^j$ and replace $\log^\tau(d(0,N))$ in the denominator of the previous steps with $\log^\tau(N)$. Moreover, we use the inequalities $d(0,2n)\leq 2d(0,n)$ and $\Psi(2x)\leq C\Psi(x)$ to show that $\frac{\Psi(d(0,N_{j+1}))}{\Psi(d(0,N_j))}$ is bounded.

Step 6. Now, we prove the strong maximal inequality. To do it, we use (6), and we take the supremum over j. This gives

$$\sup_{N\geq1} \left| \frac{\sum_{k=1}^{N} X_k}{(\Psi(d(0,N))\log^\tau(d(0,N)))^{1/p}} \right|^p \leq C\sup_{j\geq1} \sup_{N_j<N\leq N_{j+1}} \left| \frac{\sum_{k=N_j+1}^{N} X_k}{(\Psi(d(0,N_j))\log^\tau(d(0,N_j)))^{1/p}} \right|^p$$
$$+C\sup_{j\geq1} \left| \frac{\sum_{k=1}^{N_j} X_k}{(\Psi(d(0,N_j))\log^\tau(d(0,N_j)))^{1/p}} \right|^p.$$

Choosing, for instance, the case c of the first part of the theorem and integrating over μ, we can obtain

$$\left\| \sup_{N\geq1} \left| \frac{\sum_{k=1}^{N} X_k}{(\Psi(d(0,N))\log^\tau(d(0,N)))^{1/p}} \right| \right\|_{p,\mu}^p \leq C\sum_{j\geq1} \frac{\log^p(2(N_{j+1}-N_j))\Psi(d(N_j,N_{j+1}))}{\Psi(d(0,N_j))\log^\tau(d(0,N_j))}$$
$$+C\sum_{j\geq1} \frac{\Psi(d(0,N_j))}{\Psi(d(0,N_j))\log^\tau(d(0,N_j))}.$$

The last term is finite, as we showed above in *Steps* 1 and 4. If we perform the same operation on all cases, this demonstrates the strong maximal inequality and completes the proof.

3.3 Proof of Theorem 2

Let $(N_k)_{k\in\mathbb{N}}$ be such that $N_k = \inf\{N \in \mathbb{N} : d(0, N) \geq k\}$. Note that $k \leq d(0, N_k) < k+1$. Therefore, for any $m < n$,

$$d(N_m, N_n) \leq d(0, N_n) - d(0, N_m) < 2(n-m).$$

To simplify the notation, we define $X_k = Y_k - \mathbb{E}Y_k$. We will prove the theorem in two steps. In *Step 1*, we will demonstrate that the theorem holds for the sequence $(N_k)_{k\in\mathbb{N}}$. In *Step 2*, we will generalize this result.

Step 1. As in the proof of Theorem 1, we write,

$$\left|\frac{\sum_{i=1}^{N_k} X_i}{\left(d(0,N_k)\log^\tau(d(0,N_k))\right)^{1/p}}\right| \leq \left|\frac{\sum_{k=1}^{N_{2^j}} X_k}{\left(d(0,N_{2^j})\log^\tau(d(0,N_{2^j}))\right)^{1/p}}\right| + \sup_{2^j<k\leq 2^{j+1}}\left|\frac{\sum_{k=N_{2^j+1}}^{N_k} X_k}{\left(d(0,N_{2^j})\log^\tau(d(0,N_{2^j}))\right)^{1/p}}\right|. \tag{10}$$

Know we have,

$$\begin{aligned}\mathbb{E}\left|\frac{\sum_{k=1}^{N_{2^j}} X_k}{\left(d(0,N_{2^j})\log^\tau(d(0,N_{2^j}))\right)^{1/p}}\right|^p &\leq \frac{d(0,N_{2^j})}{d(0,N_{2^j})\log^\tau(d(0,N_{2^j}))}\\ &\leq \frac{1}{\log^\tau(2^j)}\\ &\leq C\frac{1}{j^\tau}.\end{aligned}$$

It follows from the assumptions of the theorem that $\sum_j \frac{1}{j^\tau} < \infty$. Thus, according to Beppo Levi's theorem,

$$\lim_{j\to\infty}\frac{\sum_{k=1}^{N_{2^j}} X_k}{\left(d(0,N_{2^j})\log^\tau(d(0,N_{2^j}))\right)^{1/p}} = 0 \quad a.s.. \tag{11}$$

Now, we have to control the second term on the right-hand side of (10). To do it, we write for any $n, m \in [2^j+1, 2^{j+1}]$ with $m < n$,

$$\mathbb{E}\left|S_{N_n} - S_{N_m}\right|^p \leq Cd(N_m, N_n) \leq C(n-m).$$

We are now in a position to apply Lemma 3, then

$$\begin{aligned}\mathbb{E}\sup_{2^j<k\leq 2^{j+1}}\left|\frac{\sum_{i=N_{2^j+1}}^{N_k} X_i}{\left(d(0,N_{2^j})\log^\tau(d(0,N_{2^j}))\right)^{1/p}}\right|^p &\leq C\frac{\log^p\left(2(2^{j+1}-2^j)\right)(2^{j+1}-2^j)}{d(0,N_{2^j})\log^\tau(d(0,N_{2^j}))}\\ &\leq C\frac{\log^p\left(2^{j+1}\right)2^j}{2^j\log^\tau(2^j)}\\ &\leq C\frac{1}{j^{\tau-p}}.\end{aligned}$$

When $\tau > p+1$, the last quantity is a term of a convergent series. By Beppo Levi's theorem

$$\lim_{j\to\infty} \sup_{2^j<k\le 2^{j+1}} \left| \frac{\sum_{i=N_{2^j+1}}^{N_k} X_i}{\left(d(0,N_{2^j}) \log^\tau(d(0,N_{2^j}))\right)^{1/p}} \right| = 0 \quad a.s.. \tag{12}$$

According now (11) and (12), this concludes that, for $\tau > p+1$,

$$\lim_{k\to\infty} \frac{\sum_{k=1}^{N_k} X_k}{\left(d(0,N_k) \log^\tau(d(0,N_k))\right)^{1/p}} = 0 \quad a.s.,$$

and yields the desired result of *Step 1.*

Step 2. In this step we treat the general case. For this, we let N and k two integers in such a way that $N_k < N \le N_{k+1}$. Then

$$\begin{aligned}\sum_{i=1}^{N} X_i = \sum_{i=1}^{N} (Y_i - \mathbb{E}Y_i) &\le \sum_{i=1}^{N_{k+1}} Y_i - \sum_{i=1}^{N_{k+1}} \mathbb{E}Y_i + \sum_{i=N+1}^{N_{k+1}} \mathbb{E}Y_i \\ &= \sum_{i=1}^{N_{k+1}} X_i + \sum_{i=N+1}^{N_{k+1}} \mathbb{E}Y_i.\end{aligned} \tag{13}$$

And, therefore

$$\frac{\sum_{i=1}^{N} X_i}{\left(d(0,N) \log^\tau(d(0,N))\right)^{1/p}} \le \frac{\sum_{i=1}^{N_{k+1}} X_i}{\left(d(0,N_k) \log^\tau(d(0,N_k))\right)^{1/p}} + \frac{\sum_{i=N+1}^{N_{k+1}} \mathbb{E}Y_i}{\left(d(0,N_k) \log^\tau(d(0,N_k))\right)^{1/p}}.$$

By *Step 1*, for $\tau > p+1$, we have

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{N_{k+1}} X_i}{\left(d(0,N_{k+1}) \log^\tau(d(0,N_{k+1}))\right)^{1/p}} = 0 \quad a.s..$$

For large enough k, there exists a constant $C > 0$ such that

$$\frac{d(0,N_{k+1}) \log^\tau(d(0,N_{k+1}))}{d(0,N_k) \log^\tau(d(0,N_k))} \le \frac{(k+2)\log^\tau(k+2)}{k \log^\tau(k)} \le C.$$

Thus

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{N_{k+1}} X_i}{\left(d(0,N_k) \log^\tau(d(0,N_k))\right)^{1/p}} = 0 \quad a.s.. \tag{14}$$

Now, from the assumptions, $\mathbb{E}Y_i \le C d^{\frac{1}{p}}(0,i) d(i-1,i)$ and $d(m,n)$ is super-additive, and from the definition of N_k, we have

$$\begin{aligned}\frac{\sum_{i=N+1}^{N_{k+1}} \mathbb{E}Y_i}{\left(d(0,N_k) \log^\tau(d(0,N_k))\right)^{1/p}} &\le C \frac{\sum_{i=N_k+1}^{N_{k+1}} d^{\frac{1}{p}}(0,i) d(i-1,i)}{\left(d(0,N_k) \log^\tau(d(0,N_k))\right)^{1/p}} \\ &\le C \frac{d(0,N_{k+1})^{\frac{1}{p}} d(N_k,N_{k+1})}{\left(d(0,N_k) \log^\tau(d(0,N_k))\right)^{1/p}} \\ &\le C \frac{1}{\log^{\frac{\tau}{p}} k}.\end{aligned}$$

This together with the equation (14) gives

$$\limsup_{k\to\infty} \frac{\sum_{i=1}^{N} X_i}{(d(0,N)\log^{\tau}(d(0,N)))^{1/p}} \leq 0 \quad a.s..$$

To conclude, we use the fact that $N_k < N \leq N_{k+1}$ and we write

$$\sum_{i=1}^{N} X_i \geq \sum_{i=1}^{N_k} X_i - \sum_{i=N_k+1}^{N} \mathbb{E}Y_i$$

And, therefore

$$\frac{\sum_{i=1}^{N} X_i}{(d(0,N)\log^{\tau}(d(0,N)))^{1/p}} \geq \frac{\sum_{i=1}^{N_k} X_i}{(d(0,N_{k+1})\log^{\tau}(d(0,N_{k+1})))^{1/p}} - \frac{\sum_{i=N_k+1}^{N} \mathbb{E}Y_i}{(d(0,N_{k+1})\log^{\tau}(d(0,N_{k+1})))^{1/p}}.$$

Now we do as we did above to prove that

$$\liminf_{k\to\infty} \frac{\sum_{i=1}^{N} X_i}{(d(0,N)\log^{\tau}(d(0,N)))^{1/p}} \geq 0 \quad a.s..$$

Consequently

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{N} X_i}{(d(0,N)\log^{\tau}(d(0,N)))^{1/p}} = 0 \quad a.s.$$

which proves our claim and finishes the proof of theorem.

3.4 Proof of Corollary 2

Using Abel's summation formula, we have

$$\begin{aligned}\sum_{k=1}^{N} \frac{X_k}{(k^{\beta}\log^{\tau}(k))^{1/p}} &= \\ &\sum_{k=1}^{N-1}\left(\frac{1}{(k^{\beta}\log^{\tau}(k))^{1/p}} - \frac{1}{((k+1)^{\beta}\log^{\tau}(k+1))^{1/p}}\right)\sum_{j=1}^{k} X_j \\ &+ \frac{1}{(N^{\beta}\log^{\tau}(N))^{1/p}}\sum_{k=1}^{N} X_k. \end{aligned} \tag{15}$$

If $\beta > 1$, then there exists $\alpha > 1$ such that $(n-m)^{\beta/\alpha}$ is super-additive. Applying Corollary 1 with $d(m,n) = (m-n)^{\beta}$, we have

$$\lim_{N\to\infty} \frac{1}{(N^{\beta}\log^{\tau}(N))^{1/p}} \sum_{k=1}^{N} X_k = 0.$$

This implies that the last term of the right-hand side of (15) converges to 0 almost surely. Moreover,

$$\mathbb{E}\left|\sum_{k=1}^{N-1}\left(\frac{1}{(k^\beta\log^\tau(k))^{1/p}}-\frac{1}{((k+1)^\beta\log^\tau(k+1))^{1/p}}\right)\sum_{j=1}^{k}X_j\right|^p$$
$$\le C\sum_{k=1}^{N-1}\frac{1}{k^{\beta+1}\log^\tau(k)}\mathbb{E}\left|\sum_{j=1}^{k}X_j\right|^p$$
$$\le C\sum_{k=1}^{N-1}\frac{k^\beta}{k^{\beta+1}\log^\tau(k)}.$$

Since we assume that $\tau>1$, the last term is finite, as desired. Then, by applying Beppo Levi's theorem, we can conclude the proof for $\beta>1$. For the case where $\beta=1$, we can follow the same steps as before and use Corollary 1 with $d(m,n)=(n-m)$.

4 Discussion and applications

Let $\{X_k, k\ge 1\}$ be a sequence of random variables. It is well-known that for $p\ge 1$,

$$\mathbb{E}\left|\sum_{k=m+1}^{n}X_i\right|^p\le (n-m)^{p-1}\sum_{k=m+1}^{n}\mathbb{E}\left|X_i\right|^p.$$

If $\{X_k, k\ge 1\}$ is a sequence of independent random variables with $\mathbb{E}X_k=0$ and $\mathbb{E}|X_k|^p<\infty$ for some $1\le p\le 2$, then $\mathbb{E}\left|\sum_{k=1}^{n}X_k\right|^p\le C_p\sum_{k=1}^{n}\mathbb{E}\left|X_k\right|^p$ (see von Bahr and Esseen [20]). In what follows we consider that the sequence $\{X_k, k\in\mathbb{N}\}$ satisfying

$$\mathbb{E}\left|\sum_{k=m+1}^{n}X_i\right|^p\le (n-m)^{\alpha-1}\sum_{k=m+1}^{n}\mathbb{E}\left|X_k\right|^p,$$

for some $\alpha\in[1,p]$. Although Corollary 1 is applicable in this case, the following theorem proves to be more effective in certain special cases of $\mathbb{E}\left|X_k\right|^p$.

Theorem 3. *Let $p>1$ and let $\phi:\mathbb{R}\to\mathbb{R}^+$ be a non-decreasing function such that $\phi(2x)\le C\phi(x)$, for any $x\in\mathbb{R}^+$. Then if $1\le\alpha\le 2$,*

$$\sum_{k\ge 1}\frac{\log^p(k)k^{\alpha-1}\mathbb{E}\left|X_k\right|^p}{\phi^p(k)}<\infty\Longrightarrow\lim_{N\to\infty}\frac{1}{\phi(N)}\sum_{k=1}^{N}X_k=0\quad a.s.. \tag{16}$$

If $\alpha>2$,

$$\sum_{k\ge 1}\frac{k^{\alpha-1}\mathbb{E}\left|X_k\right|^p}{\phi^p(k)}<\infty\Longrightarrow\lim_{N\to\infty}\frac{1}{\phi(N)}\sum_{k=1}^{N}X_k=0\quad a.s..$$

Notice that, if we take

$$\begin{cases}\mathbb{E}\left|X_k\right|^p=\frac{1}{k},\\ d(m,n)=(n-m)^{\alpha-1}\sum_{k=m+1}^{n}\mathbb{E}\left|X_k\right|^p,\\ \phi(n)=(d(0,n)\log^\tau(d(0,n)))^{1/p},\end{cases}$$

then by Corollary 1, the convergence of equation (16) is guaranteed as long as $\tau > 1 + p$, since there exists a positive σ such that $n^\sigma \leq Cn^{\alpha-1}\sum_{k=1}^n \mathbb{E}|X_k|^p$, where $1 < \alpha \leq 2$. In contrast, Theorem 3 only requires a value of τ greater than p to achieve convergence. Similarly, if $\alpha > 2$, Corollary 1 necessitates $\tau > 1$ for convergence, while the previous theorem only requires $\tau > 0$.

Proof. The proof for the first part of the theorem will be presented, and the second part can be demonstrated using the same process. From the proof of Theorem 1, specifically equation (6), *step 1* and *step 4*, to obtain (16), it remains to show that

$$\sum_{j\geq 1} \frac{2^{j(\alpha-1)}\sum_{k=1}^{2^j}\mathbb{E}|X_k|^p}{\phi^p(2^j)} < \infty \quad \text{and} \quad \sum_{j\geq 1} \frac{j^p 2^{j(\alpha-1)}\sum_{k=2^j+1}^{2^{j+1}}\mathbb{E}|X_k|^p}{\phi^p(2^j)} < \infty. \tag{17}$$

Since $\phi(2x) \leq C\phi(x)$,

$$\begin{aligned}\sum_{j\geq 1} \frac{j^p 2^{j(\alpha-1)}\sum_{k=2^j+1}^{2^{j+1}}\mathbb{E}|X_k|^p}{\phi^p(2^j)} &\leq C\sum_{j\geq 1}\sum_{k=2^j+1}^{2^{j+1}} \frac{\log^p(k)k^{\alpha-1}\mathbb{E}|X_k|^p}{\phi^p(k)} \\ &\leq C\sum_{k\geq 3}\frac{\log^p(k)k^{\alpha-1}\mathbb{E}|X_k|^p}{\phi^p(k)},\end{aligned}$$

which is finite by assumption. Now, we prove the first half of (17). Provided that $\left(\frac{\phi^p(n)}{n^{\alpha-1}\log^p(n)}\right)_{n\in\mathbb{N}}$ is a strictly increasing sequence and approaches infinity as n approaches infinity, we have from the assumption of the theorem and from Kronecker's lemma that

$$\lim_{n\to\infty} \frac{\log^p(n)n^{\alpha-1}}{\phi^p(n)}\sum_{k=1}^n \mathbb{E}|X_k|^p = 0.$$

Consequently

$$\frac{n^{\alpha-1}}{\phi^p(n)}\sum_{k=1}^n \mathbb{E}|X_k|^p \leq C\frac{1}{\log^p(n)}.$$

The proof ends with $p > 1$. $\square$

The following corollary is a direct application of the previous theorem using $\phi(n) = \left(n^{\alpha-1}\log^\tau(n)\sum_{k=1}^n \mathbb{E}|X_k|^p\right)^{1/p}$.

Corollary 3. *If $1 \leq \alpha \leq 2$, then*

$$\sum_{k\geq 1}\frac{\mathbb{E}|X_k|^p}{\log^{\tau-p}(k)\sum_{i=1}^k \mathbb{E}|X_i|^p} < \infty \Longrightarrow \lim_{N\to\infty}\frac{\sum_{k=1}^N X_k}{\left(N^{\alpha-1}\log^\tau(N)\sum_{k=1}^N \mathbb{E}|X_k|^p\right)^{1/p}} = 0 \quad a.s..$$

If $\alpha > 2$, then

$$\sum_{k\geq 1}\frac{\mathbb{E}|X_k|^p}{\log^{\tau}(k)\sum_{i=1}^k \mathbb{E}|X_i|^p} < \infty \Longrightarrow \lim_{N\to\infty}\frac{\sum_{k=1}^N X_k}{\left(N^{\alpha-1}\log^\tau(N)\sum_{k=1}^N \mathbb{E}|X_k|^p\right)^{1/p}} = 0 \quad a.s..$$

4.1 Application with slowly varying sequences

A slowly varying function $L(\cdot)$ is a positive and measurable function on $[A, \infty)$ for some $A > 0$, such that for any $\lambda > 0$,

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1.$$

The de Bruijn conjugate of a slowly varying function $L(\cdot)$ is another slowly varying function $\tilde{L}(\cdot)$ that is unique up to asymptotic equivalence. The de Bruijn conjugate satisfies two conditions:

$$\lim_{x\to\infty} L(x)\tilde{L}(xL(x)) = 1 \quad \text{and} \quad \lim_{x\to\infty} \tilde{L}(x)L(x\tilde{L}(x)) = 1.$$

Let $\beta > \frac{\alpha}{p}$ and let

$$f(x) = \left(x^{\frac{\beta p-\alpha+1}{\beta p}} L\left(x^{\frac{\beta p-\alpha+1}{\beta p}}\right)\right)^{\beta p} \quad \text{and} \quad g(x) = \left(x^{\frac{1}{\beta p}} \tilde{L}\left(x^{\frac{1}{\beta p}}\right)\right)^{\frac{\beta p}{\beta p-\alpha+1}}$$

be defined on $[A, \infty)$. The following result is a direct consequence of Theorem 1.5.12 and Proposition 1.5.15 in [2] (see also Lemma 2.1 in [1]),

$$\lim_{x\to\infty} \frac{f(g(x))}{x} = \lim_{x\to\infty} \frac{g(f(x))}{x} = 1.$$

Let now, $\{X_k, k \geq 1\}$ be an identically distributed sequence such that $\mathbb{E}\left(g(|X_1|^p)\right) < \infty$. It is known that for a non-negative random variable Y, $\mathbb{E}(Y) < \infty$ if and only if $\sum_{k\geq 1} P(Y > k) < \infty$. This gives

$$\sum_{k\geq 1} \mathbb{P}(|X_1|^p > f(k)) = \sum_{k\geq 1} \mathbb{P}(g(|X_1|^p) > k) < \infty.$$

Theorem 4. *Let $Y_k = X_k I\left(|X_k|^p \leq f(k)\right)$, then*

$$\lim_{N\to\infty} \frac{1}{N^\beta \log(N) L^\beta(N)} \sum_{k=1}^{N} Y_k \quad a.s..$$

Proof. Based on Theorem 3, it is enough to demonstrate that

$$\sum_{k\geq 2} \frac{\log^p(k) k^{\alpha-1} \mathbb{E}\, |Y_k|^p}{(k^\beta \log(k) L^\beta(k))^p} < \infty.$$

To do it, we write

$$\begin{aligned}\sum_{k\geq 2} \frac{\log^p(k) k^{\alpha-1} \mathbb{E}\, |Y_k|^p}{(k^\beta \log(k) L^\beta(k))^p} &= \sum_{k\geq 2} \frac{k^{\alpha-1} \mathbb{E}\left(|X_k|\, I\left(|X_k|^p \leq f(k)\right)\right)^p}{k^{\beta p} L^{\beta p}(k)} \\ &= \mathbb{E}\left(|X_1|^p \sum_{k\geq 2} \frac{I\left(|X_1|^p \leq f(k)\right)}{k^{\beta p-\alpha+1} L^{\beta p}(k)}\right).\end{aligned}$$

Since $\beta > \frac{\alpha}{p}$, it follows from Theorem 8.7 in [23], that $\sum_{k\geq n} \frac{1}{f(k)} \leq C \frac{n}{f(n)}$ for any $n \geq g(A)$. Thus

$$|X_1|^p \sum_{k\geq 2} \frac{I\left(g(|X_1|^p) \leq k\right)}{k^{\beta p-\alpha+1} L^{\beta p}(k)} \leq C|X_1|^p \frac{g(|X_1|^p)}{f(g(|X_1|^p)} \leq Cg(|X_1|^p)$$

where the last inequality follows from the fact that $f(g(x)) \sim x$. Since $\mathbb{E}\left[g(|X_1|^p)\right] < \infty$, the proof is complete. □

4.2 Sequence of quasi-stationary random variables.

The van der Corput lemma will come in handy later on.

Lemma 4. *(Van der Corput) Let m and n be integers such that $m < n$. Let $(u_k)_{k\in\mathbb{N}}$ be a finite sequence which takes values in $\mathbb{C}$ and H a integer such that $H < n - m$. Then*

$$\left|\sum_{k=m+1}^{n} u_k\right|^2 \leq \frac{n-m+H}{H+1}\left(\sum_{k=m+1}^{n}|u_k|^2 + 2\sum_{h=1}^{H}\left(1-\frac{h}{H+1}\right)\sum_{k=m+1}^{n-h}\Re(u_{k+h}\cdot\bar{u}_k)\right). \tag{18}$$

Let $f : \mathbb{N} \to \mathbb{R}^+$ be a non-negative function. Let $\{X_k, k \in \mathbb{N}\}$ be a sequence of random variables. We say that $\{X_k, k \geq 1\}$ is $f-$quasi-stationary if for any $h \in \mathbb{N}$,

$$\mathbb{E}(X_k) = 0, \quad \mathbb{E}(X_k^2) < \infty \quad \text{and} \quad \mathbb{E}(X_k X_{k+h}) \leq f(h).$$

According to (18) with $H = n - m - 1$,

$$\begin{aligned}
\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^2 &\leq C\sum_{k=m+1}^{n}\mathbb{E}|X_k|^2 + \frac{C}{n-m}\sum_{h=m+1}^{n-1}(n-h)\sum_{k=m+1}^{n+m-h}|\mathbb{E}(X_kX_{k+h})| \\
&\leq C\sum_{k=m+1}^{n}\mathbb{E}|X_k|^2 + \frac{C}{n-m}\sum_{h=m+1}^{n-1}(n-h)^2 f(h) \\
&\leq C\sum_{k=m+1}^{n}\mathbb{E}|X_k|^2 + C(n-m)\sum_{h=m+1}^{n-1} f(h).
\end{aligned}$$

The following theorem is a direct consequence of Theorem 3.

Theorem 5. *Let $\{X_k, k \geq 1\}$ be a $f-$quasi-stationary sequence and let ϕ be a non-decreasing function. Suppose that*

$$\sum_{k\geq 2}\frac{\log^2(k)k\sum_{i=1}^{k} f(i)}{\phi^2(k)} < \infty.$$

Then

$$\lim_{N\to\infty}\frac{1}{\phi(N)}\sum_{k=1}^{N} X_k = 0 \quad a.s..$$

It is possible to push the analysis further. Specifically, suppose that for any $h > 0$, $|\mathbb{E}(X_kX_{k+h})| \leq g(k)f(h)\mathbb{E}|X_k|^2$, where g is a positive function. Using the Van der Corput lemma allows us to obtain

$$\mathbb{E}\left|\sum_{k=m+1}^{n} X_k\right|^2 \leq C\sum_{k=m+1}^{n}\mathbb{E}|X_k|^2 + C\sum_{h=m+1}^{n-1} f(h)\sum_{k=m}^{n+m-h-1} g(k)\mathbb{E}|X_k|^2.$$

Theorem 3 can also be applied.

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