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# Towards Possibilistic Decision Theory

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**Abstract.** Fuzzy sets and possibility theory offer a unified framework where both preferences and uncertainty can be modelled. Indeed fuzzy sets can be used for describing weakly ordered sets of more or less acceptable/preferred situations, and possibility distributions represent states of information pervaded with imprecision and uncertainty. This framework relies on purely ordinal scales both for preferences and for uncertainty, due to the use of max and min operations and of an order-reversing operation for manipulating the levels of these two scales.

Recently, a qualitative counterpart to von Neumann and Morgenstern' expected utility theory has been proposed, and it has been shown that the qualitative utility function, agreeing with a set of axioms describing decision-maker's behavior in face of uncertainty, is nothing but the necessity measure of a fuzzy event.

This result is first recalled and its interpretation is discussed in the framework of possibility theory. It is related to previous proposals made by different authors in the fuzzy set literature. The necessity measure which is a pessimistic estimate is contrasted with the dual measure of possibility. This latter estimate is optimistic and deals in a symmetric way with the expression of preferences and the expression of what is known of the plausible states of the world. The intuition underlying the necessity measure is that the qualitative "expected" utility is all the greater as there is no situations with a high plausibility and low utility value. A refinement of the utility ordering is suggested in case several decisions receive the same evaluation. Then an illustrative example is provided. Lastly, applications to multistage decision making and to matrix games are briefly pointed out.

## 1 - Qualitative Utility Theory

Von Neumann and Morgenstern (1944)' expected utility theory relies on the principle that the decision maker's behavior in face of uncertainty is entirely determined by his/her preferences on the uncertainty distributions about the consequences of his/her actions. In von Neumann and Morgenstern' model these distributions are probability distributions and are called "lotteries". Preferences about lotteries should fulfil a set of axioms describing the attitude of a "rational" decision maker in the face of uncertainty. The expected utility, in von Neumann and Morgenstern' approach, provides a simple criterion to rank-order the lotteries, and thus the actions, since each lottery is associated with an action (of which the associated distribution represents the uncertainty attached to its possible consequences). The decision-maker is then "rational" if the choice of his/her actions is in agreement with his/her preferences on the lotteries.

Let  $X$  be a finite set of situations (states of the world). An action in situation  $x$  results in a consequence with which the decision-maker is concerned. The set of consequences of a decision, obtained by varying the state of the world, can be ordered in terms of preference. This induces a preference ordering over  $X$ , reflecting back the

expected pay-off of the action (the benefit of being in a precise situation). This preference relation between precisely-known situations for a given decision should be extended to incompletely described situations pervaded with uncertainty.

As already said, we use only two qualitative scales, denoted respectively by  $U$  and  $V$ , for assessing preferences on the one hand and uncertainty on the other hand. A belief state about which situation in  $X$  is the actual one, is supposed to be represented by a possibility distribution  $\pi$  from  $X$  to  $V$ .  $V$  is assumed to be bounded, and we take  $\sup V = 1$ ,  $\inf V = 0$ .  $\pi(x) \in V$  estimates the plausibility level of being in situation  $x$ . The possibility distributions we consider are normalized, i.e.,  $\exists x, \pi(x) = 1$ , which expresses that at least one situation in  $X$  is completely possible (there may be several completely possible situations however). A possibility distribution representing a belief state involves a set of mutually exclusive alternatives, where each element can be ranked according to its level of plausibility to be the true situation. Let  $x$  and  $y$  be two elements of  $X$ , the possibility distribution  $\pi$  defined by  $\pi(x) = \lambda$ ,  $\pi(y) = \mu$ ,  $\pi(z) = 0$  for  $z \neq x, z \neq y$  with  $\max(\lambda, \mu) = 1$  (in order to have  $\pi$  normalized), will be called a *qualitative binary lottery* and will be denoted by  $(\lambda/x, \mu/y)$ , which means that we are either in situation  $x$  or in situation  $y$  with the respective levels of possibility  $\lambda$  and  $\mu$ . More generally, any possibility distribution  $\pi$  can be viewed as a multiple-consequence lottery  $(\lambda_1/x_1, \dots, \lambda_m/x_m)$  where  $X = \{x_1, \dots, x_m\}$  and  $\lambda_i = \pi(x_i)$ . We will also use the notation  $(\lambda/\pi_1, \mu/\pi_2)$  (with  $\max(\lambda, \mu) = 1$ ) for denoting the compound possibility distribution  $\pi = \max(\min(\pi_1, \lambda), \min(\pi_2, \mu))$ . This can be viewed as a lottery over multiple-consequence lotteries corresponding to  $\pi_1$  and  $\pi_2$ . The lottery  $(\lambda/x, \mu/y)$  can be viewed as a particular case of it when  $\pi_1$  and  $\pi_2$  are possibility distributions focusing on singletons. The resulting possibility distribution  $\pi = \max(\min(\pi_1, \lambda), \min(\pi_2, \mu))$ , with  $\max(\lambda, \mu) = 1$ , is here the qualitative counterpart of probabilistic mixtures  $\lambda p_1 + (1-\lambda)p_2$ ; see (Dubois and Prade, 1990a; Dubois et al., 1993).

Let  $\succeq$  denote the preference relation between possibility distributions ("possibilistic lotteries") given by the decision-maker, which extends the preference ordering over  $X$  to normalized possibility distributions in  $V^X$ . A singleton  $\{x_0\}$  corresponds to the possibility distribution which is zero everywhere except in  $x_0$  where  $\pi(x_0) = 1$ . When compared through the ordering relation  $\succeq$ , a singleton, as a particular case of a possibility distribution, will be either denoted by  $\{x_0\}$  or more simply by  $x_0$  since the ordering is extended from  $X$  to  $V^X$ .  $\succeq$  is supposed to satisfy the following axioms, where  $\pi \sim \pi'$  means that both  $\pi \succeq \pi'$  and  $\pi' \succeq \pi$  hold.

**Axiom 1:**  $\succeq$  is a complete partial ordering.

**Axiom 2** (certainty equivalence):

If the belief state is a crisp set  $A \subseteq X$ ,  
then there is  $x \in A$  such that  $\{x\} \sim A$ .

**Axiom 3** (risk aversion, or "precision is safer"):

$$\pi \leq \pi' \Rightarrow \pi \succeq \pi'.$$

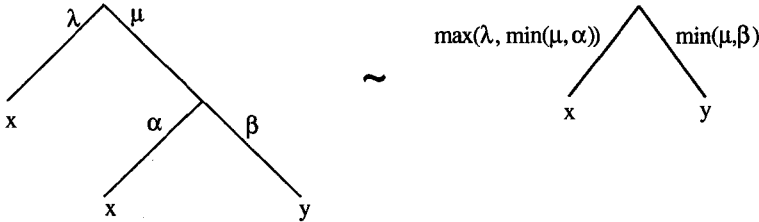
**Axiom 4** ("independence"):

$$\text{If } \pi_1 \sim \pi_2 \text{ then } (\lambda/\pi_1, \mu/\pi') \sim (\lambda/\pi_2, \mu/\pi').$$

**Axiom 5** (reduction of lotteries):

$$(\lambda/x, \mu / (\alpha/x, \beta/y)) \sim (\max(\lambda, \min(\mu, \alpha)) / x, \min(\mu, \beta) / y).$$

See Figure 1 for visualizing the tree reduction expressed by this axiom.



**Figure 1**

**Axiom 6** (continuity):

$$\text{If } \pi \succeq \pi' \text{ then } \exists \lambda \in V, \pi' \sim (1/\pi, \lambda/X).$$

Some of these axioms are similar to von Neumann and Morgenstern's axioms like the Axiom 1 that they use, or the Axioms 4, 5 and 6 which are qualitative counterparts of their axioms. Axiom 2 comes down to rejecting the notion of mean value. It is based on the idea that when the decision is made and put to work, then the state will be some  $x \in A$ , and the benefit of the decision will indeed be the one in state  $x$ . The scope of such a decision theory is thus the next decision, and not an indeterminate sequences of decision, as with expected utility. Axiom 3 expresses a form of risk aversion and more precisely, an aversion for the lack of information. Axiom 4 is self-explanatory. Axiom 5 is motivated by the particular form of mixtures in possibility theory. Axiom 6 expresses that it is possible to pass continuously from a given state of belief  $\pi$  to total ignorance by attaching a degree of uncertainty to  $\pi$  and moving it from 0 ( $\pi$  is sure) to one (total ignorance). By Axiom 3,  $x \succeq A, \forall x \in A$ , and since by Axiom 2, if  $x_A \sim A$ , then  $\forall x \in A, x \succeq x_A$ , so that  $A$  is equivalent to the worst state in  $A$ . It indicates that the proposed decision theory is cautious in essence.

Then the following theorem can be established (Dubois and Prade, 1995).

**Theorem:** Given a preference relation  $\succeq$  on the normalized possibility distributions of  $V^X$  verifying Axioms 1 to 6, there exists a fuzzy set  $F$  on  $X$  (an element of  $U^X$ , for a totally ordered set  $U$ ) and a utility function  $u$  from  $V^X$  to  $U$  representing the preference ordering  $\succeq$  such that for each normalized possibility distribution  $\pi$ , we have

$$u(\pi) = \min_{x \in X} \max(n(\pi(x)), \mu_F(x)) \tag{1}$$

where  $n$  is an order reversing function from the possibility scale  $V$  to the preference scale  $U$  such that  $n(0) = 1$  and  $n(1) = 0$  where 1 denotes the top elements of  $U$  and  $V$  and 0 their bottom elements.

Note that (1) yields  $u(x) = \mu_F(x)$ .

## 2 - Interpretation and Relations to Other Works

Interestingly enough, the qualitative utility introduced in the previous section,

$$u(\pi) = \min_{x \in X} \max(n(\pi(x)), u(x))$$

is the necessity of a fuzzy event (Dubois and Prade, 1980) in the sense of possibility theory, namely  $u(\pi) = N_{\pi}(F)$  where  $F$  is the fuzzy set of preferred situations ( $\mu_F(x) = u(x)$ ,  $\forall x \in X$ ) and  $N_{\pi}$  is the necessity measure based on the possibility distribution  $\pi$ . Usually, when  $V = U = [0,1]$ ,  $n(t) = 1 - t$  in the above expression.  $N_{\pi}(F)$  can be viewed as a degree of inclusion of the fuzzy set of more or less possible situations in the fuzzy set  $F$  of preferred outcomes, i.e., it estimates the certainty that the belief state  $\pi$  corresponds to the preferred situations described by  $F$ . It means that there is a commensurability assumption made between the uncertainty scale and the preference scale, since possibility degrees and utility degrees are aggregated in the expression of  $u(\pi)$ . Note that

$$\bullet N_{\pi}(F) = u(\pi) = 1 \text{ iff } \{x \in X, \pi(x) > 0\} \subseteq \{x \in X, u(x) = 1\}$$

i.e., the utility of  $\pi$  is maximal if all the more or less possible situations encompassed by  $\pi$  are among the most preferred ones. In this case, whatever the precise situation, its utility is maximal.

$$\bullet N_{\pi}(F) = u(\pi) = 0 \text{ iff } \{x \in X, \pi(x) = 1\} \cap \{x \in X, u(x) = 0\} \neq \emptyset$$

i.e., the utility of  $\pi$  is minimal if there is one of the most plausible situations whose pay-off is minimum (we recognize the risk-aversion of the approach).

Thus,  $N_{\pi}(F) = u(\pi)$  is all the greater as there is no situations with a high plausibility and low utility value.

When  $\pi$  is the characteristic function of an ordinary subset  $A$  of  $X$ , i.e., when all the situations encompassed by the belief state are equally plausible, the utility  $u(\pi)$  simplifies into  $u(\pi) = \min_{x \in A} u(x)$  where we recognize Wald (1950)'s pessimistic criterion which leads to decisions maximizing the minimal pay-off. In the general case,  $u(\pi)$  takes into account the fact that all the situations are not equally plausible in the set  $\{x \in X, \pi(x) > 0\}$ .

Several fuzzy set authors have proposed definitions of utility functions in the presence of possibilistic uncertainty, including the form described in the theorem. Indeed, the possibilistic counterpart of Wald maximin criterion of the form proposed here, has been introduced without any axiomatic justification by Whalen (1984), in terms of a "disutility" function  $D(\pi) = n^{-1}(u(\pi))$  where  $u(\pi)$  is given by (1).  $D(\pi)$  takes the form of the degree of possibility of the fuzzy set  $\bar{F}$ , i.e.,  $\mu_{\bar{F}} = n^{-1} \circ \mu_F$  (the fuzzy complement of  $F$  when  $V = U = [0,1]$ ) of less preferred situations, namely

$$D(\pi) = \max_{x \in X} \min(\pi(x), \mu_{\bar{F}}(x)). \quad (2)$$

Previously, Yager (1979) has introduced the possibilistic extension of the optimistic maximax criterion  $E(\pi)$  of the form dual to (1), i.e.,

$$E(\pi) = \max_{x \in X} \min(\pi(x), \mu_F(x)) \quad (3)$$

which is the degree of possibility of a fuzzy set (Zadeh, 1978). This optimistic utility has been also advocated by Mathieu-Nicot (1985). However, observe that we always have

$$u(\pi) \leq E(\pi) \quad (4)$$

and that choosing an action which maximizes  $E(\pi)$  rather than  $u(\pi)$  can be overoptimistic. Indeed consider the case where  $F$  is the crisp subset of the acceptable states of the worlds, and  $\pi$  represents an incomplete state of information corresponding to an ordinary subset  $P$  of  $X$ . Thus  $u(\pi)$  as well as  $E(\pi)$  can take only the values 0 or 1. Maximizing  $u(\pi)$  is equivalent to selecting the actions such that their (possibly ill-known) consequences, whatever they are, ensure that one stays in an acceptable state of the world, since  $P \subseteq F \Leftrightarrow u(\pi) = 1$ . Maximizing  $E(\pi)$  leads to select the larger set of actions such that  $P \cap F \neq \emptyset$ . It includes actions which may have consequences which are unacceptable if  $P \cap \bar{F} \neq \emptyset$ . It amounts to assuming that the state of the world lies in  $P \cap F$ , i.e., to take one's wishes (i.e.,  $F$ ) for the reality!...

However, we may have  $u(\pi) = 0$  for all the actions under consideration; in that case  $E(\pi)$  may be useful to make a choice; namely when all actions are equally risky it is better to select the one offering the best opportunity.

The estimates  $u(\pi)$  and  $E(\pi)$  are the basis for fuzzy pattern matching evaluations (Cayrol et al., 1982) where a fuzzy pattern expressing preferences about the values of some attribute(s) used for describing the required items, is matched against what is known about the attribute values of the items stored in a database.  $\pi$  then represents what is known about a given item and  $u(\pi)$  (resp.  $E(\pi)$ ) estimates to what extent it is certain (resp. possible) that the item satisfies the requirement expressed by the pattern, i.e., to what extent the item has to be selected.

Besides, as already pointed out (e.g., (Inuiguchi et al., 1989)), the expression of the necessity of a fuzzy event is a particular case of a fuzzy integral in the sense of Sugeno (1974). Namely  $N_\pi(F)$  can be shown to be equal to (for  $V = U = [0,1]$ )

$$N_\pi(F) = \sup_{\alpha \in (0,1]} \min(\alpha, N_\pi(F_\alpha)) \quad (5)$$

with  $F_\alpha = \{x \in X, \mu_F(x) \geq \alpha\}$ , which is a particular case of Sugeno integral

$$\int_X h(x) \circ g(\cdot) = \sup_{\alpha \in (0,1]} \min(\alpha, g(H_\alpha)) \quad (6)$$

with  $H_\alpha = \{x \in X, h(x) \geq \alpha\}$  and  $g$  is a set function monotonic with respect to set inclusion, such that  $g(\emptyset) = 0$  and  $g(X) = 1$ . Sugeno integrals can be regarded as qualitative counterparts to Choquet integrals of the form  $\int_0^1 g(H_\alpha) d\alpha$ . See Dubois and Prade (1990b) for instance.

### 3 - Refining the Utility Ordering

As already said an action, associated with a possibility distribution  $\pi$ , is evaluated on the basis of the worst resulting situation with respect to this action, namely, a situation which has a rather high plausibility degree and a low utility value, since

$$u(\pi) = \min_{x \in X} \max(n(\pi(x)), \mu_F(x)).$$

Formally speaking, each possible situation  $x$  behaves as a criterion according to which each action is evaluated. Good actions, from the point of view of situation  $x$ , are the ones that are such that if  $x$  has a low utility value,  $x$  is not plausible for these actions. Indeed  $\max(n(\pi(x)), \mu_F(x)) = \max(n(\mu_{\bar{F}}(x)), n(\pi(x))) = \mu_{\bar{F}}(x) \rightarrow n(\pi(x))$  in the sense of Dienes' implication ( $a \rightarrow b = \max(n(a), b)$ ), where  $\mu_{\bar{F}}(x)$  estimates how much  $x$  has a low utility value.

Thus  $u(\pi)$  can be formally viewed as a min-conjunctive multiple criteria evaluation. This type of evaluation may lead to actions having the same estimate, i.e., the same level for the "worst" case, but which still might be compared according to the other situations. Refinements of the min ordering have been recently discussed in the framework of the fuzzy set approach to multiple criteria decision-making; see (Dubois, Fargier and Prade, 1995b). Two refinements of the min-ordering are noticeable. Consider two vectors of grades  $\vec{u} = (u_1, \dots, u_m)$  and  $\vec{v} = (v_1, \dots, v_m)$  according to  $m$  criteria. Assume that the vectors are increasingly rearranged into  $\vec{u}^* = (u_{i_1}, \dots, u_{i_m})$  and  $\vec{v}^* = (v_{j_1}, \dots, v_{j_m})$  where  $u_{i_1} \leq \dots \leq u_{i_m}$  and  $v_{j_1} \leq \dots \leq v_{j_m}$ . Then so-called leximin and least satisfied discriminating criterion orderings can be defined:

- leximin:

$$\vec{u} >_{\text{leximin}} \vec{v} \Leftrightarrow \exists k \in \{1, \dots, m\} \text{ s.t. } \begin{array}{l} \text{i) } \forall \lambda < k, u_{i_\lambda} = v_{j_\lambda} \\ \text{ii) } u_{i_k} > v_{j_k} \end{array}$$

- least satisfied discriminating criterion (discrimin)

$$\vec{u} >_{\text{LSDC}} \vec{v} \Leftrightarrow \forall k \in \{1, \dots, m\} \text{ s.t. } \begin{array}{l} \text{i) } \forall \lambda < k, u_{i_\lambda} = v_{j_\lambda} \\ \text{ii) } u_{i_k} > v_{j_k}. \end{array}$$

Clearly, the leximin ordering refines the discrimin ordering which itself refines the min ordering ( $\vec{u} >_{\text{min}} \vec{v} \Leftrightarrow u_{i_1} > v_{j_1}$ ).

These two refinements can be applied so as to distinguish between best actions which are considered equal in the sense of  $u(\pi)$ . In particular, consider the particular case where all the actions are such that  $u(\pi) = 0$ ; it means that any action has totally plausible consequences which are completely undesirable (i.e., s.t.  $\exists x, \pi(x) = 1$  and  $\mu_F(x) = 0$ ). Then, for instance, it is natural to prefer an action such that  $u(\pi) = 0$  due to *only one* possible bad situation  $x$ , to an action such that there exist two or more  $x$  such that  $\max(n(\pi(x)), \mu_F(x)) = 0$ . In other words, preference is given to the action(s) such that there exist *almost no* situation with high plausibility and low utility.

## 4 - Illustrative Example

A slightly simpler version of the following example is briefly commented in (Dubois and Prade, 1995). It is discussed here in somewhat greater details. Assume you have to leave home so as to take the subway or your car in order to arrive on time at some meeting. In such a problem there are several criteria: i) you do not want to leave home too early (let  $M$  be the fuzzy set of departure times which are acceptable;  $M$  has a non-decreasing membership function since the later you leave, the better it is), ii) you do not want to arrive too late at the meeting (let  $N$  be the fuzzy set of arrival times which are acceptable;  $N$  has a non-increasing membership function), iii) you may have some preference between taking the subway or taking your car; let  $\mu_T(\text{subway})$  and  $\mu_T(\text{car})$  being respectively the level of acceptability of each transportation system (we assume normalization:  $\max(\mu_T(\text{subway}), \mu_T(\text{car})) = 1$ ). Besides, there is some uncertainty about the duration of the trip between your home and the meeting. If you take the subway, you may have to wait for it at that time of the day, and if you take your car there may be a traffic jam. Let  $\pi(z,t)$  be the possibility that the duration is  $z$  when the chosen transportation mode is  $t$ . Then the decision (choice of a departure time  $s$ , and of the type  $t$  of transportation) is obtained by finding  $s$  and  $t$  which maximizes the multiple criteria evaluation

$$\min(\mu_T(t), \mu_M(s), \inf_z \max(1 - \pi(z,t), \mu_N(s + z))) \quad (7)$$

using  $n(a) = 1 - a$  and  $U = V = [0,1]$ . This expression can be understood as a multiple-valued logic evaluation of the sentence

$$\exists t, t \in T \text{ and } \exists s, s \in M \text{ and } \forall z, \text{ if } z \in D(t) \text{ then } s + z \in N$$

where  $\mu_{D(t)}(z) = \pi(z,t)$  is the membership function of the fuzzy set of the possible values of the duration of the trip for the transportation mode  $t$ . Indeed, the maximization and the minimization are multiple-valued counterparts of the existential and universal quantifiers respectively and  $\max(1 - a, b)$  is a multiple-valued implication. The term  $\inf_z \max(1 - \pi(z,t), \mu_N(s + z))$  is the qualitative "expected" utility of choosing  $s$  as a starting time decision, when  $t$  is chosen as transportation mode, where  $\mu_N(s + z)$  is the preference degree in situation  $z$  for decision  $s$  (evaluating to what extent the arrival time constraint is satisfied) while  $\pi(\cdot, t)$  represents the incomplete knowledge about the situation. Note that  $z$  and  $t$  are variables of a different nature;  $z$  refers to the duration whose precise value is not under decision-maker's control, while  $t$  is a decision variable, thus under his/her control. The expression (7) thus reflects the conjunctive aggregation of the criteria, taking into account the uncertainty. See (Dubois, Fargier and Prade, 1995a) for a numerical treatment of the example and for the application of this approach to job-shop scheduling; it is also shown in this reference why the supremum on  $s$  and  $t$  of (7) is still equal to

$$\sup_{a,t} \min(\mu_T(t), \inf_z \max(1 - \pi(z,t), \mu_M(a - z)), \mu_N(a))$$

since we can also see the problem as finding out an acceptable arrival time  $a$  such that the corresponding departure time  $a - z$  is acceptable whatever the possible value of  $z$ , when the transportation mode is  $t$ . However, expression (7) is more natural since, in



practice, we are interested in the departure time. It is also shown in (Dubois et al., 1995a) that when dealing with continuous membership functions and possibility distributions, the 'inf<sub>z</sub> max' subpart of (7) can be rewritten as a 'sup<sub>z</sub> min' expression, which facilitates the computation.

## 5 - Multiplestage Decision

In their seminal paper on fuzzy set-based approach to multiple criteria decision, Bellman and Zadeh (1970) have applied their approach to multistage decision-making in order to illustrate the concepts of fuzzy goal, fuzzy constraint and fuzzy decision. See also Fung and Fu (1977). Then the decision-viewed as a decomposable fuzzy set was expressed as

$$\mu_D(u_0, \dots, u_{N-1}) = \min(\mu_{C_0}(u_0), \dots, \mu_{C_{N-1}}(u_{N-1}), \mu_{G_N}(x_N))$$

where

- $C_0, \dots, C_{N-1}$  are the fuzzy constraints at stages 0, ..., t, ..., N-1 on the applicable inputs  $u_t$
- $G_N$  is the fuzzy goal describing the desired state  $x_N$  of the system at stage N
- $x_N$  is expressible as a function of  $x_0$  and  $u_0, \dots, u_{N-1}$  through the iteration of the state equation

$$x_{t+1} = f(x_t, u_t), t = 0, 1, 2, \dots$$

Although they mentioned the idea of also dealing with a fuzzy system described by a membership function of the form  $\mu(x_{t+1} | x_t, u_t)$ , Bellman and Zadeh (1970) only considered the case of a deterministic system governed by the above state equation and of a stochastic system whose state at time  $t + 1$  is a probability distribution  $p(x_{t+1} | x_t, u_t)$  and they applied dynamic programming for finding  $u_0, \dots, u_{N-1}$  maximizing  $\mu_D$  as defined above and the probability of the fuzzy event  $G_N$ ,  $\sum_{x_N} p(x_N | x_{N-1}, u_{N-1}) \cdot \mu_{G_N}(x_N)$  respectively. Kacprzyk (1983) has studied the case of a fuzzy system, described by a fuzzy relation linking  $u_t, x_t$  and  $x_{t+1}$ . Since the precise values of the states of the system are no longer accessible, the extent to which the state of stage N satisfies the goal is only estimated in terms of the possibility of the fuzzy event  $G_N$ , namely

$$\sup_{x_N} \min(\mu_{X_N}(x_N), \mu_{G_N}(x_N))$$

in this approach where dynamic programming and branch-and-bound solutions are provided.

As already said such an estimate of the satisfaction of the goal is too optimistic and we should rather use the necessity of the fuzzy event

$$\inf_{x_N} \max(1 - \mu_{X_N}(x_N), \mu_{G_N}(x_N))$$

as first suggested in (Dubois and Prade, 1982).

Dean (1994) has recently strongly advocated Markov decision processes as a basic representation for planning under uncertainty. A Markov decision process is made of a Markov chain together with a set of actions available to the control system and a time separable expected value function (which enables us to reduce an  $n$ -dimensional problem to  $n-1$ -dimensional problems using dynamic programming). Examples of works in planning under uncertainty along this line are (Draper et al., 1994; Kushmerick et al., 1994; Thiebaut et al., 1995).

The recent introduction of possibilistic Markov chains (Dubois et al., 1994; see also Friedman and Sandler, 1994) and the foundations of a qualitative possibilistic decision theory opens the road to applications to planning under uncertainty when preferences and uncertainty are naturally assessed on ordinal scales.

## 6 - Possibilistic Matrix Games

A matrix game is a simple two-person zero-sum game in which the players have a finite number of alternatives among which they have to choose. There are two players  $P_1$  and  $P_2$  and a  $m \times n$  matrix  $A = (a_{ij})$ . For  $P_1$  the strategies (possible choices) correspond to the  $m$  rows of  $A$  and for  $P_2$  the strategies are the  $n$  columns of  $A$ . When  $P_1$  chooses the  $i^{\text{th}}$  row and  $P_2$  the  $j^{\text{th}}$  column, then  $a_{ij}$  is the outcome of the game with the convention that  $P_1$  has to pay to  $P_2$  this amount. Thus  $P_1$  tries to minimize the outcome and  $P_2$  to maximize it. This leads to the well-known inequality

$$\max_j \min_i a_{ij} \leq \min_i \max_j a_{ij} \quad (8)$$

which expresses that the optimal outcome in the case when  $P_2$  plays the first (securing his gains), is less or equal to the optimal outcome when  $P_1$  plays the first (securing his losses against any behaviour of  $P_2$ ). When (8) holds as an equality, the game is said to have (a) saddle point(s) in pure strategies. Mixed strategies have been proposed where a probability distribution is attached to the strategies of each player (which may be thought as reflecting the frequency with which each strategy is applied by the player). Then a statistical equilibrium can be reached (when the game is sufficiently repeated) under the form

$$\max_q \min_p p^t A q = \min_p \max_q p^t A q \quad (9)$$

where  $p = (p_i)_{i=1,m}$  and  $q = (q_j)_{j=1,n}$  are probability distributions over the strategies of  $P_1$  and  $P_2$  respectively. It is then always possible to get an equilibrium; see (Basar and Olsder, 1982). However Nicolas and Grabisch (1995) have pointed out that the statistical view underlying the mixed strategies is not always realistic in case the game is not repeated and some values in  $A$  strongly (and may be definitively) penalize a player. They propose to replace (8) by

$$\max_j \top_i a_{ij} \leq \min_i \perp_j a_{ij} \quad (10)$$

where  $T_i$  and  $\perp_j$  are t-norm and t-conorm operations respectively (where the  $a_{ij}$ 's are assumed to belong to  $[0,1]$ ), since decision-makers do not always obey minimax strategies under situations of risk, according to these authors. Note that (10) is a consequence of (8), since  $T_i a_{ij} \leq \min_i a_{ij}$  and  $\perp_j a_{ij} \geq \max_j a_{ij}$ .

Observe that the minimax strategies which appear in (8) are in the spirit of Wald's criterion, i.e., maximizing the minimal pay-off. Thus, it is natural to propose a possibilistic generalization of (8). Let  $(\pi_1, \dots, \pi_m)$  (resp.  $(\pi'_1, \dots, \pi'_n)$ ) be a normalized possibility distribution over the possible strategies of player  $P_1$  (resp.  $P_2$ ). Then, the min and max operations in (8) can be replaced by weighted max and min operations (see Dubois and Prade (1986) for a presentation of these weighted operations in a possibility theory perspective). A weighted min is of the form

$$\min_i \max(a_i, 1 - w_i) \tag{11}$$

where the weights  $w_i$  are normalized ( $\max_i w_i = 1$ ). Note that when  $\forall i, w_i = 1$ , it reduces to  $\min_i a_i$  as expected, while  $a_i$  is not taken into account when  $w_i = 0$ . The expression (11) was first proposed in Yager (1981) with a multiple criteria interpretation,  $w_i$  being the level of importance of criterion  $i$ ,  $a_i$  the rate of the object under consideration according to  $i$ ;  $\max(a_i, 1 - w_i)$  was introduced by Yager (1981) as a particular case of a multiple-valued implication connective  $w_i \rightarrow a_i$  (a criterion should be taken into account all the more as it is important). Dually, the weighted max is of the form

$$\max_i \min(a_i, w_i). \tag{12}$$

Again, we recover  $\max_i a_i$  when  $\forall i, w_i = 1$  and  $a_i$  is ignored if  $w_i = 0$ . Then the possibilistic generalization of (8) writes

$$\max_j \min(\pi'_j, \min_i \max(a_{ij}, 1 - \pi_i)) \leq \min_i \max(1 - \pi_i, \max_j \min(a_{ij}, \pi'_j)) \tag{13}$$

where  $a_{ij} \in [0,1], \forall i, \forall j$ . (13) can be easily checked. Indeed, it is equivalent to

$$\max_j \min_i \min(\pi'_j, \max(a_{ij}, 1 - \pi_i)) \leq \min_i \max_j \max(1 - \pi_i, \min(a_{ij}, \pi'_j))$$

and we have  $\min(\pi'_j, \max(a_{ij}, 1 - \pi_i)) = \max(\min(\pi'_j, a_{ij}), \min(\pi'_j, 1 - \pi_i)) \leq \max(1 - \pi_i, \min(a_{ij}, \pi'_j))$ . Thus (13) is formally a consequence of (8). (13) can be also rewritten in terms of the previously introduced expected utility function  $u(\pi)$  and of its optimistic counterpart  $E(\pi)$ , as

$$E(\pi'; u(\pi; A)) \leq u(\pi; E(\pi'; A)). \tag{14}$$

The interpretation of the possibilistic weights could be the following. A not completely possible action for  $P_2$  ( $\pi'_j < 1$ ) may decrease his gain and the loss of  $P_1$ , while a not completely possible action for  $P_1$  ( $\pi_i < 1$ ) may increase the gain of  $P_2$  (and the loss  $P_1$ ). In other words, the choice of a not completely possible action penalizes the player and benefits to his adversary. Thus, choosing a somewhat

impossible action results in a kind of penalty. Here, the possibility degrees refer to feasibility rather than to uncertainty, strictly speaking. The introduction of new strategies based on weighted versions of min and max might be the basis for an improved handling of tactical subjective aspects. This is a topic for further research.

## 7 - Concluding Remarks

Possibility theory and fuzzy sets together provides a framework for dealing in a qualitative way with uncertainty and preferences. Solid basis for decision theory can be offered in this framework, which provides a more qualitative and non-probabilistic view of decision processes — a view that Shackle (1961, 1985) had for a long time advocated by proposing an approach to decision based on potential degrees of surprise (corresponding to degrees of impossibility in possibility theory), much before Zadeh (1978) introduced possibility theory.

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## References

- Basar T., Olsder G.J. (1982) *Dynamic Noncooperative Game Theory*. Academic Press, London.
- Bellman R.E., Zadeh L.A. (1970) Decision-making in a fuzzy environment. *Management Science*, 17, B-141-B-164.
- Cayrol M., Farreny H., Prade H. (1982) Fuzzy pattern matching. *Kybernetes*, 11, 103-116.
- Dean T. (1994) Decision-theoretic planning and Markov decision processes. Tech. Report, Brown University.
- Draper D., Hanks S., Weld D. (1994) A probabilistic model of action for least-commitment planning with information gathering. Proc. of the 10th Conf. on Uncertainty in Artificial Intelligence (UAI'94) (R. Lopez de Mantaras, D. Poole, eds.), 178-186.
- Dubois D., Dupin de Saint-Cyr F., Prade H. (1994) Updating, transition constraints and possibilistic Markov chains. Proc. of the 5th Inter. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'94), Paris, July 4-8, 826-831. Revised version in: *Advances in Intelligent Computing — IPMU'94* (B. Bouchon-Meunier, R.R. Yager, L.A. Zadeh, eds.), Lecture Notes in Computer Science, Vol. 945, Springer Verlag, Berlin, 1995, 263-272.
- Dubois D., Fargier H., Prade H. (1995a) Fuzzy constraints in job-shop scheduling. *J. of Intelligent Manufacturing*, 6(4), 215-234.
- Dubois D., Fargier H., Prade H. (1995b) Refinements to the maximin approach to decision-making in fuzzy environment. *Fuzzy Sets and Systems*, to appear.
- Dubois D., Fodor J.C., Prade H., Roubens M. (1993) Aggregation of decomposable measures with application to utility theory. Tech. Report IRIT/93-55-R, IRIT, Universit  Paul Sabatier, Toulouse, France. To appear in *Theory and Decision*.
- Dubois D., Prade H. (1980) *Fuzzy Sets and Systems: Theory and Applications*. Academic Press, New York.
- Dubois D., Prade H. (1982) What does 'convergence' mean for fuzzy numbers? Proc. of the IFAC Symp. on Theory and Application of Digital Control, New Delhi, India, Jan. 5-7, 433-438.

- Dubois D., Prade H. (1986) Weighted minimum and maximum operations in fuzzy set theory. *Information Sciences*, 39, 205-210.
- Dubois D., Prade H. (1990a) Aggregation of possibility measures. In: *Multiperson Decision Making Using Fuzzy Sets and Possibility Theory* (J. Kacprzyk, M. Fedrizzi, eds.), Kluwer Academic Publ., Dordrecht, 55-63.
- Dubois D., Prade H. (1990b) Scalar evaluations of fuzzy sets: Overview and applications. *Appl. Math. Lett.*, 3(2), 37-42.
- Dubois D., Prade H. (1995) Possibility theory as a basis for qualitative decision theory. *Proc. of the 14th Inter. Joint Conf. on Artificial Intelligence (IJCAI'95)*, Montréal, Canada, Aug. 20-25, 1924-1930.
- Friedman Y., Sandler U. (1994) Evolution of systems under fuzzy dynamic laws. Preprint n° AM-001.94, Jerusalem College of Technology, Jerusalem, Israel.
- Fung L.W., Fu K.S. (1977) Characterization of a class of fuzzy optimal control problems. In: *Fuzzy Automata and Decision Processes* (M.M. Gupta, G.N. Saridis, B.R. Gaines, eds.), North-Holland, New York, 209-219.
- Inuiguchi M., Ichihashi H., Tanaka H. (1989) Possibilistic linear programming with measurable multiattribute value functions. *ORSA J. on Computing*, 1(3), 146-158.
- Kacprzyk J. (1983) *Multistage Decision-Making under Fuzziness*. Verlag TÜV Rheinland, Köln, Germany.
- Kushmerick N., Hanks S., Weld D. An algorithm for probabilistic least-commitment planning. *Proc. of the 12th National Conf. on Artificial Intelligence (AAAI'94)*, Seattle, WA, July 31-Aug. 4, 1073-1078.
- Mathieu-Nicot B. (1985) *Espérance Mathématique de l'Utilité Floue*. Collection de l'IME, n° 29, Librairie de l'Université, Dijon, France.
- Nicolas J.M., Grabisch M. (1995) Matrix games: A fuzzy approach. *Proc. of the 4th IEEE Inter. Conf. on Fuzzy Systems and the 2nd Inter. Fuzzy Engineering Symp. (FUZZ-IEEE'95 / IFES'95)*, Yokohama, Japn, March 20-24, Vol. IV, 2261-2266.
- Shackle G.L.S. (1961) *Decision, Order and Time in Human Affairs*. (2nd edition) Cambridge University Press, Cambridge, UK.
- Shackle G.L.S. (1985) Foreword to "Espérance Mathématique de l'Utilité Floue" by Mathieu-Nicot B.. Collection de l'IME, n° 29, Librairie de l'Université, Dijon, France
- Sugeno M. (1974) *Theory of fuzzy integrals and its applications*. Doctoral thesis, Tokyo Institute of Technology.
- Thiébaux S., Hertzberg J., Shoaff W., Schneider M. (1995) A stochastic model of actions and plans for anytime planning under uncertainty. *Int. J. of Intelligent Systems*, 10(2), 155-183.
- von Neumann J., Morgenstern O. (1944) *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.
- Wald A. (1950) *Statistical Decision Functions*. Wiley & Sons, New York.
- Whalen T. (1984) Decision making under uncertainty with various assumptions about available information. *IEEE Trans. on Systems, Man and Cybernetics*, 14, 888-900.
- Yager R.R. (1979) Possibilistic decision making. *IEEE Trans. on Systems, Man and Cybernetics*, 9, 388-392.
- Yager R.R. (1981) A new methodology for ordinal multiobjective decisions based on fuzzy sets. *Decision Sciences*, 12, 589-600.
- Zadeh L.A. (1978) Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3-28, 1978.