

# Non-formality of Voronov's Swiss-Cheese operad

```
In[1]:= SetDirectory[NotebookDirectory[]];  
      << SwissCheeseCode`
```

## 1. Helpers and structural operations

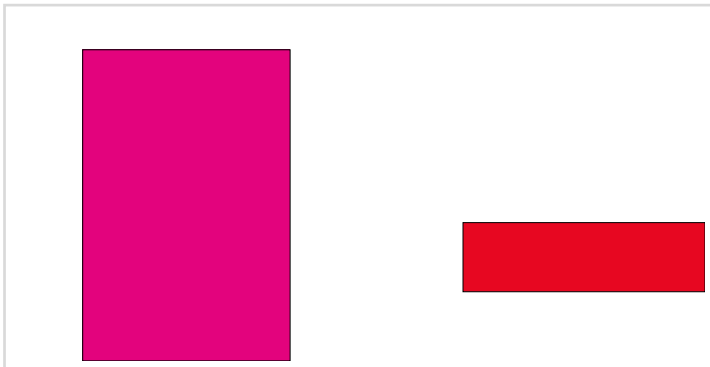
### 1.1. Composition

```
In[3]:= exampleCubes = <|c[1] → Cuboid[{0, -0.8}, {0.9, -0.2}], c[2] → Cuboid[{0.2, 0.3}, {0.4, 1}]|>
```

```
Out[3]= <|c[1] → Cuboid[{0, -0.8}, {0.9, -0.2}], c[2] → Cuboid[{0.2, 0.3}, {0.4, 1}]|>
```

```
In[4]:= export["compos-outer.png",  
             Framed[First[display2d[exampleCubes, Axes → False]], FrameStyle → LightGray]]
```

Out[4]=

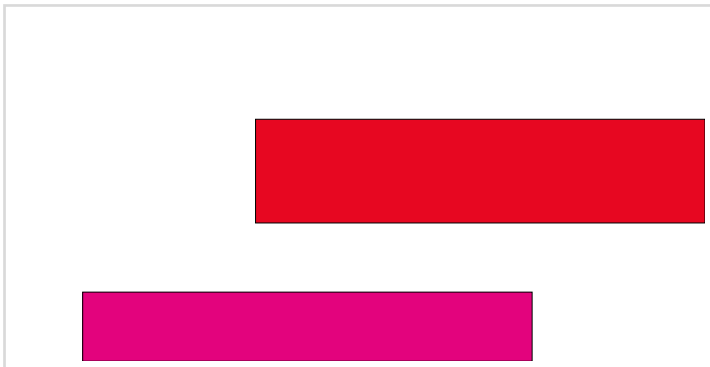


```
In[5]:= exampleCubes2 = <|c[1] → Cuboid[{0, -0.8}, {0.2, 0.5}], c[2] → Cuboid[{0.4, -0.3}, {0.7, 1}]|>
```

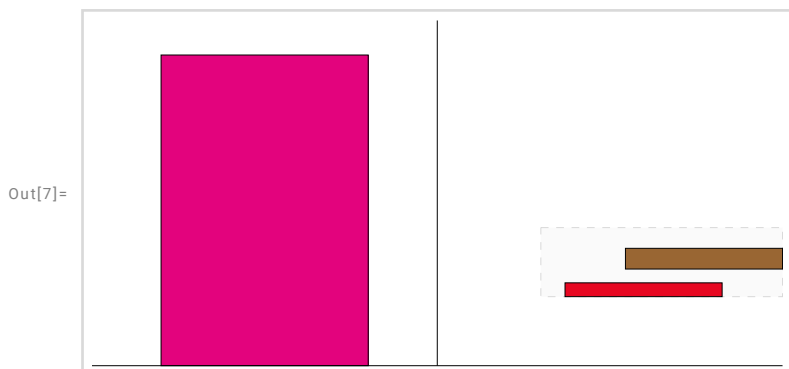
```
Out[5]= <|c[1] → Cuboid[{0, -0.8}, {0.2, 0.5}], c[2] → Cuboid[{0.4, -0.3}, {0.7, 1}]|>
```

```
In[6]:= export["compos-inner.png",  
             Framed[First[display2d[exampleCubes2, Axes → False]], FrameStyle → LightGray]]
```

Out[6]=

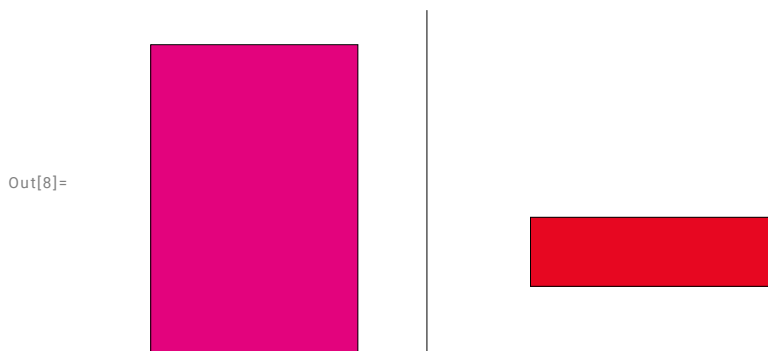


```
In[7]:= export["compos-result.png",  
  Framed[First[display2d[γ[exampleCubes, Association[c[2] → exampleCubes2]],  
    Prolog → {EdgeForm[Directive[GrayLevel[0.85], Dashed]], GrayLevel[0.98],  
    Reverse /@ exampleCubes[c[2]]}], FrameStyle → LightGray]]
```

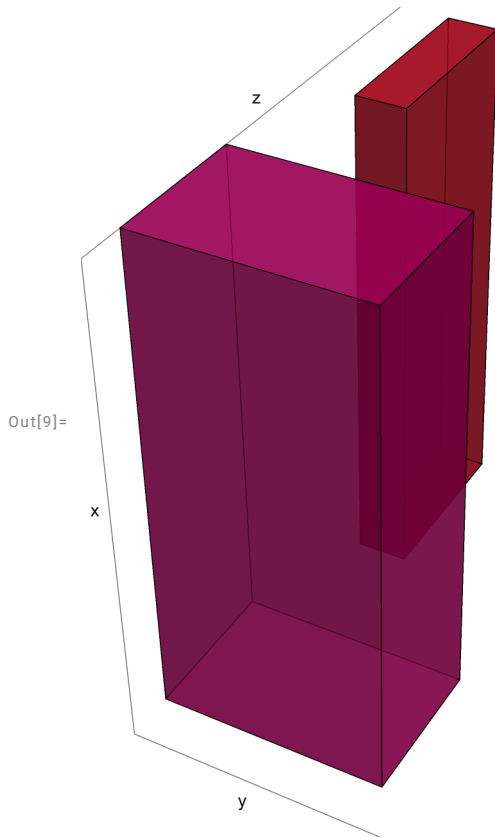


## 1.2. Inclusion

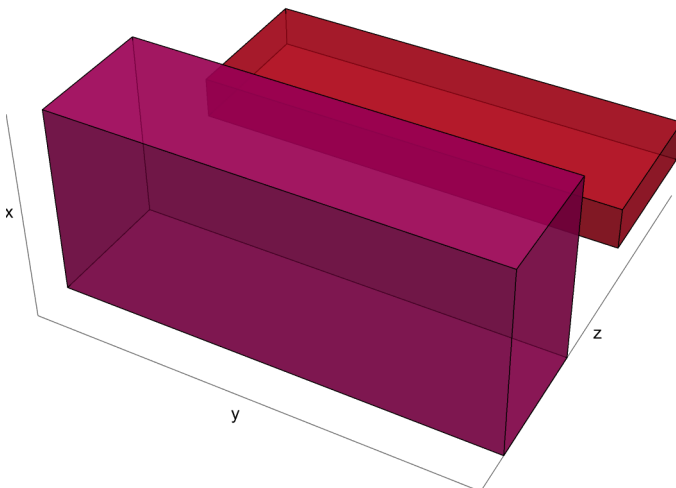
```
In[8]:= export["example-before.png", First@display2d[exampleCubes]]
```



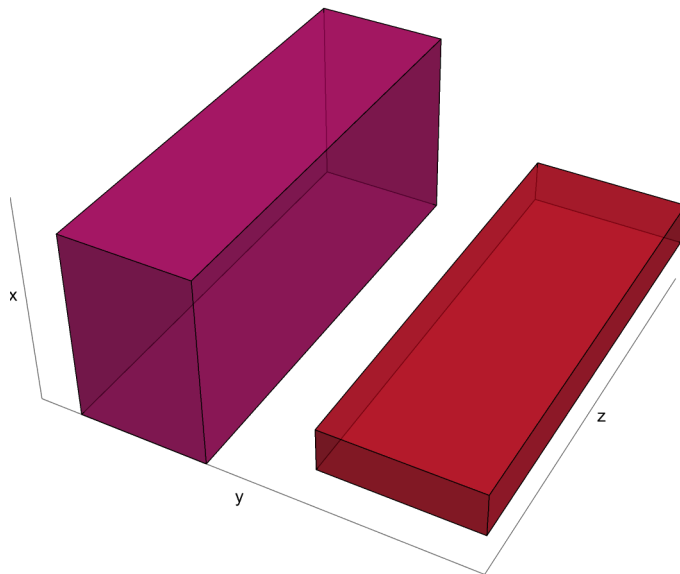
```
In[9]:= export["example-inclusion-1.png",  
            First@display[ $\mathcal{L}$ [{0}, exampleCubes], PlotRange  $\rightarrow$  {{0, 1}, {-1, 1}, {-1, 1}}]]
```



```
In[10]:= export["example-inclusion-2.png", First@display[ $\mathcal{L}$ [{1}, exampleCubes]]]  
Out[10]=
```

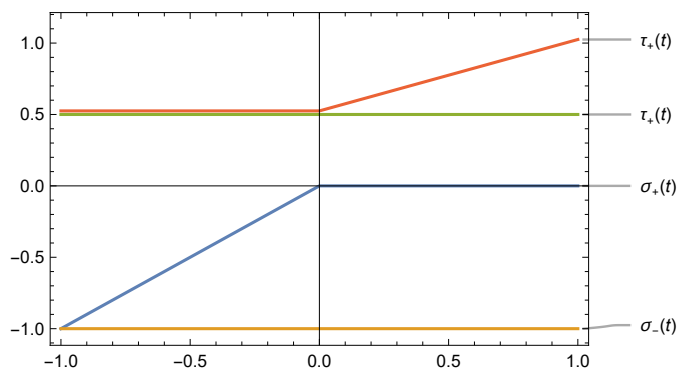


```
In[11]:= export["example-inclusion-3.png", First@display[ $\mathcal{L}$ {2}, exampleCubes]]
Out[11]=
```



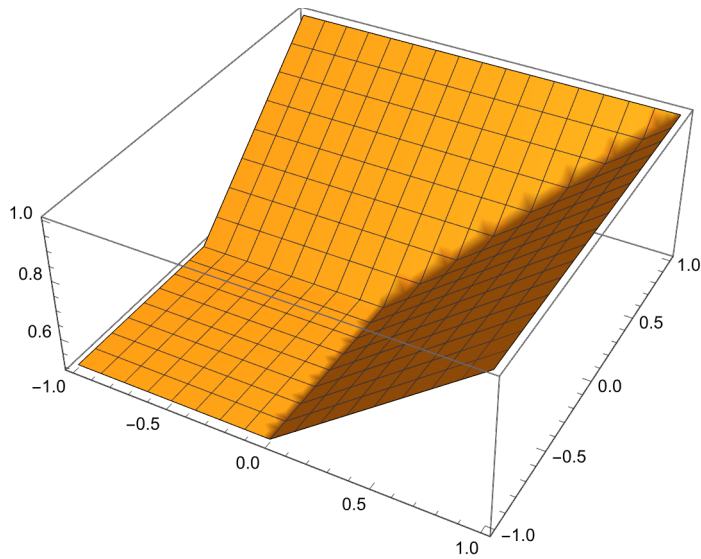
### 1.3. The maps $\sigma$ and $\tau$

```
In[12]:= export["plot-sigma-tau.png",
  Plot[{ $\sigma$ [1, t],  $\sigma$ [-1, t],  $\tau$ [{1}, {t}],  $\tau$ [-1, {t}] + 0.025}, {t, -1, 1},
  PlotLabels  $\rightarrow$  TraditionalForm /@ {" $\sigma_+$ [t]", " $\sigma_-$ [t]", " $\tau_+$ [t]", " $\tau_-$ [t]"}, Frame  $\rightarrow$  True]
Out[12]=
```



```
In[13]:= export["plot-tau-3d.png",
  Plot3D[τ[{-1, -1}, {s, t}], {s, -1, 1}, {t, -1, 1}, Exclusions → None]]
```

Out[13]=



## 2. Non-formality in dimension 2

### 2.1. Basic elements

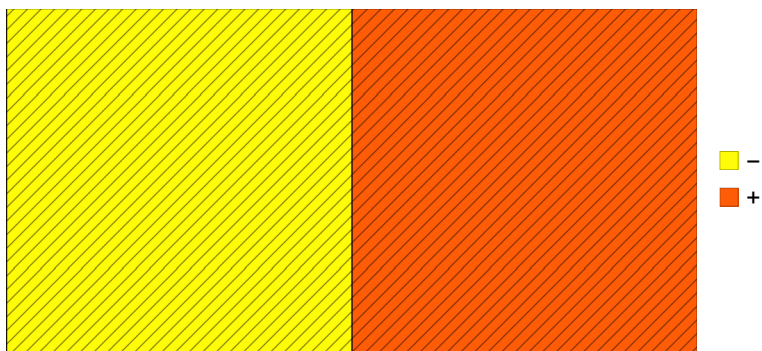
```
In[14]:= format[μ[1]]
```

Out[14]=

$\langle | - \rightarrow [0, 1] \times [-1, 0], + \rightarrow [0, 1] \times [0, 1] | \rangle$

```
In[15]:= export["mu1.png", display2d[μ[1]]]
```

Out[15]=



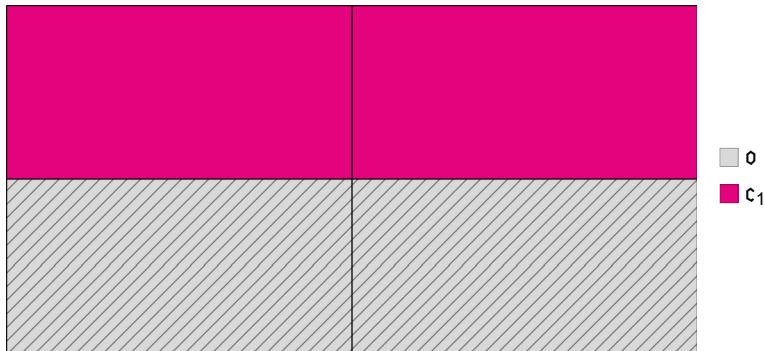
```
In[16]:= format[α[1]]
```

Out[16]=

$\langle \left| 0 \rightarrow \left[0, \frac{1}{2}\right] \times [-1, 1], c_1 \rightarrow \left[\frac{1}{2}, 1\right] \times [-1, 1] \right| \rangle$

```
In[17]:= export["act1.png", display2d[α[1]]]
```

Out[17]=



## 2.2. The loops

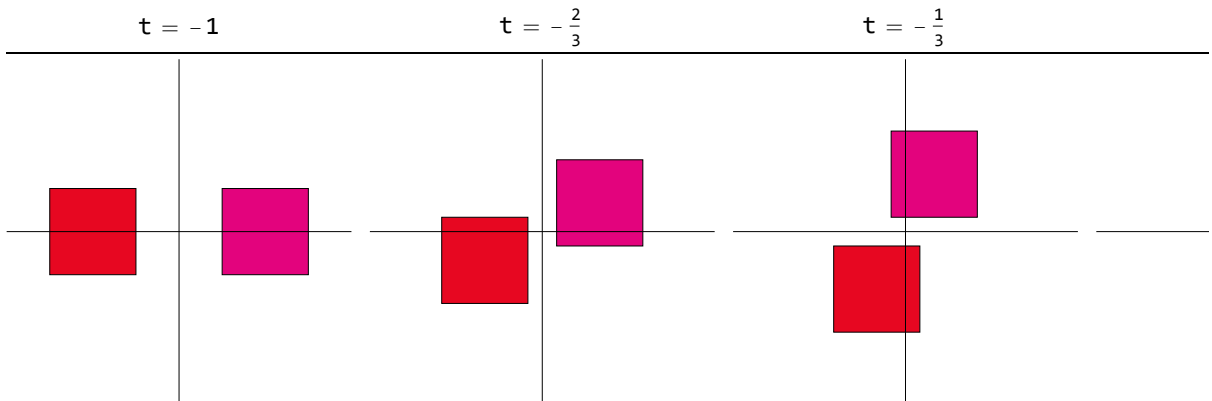
```
In[18]:= formatTable[Association["l" + "1" → l+[1][{t}], "l" + "2" → l+[2][{t1, t2}}]]
```

Out[18]//TableForm=

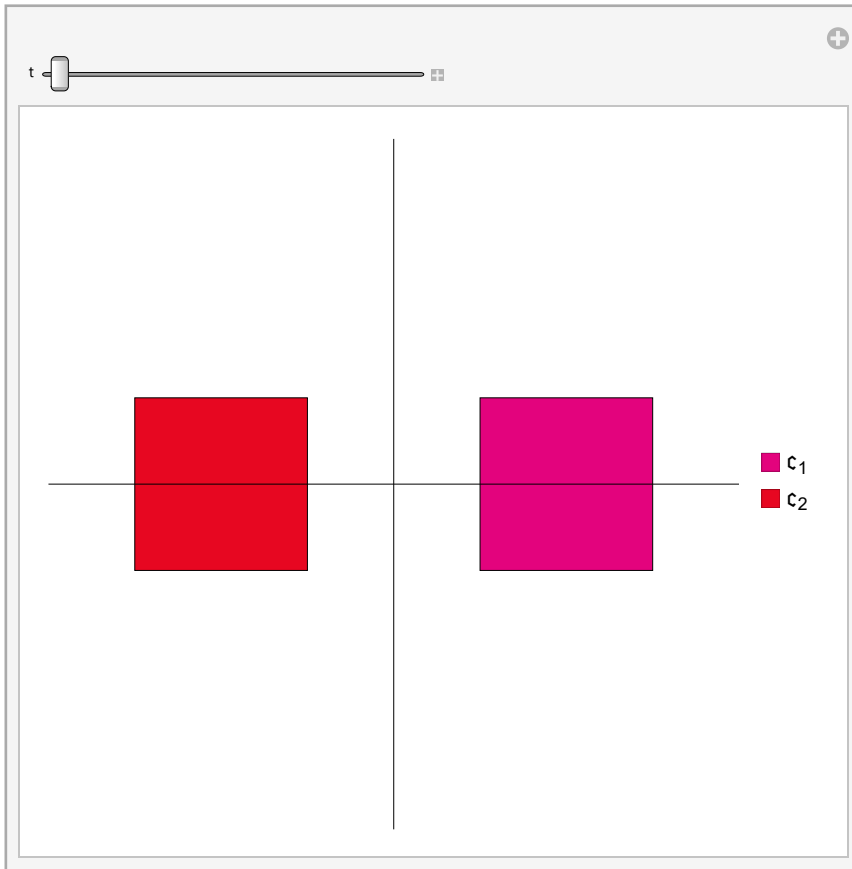
	$c_1$
$l_+^1$	$\left[ \frac{1}{4} (1 - 2 \text{Abs}[t]), \frac{1}{4} (3 - 2 \text{Abs}[t]) \right] \times \left[ \frac{1}{4} (-1 - 2t), \frac{1}{4} (1 - 2t) \right]$
$l_+^2$	$\left[ \frac{1}{4} (1 - \text{Max}[2 \text{Abs}[t_1], 2 \text{Abs}[t_2]]), \frac{1}{4} (3 - \text{Max}[2 \text{Abs}[t_1], 2 \text{Abs}[t_2]]) \right] \times \left[ \frac{1}{4} (-1 - 2t_1), \frac{1}{4} (1 - 2t_1) \right]$

```
In[19]:= export["l1plus.png", grid2d[l+[1][{#}] &, 3, PlotRange → {{-1, 1}, {-1, 1}}]]
```

Out[19]//TableForm=



In[20]:= `Manipulate[display2d[1[1][{t}], PlotRange -> {{-1, 1}, {-1, 1}}, {t, -1, 1}]`  
 Out[20]=

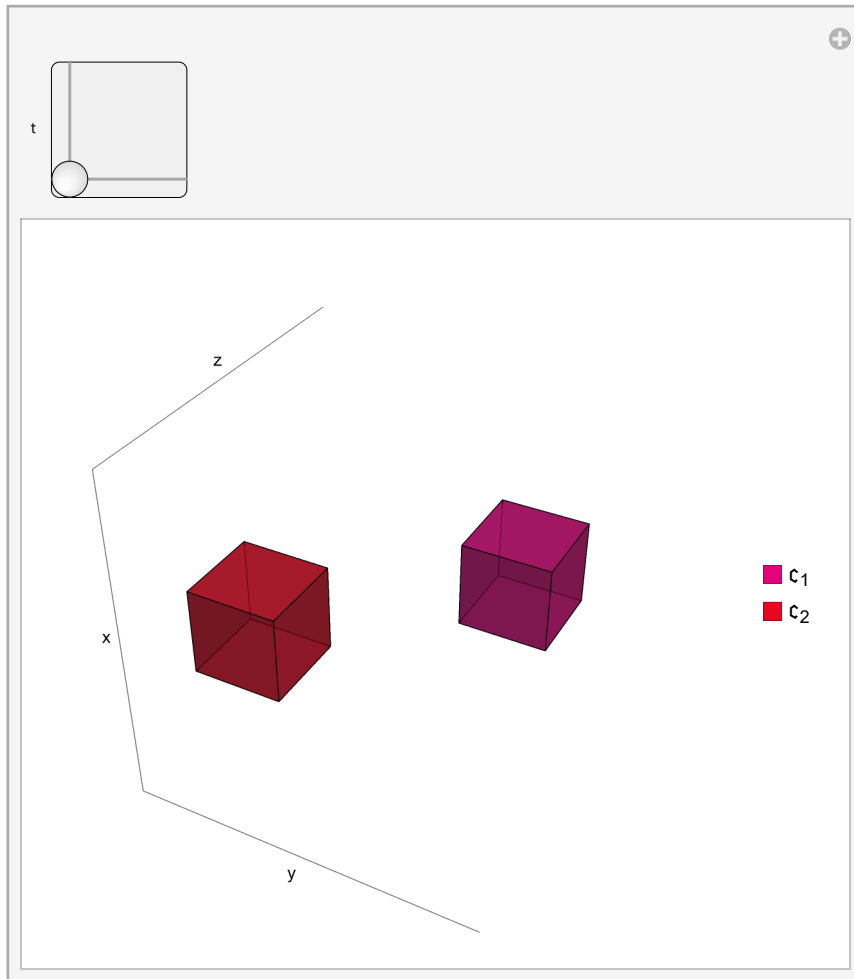


The two halves of the loops do glue (example for  $n = 2$ ):

In[21]:= `Reduce[ $\forall_s \forall_t (\text{Abs}[s] == 1 \mid \mid \text{Abs}[t] == 1) \Rightarrow \mathbf{1}_+[2][\{s, t\}][c[1]] == \mathbf{1}_+[2][\{-s, -t\}][c[2]]$ ]`  
 Out[21]=  
 True

```
In[22]:= Manipulate[display[1[2][t], PlotRange -> {{-1, 1}, {-1, 1}, {-1, 1}},
  {t, {-1, -1}, {1, 1}}]
```

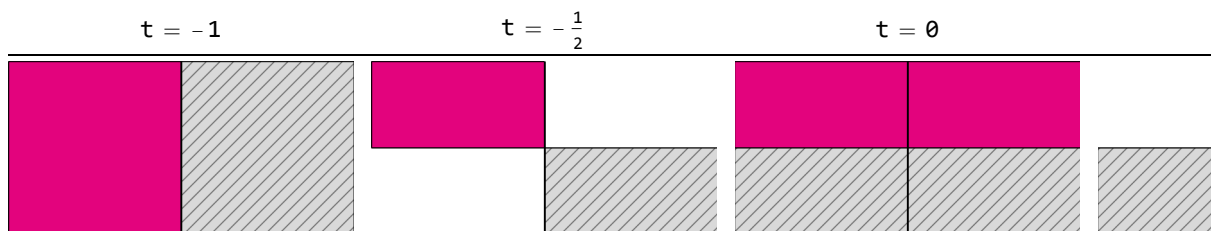
Out[22]=



## 2.3. Non-formality of larger Swiss-Cheese

```
In[23]:= export["kappa.png", grid2d[x, 2]]
```

Out[23]//TableForm=





## 2.4. Non-formality of Voronov's Swiss Cheese

### 2.4.1. The chains $\beta$

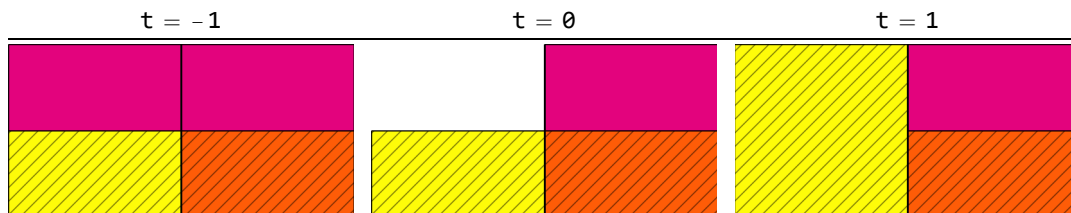
```
In[24]:= formatTable[Association[Table[" $\beta$ "o[ $\epsilon$ ]  $\rightarrow$   $\beta$ [o[ $\epsilon$ ]][{t}], { $\epsilon$ , {-1, 1}}]]]
```

Out[24]//TableForm=

	$c_1$	-	+
$\beta_-$	$\left[\frac{1}{2}, 1\right] \times [-1, -\text{Min}[0, t]]$	$\left[0, \frac{1}{2}\right] \times [-1, 0]$	$\left[0, \text{Max}\left[\frac{1}{2}, \frac{1+t}{2}\right]\right] \times [0, 1]$
$\beta_+$	$\left[\frac{1}{2}, 1\right] \times [\text{Min}[0, t], 1]$	$\left[0, \text{Max}\left[\frac{1}{2}, \frac{1+t}{2}\right]\right] \times [-1, 0]$	$\left[0, \frac{1}{2}\right] \times [0, 1]$

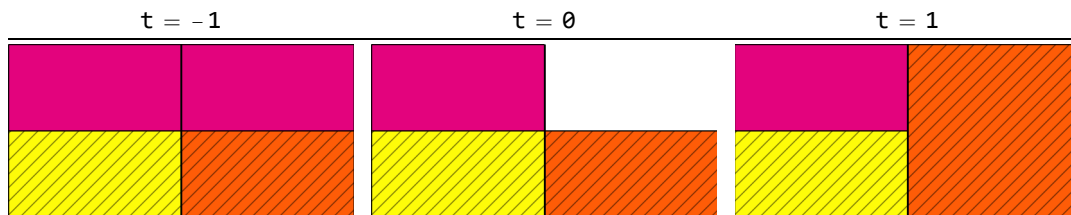
```
In[25]:= export["beta1plus.png", grid2d[ $\beta$ [o[1]][{#}] &, 1]]
```

Out[25]//TableForm=



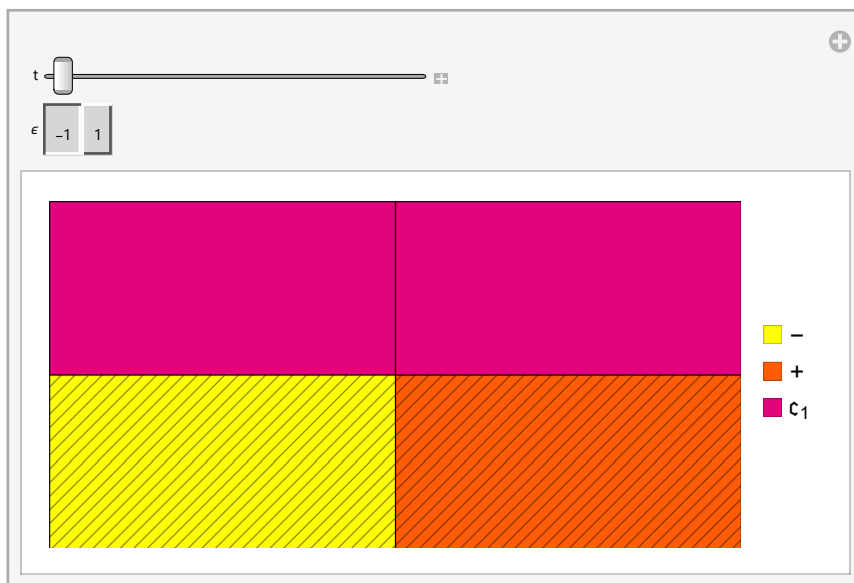
```
In[26]:= export["beta1minus.png", grid2d[ $\beta$ [o[-1]][{#}] &, 1]]
```

Out[26]//TableForm=



```
In[27]:= Manipulate[display2d[ $\beta$ [o[ $\epsilon$ ]][{t}], {t, -1, 1}, { $\epsilon$ , {-1, 1}}]
```

Out[27]=



### 2.4.2. The chain $\eta$

```
In[28]:= Column[(format[#1][t] &) /@eta1_plus_components]
```

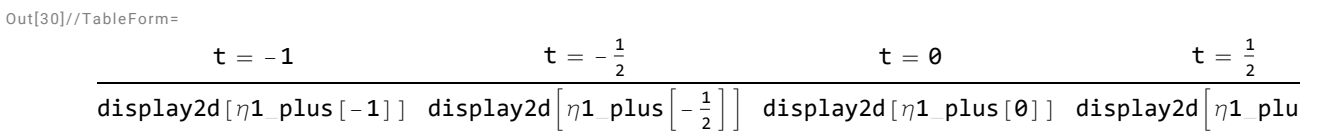
```
Out[28]=
beta_+ o_ alpha
alpha o_ beta_-
alpha o_ beta_+
beta_- o_ alpha
```

```
In[29]:= formatTable[Association["eta"+ -> eta1_+[t]]]
```

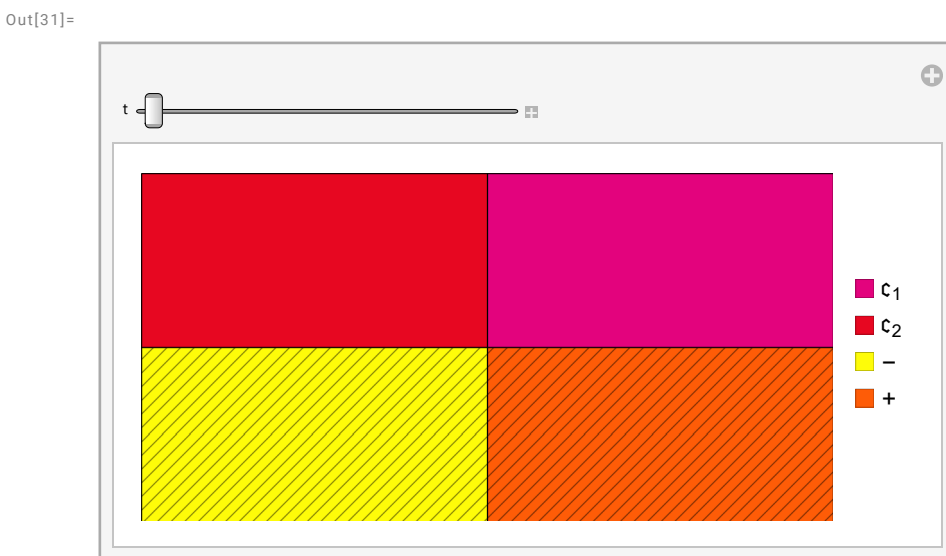
Out[29]//TableForm=

	$c_1$	$c_2$
$\eta_+$	$\left[ \frac{1}{2}, 1 \right] \times \left[ \begin{cases} -1 & t > -\frac{1}{2} \\ -3-4t & -\frac{3}{4} < t \leq -\frac{1}{2} \\ 0 & \text{True} \end{cases} \right]$	$\left[ \begin{cases} 1 & t \leq \frac{1}{2} \\ 3-4t & \frac{1}{2} < t < \frac{3}{4} \\ 0 & \text{True} \end{cases} \right]$

```
In[30]:= export["eta1plus.png", grid2d[eta1_plus, 2]]
```



```
In[31]:= Manipulate[display2d[eta1_+[t]], {t, -1, 1}]
```



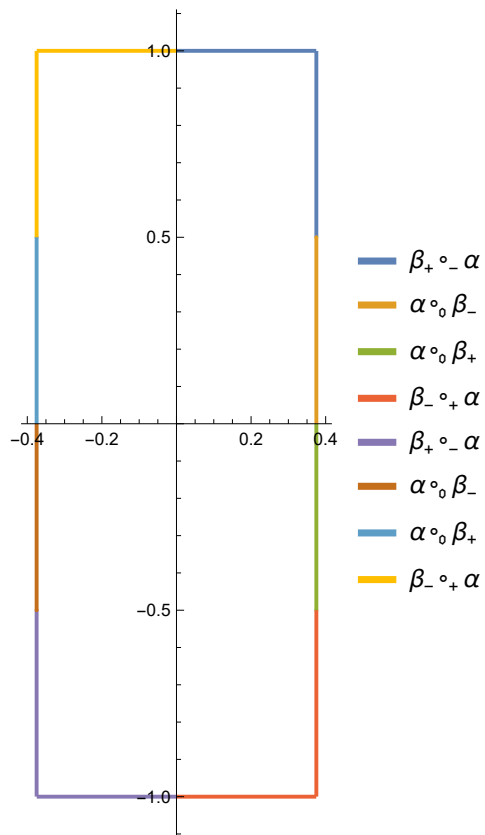
The chain  $\eta_+$  glues to itself:

```
In[32]:= eta1_+[1] == swapClosed[eta1_+[-1]]
```

```
Out[32]=
True
```

```
In[33]:= ParametricPlot[
  Evaluate[(Subtract @@ RegionCentroid /@ #1[s] [Key[c[1]], Key[c[2]]] &) /@
    η1_components], {s, -1, 1}, ...]
```

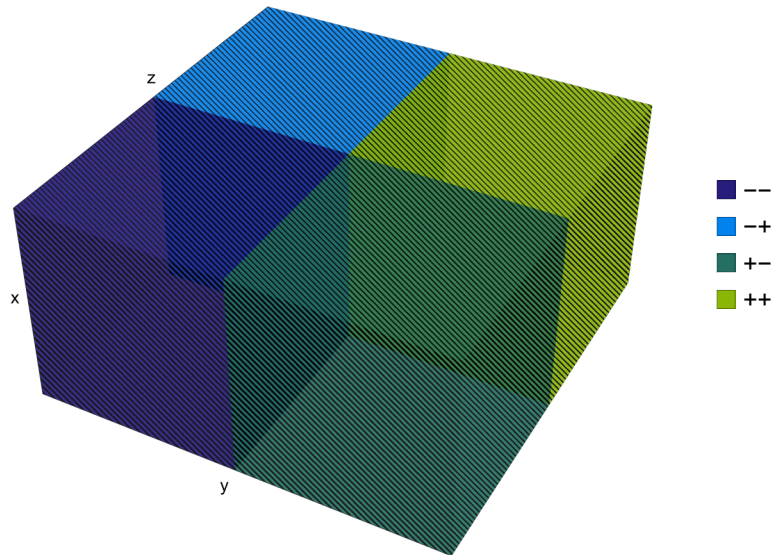
Out[33]=



### 3. Non-formality in dimension 3

```
In[34]:= export["mu-tot.png", display[μ[2]]]
```

```
Out[34]=
```



#### 3.1. The chains $\beta^2$

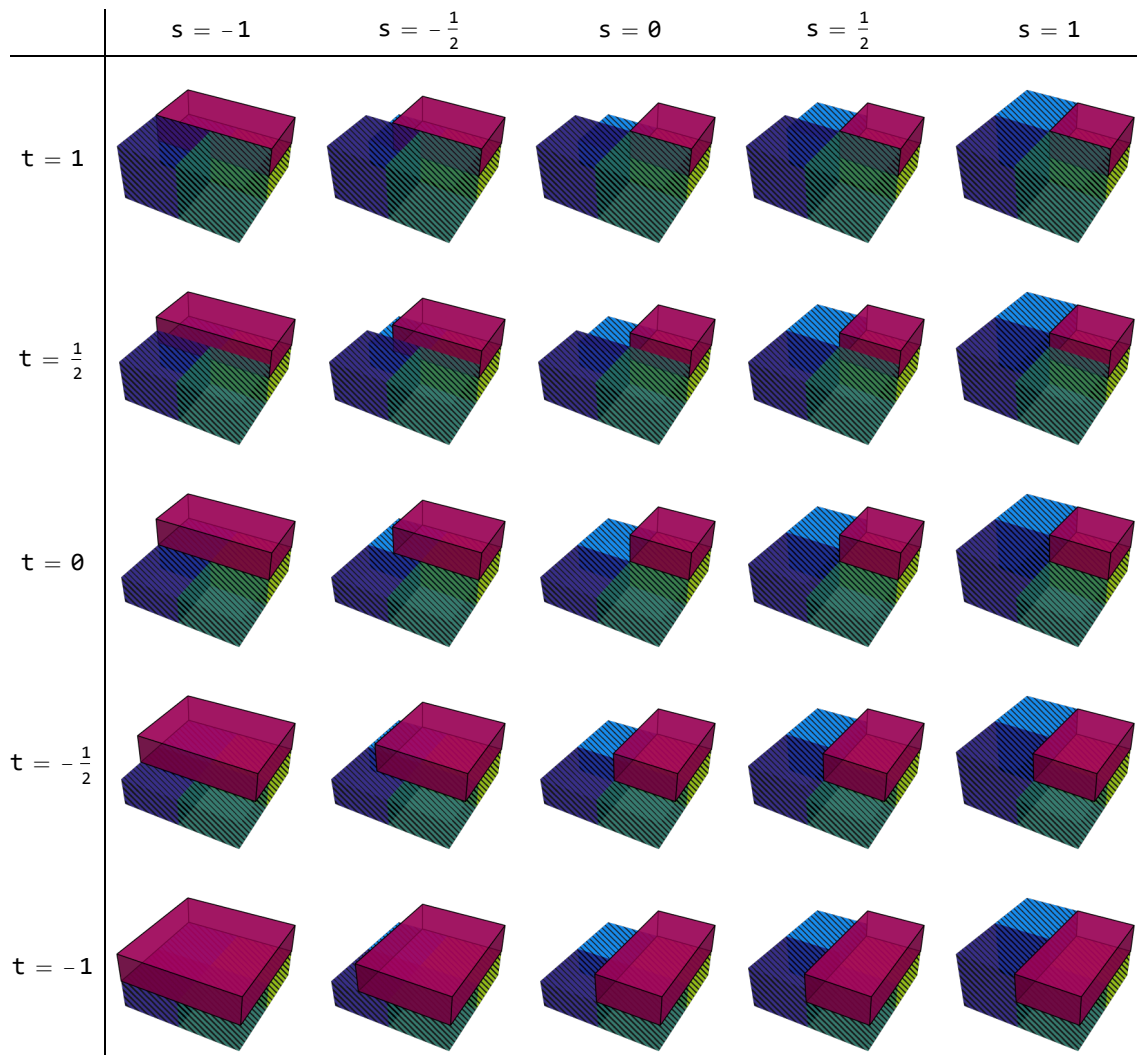
```
In[35]:= formatTable[
  Association[Table["β"o[ε1,ε2]2 → β[o[ε1, ε2]][{s, t}], {ε1, {-1, 1}}, {ε2, {-1, 1}}]]]
```

```
Out[35]//TableForm=
```

	$c_1$	--	-
$\beta_{--}^2$	$\left[\frac{1}{2}, 1\right] \times [-1, -\text{Min}[0, s]] \times [-1, -\text{Min}[0, t]]$	$\left[0, \frac{1}{2}\right] \times [-1, 0] \times [-1, 0]$	
$\beta_{-+}^2$	$\left[\frac{1}{2}, 1\right] \times [-1, -\text{Min}[0, s]] \times [\text{Min}[0, t], 1]$	$\left[0, \text{Max}\left[\frac{1}{2}, \frac{1+t}{2}\right]\right] \times [-1, 0] \times [-1, 0]$	
$\beta_{+-}^2$	$\left[\frac{1}{2}, 1\right] \times [\text{Min}[0, s], 1] \times [-1, -\text{Min}[0, t]]$	$\left[0, \text{Max}\left[\frac{1}{2}, \frac{1+s}{2}\right]\right] \times [-1, 0] \times [-1, 0]$	
$\beta_{++}^2$	$\left[\frac{1}{2}, 1\right] \times [\text{Min}[0, s], 1] \times [\text{Min}[0, t], 1]$	$\left[0, \text{Max}\left[\frac{1}{2}, \frac{1+s}{2}, \frac{1+t}{2}\right]\right] \times [-1, 0] \times [-1, 0]$	

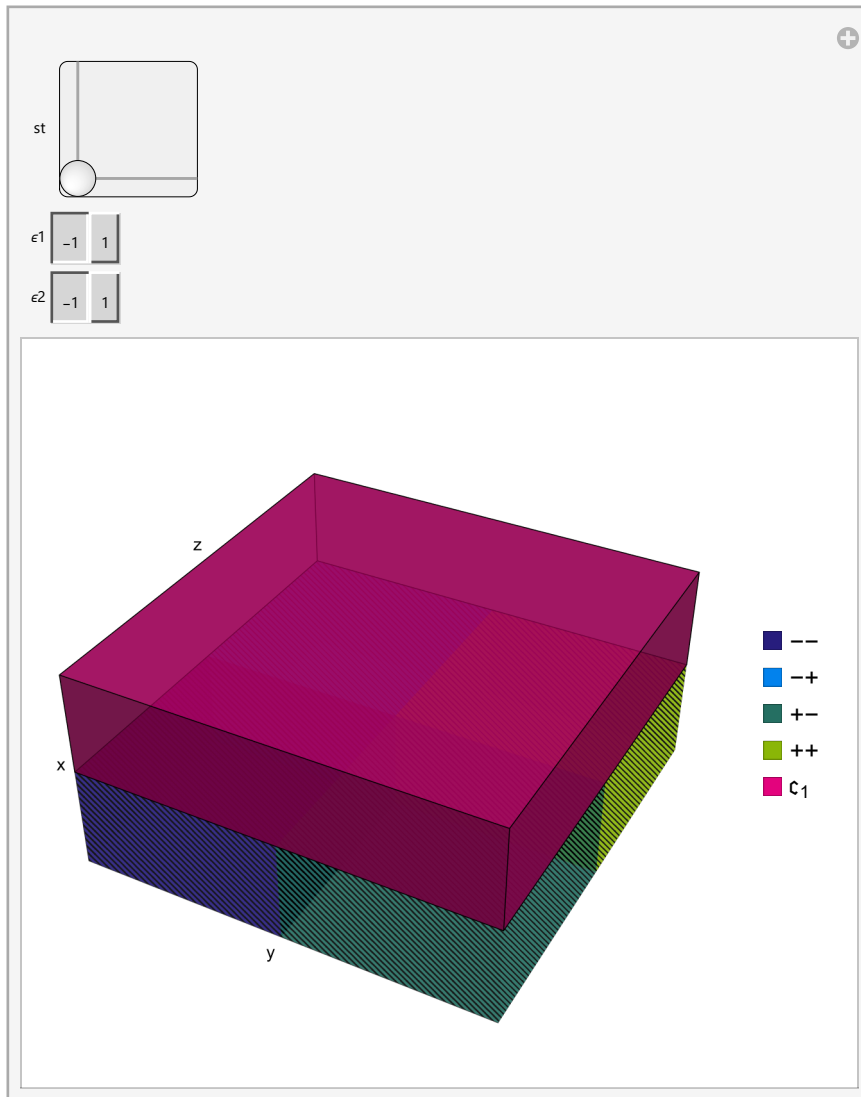
```
In[36]:= export["theta-grid.png", grid[ $\beta$ [0[1, 1]][{##1}] &, 2]]
```

```
Out[36]//TableForm=
```



```
In[37]:= Manipulate[display[ $\beta$ [ $\epsilon_1$ ,  $\epsilon_2$ ]] [st],
  {st, {-1, -1}, {1, 1}}, { $\epsilon_1$ , {-1, 1}}, { $\epsilon_2$ , {-1, 1}}
```

Out[37]=



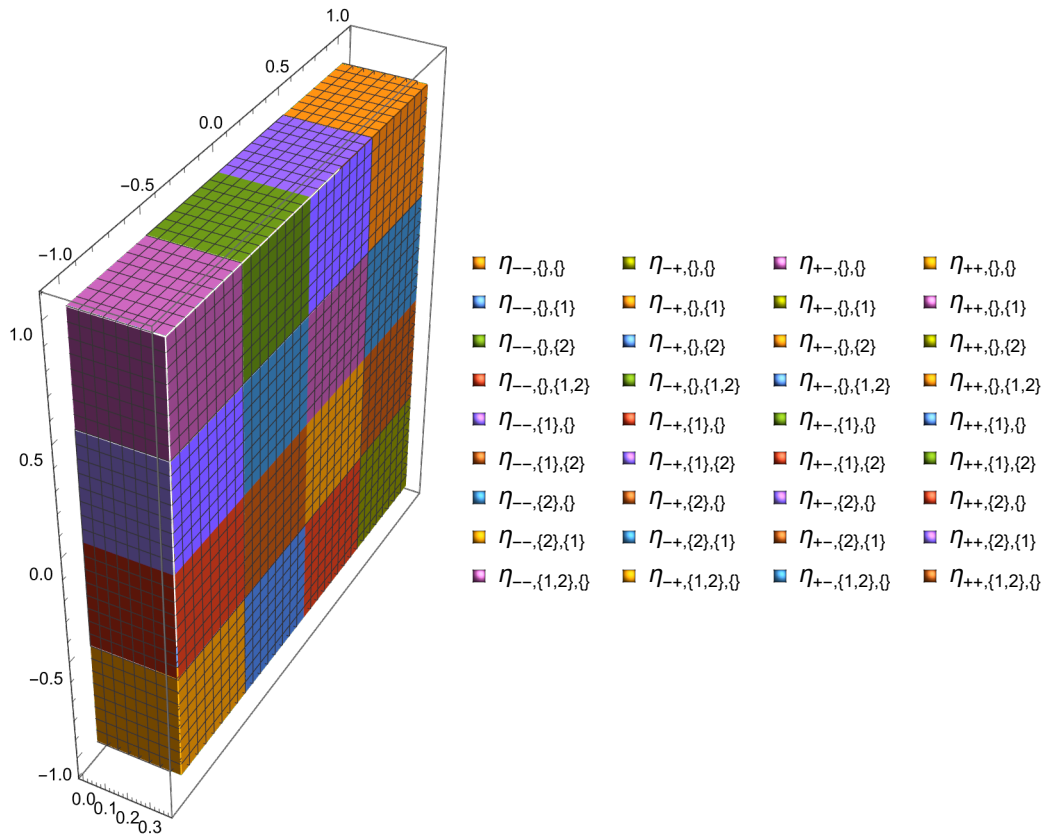
### 3.2. The hemisphere

```
In[38]:= hemisphereLabels = (ReleaseHold[format[Hold[#1[_]]]] &) /@ hemisphereComponents;
```

```
In[39]:= hemisphereCenters[s_, t_] =
  (Subtract @@ RegionCentroid /@ #1[{s, t}][[Key[c[1]], Key[c[2]]]] &) /@
  hemisphereComponents;
```

```
In[40]:= ParametricPlot3D[Evaluate[hemisphereCenters[s, t]],
  {s, -1, 1}, {t, -1, 1}, PlotLegends -> hemisphereLabels]
```

Out[40]=



## 4. General case ( $n \geq 4$ )

There are no visualizations as the dimension is too large.

```
In[41]:= MatrixForm[ $\mathfrak{B}$ [{2, 6}, {3, 5}][Array[t_#1 &, 6]]]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} \text{Max}[-\text{Max}[t_3, t_5], t_1] \\ \text{Min}[-\text{Max}[t_3, t_5], t_2] \\ \text{Max}[-\text{Max}[t_3, t_5], t_4] \\ \text{Min}[-\text{Max}[t_3, t_5], t_6] \end{pmatrix}$$

```
In[42]:= MatrixForm[ $\mathfrak{B}$ [Array[t_# &, 6]]]
```

Out[42]//MatrixForm=

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ \text{Min}[t_5, t_6] \end{pmatrix}$$

```
In[43]:=  $\mu$ [5, {3}] // format
```

Out[43]=

$$\begin{aligned} <| \rightarrow [0, 1] \times [-1, 1] \times [-1, 1] \times [-1, 0] \times [-1, 1] \times [-1, 1], \\ + \rightarrow [0, 1] \times [-1, 1] \times [-1, 1] \times [0, 1] \times [-1, 1] \times [-1, 1] |> \end{aligned}$$

In[44]:=  $\mu[3, \{1, 2\}]$  // format

Out[44]=

$\langle | \text{--} \rightarrow [0, 1] \times [-1, 0] \times [-1, 0] \times [-1, 1], \text{--} \rightarrow [0, 1] \times [-1, 0] \times [0, 1] \times [-1, 1],$   
 $\text{+-} \rightarrow [0, 1] \times [0, 1] \times [-1, 0] \times [-1, 1], \text{++} \rightarrow [0, 1] \times [0, 1] \times [0, 1] \times [-1, 1] | \rangle$

In[45]:=  $\alpha[3]$  // format

Out[45]=

$\langle | 0 \rightarrow [0, \frac{1}{2}] \times [-1, 1] \times [-1, 1] \times [-1, 1], c_1 \rightarrow [\frac{1}{2}, 1] \times [-1, 1] \times [-1, 1] \times [-1, 1] | \rangle$

In[46]:=  $\beta[6, \{2, 5\}, \circ[-1, 1]][\{t_2, t_5\}]$  // format

Out[46]=

$\langle | \text{--} \rightarrow [0, \text{Max}[\frac{1}{2}, \frac{1}{2}(1+t_5)]] \times [-1, 1] \times [-1, 0] \times [-1, 1] \times [-1, 1] \times [-1, 0] \times [-1, 1],$   
 $\text{--} \rightarrow [0, \frac{1}{2}] \times [-1, 1] \times [-1, 0] \times [-1, 1] \times [-1, 1] \times [0, 1] \times [-1, 1],$   
 $\text{+-} \rightarrow [0, \text{Max}[\frac{1}{2}, \frac{1}{2}(1+t_2), \frac{1}{2}(1+t_5)]] \times [$   
 $\quad -1, 1] \times [0, 1] \times [-1, 1] \times [-1, 1] \times [-1, 0] \times [-1, 1],$   
 $\text{++} \rightarrow [0, \text{Max}[\frac{1}{2}, \frac{1}{2}(1+t_2)]] \times [-1, 1] \times [0, 1] \times [-1, 1] \times [-1, 1] \times [0, 1] \times [-1, 1],$   
 $c_1 \rightarrow [\frac{1}{2}, 1] \times [-1, 1] \times [-1, -\text{Min}[0, t_2]] \times [-1, 1] \times [-1, 1] \times [\text{Min}[0, t_5], 1] \times [-1, 1] | \rangle$