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November 21, 2022

A MODULAR ANALOGUE OF A PROBLEM OF VINOGRADOV

R. ACHARYA, S. DRAPPEAU, S. GANGULY, AND O. RAMARÉ

ABSTRACT. Given a primitive, non-CM, holomorphic cusp form f with normalized Fourier coefficients $a(n)$ and given an interval $I \subset [-2, 2]$, we study the least prime p such that $a(p) \in I$. This can be viewed as a modular form analogue of Vinogradov's problem on the least quadratic non-residue. We obtain strong explicit bounds on p , depending on the analytic conductor of f for some specific choices of I .

1. INTRODUCTION

The present article is concerned with understanding the distribution of the initial Fourier coefficients of primitive holomorphic cusp forms at primes. Suppose f is such a form of weight k for the group $\Gamma_0(N)$. We further assume that f is non-CM and has trivial nebentypus. The normalized Fourier coefficients of f at infinity are denoted by $(a(n))_{n \geq 1}$, so that $a(1) = 1$ and

$$f(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e(nz),$$

where, as usual, $e(z)$ denotes $e^{2\pi iz}$ and with this normalization, the Ramanujan bound (proved by Deligne [6]) says $-2 \leq a(p) \leq 2$ for primes p . Furthermore, the function $n \mapsto a(n)$ is real-valued and multiplicative. We refer the reader to the text [12] for background information on holomorphic modular forms. The Sato-Tate conjecture for distribution of the angles θ_p , defined by $a(p) = 2 \cos \theta_p$, as p runs over primes, which is now a theorem of Clozel, Harris, Shepherd-Barron and Taylor [3, 31, 9], implies, in particular, that any interval of positive measure within $[-2, 2]$ contains infinitely many values of $a(p)$. The goal of this article is to obtain bounds for the least prime p such that $a(p)$ lies in a fixed interval $I \subset [-2, 2]$. This can be considered as an analogue of Vinogradov's problem of estimating, given a modulus $q \geq 1$, the size of the least quadratic non-residue modulo q (see [2], [32]). The quality of our bounds will be measured in terms of the analytic conductor $q(f) = Nk^2$ of the form f (see §2.1), and also separately in term of the weight k of the form, considering the level N to be fixed and in terms of the level N , considering the weight k to be fixed. We restrict our attention to forms with trivial nebentypus in order to clarify the presentation but the methods presented here can be extended to a more general setting.

Let $I \subset [-2, 2]$. Theorem 1.6 of the paper [22] of Lemke-Oliver and Thorner implies that there exists a constant A depending only on I such that $a(p) \in I$ for some prime $p \leq q^A$. Their method relies on effective log-free zero density estimates for the L -function associated with f , and the Turán power-sum method. The value of the constant A is not stated explicitly in their paper but it is not hard to see that the constant is effective and can be worked out explicitly. However

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the method is likely to produce quite large values of A . Our aim in the present work is to make the value of A as small as possible for some specific intervals.

We define, when κ is positive and $x \in [0, 1]$:

$$\mathcal{F}(x; \kappa) = \int_0^{x/(1+x)} \frac{h^{\kappa-1} dh}{1-h} = \sum_{k \geq 0} \frac{1}{\kappa+k} \left(\frac{x}{1+x} \right)^{\kappa+k}. \quad (1.1)$$

Note that $\mathcal{F}(\cdot; \kappa)$ is increasing between $\mathcal{F}(0; \kappa) = 0$ and $\mathcal{F}(1; \kappa) = \int_0^{1/2} \frac{h^{\kappa-1} dh}{1-h}$. We thus define a function $\mathcal{G}(\cdot; \kappa)$ with value in $[0, 1]$ by

$$\mathcal{G}(y; \kappa) = \max\{x \in [0, 1] : \mathcal{F}(x; \kappa) \leq 1/y\}. \quad (1.2)$$

The function \mathcal{G} is non-increasing and we have $\mathcal{G}(y; \kappa) = 1$ when $y \leq 1/\mathcal{F}(1; \kappa)$ and by convention $\mathcal{G}(\infty; \kappa) = 0$.

We now state our main results which depend crucially on knowledge about the analytic properties of the symmetric power L functions associated to f (see §2.1 for definition). This is likely to change in the future; only small changes would be required in our proofs to reflect any such improvement. Here is the assumption we rely on.

Hypothesis \mathcal{H}_ℓ : The L -function $L(s, \text{sym}^\ell(f))$ has analytic continuation to the entire complex plane and it satisfies the bound

$$L(1/2 + it, \text{sym}^\ell(f)) \ll_\varepsilon q(\text{sym}^\ell(f), s)^{\lambda_\ell + \varepsilon}$$

for any $\varepsilon > 0$.

For holomorphic forms, the automorphy of $L(s, \text{sym}^\ell f)$ has been known for $\ell \leq 8$ by [7, 17, 18, 16, 4, 5], and has recently been proved for *all* ℓ when N is squarefree by Newton-Thorne [28]. As a result, these L -functions admit holomorphic continuation to the entire complex plane and by the convexity principle, \mathcal{H}_ℓ holds with $\lambda_\ell = 1/4$ (known as the convexity bound) for $\ell \leq 8$ unconditionally and for all ℓ when N is squarefree.

Our results are the following.

Theorem 1.1. *For any $\delta \in (0, 2]$, let $\theta_1(\delta) = \mathcal{G}(2 + \delta; \delta)$. The function θ_1 is increasing and we have $\theta_1(0+) = 0$ and $\theta_1(1) = 0.3956 \dots$. Suppose $\lambda_1 > 0$ is an exponent that satisfies the hypothesis \mathcal{H}_ℓ below for $\ell = 1$, and let $\varepsilon > 0$. Then for $q = N$ or k^2 sufficiently large, there exists a prime*

$$p \ll_\varepsilon q^{\frac{2\lambda_1}{1+\theta_1(\delta)} + \varepsilon}$$

with $a(p) \leq \delta$.

Remark 1.2. The convexity bound (Phragmén-Lindelöf principle) allows taking $\lambda_1 = \frac{1}{4}$ but better exponents, called subconvex exponents are known in both the weight and the level aspects. For example, one may take $\lambda_1 = \frac{1}{6}$ when $N = 1$ by a result of Jutila and Motohashi [15].

Theorem 1.3. *For any $\delta \in (0, 1]$, let $\theta_2(\delta) = \mathcal{G}((1 + \delta)^2; 2\delta + \delta^2)$. The function θ_2 is increasing when $\delta \leq 0.5305 \dots$, and constant equal to 1 afterwards. We have $\theta_2(0+) = 0$, $\theta_2(1/2) = 0.9093 \dots$, $\theta_2(1) = 1$. Suppose $\lambda_2 > 0$ is an exponent that satisfies the hypothesis \mathcal{H}_ℓ below for $\ell = 2$, and let $\varepsilon > 0$. For any $\delta \in [0, 1]$, and for $q = N$ or k^2 sufficiently large, there exists a prime*

$$p \ll_\varepsilon q^{\frac{4\lambda_2}{1+\theta_2(\delta)} + \varepsilon}$$

with $|a(p)| \leq 1 + \delta$.

Remark 1.4. The convexity bound allows the choice $\lambda_2 = \frac{1}{4}$ and currently this is the best known exponent. Obtaining a subconvex estimate for the symmetric square L -function in the level or the weight aspect is a challenging problem.

It turns out that showing the existence of primes p of small size in terms of the conductor (i.e., weight and level) such that $a(p) \geq 0$ is rather difficult. By utilizing the fact that hypotheses \mathcal{H}_ℓ holds true for $1 \leq \ell \leq 5$, we are able to show the following result:

Theorem 1.5. *There is a prime $p \ll k^{24}N^{21}$ such that $a(p) \geq 0$.*

The results above are all obtained using a similar strategy and this is summarized in Theorem 1.11 below. For some specific intervals, however, we obtain better bounds by employing ad hoc techniques using L -functions as we now describe.

Theorem 1.6. *For any $\varepsilon > 0$, there is a prime $p = \mathcal{O}_\varepsilon(kN)^{1+\varepsilon}$ such that $a(p) < 0$.*

Corollary. *The least prime such that $a(p) \neq 0$ is $\ll_\varepsilon (kN)^{1+\varepsilon}$, for any $\varepsilon > 0$.*

Remark 1.7. As the proof of the above theorem shows, the exponent 1 can be replaced by $4\lambda_2$ and any subconvex estimate $\lambda_2 < 1/4$ for the symmetric square L -function will lead to an improvement of the above result.

The next result relates the possibility of the initial coefficients at primes assuming extreme values with the size of $L(1, f)$. For $q = Nk^2$, let

$$\gamma^- := \liminf_{q \rightarrow \infty} \frac{\log L(1, f)}{\log \log q}, \quad \gamma^+ := \limsup_{q \rightarrow \infty} \frac{\log L(1, f)}{\log \log q}.$$

From the zero-free region of $L(s, f)$ (See [11]), the standard techniques yield

$$-2 \leq \gamma^- \leq \gamma^+ \leq 2. \tag{1.3}$$

Theorem 1.8. *For any $\delta, \varepsilon > 0$, the least prime p such that $a(p) > \gamma^- - \delta$ is $\mathcal{O}(q^\varepsilon)$. Similarly, the least prime p such that $a(p) < \gamma^+ + \delta$ is $\mathcal{O}(q^\varepsilon)$.*

Remark 1.9. The bounds (1.3) seem to be the best known, and any improvement would yield a non-trivial result in Theorem 1.8. The quality of the upper-bound on p , namely $\mathcal{O}(q^\varepsilon)$, compared to the above results, suggests that improving the bounds (1.3) is a difficult task. Under the Riemann Hypothesis for $L(s, f)$, one has the bounds

$$(\log \log q)^{-2} \ll L(1, f) \ll (\log \log q)^2,$$

at least in the case $N = 1$ (see [23, Thm. 3] for a precise and stronger statement), which yields conjecturally $\gamma^- = \gamma^+ = 0$. Furthermore, it is known that these bounds hold for almost all forms (see [24, Cor. 2] for a precise statement).

Several authors investigated the smallest *integer* n such that $a(n) < 0$, see for instance [13], [19], [21] or [25]. It follows from [25] that the least such n is $\mathcal{O}(q^{3/8})$, where $q = Nk^2$. A closer scrutiny of their proofs reveals that the integer n they produce is either a prime or the square of a prime. Indeed, all the above works make use of the contrast between the sizes of $a(p)$ and $a(p^2)$ forced by the Hecke relation $a(p)^2 - 1 = a(p^2)$ for primes p . Since we aim at localizing only $a(p)$'s, the coefficients at primes, we cannot rely on such procedures. In fact, the two methods we propose are *reverse*: from a localization on $a(p)$, we show that some polynomial in $a(p)$ has to be large for many primes p . This polynomial defines the

value at p of a new function whose Dirichlet series we approximate with products of $L(s, \text{sym}^\ell f)$ and it is by using the analytic properties of these latter that we reach a contradiction. To find an integer n such that $a(n) < 0$, only the analytic properties of $L(s, f)$ are required.

Regarding bounds conditional on the Riemann Hypothesis, Ankeny [1] has proved that for any non-trivial character $\chi \pmod q$, if the Riemann hypothesis is true for $L(s, \chi)$, then the least n such that $\chi(n) \neq 1$ is $\mathcal{O}((\log q)^2)$. It is not difficult to show that the analogous phenomenon holds in our setting:

Theorem 1.10. *Assume that for all $\ell \geq 1$, the function $L(s, \text{sym}^\ell f)$ is entire and satisfies the Riemann hypothesis. Then for any interval $I \subseteq [-2, 2]$ of positive measure, the least prime p such that $a(p) \in I$ satisfies $p \ll_I (\log q)^2$.*

Let us now state our general theorem depending on the hypothesis \mathcal{H}_ℓ . Note that this result implies Theorems 1.1, 1.3 and 1.5.

Theorem 1.11 (Generic theorem). *Let $(b_\ell)_{1 \leq \ell \leq L}$ be non-negative integers, Let $\kappa > 0$ and F be real, and let $I \subset [-2, 2]$ be such that*

$$\begin{cases} \forall x \in [-2, 2] \setminus I, & \sum_{1 \leq \ell \leq L} b_\ell U_\ell(x/2) \geq \kappa > 0, \\ \forall x \in [-2, 2], & \sum_{1 \leq \ell \leq L} b_\ell U_\ell(x/2) \geq F, \end{cases} \quad (1.4)$$

where U_ℓ are the Chebyshev polynomials of the second kind. Then, on assuming $(\mathcal{H}_\ell)_{\ell \leq L}$, the least prime p such that $a(p) \in I$ satisfies

$$\frac{\log p}{\log N} \leq \frac{2 \sum_{\ell} \ell b_\ell \lambda_\ell}{1 + \mathcal{G}(\kappa - F; \kappa)} + \varepsilon, \quad (1.5)$$

for any $\varepsilon > 0$ and N large enough with respect to the weight k and ε ; and

$$\frac{\log p}{\log k} \leq \frac{2 \sum_{\ell} (\ell + \epsilon(\ell)) b_\ell \lambda_\ell}{1 + \mathcal{G}(\kappa - F; \kappa)} + \varepsilon. \quad (1.6)$$

for any $\varepsilon > 0$ and k large enough with respect to the level N and ε . Here $\epsilon(\ell) = \frac{1 - (-1)^\ell}{2} \in \{0, 1\}$ is the parity of ℓ .

The intervals $[\alpha, \beta]$ for which there is a linear combination with non-negative coefficients of U_1, \dots, U_8 which takes positive values outside $[\alpha, \beta]$ delimit a curve in (α, β) , whose exact determination is an interesting question (without the non-negativity condition, the analogue for U_1, \dots, U_4 was solved in Appendix A of [22]). Between this curve and the diagonal $\alpha = \beta$, Theorem 1.11 yields an upper-bound on $\frac{\log p}{\log q}$, which gets smaller as one moves away from the diagonal. This is represented in Figure 1, which was obtained by case-by-case analysis of all linear combinations with $\sum_{\ell \leq 8} \ell b_\ell \leq 42$. On the left, darker colors indicate a larger upper-bound.

Theorem 1.11 should be compared with Theorem 1.8 of [22]. In both cases, we are given an interval $I \subset [-2, 2]$, and we are looking for the least prime p such that $a(p) \in I$. In Theorem 1.8 of [22], the authors obtain an exponent depending on the quality with which the indicator function $\mathbb{1}_I$ can be minorized by a linear combination of U_0, U_1, U_2, \dots . In Theorem 1.11, we obtain an exponent depending on the quality with which the complementary indicator function $\mathbb{1}_{[-2, 2] \setminus I}$ is minorized by a linear combination *with non-negative coefficients* of U_1, U_2, \dots . An inconvenient of our method is that there is no clear description of the allowable

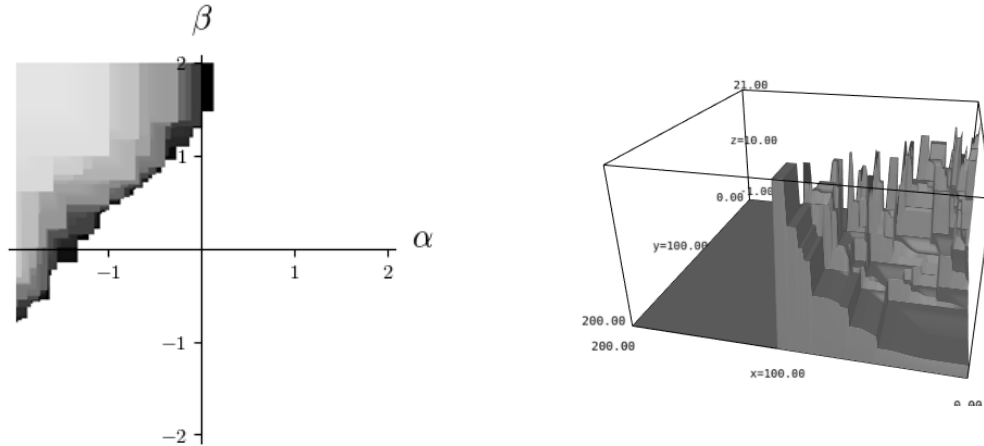


FIGURE 1. Upper-bound on $\frac{\log p}{\log N}$ in Theorem 1.11 for $I = [\alpha, \beta]$.

intervals I . Theorems 1.1-1.5 indicate that, when it can be applied, the method described here yields non-trivial numerical results.

Notation. Our notation is quite standard. We follow the usual practice of denoting by p an arbitrary prime and by ε an arbitrarily small positive real number which need not be the same in every occurrence. For any set $X \subset \mathbb{R}$ and maps $F : X \mapsto \mathbb{C}$ and $G : X \mapsto [0, \infty)$, we write

$$F(x) \ll G(x) \text{ or } F(x) = \mathcal{O}(G(x))$$

if there exists a $C > 0$ such that $|F(x)| \leq CG(x)$ for all $x \in X$. Sometimes, the implied constant C depends on some parameters and this dependence is shown in the subscript. For example, often the implied constant depends on the parameter ε , an arbitrarily small positive real number and we display this dependence by writing \ll_ε or \mathcal{O}_ε . Sometimes, the dependence is not shown when it is clear from the context in order to avoid making the notation too cumbersome. By $\check{\eta}$, we denote the Mellin transform of a function η :

$$\check{\eta}(s) = \int_0^\infty \eta(t)t^{s-1}dt. \quad (1.7)$$

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2. BACKGROUND ON MODULAR FORMS AND L -FUNCTIONS

2.1. Symmetric power L -functions. For a primitive form f , as in the introduction, its normalized coefficients $a_f(p) = a(p)$ can be written as

$$a(p) = \alpha_f(p) + \beta_f(p)$$

where, for $p \nmid N$, $\alpha_f(p) = 1/\beta_f(p)$ and both are complex numbers of absolute value 1. For each $\ell \in \mathbb{N}$, the ℓ -th symmetric power L -function of f is defined, for $\Re s > 1$, by

$$L(s, \text{sym}^\ell f) = \prod_p \prod_{0 \leq j \leq \ell} (1 - \alpha_f(p)^{\ell-j} \beta_f(p)^j / p^s)^{-1} =: \sum_{n \geq 1} \frac{a_{\text{sym}^\ell f}(n)}{n^s}. \quad (2.1)$$

We have $\text{sym}^1 f = f$ and it is convenient to set $\text{sym}^0 f = \mathbb{1}$ so that $L(s, \text{sym}^0 f) = \zeta(s)$. It is expected from a general conjecture of Langlands [20] that for every ℓ , there is a cuspidal automorphic representation of $GL_{\ell+1}(\mathbb{A}_{\mathbb{Q}})$ that corresponds to the L -function $L(s, \text{sym}^\ell f)$. For $1 \leq \ell \leq 8$, this was shown in [7] (for $\ell = 2$), [17] (for $\ell = 3$), [18, 16] (for $\ell = 4$) and [4, 5] (for $5 \leq \ell \leq 8$). When N is squarefree, this has been announced for all $\ell \geq 0$ in [28].

Following [14, Eq.(5.5)], we define the analytic conductor of $L(s, \text{sym}^\ell f)$ as

$$q(s, \text{sym}^\ell(f)) = N^\ell (|t| + 2)^{\ell+1} k^{\ell+\epsilon(\ell)}, \quad (2.2)$$

with $\epsilon(\ell) = \frac{1-(-1)^\ell}{2}$ being 1 or 0 according as ℓ is odd or even, as in the statement of Theorem 1.11.

Once we know that a symmetric power L -function comes from an automorphic representation, the analytic continuation and functional equation for that L -functions follows from [8] and thus the Phragmén-Lindelöf convexity principle (or the approximate functional equation [14, eq. (5.20)]) implies that for $1 \leq \ell \leq 8$, the hypothesis \mathcal{H}_ℓ holds with the value $\lambda_\ell = 1/4$, even for $\delta = 0$. This is known as the convexity bound. Giving a bound on an L -function that is stronger than the convexity bound is a challenging problem which has been solved in a few cases (see [27] and the references therein) and this is known as the subconvexity problem. Sometimes we are interested in the size of the L -functions in terms of only the size of the variable t , or the weight k or the level N . A result of Jutila and Motohashi [15] says that taking $\lambda_1 = 1/6$ is permissible in the weight and the t -aspect. We further define

$$q(\text{sym}^\ell(f)) := N^\ell k^{\ell+\epsilon(\ell)}. \quad (2.3)$$

In particular, $q(f) = Nk^2$ and $q(\text{sym}^2(f)) = N^2k^2$. Note that in the weight aspect, $q(f)$ and $q(\text{sym}^2(f))$ are of the same order.

For the coefficients of the symmetric ℓ -th power L -function of f , we have the following relation for every prime p :

$$a_{\text{sym}^\ell f}(p) = a(p^\ell) = U_\ell(\cos \theta(p)) = U_\ell(a(p)/2) = \frac{\sin((\ell+1)\theta(p))}{\sin \theta(p)}, \quad (2.4)$$

where U_ℓ is the Chebyshev polynomial of second kind, whose properties we recall next.

2.2. Chebyshev polynomials of the second kind. We recall that the Chebyshev polynomial of second kind $(U_\ell)_{\ell \geq 0}$ are defined by

$$U_0 = 1, \quad U_1 = 2x, \quad U_{\ell+1} - 2xU_\ell + U_{\ell-1} = 0. \quad (2.5)$$

These polynomials form an orthonormal basis in the space of polynomials on the interval $[-1, 1]$ relative to the Hermitian product

$$\langle f, g \rangle = \int_{-1}^1 f(x) \overline{g(x)} \frac{2}{\pi} \sqrt{1-x^2} dx. \quad (2.6)$$

The first few are given by

$$\begin{aligned} U_2 &= 4x^2 - 1, \\ U_3 &= 8x^3 - 4x, \\ U_4 &= 16x^4 - 12x^2 + 1, \\ U_5 &= 32x^5 - 32x^3 + 6x, \\ U_6 &= 64x^6 - 80x^4 + 24x^2 - 1, \\ U_7 &= 128x^7 - 192x^5 + 80x^3 - 8x, \\ U_8 &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1. \end{aligned}$$

The last equality in Eq. (2.4) comes from the relation

$$U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}.$$

3. AUXILIARY LEMMAS

3.1. Convolutions.

Lemma 3.1. *Assume $(\mathcal{H}_\ell)_{1 \leq \ell \leq L}$. Let $L \geq 1$ be an integer and let $(b_\ell)_{0 \leq \ell \leq L}$ be a collection of non-negative integers. Then, we have the equality*

$$\prod_p \left(1 + \frac{\sum_\ell b_\ell a(p^\ell)}{p^s} \right) = \prod_{0 \leq \ell \leq L} L(s, \text{sym}^\ell f)^{b_\ell} H(s),$$

where H is a function that is holomorphic and bounded by a constant in the region $\Re s \geq \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$.

Proof. This follows easily by comparing the p -th Euler factors. \square

We recall that, in the half-plane of absolute convergence, we have

$$L(s, f) = \prod_p \left(1 - \frac{a(p)}{p^s} + \frac{1}{p^{2s}} \right)^{-1} = \prod_p \left(1 - \frac{a(p)}{p^s} \right)^{-1} \left(1 - \frac{\beta(p)}{p^s} \right)^{-1} \quad (3.1)$$

as well as

$$L(s, \text{sym}^2 f) = \prod_p \left(1 - \frac{a(p)^2 - 1}{p^s} + \frac{a(p)^2 - 1}{p^{2s}} - \frac{1}{p^{3s}} \right)^{-1}. \quad (3.2)$$

3.2. Averages of multiplicative functions. We quote Theorem 21.2 of [29] which follows an idea of Wirsing [33].

Lemma 3.2. *Let f be a non-negative multiplicative function and κ be a non-negative real parameter such that*

$$\left\{ \begin{array}{l} \sum_{\substack{p \geq 2, \nu \geq 1 \\ p^\nu \leq Q}} f(p^\nu) \log(p^\nu) = \kappa Q + \mathcal{O}(Q/\log(2Q)) \quad (Q \geq 1), \\ \sum_{p \geq 2} \sum_{\substack{\nu, \ell \geq 1, \\ p^{\nu+\ell} \leq Q}} f(p^\ell) f(p^\nu) \log(p^\nu) \ll \sqrt{Q}, \end{array} \right.$$

then we have

$$\sum_{d \leq D} f(d) = \kappa C \cdot D (\log D)^{\kappa-1} (1 + o(1)),$$

where

$$C = \frac{1}{\Gamma(\kappa + 1)} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^\kappa \sum_{\nu \geq 0} f(p^\nu) \right\}. \quad (3.3)$$

Lemma 3.3. *Under the same hypotheses of Lemma 3.2 we have, for any continuously differentiable function η with $\int_0^1 \eta(u) du \neq 0$:*

$$\sum_{d \leq D} f(d) \eta(d/D) = \kappa C (1 + o(1)) \int_2^D (\log u)^{\kappa-1} \eta(u/D) du$$

as $D \rightarrow \infty$.

The condition on η is obviously satisfied if, as will be the case for us, η is non-negative with support inside the interval $[0, 1]$.

Proof. Using Lemma 3.2, we find that

$$\begin{aligned} \sum_{d \leq D} f(d) \eta(d/D) &= \sum_{d \leq D} f(d) \eta(1) - \sum_{d \leq D} f(d) \int_{d/D}^1 \eta'(t) dt \\ &= \sum_{d \leq D} f(d) \eta(1) - \int_0^1 \sum_{d \leq tD} f(d) \eta'(t) dt \\ &= \kappa \eta(1) C \cdot D (\log D)^{\kappa-1} \\ &\quad - \int_2^D \kappa C u (\log u)^{\kappa-1} \eta'(u/D) du / D + o(D (\log D)^{\kappa-1}) \end{aligned}$$

as $D \rightarrow \infty$. Hence, by partial summation,

$$\begin{aligned} \sum_{d \leq D} f(d) \eta(d/D) (\kappa C)^{-1} &= \int_2^D (\kappa - 1 + \log u) (\log u)^{\kappa-2} \eta(u/D) du + o(D (\log D)^{\kappa-1}) \\ &= \int_2^D (\log u)^{\kappa-1} \eta(u/D) du + o(D (\log D)^{\kappa-1}). \end{aligned}$$

However we also have

$$\begin{aligned} \int_2^D (\log u)^{\kappa-1} \eta(u/D) du &= \mathcal{O} \left(\int_2^{D/\log D} (\log u)^{\kappa-1} du \right) + \int_{D/\log D}^D (\log u)^{\kappa-1} \eta(u/D) du \\ &= \mathcal{O} (D (\log D)^{\kappa-2}) + D (\log D)^{\kappa-1} \int_{1/\log D}^1 \left(1 + \frac{\log v}{\log D}\right)^{\kappa-1} \eta(v) dv \\ &= \mathcal{O} (D (\log D)^{\kappa-2}) + D (\log D)^{\kappa-1} \int_{1/\log D}^1 \eta(v) dv \\ &\sim \left(\int_0^1 \eta(v) dv \right) D (\log D)^{\kappa-1} \end{aligned}$$

as $D \rightarrow \infty$, since $\int_0^1 \eta(u) du \neq 0$. In the third line we have used the uniform estimate $(1 + (\log v)/\log D)^{\kappa-1} = 1 + O(\log(1/v)/\log D)$ for $1/\log D < v < 1$. Hence our claimed estimate

$$\sum_{d \leq D} f(d) \eta(d/D) = \kappa C (1 + o(1)) \int_2^D (\log u)^{\kappa-1} \eta(u/D) du$$

follows. □

4. A GENERAL AVERAGE BOUND

Lemma 4.1. *Let $L \in \mathbb{N}_{>0}$, and assume $(\mathcal{H}_\ell)_{1 \leq \ell \leq L}$. Let $(b_\ell)_{0 \leq \ell \leq L}$ be a collection of non-negative integers. Given a primitive form $f(z) = \sum_{n \geq 1} a(n)e(nz)$ as in the introduction, let us define a multiplicative function h_f by the equality*

$$\sum_n \frac{h_f(n)}{n^s} = \prod_p \left(1 + \frac{\sum_\ell b_\ell a(p^\ell)}{p^s} \right)$$

Then h_f is supported on square-free integers and there exists a polynomial P_L of degree at most $b_0 - 1$ such that, for any $\varepsilon > 0$, we have

$$\sum_{n \geq 1} h_f(n) \eta(n/X) = X P_L(\log X) + \mathcal{O}\left(X^{\frac{1}{2} + \varepsilon} \prod_{1 \leq \ell \leq L} q(\text{sym}^\ell(f))^{b_\ell \lambda_\ell + \varepsilon}\right) \quad (4.1)$$

for any compactly supported twice continuously differentiable non-negative function η .

Proof. Let us denote by S the left-hand side of (4.1). By taking Mellin transforms (e.g. p.90 of [14]), we get

$$S = \frac{1}{2i\pi} \int_{2-i\infty}^{2+i\infty} X^s \check{\eta}(s)^s \sum_{n \geq 1} \frac{h_f(n)}{n^s} ds.$$

The fact that η is twice continuously differentiable ensures us that its Mellin transform verifies $\check{\eta}(s) \ll 1/(1 + |s|^2)$ uniformly in any closed vertical strip in the half plane $\Re s > 0$. Lemma 3.1 gives us an expression for the Dirichlet series $\sum_{n \geq 1} h_f(n)/n^s$ from which we see that we can shift the line of integration to $\Re s = \frac{1}{2} + \varepsilon$ obtaining that the error term is at most

$$\mathcal{O}\left(X^{\frac{1}{2} + \varepsilon} \prod_{1 \leq \ell \leq L} q(\text{sym}^\ell(f))^{b_\ell \lambda_\ell + \varepsilon}\right),$$

by our hypothesis $(\mathcal{H}_\ell)_{1 \leq \ell \leq L}$ and the convexity principle. The residue at 1 gives the claimed main term, and the lemma follows readily. □

5. A GENERAL LEMMA AROUND VINOGRADOV'S TRICK

Lemma 5.1. *Let g be a real-valued multiplicative function supported on the squarefree integers. We assume further that $g(p) \geq F$ for every prime p , and that for every prime $p \leq P$, we have $g(p) \geq \kappa > 0$. Let η be a non-negative, continuously differentiable function with support within $[0, 1]$ such that $\int_0^1 \eta(v) dv = 1$. We have, for $M = P^\theta$ for some $\theta \in [0, 1]$,*

$$\sum_{n \geq 1} \mu^2(n) g(n) \eta\left(\frac{n}{PM}\right) \geq (1 + o(1)) \kappa C M P (\log MP)^{\kappa-1} (1 - (\kappa - F) \mathcal{F}(\theta; \kappa))$$

where C is given by (3.3) and \mathcal{F} is defined in (1.1)

The factor $\mu^2(n)$ is only here to remind the reader that the variable n is restricted to squarefree values. It can be omitted!

Proof. We set

$$S = \sum_{n \geq 1} g(n) \eta\left(\frac{n}{PM}\right). \quad (5.1)$$

By our hypotheses, we find that

$$\begin{aligned}
S &= \sum_{\substack{n \leq PM, \\ P^+(n) \leq P}} g(n) \eta\left(\frac{n}{PM}\right) + \sum_{P < p \leq PM} g(p) \sum_{n \leq PM/p} g(n) \eta\left(\frac{pn}{PM}\right) \\
&\geq \sum_{\substack{n \leq PM, \\ P^+(n) \leq P}} g(n) \eta\left(\frac{n}{PM}\right) + F \sum_{P < p \leq PM} \sum_{n \leq PM/p} \mu^2(n) \kappa^{\omega(n)} \eta\left(\frac{pn}{PM}\right) \\
&\geq \sum_{n \leq PM} \mu^2(n) \kappa^{\omega(n)} \eta\left(\frac{n}{PM}\right) + (F - \kappa) \sum_{P < p \leq PM} \sum_{n \leq PM/p} \mu^2(n) \kappa^{\omega(n)} \eta\left(\frac{pn}{PM}\right).
\end{aligned}$$

Here $P^+(n)$ denotes the greatest prime divisor of n . We appeal to Lemma 3.3 with $f(n) = \mu^2(n) \kappa^{\omega(n)}$ and get

$$\begin{aligned}
S/(C\kappa) &\geq (1 + o(1)) \int_2^{PM} (\log u)^{\kappa-1} \eta\left(\frac{u}{PM}\right) du \\
&\quad + (F - \kappa + o(1)) \sum_{N < p \leq PM} \int_2^{PM/p} (\log u)^{\kappa-1} \eta\left(\frac{up}{PM}\right) du.
\end{aligned}$$

Note that the change of variable $vPM = u$ shows that

$$\int_2^{PM} (\log u)^{\kappa-1} \eta\left(\frac{u}{PM}\right) du = PM (\log PM)^{\kappa-1} \int_0^1 \eta(v) dv (1 + o(1)).$$

We use this estimate with M replaced by M/t and the prime number theorem to infer that

$$\begin{aligned}
\sum_{N < p \leq PM} \int_2^{PM/p} (\log u)^{\kappa-1} \eta\left(\frac{up}{PM}\right) du \\
= PM(1 + o(1)) \int_0^1 \eta(v) dv \int_N^{PM} \left(\log \frac{PM}{t}\right)^{\kappa-1} \frac{dt}{t \log t}
\end{aligned}$$

while this last integral equals, with the change of variable $v = (PM)^h$ and $M = N^\theta$,

$$\int_1^M \frac{(\log v)^{\kappa-1} dv}{v(\log(PM) - \log v)} = (\log PM)^{\kappa-1} \int_0^{\theta/(1+\theta)} \frac{h^{\kappa-1} dh}{1-h}.$$

Recall that $\int_0^1 \eta(v) dv = 1$. We thus find that

$$\begin{aligned}
\frac{(1 + o(1))S}{C\kappa PM (\log PM)^{\kappa-1}} &\geq 1 + (F - \kappa) \int_0^{\theta/(1+\theta)} \frac{h^{\kappa-1} dh}{1-h} \\
&= 1 - (\kappa - F) \mathcal{F}(\theta, \kappa).
\end{aligned}$$

□

6. PROOF OF THEOREMS 1.11, 1.1, 1.3, 1.5

Suppose $a(p) \notin I$ for every $p \leq P$. Under the assumptions of Theorem 1.11, let $\theta \in [0, 1]$ be such that

$$\frac{1}{\kappa - F} > \mathcal{F}(\theta; \kappa); \tag{6.1}$$

for instance, we may take $\theta = \max(\mathcal{G}(\kappa - F; \kappa) - \varepsilon, 0)$. Consider the sum

$$S = \sum_{n \geq 1} h_f(n) \eta(n/PM)$$

where $M \in [1, P]$. From the upper and the lower bound of S as given by Lemma 4.1 and 5.1 respectively and noting that $b_0 = 0$, we obtain,

$$(PM)^{\frac{1}{2} + \varepsilon} \prod_{1 \leq \ell \leq L} q(\text{sym}^\ell(f))^{b_\ell \lambda_\ell + \varepsilon} \gg PM.$$

Therefore, with $M = P^\theta$ for some $\theta \in [0, 1]$ satisfying (6.1), we have

$$P \ll_k N^{\frac{2 \sum_{\ell \geq 1} \ell b_\ell \lambda_\ell}{1 + \theta} + \varepsilon}.$$

This leads to the estimate (1.5) and the other estimate (1.6) is proved in a similar manner.

Let us inspect what this gives to us under the convexity bound for $\lambda_\ell = 1/4$. Since the quantity $2 \sum_{\ell \geq 1} \ell b_\ell \lambda_\ell$ takes all the values that are half-positive integers, we may inspect the first of them one by one. As we did above, we focus on the level N .

First case $(1/2) \sum_{\ell \geq 1} \ell b_\ell = 1/2$. This is only possible with the choice $b_1 = 1$, all other b_ℓ 's being 0. We have $\sum_{1 \leq \ell \leq L} b_\ell U_\ell(x/2) = x$ which is positive when $x = a(p) > 0$. On assuming $a(p) \geq \delta$ when $p \leq P$, we see that we may take $\kappa = \delta$ and $F = -2$ and get, for $N \geq N_0(\varepsilon)$,

$$\frac{\log P}{\log N} \leq \frac{2\lambda_1}{1 + \mathcal{G}(2 + \delta; \delta)} + \varepsilon. \quad (6.2)$$

Hence Theorem 1.1.

Second case $(1/2) \sum_{\ell \geq 1} \ell b_\ell = 1$. This is only possible with the choice $b_2 = 1$, all other b_ℓ 's being 0. We have $\sum_{1 \leq \ell \leq L} b_\ell U_\ell(x/2) = x^2 - 1$ which is positive when $x = a(p) \notin [-1, 1]$. On assuming $|a(p)| \geq 1 + \delta$ when $p \leq P$, we see that we may take $\kappa = 2\delta + \delta^2$ and $F = -1$ and get, for $N \geq N_0(\varepsilon)$,

$$\frac{\log P}{\log N} \leq \frac{4\lambda_2}{1 + \mathcal{G}(1 + 2\delta + \delta^2; 2\delta + \delta^2)} + \varepsilon. \quad (6.3)$$

Hence Theorem 1.3.

Finding non-negative values. Let $I = [0, 2]$. A numerical computation found the coefficients $(b_\ell)_{0 \leq \ell \leq 5} = (0, 0, 3, 5, 4, 1)$, which satisfy (1.4) with $\kappa \geq 1/3$ and $F = -10$. Then Theorem 1.5 follows from the bounds (1.5) and (1.6).

7. PROOF OF THEOREM 1.6

Let $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be smooth, compactly supported and such that $\mathbb{1}_{[0,1]} \geq \eta \geq \mathbb{1}_{[1/3, 2/3]}$, let $\varepsilon > 0$, and consider

$$T(X) = \sum_n \mu^2(n) a(n) \eta(n/X) \quad (X \geq 1).$$

By Lemma 4.1, we get

$$T(X) \ll X^{1/2} (k^2 N)^{1/4 + \varepsilon}. \quad (7.1)$$

Suppose that $a(p) \geq 0$ for all primes $p \leq X$. If the inequality

$$\sum_{\substack{n: \\ a(n) \geq 1}} \mu^2(n) a(n) \eta(n/X) \geq X^{1-\varepsilon} \quad (7.2)$$

holds then we easily have

$$T(X) \geq X^{1-\varepsilon}. \quad (7.3)$$

Otherwise, suppose that (7.2) does not hold. We write

$$\begin{aligned} T(X) &= \sum_{\substack{n: \\ 0 \leq a(n) < 1}} \mu^2(n) a(n) \eta(n/X) + \sum_{\substack{n: \\ a(n) \geq 1}} \mu^2(n) a(n) \eta(n/X) \\ &\geq \sum_{\substack{n: \\ 0 \leq a(n) < 1}} \mu^2(n) a(n)^2 \eta(n/X) + \sum_{\substack{n: \\ a(n) \geq 1}} \mu^2(n) a(n) \eta(n/X) \\ &= \sum_n \mu^2(n) a(n)^2 \eta(n/X) + \sum_{\substack{n: \\ a(n) \geq 1}} \mu^2(n) a(n) (1 - a(n)) \eta(n/X). \end{aligned}$$

Now the last sum is $\mathcal{O}(X^{1-\varepsilon/2})$ by Deligne's bound $|a(p)| \leq 2$ and the negation of (7.2). The first sum can be handled by Rankin-Selberg method (Lemma 4.1) and is $\gg L(1, \text{sym}^2 f)X + \mathcal{O}(X^{1/2}(k^2 N^2)^{1/4+\varepsilon})$. Thus we have, using the lower bound $L(1, \text{sym}^2 f) \gg 1/\log(kN)$ due to Hoffstein and Lockhart [10],

$$T(X) \gg X/\log(kN) + \mathcal{O}(X^{1/2}(k^2 N^2)^{1/4+\varepsilon}) + \mathcal{O}(X^{1-\varepsilon/2}). \quad (7.4)$$

One of the equations (7.3) and (7.4) must hold and either, in conjunction with equation (7.1), imply the theorem.

8. PROOF OF THEOREM 1.8

By equation (3) of [30], the Deligne bound $|a(p)| \leq 2$ and Mertens' theorem (see [14, Eq. (2.15)]), we have

$$\log L(1, f) = \mathcal{O}_\varepsilon(1) + \sum_{p \leq q^\varepsilon} \frac{a(p)}{p},$$

and therefore

$$\sum_{p \leq q^\varepsilon} \frac{1}{p} \left(a(p) - \frac{\log L(1, f)}{\log \log q} \right) = \mathcal{O}_\varepsilon(1).$$

However, if we had $a(p) < \gamma^- - \delta$ for $p \leq q^\varepsilon$, then we would also have

$$\sum_{p \leq q^\varepsilon} \frac{1}{p} \left(a(p) - \frac{\log L(1, f)}{\log \log q} \right) \leq \mathcal{O}_{\varepsilon, \delta}(1) - \frac{\delta}{2} \log \log q,$$

which is a contradiction for q large enough, and therefore there must be a prime $p \leq q^\varepsilon$ such that $a(p) \geq \gamma^- - \delta$. An identical argument shows the existence of $p \leq q^\varepsilon$ such that $a(p) \leq \gamma_+ + \delta$.

9. CONDITIONAL BOUNDS: PROOF OF THEOREM 1.10

By the Stone-Weierstrass theorem, the fact that (U_ℓ) forms a basis of $\mathbb{R}[X]$, and the relation (2.4), we may find $L \geq 1$ and real coefficients b_0, \dots, b_L depending on I , with $b_0 > 0$, such that

$$\sum_{p \leq x} \mathbb{1}(a(p) \in I) \left(1 - \frac{p}{x}\right) \log p \geq \sum_{\ell=0}^L b_\ell \sum_{p \leq x} a_{\text{sym}^\ell f}(p) \left(1 - \frac{p}{x}\right) \log p. \quad (9.1)$$

By Chebyshev's estimate, the contribution of the term $\ell = 0$ is

$$b_0 \sum_{p \leq x} \left(1 - \frac{p}{x}\right) \log p \gg_I x$$

with an absolute constant. To show that the right-hand side of (9.1) is positive for some $x = \mathcal{O}_I((\log q)^2)$, it therefore suffices to show that for all integer $\ell \geq 1$ and all real $x \geq 1$, we have

$$\sum_{p \leq x} a_{\text{sym}^\ell f}(p) \left(1 - \frac{p}{x}\right) \log p = \mathcal{O}_\ell(x^{1/2} \log q).$$

This is an immediate consequence of the explicit formula [14, eq. (5.33)] (with an additional smoothing, as in [26, eq. (13.28)]) along with classical zero density estimates [14, Theorem 5.8].

REFERENCES

1. N. C. Ankeny, *The least quadratic non residue*, Ann. of Math. (2) **55** (1952), 65–72.
2. D. A. Burgess, *The distribution of quadratic residues and non-residues*, Mathematika **4** (1957), 106–112.
3. L. Clozel, M. Harris, and R. Taylor, *Automorphy for some l -adic lifts of automorphic mod l Galois representations*, Inst. Hautes Études Sci. Publ. Math. (2008), no. 108, 1–181.
4. L. Clozel and J. A. Thorne, *Level raising and symmetric power functoriality, II*, Ann. of Math. (2) **181** (2015), no. 1, 303–359.
5. ———, *Level-raising and symmetric power functoriality, III*, Duke Math. J. **166** (2017), no. 2, 325–402.
6. P. Deligne, *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 43, 273–307.
7. S. Gelbart and H. Jacquet, *A relation between automorphic representations of $\text{GL}(2)$ and $\text{GL}(3)$* , Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542.
8. R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics, Vol. 260, Springer-Verlag, Berlin-New York, 1972.
9. M. Harris, N. Shepherd-Barron, and R. Taylor, *A family of Calabi-Yau varieties and potential automorphy*, Ann. of Math. (2) **171** (2010), no. 2, 779–813.
10. J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegel zero. With an appendix by Dorian Goldfeld, Hoffstein and Daniel Lieman*. Ann. of Math. (2) **140** (1994), no. 1, 161–181.
11. J. Hoffstein, D. Ramakrishnan, *Siegel zeros and cusp forms*, Int. Math. Res. Not., Volume 1995, Issue 6, 1995, Pages 279–308, 1995.
12. H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997.
13. H. Iwaniec, W. Kohnen, and J. Sengupta, *The first negative Hecke eigenvalue*, Int. J. Number Theory **3** (2007), no. 3, 355–363.
14. H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2004, xii+615 pp.
15. M. Jutila and Y. Motohashi, *Uniform bounds for Hecke L -functions*, Acta Math., Vol. 195, No. 1, 2005, pp. 61–115
16. H. H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J. **112** (2002), no. 1, 177–197.
17. ———, *Functorial products for $\text{GL}_2 \times \text{GL}_3$ and the symmetric cube for GL_2* , Ann. of Math. (2) **155** (2002), no. 3, 837–893, With an appendix by Colin J. Bushnell and Guy Henniart.
18. H. H. Kim, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2* , J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.
19. E. Kowalski, Y.-K. Lau, K. Soundararajan, and J. Wu, *On modular signs*, Math. Proc. Cambridge Philos. Soc. **149** (2010), no. 3, 389–411. MR 2726725
20. R.P. Langlands, *Problems in the Theory of Automorphic Forms*, Lecture Notes in Mathematics, Vol. 170, Springer-Verlag, New York, 1970, pp. 18–86.
21. Y. K. Lau, J. Y. Liu, and J. Wu, *The first negative coefficients of symmetric square L -functions*, Ramanujan J. **27** (2012), no. 3, 419–441.
22. R. J. Lemke Oliver and J. Thorner, *Effective log-free zero density estimates for automorphic L -functions and the Sato-Tate conjecture*, Internat. Math. Res. Notices (2018), 30pp.

23. Y.-K. Lau and J. Wu, *A density theorem on automorphic L -functions and some applications*, Trans. Amer. Math. Soc. **358** (2006), 441–472.
24. Y.-K. Lau and J. Wu, *Extreme values of symmetric power L -functions at 1*, Acta Arith. **126** (2007), No. 1, 57–76.
25. K. Matomäki, *On signs of Fourier coefficients of cusp forms*, Math. Proc. Cambridge Philos. Soc. **152** (2012), no. 2, 207–222. MR 2887873
26. H.L. Montgomery and R.C. Vaughan, *Multiplicative Number Theory: I. Classical Theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, 2006.
27. R. Munshi, *The subconvexity problem for L -functions*, Proc. Int. Cong. of Math. – 2018 Rio de Janeiro, Vol. 1 (361–374), available at <https://eta.impa.br/dl/164.pdf>
28. J. Newton, J. A. Thorne : *Symmetric power functoriality for holomorphic modular forms*, Publ. Math. Inst. Hautes Études Sci. **134** (2021), 1–116, <https://doi.org/10.1007/s10240-021-00127-3>.
29. O. Ramaré, *Arithmetical aspects of the large sieve inequality*, Harish-Chandra Research Institute Lecture Notes, vol. 1, Hindustan Book Agency, New Delhi, 2009, With the collaboration of D. S. Ramana. MR MR2493924
30. O. Ramaré, *Comparing $L(s, \chi)$ with its truncated Euler product and generalization*, Funct. Approximatio, Comment. Math. **42** (2010), No. 2, 145–151.
31. R. Taylor, *Automorphy for some l -adic lifts of automorphic mod l Galois representations. II*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 183–239.
32. I. M. Vinogradov, *On a general theorem concerning the distribution of the residues and non-residues of powers*, Trans. Amer. Math. Soc. **29** (1927), no. 1, 209–217.
33. E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann. **143** (1961), 75–102.

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