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# A MODULAR ANALOGUE OF A PROBLEM OF VINOGRADOV

R. ACHARYA, S. DRAPPEAU, S. GANGULY, AND O. RAMARÉ

**ABSTRACT.** Given a primitive, non-CM, holomorphic cusp form  $f$  with normalized Fourier coefficients  $a(n)$  and given an interval  $I \subset [-2, 2]$ , we study the least prime  $p$  such that  $a(p) \in I$ . This can be viewed as a modular form analogue of Vinogradov's problem on the least quadratic non-residue. We obtain strong explicit bounds on  $p$ , depending on the analytic conductor of  $f$  for some specific choices of  $I$ .

## 1. INTRODUCTION

The present article is concerned with understanding the distribution of the initial Fourier coefficients of primitive holomorphic cusp forms at primes. Suppose  $f$  is such a form of weight  $k$  for the group  $\Gamma_0(N)$ . We further assume that  $f$  is non-CM and has trivial nebentypus. The normalized Fourier coefficients of  $f$  at infinity are denoted by  $(a(n))_{n \geq 1}$ , so that  $a(1) = 1$  and

$$f(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} e(nz),$$

where, as usual,  $e(z)$  denotes  $e^{2\pi iz}$  and with this normalization, the Ramanujan bound (proved by Deligne [6]) says  $-2 \leq a(p) \leq 2$  for primes  $p$ . Furthermore, the function  $n \mapsto a(n)$  is real-valued and multiplicative. We refer the reader to the text [12] for background information on holomorphic modular forms. The Sato-Tate conjecture for distribution of the angles  $\theta_p$ , defined by  $a(p) = 2 \cos \theta_p$ , as  $p$  runs over primes, which is now a theorem of Clozel, Harris, Shepherd-Barron and Taylor [3, 31, 9], implies, in particular, that any interval of positive measure within  $[-2, 2]$  contains infinitely many values of  $a(p)$ . The goal of this article is to obtain bounds for the least prime  $p$  such that  $a(p)$  lies in a fixed interval  $I \subset [-2, 2]$ . This can be considered as an analogue of Vinogradov's problem of estimating, given a modulus  $q \geq 1$ , the size of the least quadratic non-residue modulo  $q$  (see [2], [32]). The quality of our bounds will be measured in terms of the analytic conductor  $q(f) = Nk^2$  of the form  $f$  (see §2.1), and also separately in term of the weight  $k$  of the form, considering the level  $N$  to be fixed and in terms of the level  $N$ , considering the weight  $k$  to be fixed. We restrict our attention to forms with trivial nebentypus in order to clarify the presentation but the methods presented here can be extended to a more general setting.

Let  $I \subset [-2, 2]$ . Theorem 1.6 of the paper [22] of Lemke-Oliver and Thorner implies that there exists a constant  $A$  depending only on  $I$  such that  $a(p) \in I$  for some prime  $p \leq q^A$ . Their method relies on effective log-free zero density estimates for the  $L$ -function associated with  $f$ , and the Turán power-sum method. The value of the constant  $A$  is not stated explicitly in their paper but it is not hard to see that the constant is effective and can be worked out explicitly. However

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the method is likely to produce quite large values of  $A$ . Our aim in the present work is to make the value of  $A$  as small as possible for some specific intervals.

We define, when  $\kappa$  is positive and  $x \in [0, 1]$ :

$$\mathcal{F}(x; \kappa) = \int_0^{x/(1+x)} \frac{h^{\kappa-1} dh}{1-h} = \sum_{k \geq 0} \frac{1}{\kappa+k} \left( \frac{x}{1+x} \right)^{\kappa+k}. \quad (1.1)$$

Note that  $\mathcal{F}(\cdot; \kappa)$  is increasing between  $\mathcal{F}(0; \kappa) = 0$  and  $\mathcal{F}(1; \kappa) = \int_0^{1/2} \frac{h^{\kappa-1} dh}{1-h}$ . We thus define a function  $\mathcal{G}(\cdot; \kappa)$  with value in  $[0, 1]$  by

$$\mathcal{G}(y; \kappa) = \max\{x \in [0, 1] : \mathcal{F}(x; \kappa) \leq 1/y\}. \quad (1.2)$$

The function  $\mathcal{G}$  is non-increasing and we have  $\mathcal{G}(y; \kappa) = 1$  when  $y \leq 1/\mathcal{F}(1; \kappa)$  and by convention  $\mathcal{G}(\infty; \kappa) = 0$ .

We now state our main results which depend crucially on knowledge about the analytic properties of the symmetric power  $L$  functions associated to  $f$  (see §2.1 for definition). This is likely to change in the future; only small changes would be required in our proofs to reflect any such improvement. Here is the assumption we rely on.

**Hypothesis  $\mathcal{H}_\ell$ :** The  $L$ -function  $L(s, \text{sym}^\ell(f))$  has analytic continuation to the entire complex plane and it satisfies the bound

$$L(1/2 + it, \text{sym}^\ell(f)) \ll_\varepsilon q(\text{sym}^\ell(f), s)^{\lambda_\ell + \varepsilon}$$

for any  $\varepsilon > 0$ .

For holomorphic forms, the automorphy of  $L(s, \text{sym}^\ell f)$  has been known for  $\ell \leq 8$  by [7, 17, 18, 16, 4, 5], and has recently been proved for *all*  $\ell$  when  $N$  is squarefree by Newton-Thorne [28]. As a result, these  $L$ -functions admit holomorphic continuation to the entire complex plane and by the convexity principle,  $\mathcal{H}_\ell$  holds with  $\lambda_\ell = 1/4$  (known as the convexity bound) for  $\ell \leq 8$  unconditionally and for all  $\ell$  when  $N$  is squarefree.

Our results are the following.

**Theorem 1.1.** *For any  $\delta \in (0, 2]$ , let  $\theta_1(\delta) = \mathcal{G}(2 + \delta; \delta)$ . The function  $\theta_1$  is increasing and we have  $\theta_1(0+) = 0$  and  $\theta_1(1) = 0.3956 \dots$ . Suppose  $\lambda_1 > 0$  is an exponent that satisfies the hypothesis  $\mathcal{H}_\ell$  below for  $\ell = 1$ , and let  $\varepsilon > 0$ . Then for  $q = N$  or  $k^2$  sufficiently large, there exists a prime*

$$p \ll_\varepsilon q^{\frac{2\lambda_1}{1+\theta_1(\delta)} + \varepsilon}$$

with  $a(p) \leq \delta$ .

**Remark 1.2.** The convexity bound (Phragmén-Lindelöf principle) allows taking  $\lambda_1 = \frac{1}{4}$  but better exponents, called subconvex exponents are known in both the weight and the level aspects. For example, one may take  $\lambda_1 = \frac{1}{6}$  when  $N = 1$  by a result of Jutila and Motohashi [15].

**Theorem 1.3.** *For any  $\delta \in (0, 1]$ , let  $\theta_2(\delta) = \mathcal{G}((1 + \delta)^2; 2\delta + \delta^2)$ . The function  $\theta_2$  is increasing when  $\delta \leq 0.5305 \dots$ , and constant equal to 1 afterwards. We have  $\theta_2(0+) = 0$ ,  $\theta_2(1/2) = 0.9093 \dots$ ,  $\theta_2(1) = 1$ . Suppose  $\lambda_2 > 0$  is an exponent that satisfies the hypothesis  $\mathcal{H}_\ell$  below for  $\ell = 2$ , and let  $\varepsilon > 0$ . For any  $\delta \in [0, 1]$ , and for  $q = N$  or  $k^2$  sufficiently large, there exists a prime*

$$p \ll_\varepsilon q^{\frac{4\lambda_2}{1+\theta_2(\delta)} + \varepsilon}$$

with  $|a(p)| \leq 1 + \delta$ .

**Remark 1.4.** The convexity bound allows the choice  $\lambda_2 = \frac{1}{4}$  and currently this is the best known exponent. Obtaining a subconvex estimate for the symmetric square  $L$ -function in the level or the weight aspect is a challenging problem.

It turns out that showing the existence of primes  $p$  of small size in terms of the conductor (i.e., weight and level) such that  $a(p) \geq 0$  is rather difficult. By utilizing the fact that hypotheses  $\mathcal{H}_\ell$  holds true for  $1 \leq \ell \leq 5$ , we are able to show the following result:

**Theorem 1.5.** *There is a prime  $p \ll k^{24}N^{21}$  such that  $a(p) \geq 0$ .*

The results above are all obtained using a similar strategy and this is summarized in Theorem 1.11 below. For some specific intervals, however, we obtain better bounds by employing ad hoc techniques using  $L$ -functions as we now describe.

**Theorem 1.6.** *For any  $\varepsilon > 0$ , there is a prime  $p = \mathcal{O}_\varepsilon(kN)^{1+\varepsilon}$  such that  $a(p) < 0$ .*

**Corollary.** *The least prime such that  $a(p) \neq 0$  is  $\ll_\varepsilon (kN)^{1+\varepsilon}$ , for any  $\varepsilon > 0$ .*

**Remark 1.7.** As the proof of the above theorem shows, the exponent 1 can be replaced by  $4\lambda_2$  and any subconvex estimate  $\lambda_2 < 1/4$  for the symmetric square  $L$ -function will lead to an improvement of the above result.

The next result relates the possibility of the initial coefficients at primes assuming extreme values with the size of  $L(1, f)$ . For  $q = Nk^2$ , let

$$\gamma^- := \liminf_{q \rightarrow \infty} \frac{\log L(1, f)}{\log \log q}, \quad \gamma^+ := \limsup_{q \rightarrow \infty} \frac{\log L(1, f)}{\log \log q}.$$

From the zero-free region of  $L(s, f)$  (See [11]), the standard techniques yield

$$-2 \leq \gamma^- \leq \gamma^+ \leq 2. \quad (1.3)$$

**Theorem 1.8.** *For any  $\delta, \varepsilon > 0$ , the least prime  $p$  such that  $a(p) > \gamma^- - \delta$  is  $\mathcal{O}(q^\varepsilon)$ . Similarly, the least prime  $p$  such that  $a(p) < \gamma^+ + \delta$  is  $\mathcal{O}(q^\varepsilon)$ .*

**Remark 1.9.** The bounds (1.3) seem to be the best known, and any improvement would yield a non-trivial result in Theorem 1.8. The quality of the upper-bound on  $p$ , namely  $\mathcal{O}(q^\varepsilon)$ , compared to the above results, suggests that improving the bounds (1.3) is a difficult task. Under the Riemann Hypothesis for  $L(s, f)$ , one has the bounds

$$(\log \log q)^{-2} \ll L(1, f) \ll (\log \log q)^2,$$

at least in the case  $N = 1$  (see [23, Thm. 3] for a precise and stronger statement), which yields conjecturally  $\gamma^- = \gamma^+ = 0$ . Furthermore, it is known that these bounds hold for almost all forms (see [24, Cor. 2] for a precise statement).

Several authors investigated the smallest *integer*  $n$  such that  $a(n) < 0$ , see for instance [13], [19], [21] or [25]. It follows from [25] that the least such  $n$  is  $\mathcal{O}(q^{3/8})$ , where  $q = Nk^2$ . A closer scrutiny of their proofs reveals that the integer  $n$  they produce is either a prime or the square of a prime. Indeed, all the above works make use of the contrast between the sizes of  $a(p)$  and  $a(p^2)$  forced by the Hecke relation  $a(p)^2 - 1 = a(p^2)$  for primes  $p$ . Since we aim at localizing only  $a(p)$ 's, the coefficients at primes, we cannot rely on such procedures. In fact, the two methods we propose are *reverse*: from a localization on  $a(p)$ , we show that some polynomial in  $a(p)$  has to be large for many primes  $p$ . This polynomial defines the

value at  $p$  of a new function whose Dirichlet series we approximate with products of  $L(s, \text{sym}^\ell f)$  and it is by using the analytic properties of these latter that we reach a contradiction. To find an integer  $n$  such that  $a(n) < 0$ , only the analytic properties of  $L(s, f)$  are required.

Regarding bounds conditional on the Riemann Hypothesis, Ankeny [1] has proved that for any non-trivial character  $\chi \pmod{q}$ , if the Riemann hypothesis is true for  $L(s, \chi)$ , then the least  $n$  such that  $\chi(n) \neq 1$  is  $\mathcal{O}((\log q)^2)$ . It is not difficult to show that the analogous phenomenon holds in our setting:

**Theorem 1.10.** *Assume that for all  $\ell \geq 1$ , the function  $L(s, \text{sym}^\ell f)$  is entire and satisfies the Riemann hypothesis. Then for any interval  $I \subseteq [-2, 2]$  of positive measure, the least prime  $p$  such that  $a(p) \in I$  satisfies  $p \ll_I (\log q)^2$ .*

Let us now state our general theorem depending on the hypothesis  $\mathcal{H}_\ell$ . Note that this result implies Theorems 1.1, 1.3 and 1.5.

**Theorem 1.11** (Generic theorem). *Let  $(b_\ell)_{1 \leq \ell \leq L}$  be non-negative integers, Let  $\kappa > 0$  and  $F$  be real, and let  $I \subset [-2, 2]$  be such that*

$$\begin{cases} \forall x \in [-2, 2] \setminus I, & \sum_{1 \leq \ell \leq L} b_\ell U_\ell(x/2) \geq \kappa > 0, \\ \forall x \in [-2, 2], & \sum_{1 \leq \ell \leq L} b_\ell U_\ell(x/2) \geq F, \end{cases} \quad (1.4)$$

where  $U_\ell$  are the Chebyshev polynomials of the second kind. Then, on assuming  $(\mathcal{H}_\ell)_{\ell \leq L}$ , the least prime  $p$  such that  $a(p) \in I$  satisfies

$$\frac{\log p}{\log N} \leq \frac{2 \sum_{\ell} \ell b_\ell \lambda_\ell}{1 + \mathcal{G}(\kappa - F; \kappa)} + \varepsilon, \quad (1.5)$$

for any  $\varepsilon > 0$  and  $N$  large enough with respect to the weight  $k$  and  $\varepsilon$ ; and

$$\frac{\log p}{\log k} \leq \frac{2 \sum_{\ell} (\ell + \epsilon(\ell)) b_\ell \lambda_\ell}{1 + \mathcal{G}(\kappa - F; \kappa)} + \varepsilon. \quad (1.6)$$

for any  $\varepsilon > 0$  and  $k$  large enough with respect to the level  $N$  and  $\varepsilon$ . Here  $\epsilon(\ell) = \frac{1 - (-1)^\ell}{2} \in \{0, 1\}$  is the parity of  $\ell$ .

The intervals  $[\alpha, \beta]$  for which there is a linear combination with non-negative coefficients of  $U_1, \dots, U_8$  which takes positive values outside  $[\alpha, \beta]$  delimit a curve in  $(\alpha, \beta)$ , whose exact determination is an interesting question (without the non-negativity condition, the analogue for  $U_1, \dots, U_4$  was solved in Appendix A of [22]). Between this curve and the diagonal  $\alpha = \beta$ , Theorem 1.11 yields an upper-bound on  $\frac{\log p}{\log q}$ , which gets smaller as one moves away from the diagonal. This is represented in Figure 1, which was obtained by case-by-case analysis of all linear combinations with  $\sum_{\ell \leq 8} \ell b_\ell \leq 42$ . On the left, darker colors indicate a larger upper-bound.

Theorem 1.11 should be compared with Theorem 1.8 of [22]. In both cases, we are given an interval  $I \subset [-2, 2]$ , and we are looking for the least prime  $p$  such that  $a(p) \in I$ . In Theorem 1.8 of [22], the authors obtain an exponent depending on the quality with which the indicator function  $\mathbb{1}_I$  can be minorized by a linear combination of  $U_0, U_1, U_2, \dots$ . In Theorem 1.11, we obtain an exponent depending on the quality with which the complementary indicator function  $\mathbb{1}_{[-2, 2] \setminus I}$  is minorized by a linear combination *with non-negative coefficients* of  $U_1, U_2, \dots$ . An inconvenient of our method is that there is no clear description of the allowable

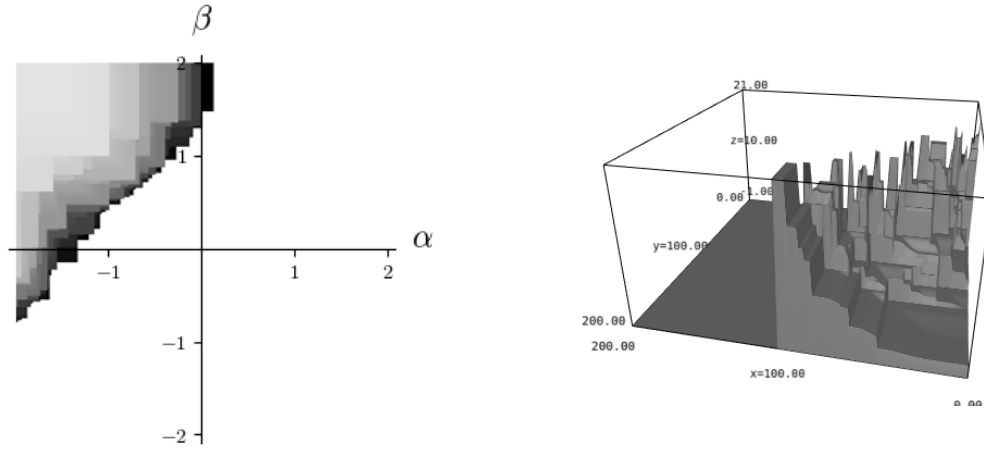


FIGURE 1. Upper-bound on  $\frac{\log p}{\log N}$  in Theorem 1.11 for  $I = [\alpha, \beta]$ .

intervals  $I$ . Theorems 1.1-1.5 indicate that, when it can be applied, the method described here yields non-trivial numerical results.

**Notation.** Our notation is quite standard. We follow the usual practice of denoting by  $p$  an arbitrary prime and by  $\varepsilon$  an arbitrarily small positive real number which need not be the same in every occurrence. For any set  $X \subset \mathbb{R}$  and maps  $F : X \mapsto \mathbb{C}$  and  $G : X \mapsto [0, \infty)$ , we write

$$F(x) \ll G(x) \text{ or } F(x) = \mathcal{O}(G(x))$$

if there exists a  $C > 0$  such that  $|F(x)| \leq CG(x)$  for all  $x \in X$ . Sometimes, the implied constant  $C$  depends on some parameters and this dependence is shown in the subscript. For example, often the implied constant depends on the parameter  $\varepsilon$ , an arbitrarily small positive real number and we display this dependence by writing  $\ll_\varepsilon$  or  $\mathcal{O}_\varepsilon$ . Sometimes, the dependence is not shown when it is clear from the context in order to avoid making the notation too cumbersome. By  $\check{\eta}$ , we denote the Mellin transform of a function  $\eta$ :

$$\check{\eta}(s) = \int_0^\infty \eta(t)t^{s-1}dt. \quad (1.7)$$

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## 2. BACKGROUND ON MODULAR FORMS AND $L$ -FUNCTIONS

**2.1. Symmetric power  $L$ -functions.** For a primitive form  $f$ , as in the introduction, its normalized coefficients  $a_f(p) = a(p)$  can be written as

$$a(p) = \alpha_f(p) + \beta_f(p)$$

where, for  $p \nmid N$ ,  $\alpha_f(p) = 1/\beta_f(p)$  and both are complex numbers of absolute value 1. For each  $\ell \in \mathbb{N}$ , the  $\ell$ -th symmetric power  $L$ -function of  $f$  is defined, for  $\Re s > 1$ , by

$$L(s, \text{sym}^\ell f) = \prod_p \prod_{0 \leq j \leq \ell} (1 - \alpha_f(p)^{\ell-j} \beta_f(p)^j / p^s)^{-1} =: \sum_{n \geq 1} \frac{a_{\text{sym}^\ell f}(n)}{n^s}. \quad (2.1)$$

We have  $\text{sym}^1 f = f$  and it is convenient to set  $\text{sym}^0 f = \mathbb{1}$  so that  $L(s, \text{sym}^0 f) = \zeta(s)$ . It is expected from a general conjecture of Langlands [20] that for every  $\ell$ , there is a cuspidal automorphic representation of  $GL_{\ell+1}(\mathbb{A}_{\mathbb{Q}})$  that corresponds to the  $L$ -function  $L(s, \text{sym}^\ell f)$ . For  $1 \leq \ell \leq 8$ , this was shown in [7] (for  $\ell = 2$ ), [17] (for  $\ell = 3$ ), [18, 16] (for  $\ell = 4$ ) and [4, 5] (for  $5 \leq \ell \leq 8$ ). When  $N$  is squarefree, this has been announced for all  $\ell \geq 0$  in [28].

Following [14, Eq.(5.5)], we define the analytic conductor of  $L(s, \text{sym}^\ell f)$  as

$$q(s, \text{sym}^\ell(f)) = N^\ell (|t| + 2)^{\ell+1} k^{\ell+\epsilon(\ell)}, \quad (2.2)$$

with  $\epsilon(\ell) = \frac{1-(-1)^\ell}{2}$  being 1 or 0 according as  $\ell$  is odd or even, as in the statement of Theorem 1.11.

Once we know that a symmetric power  $L$ -function comes from an automorphic representation, the analytic continuation and functional equation for that  $L$ -functions follows from [8] and thus the Phragmén-Lindelöf convexity principle (or the approximate functional equation [14, eq. (5.20)]) implies that for  $1 \leq \ell \leq 8$ , the hypothesis  $\mathcal{H}_\ell$  holds with the value  $\lambda_\ell = 1/4$ , even for  $\delta = 0$ . This is known as the convexity bound. Giving a bound on an  $L$ -function that is stronger than the convexity bound is a challenging problem which has been solved in a few cases (see [27] and the references therein) and this is known as the subconvexity problem. Sometimes we are interested in the size of the  $L$ -functions in terms of only the size of the variable  $t$ , or the weight  $k$  or the level  $N$ . A result of Jutila and Motohashi [15] says that taking  $\lambda_1 = 1/6$  is permissible in the weight and the  $t$ -aspect. We further define

$$q(\text{sym}^\ell(f)) := N^\ell k^{\ell+\epsilon(\ell)}. \quad (2.3)$$

In particular,  $q(f) = Nk^2$  and  $q(\text{sym}^2(f)) = N^2k^2$ . Note that in the weight aspect,  $q(f)$  and  $q(\text{sym}^2(f))$  are of the same order.

For the coefficients of the symmetric  $\ell$ -th power  $L$ -function of  $f$ , we have the following relation for every prime  $p$ :

$$a_{\text{sym}^\ell f}(p) = a(p^\ell) = U_\ell(\cos \theta(p)) = U_\ell(a(p)/2) = \frac{\sin((\ell+1)\theta(p))}{\sin \theta(p)}, \quad (2.4)$$

where  $U_\ell$  is the Chebyshev polynomial of second kind, whose properties we recall next.

**2.2. Chebyshev polynomials of the second kind.** We recall that the Chebyshev polynomial of second kind  $(U_\ell)_{\ell \geq 0}$  are defined by

$$U_0 = 1, \quad U_1 = 2x, \quad U_{\ell+1} - 2xU_\ell + U_{\ell-1} = 0. \quad (2.5)$$

These polynomials form an orthonormal basis in the space of polynomials on the interval  $[-1, 1]$  relative to the Hermitian product

$$\langle f, g \rangle = \int_{-1}^1 f(x) \overline{g(x)} \frac{2}{\pi} \sqrt{1-x^2} dx. \quad (2.6)$$

The first few are given by

$$\begin{aligned} U_2 &= 4x^2 - 1, \\ U_3 &= 8x^3 - 4x, \\ U_4 &= 16x^4 - 12x^2 + 1, \\ U_5 &= 32x^5 - 32x^3 + 6x, \\ U_6 &= 64x^6 - 80x^4 + 24x^2 - 1, \\ U_7 &= 128x^7 - 192x^5 + 80x^3 - 8x, \\ U_8 &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1. \end{aligned}$$

The last equality in Eq. (2.4) comes from the relation

$$U_n(\cos \theta) = \frac{\sin((n+1)\theta)}{\sin \theta}.$$

### 3. AUXILIARY LEMMAS

#### 3.1. Convolutions.

**Lemma 3.1.** *Assume  $(\mathcal{H}_\ell)_{1 \leq \ell \leq L}$ . Let  $L \geq 1$  be an integer and let  $(b_\ell)_{0 \leq \ell \leq L}$  be a collection of non-negative integers. Then, we have the equality*

$$\prod_p \left( 1 + \frac{\sum_\ell b_\ell a(p^\ell)}{p^s} \right) = \prod_{0 \leq \ell \leq L} L(s, \text{sym}^\ell f)^{b_\ell} H(s),$$

where  $H$  is a function that is holomorphic and bounded by a constant in the region  $\Re s \geq \frac{1}{2} + \varepsilon$  for any  $\varepsilon > 0$ .

*Proof.* This follows easily by comparing the  $p$ -th Euler factors.  $\square$

We recall that, in the half-plane of absolute convergence, we have

$$L(s, f) = \prod_p \left( 1 - \frac{a(p)}{p^s} + \frac{1}{p^{2s}} \right)^{-1} = \prod_p \left( 1 - \frac{a(p)}{p^s} \right)^{-1} \left( 1 - \frac{\beta(p)}{p^s} \right)^{-1} \quad (3.1)$$

as well as

$$L(s, \text{sym}^2 f) = \prod_p \left( 1 - \frac{a(p)^2 - 1}{p^s} + \frac{a(p)^2 - 1}{p^{2s}} - \frac{1}{p^{3s}} \right)^{-1}. \quad (3.2)$$

**3.2. Averages of multiplicative functions.** We quote Theorem 21.2 of [29] which follows an idea of Wirsing [33].

**Lemma 3.2.** *Let  $f$  be a non-negative multiplicative function and  $\kappa$  be a non-negative real parameter such that*

$$\left\{ \begin{array}{l} \sum_{\substack{p \geq 2, \nu \geq 1 \\ p^\nu \leq Q}} f(p^\nu) \log(p^\nu) = \kappa Q + \mathcal{O}(Q/\log(2Q)) \quad (Q \geq 1), \\ \sum_{p \geq 2} \sum_{\substack{\nu, \ell \geq 1, \\ p^{\nu+\ell} \leq Q}} f(p^\ell) f(p^\nu) \log(p^\nu) \ll \sqrt{Q}, \end{array} \right.$$



then we have

$$\sum_{d \leq D} f(d) = \kappa C \cdot D (\log D)^{\kappa-1} (1 + o(1)),$$

where

$$C = \frac{1}{\Gamma(\kappa + 1)} \prod_p \left\{ \left(1 - \frac{1}{p}\right)^\kappa \sum_{\nu \geq 0} f(p^\nu) \right\}. \quad (3.3)$$

**Lemma 3.3.** *Under the same hypotheses of Lemma 3.2 we have, for any continuously differentiable function  $\eta$  with  $\int_0^1 \eta(u) du \neq 0$ :*

$$\sum_{d \leq D} f(d) \eta(d/D) = \kappa C (1 + o(1)) \int_2^D (\log u)^{\kappa-1} \eta(u/D) du$$

as  $D \rightarrow \infty$ .

The condition on  $\eta$  is obviously satisfied if, as will be the case for us,  $\eta$  is non-negative with support inside the interval  $[0, 1]$ .

*Proof.* Using Lemma 3.2, we find that

$$\begin{aligned} \sum_{d \leq D} f(d) \eta(d/D) &= \sum_{d \leq D} f(d) \eta(1) - \sum_{d \leq D} f(d) \int_{d/D}^1 \eta'(t) dt \\ &= \sum_{d \leq D} f(d) \eta(1) - \int_0^1 \sum_{d \leq tD} f(d) \eta'(t) dt \\ &= \kappa \eta(1) C \cdot D (\log D)^{\kappa-1} \\ &\quad - \int_2^D \kappa C u (\log u)^{\kappa-1} \eta'(u/D) du / D + o(D (\log D)^{\kappa-1}) \end{aligned}$$

as  $D \rightarrow \infty$ . Hence, by partial summation,

$$\begin{aligned} \sum_{d \leq D} f(d) \eta(d/D) (\kappa C)^{-1} &= \int_2^D (\kappa - 1 + \log u) (\log u)^{\kappa-2} \eta(u/D) du + o(D (\log D)^{\kappa-1}) \\ &= \int_2^D (\log u)^{\kappa-1} \eta(u/D) du + o(D (\log D)^{\kappa-1}). \end{aligned}$$

However we also have

$$\begin{aligned} \int_2^D (\log u)^{\kappa-1} \eta(u/D) du &= \mathcal{O} \left( \int_2^{D/\log D} (\log u)^{\kappa-1} du \right) + \int_{D/\log D}^D (\log u)^{\kappa-1} \eta(u/D) du \\ &= \mathcal{O} (D (\log D)^{\kappa-2}) + D (\log D)^{\kappa-1} \int_{1/\log D}^1 \left(1 + \frac{\log v}{\log D}\right)^{\kappa-1} \eta(v) dv \\ &= \mathcal{O} (D (\log D)^{\kappa-2}) + D (\log D)^{\kappa-1} \int_{1/\log D}^1 \eta(v) dv \\ &\sim \left( \int_0^1 \eta(v) dv \right) D (\log D)^{\kappa-1} \end{aligned}$$

as  $D \rightarrow \infty$ , since  $\int_0^1 \eta(u) du \neq 0$ . In the third line we have used the uniform estimate  $(1 + (\log v)/\log D)^{\kappa-1} = 1 + O(\log(1/v)/\log D)$  for  $1/\log D < v < 1$ . Hence our claimed estimate

$$\sum_{d \leq D} f(d) \eta(d/D) = \kappa C (1 + o(1)) \int_2^D (\log u)^{\kappa-1} \eta(u/D) du$$

follows.  $\square$

#### 4. A GENERAL AVERAGE BOUND

**Lemma 4.1.** *Let  $L \in \mathbb{N}_{>0}$ , and assume  $(\mathcal{H}_\ell)_{1 \leq \ell \leq L}$ . Let  $(b_\ell)_{0 \leq \ell \leq L}$  be a collection of non-negative integers. Given a primitive form  $f(z) = \sum_{n \geq 1} a(n)e(nz)$  as in the introduction, let us define a multiplicative function  $h_f$  by the equality*

$$\sum_n \frac{h_f(n)}{n^s} = \prod_p \left( 1 + \frac{\sum_\ell b_\ell a(p^\ell)}{p^s} \right)$$

*Then  $h_f$  is supported on square-free integers and there exists a polynomial  $P_L$  of degree at most  $b_0 - 1$  such that, for any  $\varepsilon > 0$ , we have*

$$\sum_{n \geq 1} h_f(n) \eta(n/X) = X P_L(\log X) + \mathcal{O} \left( X^{\frac{1}{2} + \varepsilon} \prod_{1 \leq \ell \leq L} q(\text{sym}^\ell(f))^{b_\ell \lambda_\ell + \varepsilon} \right) \quad (4.1)$$

*for any compactly supported twice continuously differentiable non-negative function  $\eta$ .*

*Proof.* Let us denote by  $S$  the left-hand side of (4.1). By taking Mellin transforms (e.g. p.90 of [14]), we get

$$S = \frac{1}{2i\pi} \int_{2-i\infty}^{2+i\infty} X^s \check{\eta}(s)^s \sum_{n \geq 1} \frac{h_f(n)}{n^s} ds.$$

The fact that  $\eta$  is twice continuously differentiable ensures us that its Mellin transform verifies  $\check{\eta}(s) \ll 1/(1 + |s|^2)$  uniformly in any closed vertical strip in the half plane  $\Re s > 0$ . Lemma 3.1 gives us an expression for the Dirichlet series  $\sum_{n \geq 1} h_f(n)/n^s$  from which we see that we can shift the line of integration to  $\Re s = \frac{1}{2} + \varepsilon$  obtaining that the error term is at most

$$\mathcal{O} \left( X^{\frac{1}{2} + \varepsilon} \prod_{1 \leq \ell \leq L} q(\text{sym}^\ell(f))^{b_\ell \lambda_\ell + \varepsilon} \right),$$

by our hypothesis  $(\mathcal{H}_\ell)_{1 \leq \ell \leq L}$  and the convexity principle. The residue at 1 gives the claimed main term, and the lemma follows readily.  $\square$

#### 5. A GENERAL LEMMA AROUND VINOGRADOV'S TRICK

**Lemma 5.1.** *Let  $g$  be a real-valued multiplicative function supported on the squarefree integers. We assume further that  $g(p) \geq F$  for every prime  $p$ , and that for every prime  $p \leq P$ , we have  $g(p) \geq \kappa > 0$ . Let  $\eta$  be a non-negative, continuously differentiable function with support within  $[0, 1]$  such that  $\int_0^1 \eta(v) dv = 1$ . We have, for  $M = P^\theta$  for some  $\theta \in [0, 1]$ ,*

$$\sum_{n \geq 1} \mu^2(n) g(n) \eta \left( \frac{n}{PM} \right) \geq (1 + o(1)) \kappa C M P (\log MP)^{\kappa-1} (1 - (\kappa - F) \mathcal{F}(\theta; \kappa))$$

*where  $C$  is given by (3.3) and  $\mathcal{F}$  is defined in (1.1)*

The factor  $\mu^2(n)$  is only here to remind the reader that the variable  $n$  is restricted to squarefree values. It can be omitted!

*Proof.* We set

$$S = \sum_{n \geq 1} g(n) \eta \left( \frac{n}{PM} \right). \quad (5.1)$$

By our hypotheses, we find that

$$\begin{aligned}
S &= \sum_{\substack{n \leq PM, \\ P^+(n) \leq P}} g(n) \eta\left(\frac{n}{PM}\right) + \sum_{P < p \leq PM} g(p) \sum_{n \leq PM/p} g(n) \eta\left(\frac{pn}{PM}\right) \\
&\geq \sum_{\substack{n \leq PM, \\ P^+(n) \leq P}} g(n) \eta\left(\frac{n}{PM}\right) + F \sum_{P < p \leq PM} \sum_{n \leq PM/p} \mu^2(n) \kappa^{\omega(n)} \eta\left(\frac{pn}{PM}\right) \\
&\geq \sum_{n \leq PM} \mu^2(n) \kappa^{\omega(n)} \eta\left(\frac{n}{PM}\right) + (F - \kappa) \sum_{P < p \leq PM} \sum_{n \leq PM/p} \mu^2(n) \kappa^{\omega(n)} \eta\left(\frac{pn}{PM}\right).
\end{aligned}$$

Here  $P^+(n)$  denotes the greatest prime divisor of  $n$ . We appeal to Lemma 3.3 with  $f(n) = \mu^2(n) \kappa^{\omega(n)}$  and get

$$\begin{aligned}
S/(C\kappa) &\geq (1 + o(1)) \int_2^{PM} (\log u)^{\kappa-1} \eta\left(\frac{u}{PM}\right) du \\
&\quad + (F - \kappa + o(1)) \sum_{N < p \leq PM} \int_2^{PM/p} (\log u)^{\kappa-1} \eta\left(\frac{up}{PM}\right) du.
\end{aligned}$$

Note that the change of variable  $vPM = u$  shows that

$$\int_2^{PM} (\log u)^{\kappa-1} \eta\left(\frac{u}{PM}\right) du = PM (\log PM)^{\kappa-1} \int_0^1 \eta(v) dv (1 + o(1)).$$

We use this estimate with  $M$  replaced by  $M/t$  and the prime number theorem to infer that

$$\begin{aligned}
\sum_{N < p \leq PM} \int_2^{PM/p} (\log u)^{\kappa-1} \eta\left(\frac{up}{PM}\right) du \\
= PM(1 + o(1)) \int_0^1 \eta(v) dv \int_N^{PM} \left(\log \frac{PM}{t}\right)^{\kappa-1} \frac{dt}{t \log t}
\end{aligned}$$

while this last integral equals, with the change of variable  $v = (PM)^h$  and  $M = N^\theta$ ,

$$\int_1^M \frac{(\log v)^{\kappa-1} dv}{v(\log(PM) - \log v)} = (\log PM)^{\kappa-1} \int_0^{\theta/(1+\theta)} \frac{h^{\kappa-1} dh}{1-h}.$$

Recall that  $\int_0^1 \eta(v) dv = 1$ . We thus find that

$$\begin{aligned}
\frac{(1 + o(1))S}{C\kappa PM (\log PM)^{\kappa-1}} &\geq 1 + (F - \kappa) \int_0^{\theta/(1+\theta)} \frac{h^{\kappa-1} dh}{1-h} \\
&= 1 - (\kappa - F) \mathcal{F}(\theta, \kappa).
\end{aligned}$$

□

## 6. PROOF OF THEOREMS 1.11, 1.1, 1.3, 1.5

Suppose  $a(p) \notin I$  for every  $p \leq P$ . Under the assumptions of Theorem 1.11, let  $\theta \in [0, 1]$  be such that

$$\frac{1}{\kappa - F} > \mathcal{F}(\theta; \kappa); \tag{6.1}$$

for instance, we may take  $\theta = \max(\mathcal{G}(\kappa - F; \kappa) - \varepsilon, 0)$ . Consider the sum

$$S = \sum_{n \geq 1} h_f(n) \eta(n/PM)$$

where  $M \in [1, P]$ . From the upper and the lower bound of  $S$  as given by Lemma 4.1 and 5.1 respectively and noting that  $b_0 = 0$ , we obtain,

$$(PM)^{\frac{1}{2} + \varepsilon} \prod_{1 \leq \ell \leq L} q(\text{sym}^\ell(f))^{b_\ell \lambda_\ell + \varepsilon} \gg PM.$$

Therefore, with  $M = P^\theta$  for some  $\theta \in [0, 1]$  satisfying (6.1), we have

$$P \ll_k N^{\frac{2 \sum_{\ell} \ell b_\ell \lambda_\ell}{1 + \theta} + \varepsilon}.$$

This leads to the estimate (1.5) and the other estimate (1.6) is proved in a similar manner.

Let us inspect what this gives to us under the convexity bound for  $\lambda_\ell = 1/4$ . Since the quantity  $2 \sum_{\ell \geq 1} \ell b_\ell \lambda_\ell$  takes all the values that are half-positive integers, we may inspect the first of them one by one. As we did above, we focus on the level  $N$ .

**First case**  $(1/2) \sum_{\ell \geq 1} \ell b_\ell = 1/2$ . This is only possible with the choice  $b_1 = 1$ , all other  $b_\ell$ 's being 0. We have  $\sum_{1 \leq \ell \leq L} b_\ell U_\ell(x/2) = x$  which is positive when  $x = a(p) > 0$ . On assuming  $a(p) \geq \delta$  when  $p \leq P$ , we see that we may take  $\kappa = \delta$  and  $F = -2$  and get, for  $N \geq N_0(\varepsilon)$ ,

$$\frac{\log P}{\log N} \leq \frac{2\lambda_1}{1 + \mathcal{G}(2 + \delta; \delta)} + \varepsilon. \quad (6.2)$$

Hence Theorem 1.1.

**Second case**  $(1/2) \sum_{\ell \geq 1} \ell b_\ell = 1$ . This is only possible with the choice  $b_2 = 1$ , all other  $b_\ell$ 's being 0. We have  $\sum_{1 \leq \ell \leq L} b_\ell U_\ell(x/2) = x^2 - 1$  which is positive when  $x = a(p) \notin [-1, 1]$ . On assuming  $|a(p)| \geq 1 + \delta$  when  $p \leq P$ , we see that we may take  $\kappa = 2\delta + \delta^2$  and  $F = -1$  and get, for  $N \geq N_0(\varepsilon)$ ,

$$\frac{\log P}{\log N} \leq \frac{4\lambda_2}{1 + \mathcal{G}(1 + 2\delta + \delta^2; 2\delta + \delta^2)} + \varepsilon. \quad (6.3)$$

Hence Theorem 1.3.

**Finding non-negative values.** Let  $I = [0, 2]$ . A numerical computation found the coefficients  $(b_\ell)_{0 \leq \ell \leq 5} = (0, 0, 3, 5, 4, 1)$ , which satisfy (1.4) with  $\kappa \geq 1/3$  and  $F = -10$ . Then Theorem 1.5 follows from the bounds (1.5) and (1.6).

## 7. PROOF OF THEOREM 1.6

Let  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be smooth, compactly supported and such that  $\mathbb{1}_{[0,1]} \geq \eta \geq \mathbb{1}_{[1/3, 2/3]}$ , let  $\varepsilon > 0$ , and consider

$$T(X) = \sum_n \mu^2(n) a(n) \eta(n/X) \quad (X \geq 1).$$

By Lemma 4.1, we get

$$T(X) \ll X^{1/2} (k^2 N)^{1/4 + \varepsilon}. \quad (7.1)$$

Suppose that  $a(p) \geq 0$  for all primes  $p \leq X$ . If the inequality

$$\sum_{\substack{n: \\ a(n) \geq 1}} \mu^2(n) a(n) \eta(n/X) \geq X^{1-\varepsilon} \quad (7.2)$$

holds then we easily have

$$T(X) \geq X^{1-\varepsilon}. \quad (7.3)$$

Otherwise, suppose that (7.2) does not hold. We write

$$\begin{aligned} T(X) &= \sum_{\substack{n: \\ 0 \leq a(n) < 1}} \mu^2(n) a(n) \eta(n/X) + \sum_{\substack{n: \\ a(n) \geq 1}} \mu^2(n) a(n) \eta(n/X) \\ &\geq \sum_{\substack{n: \\ 0 \leq a(n) < 1}} \mu^2(n) a(n)^2 \eta(n/X) + \sum_{\substack{n: \\ a(n) \geq 1}} \mu^2(n) a(n) \eta(n/X) \\ &= \sum_n \mu^2(n) a(n)^2 \eta(n/X) + \sum_{\substack{n: \\ a(n) \geq 1}} \mu^2(n) a(n) (1 - a(n)) \eta(n/X). \end{aligned}$$

Now the last sum is  $\mathcal{O}(X^{1-\varepsilon/2})$  by Deligne's bound  $|a(p)| \leq 2$  and the negation of (7.2). The first sum can be handled by Rankin-Selberg method (Lemma 4.1) and is  $\gg L(1, \text{sym}^2 f) X + \mathcal{O}(X^{1/2} (k^2 N^2)^{1/4+\varepsilon})$ . Thus we have, using the lower bound  $L(1, \text{sym}^2 f) \gg 1/\log(kN)$  due to Hoffstein and Lockhart [10],

$$T(X) \gg X/\log(kN) + \mathcal{O}(X^{1/2} (k^2 N^2)^{1/4+\varepsilon}) + \mathcal{O}(X^{1-\varepsilon/2}). \quad (7.4)$$

One of the equations (7.3) and (7.4) must hold and either, in conjunction with equation (7.1), imply the theorem.

## 8. PROOF OF THEOREM 1.8

By equation (3) of [30], the Deligne bound  $|a(p)| \leq 2$  and Mertens' theorem (see [14, Eq. (2.15)]), we have

$$\log L(1, f) = \mathcal{O}_\varepsilon(1) + \sum_{p \leq q^\varepsilon} \frac{a(p)}{p},$$

and therefore

$$\sum_{p \leq q^\varepsilon} \frac{1}{p} \left( a(p) - \frac{\log L(1, f)}{\log \log q} \right) = \mathcal{O}_\varepsilon(1).$$

However, if we had  $a(p) < \gamma^- - \delta$  for  $p \leq q^\varepsilon$ , then we would also have

$$\sum_{p \leq q^\varepsilon} \frac{1}{p} \left( a(p) - \frac{\log L(1, f)}{\log \log q} \right) \leq \mathcal{O}_{\varepsilon, \delta}(1) - \frac{\delta}{2} \log \log q,$$

which is a contradiction for  $q$  large enough, and therefore there must be a prime  $p \leq q^\varepsilon$  such that  $a(p) \geq \gamma_- - \delta$ . An identical argument shows the existence of  $p \leq q^\varepsilon$  such that  $a(p) \leq \gamma_+ + \delta$ .

## 9. CONDITIONAL BOUNDS: PROOF OF THEOREM 1.10

By the Stone-Weierstrass theorem, the fact that  $(U_\ell)$  forms a basis of  $\mathbb{R}[X]$ , and the relation (2.4), we may find  $L \geq 1$  and real coefficients  $b_0, \dots, b_L$  depending on  $I$ , with  $b_0 > 0$ , such that

$$\sum_{p \leq x} \mathbb{1}(a(p) \in I) \left(1 - \frac{p}{x}\right) \log p \geq \sum_{\ell=0}^L b_\ell \sum_{p \leq x} a_{\text{sym}^\ell f}(p) \left(1 - \frac{p}{x}\right) \log p. \quad (9.1)$$

By Chebyshev's estimate, the contribution of the term  $\ell = 0$  is

$$b_0 \sum_{p \leq x} \left(1 - \frac{p}{x}\right) \log p \gg_I x$$

with an absolute constant. To show that the right-hand side of (9.1) is positive for some  $x = \mathcal{O}_I((\log q)^2)$ , it therefore suffices to show that for all integer  $\ell \geq 1$  and all real  $x \geq 1$ , we have

$$\sum_{p \leq x} a_{\text{sym}^\ell f}(p) \left(1 - \frac{p}{x}\right) \log p = \mathcal{O}_\ell(x^{1/2} \log q).$$

This is an immediate consequence of the explicit formula [14, eq. (5.33)] (with an additional smoothing, as in [26, eq. (13.28)]) along with classical zero density estimates [14, Theorem 5.8].

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