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# Bernstein adaptive nonparametric conditional sampling: a new method for rare event probability estimation

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## Abstract

In the context of reliability assessment, estimating a failure probability associated to a rare event is a common task. To do so, various techniques have been proposed to overcome traditional crude Monte Carlo which becomes intractable in such a context. Among others, Subset Simulation is a widely used technique which relies on “splitting” the rare event probability into a sequence (i.e., a product) of less rare conditional probabilities associated to nested failure events, easier to estimate. However, this technique relies on simulating samples conditionally to the failure event by means of Monte Carlo Markov chain algorithms. These algorithms enable, at convergence, to simulate according to the target density. However, in practice, it often produces non-independent and identically distributed (i.i.d.) samples due to the correlation between Markov chains. In the present work, we propose another way to sample conditionally to the nested failure events in order to get i.i.d. samples which can be required (e.g., to perform dedicated sensitivity analysis). The proposed algorithm relies on a nonparametric fit of the conditional joint distribution using a combined kernel density estimation for marginals fitting and the Empirical Bernstein Copula (EBC). Thus, this new method presents some similarities with “Nonparametric Adaptive Importance Sampling” but addresses the problem of copula fitting by means of EBC. The proposed algorithm is tested on three toy-cases and its performances are compared with those obtained from Subset Sampling.

*Keywords:* Reliability analysis, Uncertainty propagation, Copulas, Subset Sampling

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## 1. INTRODUCTION

Reliability analysis of a system is often associated with rare event probability estimation. Considering that the system’s performance is modeled by a deterministic scalar function  $g$  :

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9  $\mathcal{D}_{\mathbf{x}} \subseteq \mathbb{R}^d \rightarrow \mathbb{R}$ , called *limit-state function* and a critical threshold on the system’s output  
 10  $y_{\text{th}} \in \mathbb{R}$ , one can define the *failure domain* as  $\mathcal{F}_{\mathbf{x}} := \{\mathbf{x} \in \mathcal{D}_{\mathbf{x}} | g(\mathbf{x}) \leq y_{\text{th}}\}$ . Uncertain inputs  
 11 are represented by a continuous random vector  $\mathbf{X} \in \mathcal{D}_{\mathbf{x}}$  assumed to be distributed according  
 12 to its joint probability density function (PDF)  $f_{\mathbf{x}}$ . In this context, uncertainty propagation  
 13 consists in composing the random vector  $\mathbf{X}$  by the function  $g$  to get an output variable of  
 14 interest  $Y = g(\mathbf{X}) \in \mathbb{R}$ . A usual risk measure in reliability analysis is the *failure probability*,  
 15 denoted by  $p_{\text{f}}$ , and defined as the probability that the system exceeds the threshold  $y_{\text{th}}$ :

$$p_{\text{f}} := \mathbb{P}(g(\mathbf{X}) \leq y_{\text{th}}) = \int_{\mathcal{D}_{\mathbf{x}}} \mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) \, \text{d}\mathbf{x} \quad (1)$$

16 where  $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\cdot)$  is the indicator function of the failure domain such that  $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) = 1$  if  $\mathbf{x} \in \mathcal{F}_{\mathbf{x}}$  and  
 17  $\mathbb{1}_{\mathcal{F}_{\mathbf{x}}}(\mathbf{x}) = 0$  otherwise. Rare event problems are usually solved in the so-called *standard normal*  
 18 *space* after applying an “iso-probabilistic transformation” which can be either the Rosenblatt  
 19 or the generalized Nataf one (Lebrun, 2013). Additionally, the limit-state function  $g$  can be  
 20 viewed as an input-output “black-box” model which can be costly to evaluate (e.g., a complex  
 21 numerical model), making the failure probability estimation nontrivial. When the limit-state  
 22 function is a costly computer model, one can build a surrogate model and use specific active  
 23 learning methods (see, e.g., Moustapha et al. (2022)). However, using surrogate models is not  
 24 always possible for practical engineering applications as they might introduce another level  
 25 of approximation, which can be prohibitive from safety auditing. Moreover, their validation  
 26 as well as their behavior with respect to large input dimension case make also their use quite  
 27 complex (see, e.g., (Marrel et al., 2022)).

28 Going back to the rare event estimation literature, one can consider two major types  
 29 of techniques for failure probability calculation (Morio and Balesdent, 2015): (i) Geomet-  
 30 ric approaches, such as the *first-/second-order reliability method* (FORM/SORM) whose aim  
 31 is to approximate the limit-state function by a first-/second-order Taylor expansion at the  
 32 most probable failure point; (ii) Simulation-based techniques such as the *crude Monte Carlo*  
 33 method. Unfortunately, FORM/SORM methods do not provide a lot of statistical information  
 34 as they are purely geometric approaches. Meanwhile, estimating a rare event probability by  
 35 crude Monte Carlo becomes rapidly intractable. To overcome this limit, advanced simulation  
 36 techniques have been developed: among others, one can mention several “variance reduction  
 37 methods” such as the non-adaptive and adaptive versions of the *Importance Sampling* (Ru-

38 binstein and Kroese, 2008) (either parametric, using the Cross-Entropy method Kurtz and  
39 Song (2013), or nonparametric Morio (2011)) and splitting techniques (C erou et al., 2012)  
40 such as the *Subset Simulation* (SS) Au and Beck (2001). In these techniques, the idea is to  
41 write the rare event  $p_f$  as a product of larger conditional probabilities, each one of them being  
42 easier to estimate. To generate intermediary conditional samples, this method uses Markov  
43 chain Monte Carlo (MCMC) sampling, which presents numerous versions (Papaioannou et al.,  
44 2015). However, MCMC algorithms are known to be highly tunable algorithms which produce  
45 non-i.i.d. samples, which consequently, cannot be used for direct statistical estimation (e.g.,  
46 failure probability or sensitivity indices (Da Veiga et al., 2021)).

47 The present work proposes a new rare event estimation method, adopting the same sequen-  
48 tial structure as SS while using a strictly different sampling mechanism to generate conditional  
49 samples. This method intends to fit the intermediary conditional distributions with a nonpara-  
50 metric tool called the *Empirical Bernstein Copula*. Contrarily to SS, the proposed method  
51 named ‘‘Bernstein adaptive nonparametric conditional sampling’’ (BANCS), generates i.i.d.  
52 samples of the intermediary conditional distributions. For instance, a practical use of such  
53 i.i.d. samples can be to estimate dedicated reliability-oriented sensitivity indices (see, e.g.,  
54 Chabridon et al. (2021); Marrel and Chabridon (2021)).

55 In this paper, Section 2 will recall the methodology of subset sampling and probabilistic  
56 modeling. Then, Section 3 will introduce the BANCS method for rare event estimation.  
57 Section 4 will apply this method to three toy-cases and analyze the results with respect to SS  
58 performances. Then, the last section present some conclusions and research perspectives.

## 59 2. BACKGROUND

### 60 2.1. Subset sampling

61 Subset sampling splits the failure event  $\mathcal{F}_x$  into an intersection of  $k_{\#}$  intermediary events  
62  $\mathcal{F}_x = \bigcap_{k=1}^{k_{\#}} \mathcal{F}_{[k]}$ . Each are nested such that  $\mathcal{F}_{[1]} \supset \dots \supset \mathcal{F}_{[k_{\#}]} = \mathcal{F}_x$ . The failure probability is  
63 then expressed as a product of conditional probabilities:

$$p_f = \mathbb{P}(\mathcal{F}_x) = \mathbb{P}(\bigcap_{k=1}^{k_{\#}} \mathcal{F}_{[k]}) = \prod_{k=1}^{k_{\#}} \mathbb{P}(\mathcal{F}_{[k]} | \mathcal{F}_{[k-1]}). \quad (2)$$

64 From a practical point of view, the analyst tunes the algorithm by setting the intermediary  
65 probabilities  $\mathbb{P}(\mathcal{F}_{[k]}|\mathcal{F}_{[k-1]}) = p_0, \forall k \in \{1, \dots, k_{\#}\}$ . Then, the corresponding quantiles  $q_{[1]}^{p_0} >$   
66  $\dots > q_{[k_{\#}]}^{p_0}$  are estimated for each conditional subset samples  $\mathbf{X}_{[k],N}$  of size  $N$ . Note that the  
67 initial quantile is estimated by crude Monte Carlo sampling on the input PDF  $f_{\mathbf{X}}$ . Following  
68 conditional subset samples are generated by MCMC sampling of  $f_{\mathbf{X}}(\mathbf{x}|\mathcal{F}_{[k-1]})$ , using as seeds  
69 initialisation points the  $n = Np_0$  samples given by  $\mathbf{A}_{[k],n} = \{\mathbf{X}_{[k-1]}^{(j)} \subset \mathbf{X}_{[k-1],N} | g(\mathbf{X}_{[k-1]}^{(j)}) >$   
70  $\widehat{q}_{[k-1]}^{\alpha}\}_{j=1}^n$ . This process is repeated until an intermediary quantile exceeds the threshold:  
71  $\widehat{q}_{[k_{\#}]}^{p_0} < y_{\text{th}}$ . Finally, the failure probability is estimated by:

$$p_f \approx \widehat{p}_f^{\text{SS}} = p_0^{k_{\#}-1} \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{g(\mathbf{x}) \leq y_{\text{th}}\}}(\mathbf{X}_{[k_{\#}],N}^{(j)}). \quad (3)$$

72 In practice, the subset sample size should be large enough to properly estimate intermediary  
73 quantiles, which leads Au and Beck (2001) to recommend setting  $p_0 = 0.1$ . SS efficiency  
74 depends on the proper choice and tuning of the MCMC algorithm (Papaioannou et al., 2015).  
75 Our work uses the SS implementation from `OpenTURNS`<sup>1</sup> (Baudin et al., 2017) which integrates  
76 a component-wise Metropolis-Hastings algorithm. As an alternative to generating samples on  
77 a conditional distribution by MCMC, one could try to fit this conditional distribution.

## 78 2.2. Multivariate modeling using copulas

79 The Sklar theorem (Joe, 1997) affirms that the multivariate distribution of any random  
80 vector  $\mathbf{X} \in \mathbb{R}^d$  can be broken down into two objects:

- 81 1. A set of univariate marginal distributions to describe the behavior of the individual  
82 variables;
- 83 2. A function describing the dependence structure between all variables, called a copula.

84 This theorem states that considering a random vector  $\mathbf{X} \in \mathbb{R}^d$ , with its distribution  $F$  and its  
85 marginals  $\{F_i\}_{i=1}^d$ , there exists a copula  $C : [0, 1]^d \rightarrow [0, 1]$ , such that:

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_p(x_d)). \quad (4)$$

86 It allows us to divide the problem of fitting a joint distribution into two independent  
87 problems: fitting the marginals and fitting the copula. Note that when the joint distribution

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<sup>1</sup><https://openturns.github.io/www/index.html>

88 is continuous, this copula is unique. Provided a dataset, this framework allows to combine a  
 89 parametric (or nonparametric) fit of marginals with a parametric (or nonparametric) fit of the  
 90 copula. When the distribution’s dimension is higher than two, one can perform a parametric  
 91 fit using vine copulas (Joe and Kurowicka, 2011), implying the choice of multiple types of  
 92 parametric copulas. Otherwise, nonparametric fit by multivariate kernel density estimation  
 93 (KDE) presents a computational burden as soon as the dimension increases (Chabridon et al.,  
 94 2021). Since univariate marginals are usually well-fitted with nonparametric tools (e.g., KDE),  
 95 let us introduce an effective nonparametric method for copula fitting.

### 96 3. A NEW COPULA-BASED CONDITIONAL SAMPLING METHOD

#### 97 3.1. Empirical Bernstein copula

98 Copulas are continuous and bounded functions defined on a compact set (the unit hyper-  
 99 cube). Bernstein polynomials allow to uniformly approximate as closely as desired any contin-  
 100 uous and real-valued function defined on a compact set (Weierstrass approximation theorem).  
 101 Therefore, they are good candidates to approximate unknown copulas. This concept was intro-  
 102 duced as *empirical Bernstein copula* (EBC) by Sancetta and Satchell (2004) for applications in  
 103 economics and risk management. Later on, Segers et al. (2017) offered further asymptotic stud-  
 104 ies. Formally, the multivariate Bernstein polynomial for a function  $C : [0, 1]^d \rightarrow \mathbb{R}$  on a grid  
 105 over the unit hypercube  $G := \left\{ \frac{0}{m_1}, \dots, \frac{m_1}{m_1} \right\} \times \dots \times \left\{ \frac{0}{m_d}, \dots, \frac{m_d}{m_d} \right\}$ ,  $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ ,  
 106 writes:

$$B_{\mathbf{m}}(C)(\mathbf{u}) := \sum_{t_1=0}^{m_1} \dots \sum_{t_d=0}^{m_d} C \left( \frac{t_1}{m_1}, \dots, \frac{t_d}{m_d} \right) \prod_{j=1}^d P_{m_j, t_j}(u_j), \quad (5)$$

107 with  $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$ , and the Bernstein polynomial  $P_{m,t}(u) := \frac{t!}{m!(t-m)!} u^m (1-u)^{t-m}$ .  
 108 Notice how the grid definition implies the polynomial’s order. When  $C$  is a copula, then  $B_{\mathbf{m}}(C)$   
 109 is called “Bernstein copula”. Therefore, the empirical Bernstein copula is an application of the  
 110 Bernstein polynomial in Eq. (5) to the so-called “empirical copula”.

111 In practice, considering a sample  $\mathbf{X}_n = \{\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}\} \in \mathbb{R}^{np}$  and the associated ranked  
 112 sample  $\mathbf{R}_n = \{\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(n)}\}$ , the corresponding empirical copula writes:

$$C_n(\mathbf{u}) := \frac{1}{n} \sum_{i=0}^n \prod_{j=1}^p \mathbb{1} \left\{ \frac{r_j^{(i)}}{n} \leq u_j \right\}, \quad (6)$$

113 with  $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$ . In the following, the polynomial order is set as equal in each  
 114 dimension:  $\{m_i = m\}_{i=1}^d$ . Theoretically, the tuning parameter can be optimized to minimize  
 115 an ‘‘Mean Integrated Squared Error’’ (MISE), leading to a bias-variance tradeoff. Formally,  
 116 the MISE of the empirical Bernstein copula  $B_{\mathbf{m}}(C_n)$  is defined as follows:

$$\mathbb{E}[\|B_{\mathbf{m}}(C_n) - C\|_2^2] = \mathbb{E}\left[\int_{\mathbb{R}^d} (B_{\mathbf{m}}(C_n)(\mathbf{u}) - C(\mathbf{u}))^2 d\mathbf{u}\right]. \quad (7)$$

117 Then, Sancetta and Satchell (2004) prove in their Theorem 3 that:

- 118 •  $B_{\mathbf{m}}(C_n)(\mathbf{u}) \rightarrow C(\mathbf{u})$  for any  $u_j \in ]0, 1[$  if  $\frac{m^{d/2}}{n} \rightarrow 0$ , when  $m, n \rightarrow \infty$ .
- 119 • The optimal order of the polynomial in terms of MISE is:  $m \lesssim m_{\text{IMSE}} = n^{2/(d+4)}$ ,  $\forall u_j \in$   
 120  $]0, 1[$ . The sign  $\lesssim$  means ‘‘less than or approximately’’.

121 Let us remark that in the special case  $m = n$ , also called the ‘‘Beta copula’’ in Segers  
 122 et al. (2017), the bias is very small while the variance gets large. To illustrate the previous  
 123 theorem, Lasserre (2022) represents the evolution of the  $m_{\text{IMSE}}$  for different dimensions and  
 124 sample sizes (see Fig. 1). In high dimension, the values of  $m_{\text{IMSE}}$  tend towards one, which is  
 125 equivalent to the independent copula. Therefore, high-dimensional problems should be divided  
 126 into a product of smaller problems on which the EBC is tractable. Provided a large enough  
 127 learning set  $\mathbf{X}_n$ , KDE fitting of marginals combined with EBC fitting of the copula delivers  
 128 good results even on complex dependence structures. Moreover, EBC provides an explicit  
 129 expression, making a Monte Carlo generation of i.i.d. samples simple. In the following, this  
 130 nonparametric tool is used to fit the intermediary conditional distributions present in subset  
 131 sampling.

### 132 3.2. Bernstein adaptive nonparametric conditional sampling (BANCS) method

133 This new method reuses the main idea from SS while employing a different approach to  
 134 generate conditional samples. Instead of using MCMC sampling, the conditional distribution is  
 135 firstly fitted by a nonparametric procedure, before sampling on this nonparametric model. As  
 136 described in Algorithm 1, conditional sampling is done on a distribution composed by merging  
 137 marginals  $\{\widehat{F}_i\}_{i=1}^d$  fitted by KDE, with a copula  $B_{\mathbf{m}}(C_n)$  fitted by EBC. Fig. 2 illustrate the  
 138 nonparametric fit and conditional sampling in BANCS method on a two-dimensional reliability  
 139 problem (later introduced as ‘‘toy-case #1’’). At iteration  $k$ , after estimating the intermediary

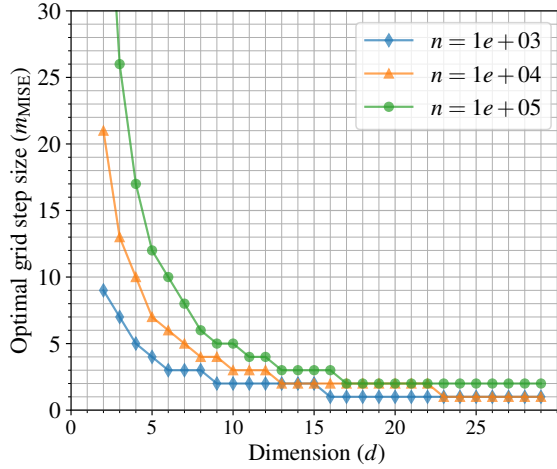


Figure 1: Evolution of  $m_{\text{IMSE}}$  for different dimensions and sample sizes.

140 quantile  $\hat{q}_{[k]}^{p_0}$ , a nonparametric model is fitted on  $\mathbf{A}_{[k+1],n}$  and used to generate the next  $N$ -  
 141 sized subset sample  $\mathbf{X}_{[k+1],N}$ . Note that the BANCS method does not require iso-probabilistic  
 142 transform.

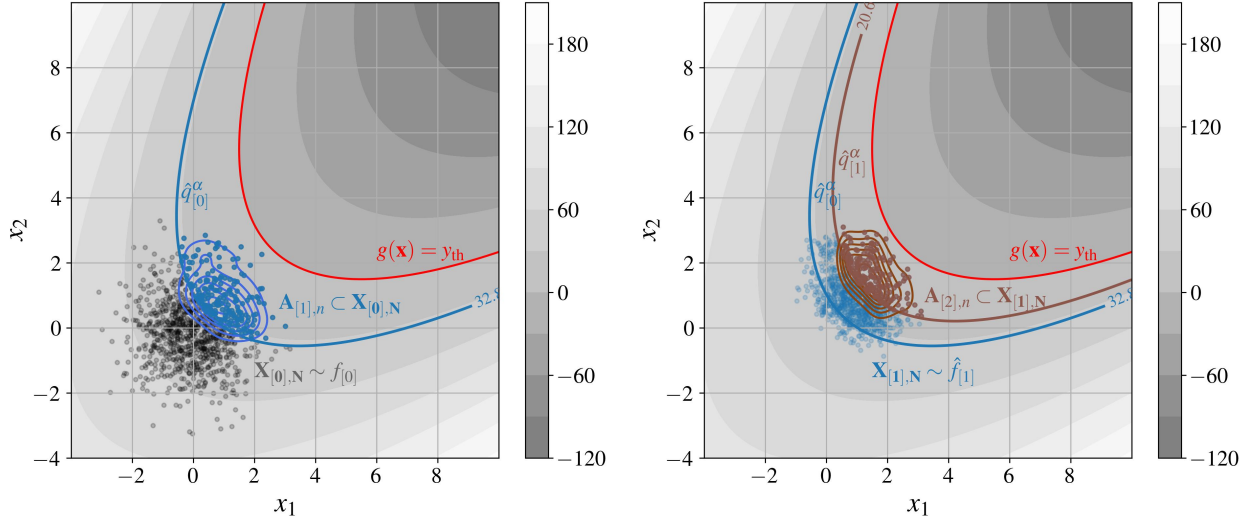


Figure 2: BANCS on toy-case #1: illustration of nonparametric fit at the first iteration.

143 As discussed in the previous section, EBC fitting is tuned by the Bernstein polynomial  
 144 of order  $m$ , implying a bias-variance trade off. In Fig. 2, conditional distributions fitted by  
 145 EBC (blue and brown isolines) seem to present a slight bias since they overlay the quantiles.  
 146 However, reducing this bias implies decreasing the tuning parameter  $m$ , until  $m = 1$ , which



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**Algorithm 1** Bernstein adaptive nonparametric conditional sampling (BANCS).

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▷ *Inputs:* ◁

$f_{\mathbf{X}}$ , joint PDF of the inputs

$g(\cdot)$ , limit-state function

$y_{\text{th}} \in \mathbb{R}$ , threshold defining the failure event

$N$ , number of samples per iteration

$m \in \mathbb{N}$ , parameter of the EBC fitting

$p_0 \in ]0, 1[$ , empirical quantile order (rarity parameter)

▷ *Algorithm:* ◁

Set  $k = 0$  and  $f_{[0]} = f_{\mathbf{X}}$

Sample  $\mathbf{X}_{[0],N} = \{\mathbf{X}_{[0]}^{(j)}\}_{j=1}^N \stackrel{\text{i.i.d.}}{\sim} f_{[0]}$

Evaluate  $G_{[0],N} = \{g(\mathbf{X}_{[0]}^{(j)})\}_{j=1}^N$

Estimate the empirical  $p_0$ -quantile  $\hat{q}_{[0]}^{p_0}$  of the set  $G_{[0],N}$

**while**  $\hat{q}_{[k]}^{p_0} > y_{\text{th}}$  **do**

    Subsample  $\mathbf{A}_{[k+1],n} = \{\mathbf{X}_{[k]}^{(j)} \subset \mathbf{X}_{[k],N} | g(\mathbf{X}_{[k]}^{(j)}) > \hat{q}_{[k]}^{p_0}\}_{j=1}^n$

    Fit marginals of the subset  $\mathbf{A}_{[k+1],n}$  by KDE  $\{\hat{F}_i\}_{i=1}^d$

    Fit the copula of the subset  $\mathbf{A}_{[k+1],n}$  by EBC  $B_{\mathbf{m}}(C_n)$

    Build a CDF  $\hat{F}_{[k+1]}(\mathbf{x}) = B_{\mathbf{m}}(C_n)(\hat{F}_1(x_1), \dots, \hat{F}_d(x_d))$

    Sample  $\mathbf{X}_{[k+1],N} = \{\mathbf{X}_{[k+1]}^{(j)}\}_{j=1}^N \stackrel{\text{i.i.d.}}{\sim} \hat{f}_{[k+1]}$

    Evaluate  $G_{[k+1],N} = \{g(\mathbf{X}_{[k+1]}^{(j)})\}_{j=1}^N$

    Estimate the empirical  $p_0$ -quantile  $\hat{q}_{[k+1]}^{p_0}$  of  $G_{[k+1],N}$

    Set  $k = k + 1$

Set total iteration number  $k_{\#} = k - 1$

Estimate  $\hat{p}_{\text{f}} = (1 - p_0)^{k_{\#}} \cdot \frac{1}{N} \sum_{j=1}^N \mathbb{1}_{\{g(\mathbf{X}_{[k_{\#}]^{(j)}}) \geq y_{\text{th}}\}}(\mathbf{X}_{[k_{\#}]^{(j)}})$

▷ *Outputs:* ◁

$\hat{p}_{\text{f}}$ , estimate of  $p_{\text{f}}$

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147 is equivalent to an independent copula. Tools to control the goodness of fit of nonparametric  
148 conditional distributions are also available. As an example, let us consider the fitted condi-  
149 tional distribution at the first iteration (visible in Fig. 2). Its quantile-quantile plot in Fig. 3  
150 (left) shows a good fit of the two marginals by KDE. Then, the goodness of fit of copulas can  
151 be evaluated by Kendall's plot, represented in Fig. 3 (right). This fit is also good, even if a  
152 slight bias is again visible.

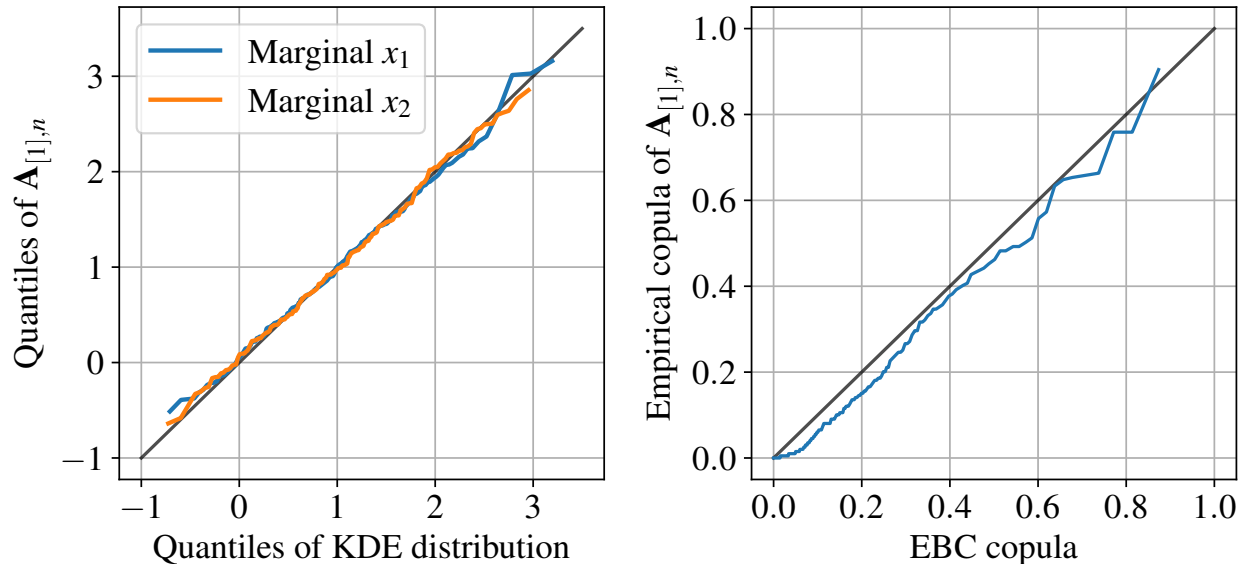


Figure 3: QQ-plot (left) for KDE on marginals and Kendall plot (right) for EBC on the copula of the conditional distribution from Fig. 2.

#### 153 4. NUMERICAL EXPERIMENTS

154 In the following analytical numerical experiments, the intermediary probabilities were set  
 155 to  $p_0 = 0.1$ , allowing a fair comparison with subset sampling. Then, the subset sample size  
 156 is set to  $N = 10^4$ , in order to get a reasonable sample size  $n = Np_0 = 10^3$  to perform the  
 157 nonparametric fitting. EBC tuning is setup to minimize the MISE in Eq. (7):  $m = 1 + n^{\frac{2}{d+4}}$ . In  
 158 order to take into account the variability of the method's results, each experiment is repeated  
 159 100 times, allowing the computation of a coefficient of variation  $\hat{\delta} = \frac{\sigma_{\widehat{p}_i}}{\mu_{\widehat{p}_i}}$ . Note that an  
 160 implementation of the BANCS method and the following numerical experiments are available  
 161 in a Git repository<sup>2</sup>.

##### 162 4.1. Toy-case #1: Parabolic reliability problem

163 Let us define the parabolic reliability problem, considering the function  $g_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ :

$$g_1(\mathbf{x}) = (x_1 - x_2)^2 - 8(x_1 + x_2 - 5), \quad (8)$$

164 with the input random vector  $\mathbf{X} = (X_1, X_2)$  following a standard 2-dimensional normal dis-  
 165 tribution. The reliability problem consists in evaluating:  $p_{f,1} = \mathbb{P}(g_1(\mathbf{X}) \leq 0) = 1.31 \times 10^{-4}$ .

<sup>2</sup><https://github.com/efekhari27/icasp14>

166 *4.2. Toy-case #2: Four-branch reliability problem*

Let us define the four-branch reliability problem (originally proposed by Waarts (2000)), considering the following function  $g_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ :

$$g_2(\mathbf{x}) = \min \begin{pmatrix} 5 + 0.1(x_1 - x_2)^2 - \frac{(x_1+x_2)}{\sqrt{2}} \\ 5 + 0.1(x_1 - x_2)^2 + \frac{(x_1+x_2)}{\sqrt{2}} \\ (x_1 - x_2) + \frac{9}{\sqrt{2}} \\ (x_2 - x_1) + \frac{9}{\sqrt{2}} \end{pmatrix}, \quad (9)$$

167 with the input random vector  $\mathbf{X} = (X_1, X_2)$  following a standard 2-dimensional normal dis-  
 168 tribution. The reliability problem consists in evaluating:  $p_{f,2} = \mathbb{P}(g_2(\mathbf{X}) \leq 0) = 2.21 \times 10^{-4}$ .

169 *4.3. Toy-case #3: high-dimensional reliability problem*

170 Let us define the higher-dimensional reliability problem (proposed by Yun et al. (2018)),  
 171 considering the following function  $g_3 : \mathbb{R}^7 \rightarrow \mathbb{R}$ :

$$g_3(\mathbf{x}) = 15.59 \times 10^4 - \frac{x_1 x_3^2 x_2^4 - 4x_5 x_6 x_7^2 + x_4(x_6 + 4x_5 + 2x_6 x_7)}{2x_3^2 x_4 x_5 (x_4 + x_6 + 2x_6 x_7)}, \quad (10)$$

172 with the input random vector  $\mathbf{X} = (X_1, \dots, X_7)$ , following a product of normal distributions  
 173 defined in Yun et al. (2018). The reliability problem consists in evaluating:  $p_{f,3} = \mathbb{P}(g_3(\mathbf{X}) \leq$   
 174  $0) = 8.10 \times 10^{-3}$ .

175 *4.4. Results analysis*

176 Results of our numerical experiments are presented graphically (for 2-dimensional prob-  
 177 lems) in Figure 4, and numerically in Table 1. In the same fashion as the previous illustrations,  
 178 the figures represent the intermediary quantiles  $\hat{q}_{[k]}^{p_0}$  estimated over conditional samples of size  
 179  $N = 10^4$ . Moreover, samples  $\mathbf{A}_{[k+1],n}$  exceeding these quantiles are also represented in the  
 180 same color. Notice how the last estimated quantile is set to the problem threshold  $y_{\text{th}} = 0$ . To  
 181 capture the dispersion of BANCS estimation, 100 repetitions were realized. Let us notice that  
 182 for each toy-case, BANCS well estimates the failure probabilities' orders of magnitude. Yet the  
 183 numerical values in Table 1 consistently present a positive bias, leading to an overestimated  
 184 failure probability. This bias is partially explained by the EBC tuning choice and could be  
 185 reduced at the expense of a slightly higher variance.

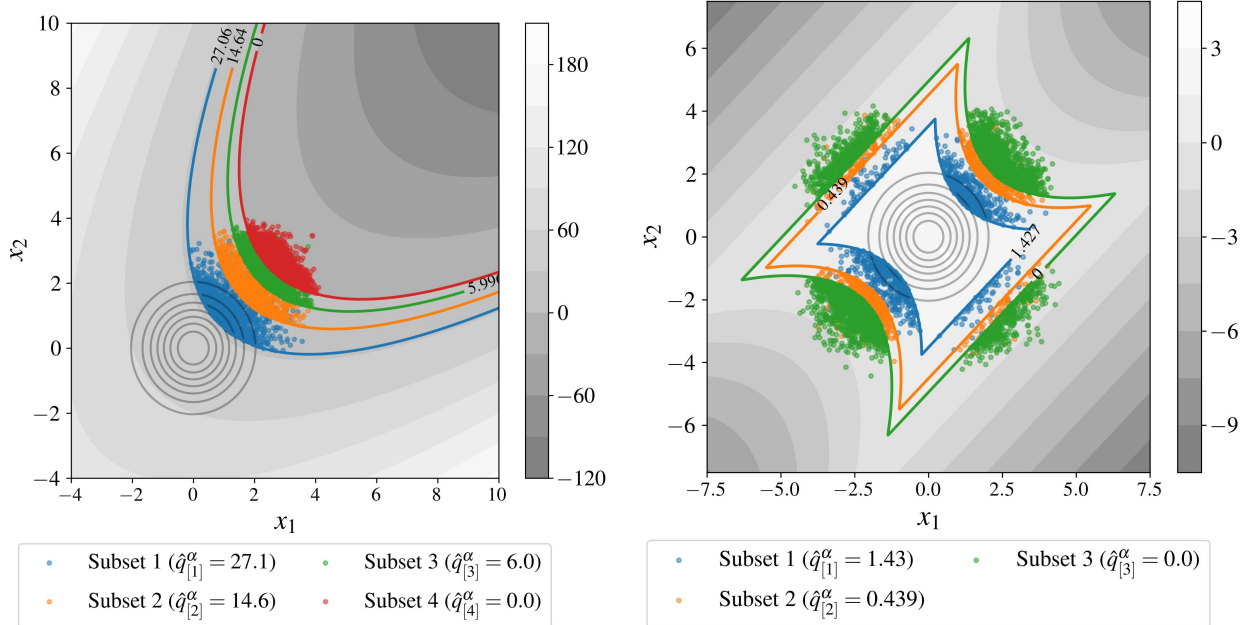


Figure 4: BANCS sampling steps on toy-case #1 (left) and #2 (right).

186 The variance obtained with the repetitions is quite large. Although, part of it is due to  
 187 the fact that the algorithm might compute a different total number of subsets (e.g., toy-case  
 188 #1 is either solved in four or five subsets). Overall, considering the EBC tuning from Eq. (7),  
 189 BANCS performs worst than SS on toy-cases #1 and #2 but performs as well as SS on the  
 190 toy-case #3. This might be due to the fact that toy-case #3 has a higher input dimension.  
 191 However, one can note that SS coefficient of variation is computed by an approximation,  
 192 tending to underestimate the true coefficient of variation (see e.g., Papaioannou et al. (2015)).

Table 1: Results of the numerical experiments (subset sample size  $N = 10^4$ ,  $p_0 = 0.1$ ).

	$d$	$p_f^{\text{ref}}$	$\hat{p}_f^{\text{BANCS}}$	$\hat{\delta}^{\text{BANCS}}$	$\hat{p}_f^{\text{SS}}$	$\hat{\delta}^{\text{SS}}$
Toy-case #1	2	$1.31 \times 10^{-4}$	$2.67 \times 10^{-4}$	24%	$1.30 \times 10^{-4}$	9%
Toy-case #2	2	$2.21 \times 10^{-4}$	$4.23 \times 10^{-4}$	7%	$2.24 \times 10^{-4}$	6%
Toy-case #3	7	$8.10 \times 10^{-3}$	$9.32 \times 10^{-3}$	15%	$8.92 \times 10^{-3}$	6%

## 193 5. CONCLUSION

194 Subset Simulation uses MCMC sampling to generate its intermediary conditional samples.  
 195 However, MCMC algorithms tends to be complex to tune and does not generate i.i.d. condi-

196 tional samples. In this work, a new method is proposed, replacing MCMC sampling with a  
197 simpler procedure. An intermediary conditional distribution is first fitted by a nonparametric  
198 approach, mixing kernel density estimation for fitting the marginals and Empirical Bernstein  
199 Copula (EBC) for fitting the copula. Then, the resulting allows to perform direct Monte Carlo  
200 sampling. This method is named “Bernstein adaptive nonparametric conditional sampling”  
201 (BANCS) and is applied to three toy-cases (two 2-dimensional and one 7-dimensional) and  
202 compared with SS.

203 The method shows promising results, even though a small positive bias consistently ap-  
204 pears. This issue results from EBC tuning, creating a bias-variance tradeoff in the copula  
205 fit. Theoretical works offer optimal tuning, allowing us to find the optimal compromise. In  
206 our numerical experiments, an empirical estimation of BANCS variance is computed over a  
207 set of repetitions. BANCS estimated coefficient of variation is higher than SS approximated  
208 coefficient of variation. This work can be further explored by building an approximation of  
209 BANCS variance and confidence interval. One major advantage remains that the samples  
210 generated at each iteration are i.i.d. leading to a possible use of these samples to perform  
211 global reliability-oriented sensitivity analysis (Marrel and Chabridon, 2021) in order to detect  
212 and analyze the most influential input variables leading to failure.

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