Sliding Mode Observer for Set-valued Lur'e Systems and Chattering Removing

Samir Adly, Ba Khiet Le

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Samir Adly* Ba Khiet Le †

Abstract

In this paper, we study a sliding mode observer for a class of set-valued Lur'e systems subject to uncertainties. We show that our approach has obvious advantages than the existing Luenberger-like observers. Furthermore, we provide an effective continuous approximation to eliminate the chattering effect in the sliding mode technique.

Keywords. Sliding mode observer; set-valued Lur'e systems; chattering effect

AMS Subject Classification. 28B05, 34A36, 34A60, 49J52, 49J53, 93D20

1 Introduction

Hybrid systems are a class of dynamic systems that may present both continuous and discrete behavior. They are characterized by the presence of continuous state variables, inputs/outputs, as well as discrete state variables, inputs/outputs. On the other hand, Lur'e-type nonlinear systems [21] are represented by the combination of a linear time-invariant system and a memoryless nonlinearity through a feedback connection. The link between Lur'e systems and hybrid systems lies in the coupling of the continuous and discrete dynamics through the nonlinear state equation and the linear output equation. This coupling makes Lur'e systems well-suited for modeling complex physical, biological, or social systems that exhibit both continuous and discrete behavior, and have found applications in areas such as control systems, signal processing, and system identification. For a comprehensive guide to hybrid dynamical systems from the modeling, stability and robustness point of view, we refer to [14]. In this paper we focus on Lur'e set-valued dynamical systems where the nonlinear feedback is given by a set-valued relation. The Lur'e set-valued dynamical system is a widely studied model in applied mathematics and control theory. In recent decades, it has received significant attention, as evidenced by the numerous studies found in references such as [1, 2, 3, 7, 8, 9, 10, 11, 13, 15, 16, 17, 18, 19, 20, 26]. Despite advancements in the understanding of the system's existence and uniqueness, stability and asymptotic analysis, the design of control and observers still poses many interesting open questions. Specifically, most observers for Lur'e set-valued systems follow the Luenberger design, which has limitations when the system is subjected to

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uncertainty, as discussed in references such as [15, 16, 17, 26]. Recently, using the powerful sliding mode technique, B. K. Le in [20] proposed a sliding mode observer for a general class of set-valued Lur'e systems subject to uncertainties as follows

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + Eu(t) + G\xi(t,u,x) \text{ for a.e. } t \in [0,+\infty), \\ w(t) = Cx(t) + D\lambda(t), \\ \lambda(t) \in -\mathcal{F}_t(w(t)),\ t \geq 0, \\ y(t) = Fx(t), \\ x(0) = x_0, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}, E \in \mathbb{R}^{n\times l}, F \in \mathbb{R}^{p\times n}, G \in \mathbb{R}^{n\times k}$ are given matrices, λ and w are two connecting variables, u is the control input, y is the physically measurable output and ξ is some uncertainty. The set-valued operator $\mathcal{F}_t : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is time-dependent and assumed to be maximal monotone. Note that (1) can be rewritten into a first order time-dependent differential inclusion as follows

$$\dot{x} \in Ax - B(\mathcal{F}_t^{-1} + D)^{-1}Cx + Eu + G\xi, \quad x(0) = x_0. \tag{2}$$

If $D = 0$ and $\mathcal{F}_t = \mathcal{F}$, we obtain the following classical case

$$\dot{x} \in Ax - B\mathcal{F}Cx + Eu + G\xi.$$

The proposed sliding mode observer for (1) is

$$\begin{cases} \dot{\tilde{x}} \in A\tilde{x} + B\tilde{\lambda} - Le_y + Eu - P^{-1}F^T\Psi(e_y), \\ \tilde{w} = C\tilde{x} + D\tilde{\lambda}, \\ \tilde{\lambda} \in -\mathcal{F}(\tilde{w} + Ke_y), \\ \tilde{y} = F\tilde{x}, \end{cases} \tag{3}$$

where

$$e = \tilde{x} - x, e_y = \tilde{y} - y = Fe,$$

$$\Psi(e_y) = \sigma_1 e_y + (\|J\|\ \rho(t,u,y) + \sigma_2)\text{Sign}(e_y)$$

and ρ is a bound of the uncertainty for some suitable matrix J and real number $\sigma_1 \geq 0, \sigma_2 > 0$. Then under some mild conditions, the observer state converges to the original state asymptotically and the observation error converges to zero in finite time (we refer to [20] for more details).

In this paper, our first contribution is to provide a sliding mode observer for the system considered in [15] as follows

$$\begin{cases} \dot{x} = Ax + B\omega + f_1(x,u) + f_2(x,u)\theta, \\ \omega \in -\mathcal{F}(Cx), \\ y = Fx, \end{cases} \tag{4}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, F \in \mathbb{R}^{p\times n}$ are given matrix, $x \in \mathbb{R}^n$ is the state, $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a maximal monotone operator, $u \in \mathbb{R}^r$ is the control input and $y \in \mathbb{R}^p$ is the measurable output. The functions f_1, f_2 are known smooth while $\theta \in \mathbb{R}^l$ is an unknown constant parameter vector. An adaptive Luenberger-like observer was proposed by Huang et al in [15]. In our paper, the parameter θ may not always be constant, such as in the case of perturbations. Our sliding mode approach is efficient and straightforward, without the need to solve an additional ordinary differential equation, as seen in [15]. Our results show that the convergence of the observer state to the actual state is exponential, while the rate of convergence in [15] is unknown. Additionally, if the matrix function f_2 is bounded, the conditions for solvability of the associated LMI are significantly improved, leading to finite time convergence of the observation error under more favorable assumptions than in [20]. Our contributions also extend to the reduced-order case, where we have improved conditions. Lastly, we propose a new smooth approximation of the sliding mode technique that reduces the chattering effect while still effectively managing uncertainty.

The paper is organized as follows. In Section 2, we revisit some key concepts. In Section 3, we introduce a sliding mode observer for (4) and demonstrate that it holds more benefits compared to the Luenberger-like observer discussed in [15]. Section 4 focuses on the reduced-order observer. In Section 5, we present a new and refined version of the sliding mode technique. Numerical examples to support the results are provided in Section 6. Finally, Section 7 concludes the paper and highlights future possibilities.

2 Notations and mathematical background

We denote the scalar product and the corresponding norm of Euclidean spaces by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. A matrix $P \in \mathbb{R}^{n\times n}$ is called positive definite, written $P > 0$, if there exists $a > 0$ such that

$$\langle Px, x\rangle \geq a\|x\|^2, \quad \forall\, x \in \mathbb{R}^n.$$

The Sign function in $\mathbb{R}^m$ is defined by

$$\mathrm{Sign}(x) = \begin{cases} \frac{x}{\|x\|} & \text{if} \quad x \neq 0 \\ \mathbb{B} & \text{if} \quad x = 0, \end{cases}$$

where $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^m$.

A set-valued mapping $\mathcal{F} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is called *monotone* if for all $x, y \in \mathbb{R}^m$ and $x^* \in \mathcal{F}(x), y^* \in \mathcal{F}(y)$, one has

$$\langle x^* - y^*, x - y\rangle \geq 0.$$

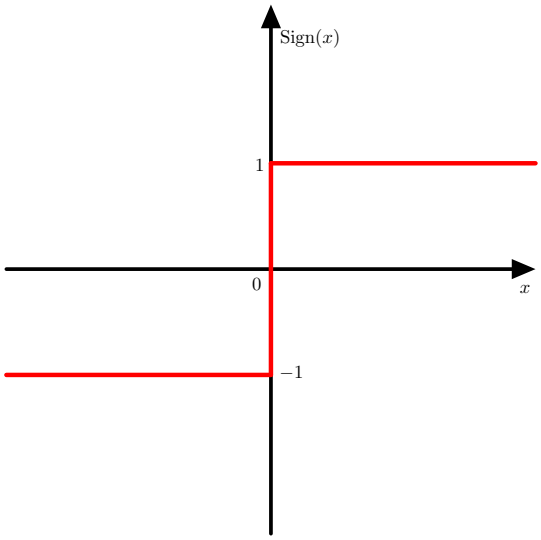

Figure 1: Sign function in $\mathbb{R}$

Furthermore, $\mathcal{F}$ is called *maximal monotone* if there is no monotone operator $\mathcal{G}$ such that the graph of $\mathcal{F}$ is contained strictly in the graph of $\mathcal{G}$.

Lemma 1. *[20] Let $F \in \mathbb{R}^{p\times n}$ be a full row rank matrix ($p \leq n$) and $P \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. If $x \in \mathrm{im}(P^{-1}F^T)$ then*

$$F^T(FP^{-1}F^T)^{-1}Fx = Px, \tag{5}$$

where $\mathrm{im}(A)$ *denotes the range of* A.

3 Exponential convergence of the sliding mode observer

In this section, we will propose a sliding mode observer for the system (4) under the following assumptions:

Assumption 1: The set-valued operator $\mathcal{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a monotone, upper semi-continuous with non-empty, closed convex and bounded values.

Assumption 2: The functions $f_1 : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ and $f_2 : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{n\times l}$ are Lipschitz continuous w.r.t x, i. e., there exist $L_1 > 0$ and $L_2 > 0$ such that for all $x_1, x_2 \in \mathbb{R}^n, u \in \mathbb{R}^r$, we have

$$\|f_1(x_1,u) - f_2(x_2,u)\| \leq L_1\|x_1 - x_2\| \text{ and } \|f_2(x_1,u) - f_2(x_2,u)\| \leq L_2\|x_1 - x_2\|.$$

Assumption 3: The unknown $\theta(t,x,u)$ is bounded by $L_3 > 0$.

Assumption 4: Let $\gamma = L_1 + L_2 L_3$. There exist $\epsilon > 0$, $P \in \mathbb{R}^{n\times n} > 0$, $L \in \mathbb{R}^{n\times p}$, $K \in \mathbb{R}^{m\times p}$ and the matrix function $h : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{l\times p}$ such that

$$P(A - LF) + (A - LF)^T P + \gamma P^2 + \gamma I + \epsilon I \leq 0, \tag{6}$$
$$B^T P = C - KF, \tag{7}$$
$$f_2^T(x,u)P = h(x,u)F. \tag{8}$$

Remark 1. From (8), we have $f_2(x,u) = P^{-1}F^T h^T(x,u)$. It means that $\mathrm{im}(f_2(x,u)) \in \mathrm{im}(P^{-1}F^T)$ for all (x,u).

The proposed sliding mode observer for (4) is

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{\omega} - Le_y + f_1(\tilde{x},u) - \beta P^{-1}F^T\|h(\tilde{x},u)\|\mathrm{Sign}(e_y), \\ \tilde{\omega} \in -\mathcal{F}(C\tilde{x} - Ke_y), \\ \tilde{y} = F\tilde{x}, \end{cases} \tag{9}$$

where $\beta \geq L_3$.

Remark 2. Under the conditions outlined in Assumptions 1 to 4, it has been established that solutions for both the original system and the observer system exist. This is due to the fact that the right-hand side operators in the corresponding differential inclusions are upper semi-continuous and have non-empty, closed, convex, and bounded values (see for instance [5, 12, 15]).The following result confirms the exponential convergence of the observer state to the original state.

Theorem 2. *Let Assumptions 1–4 hold. Then the observer state $\tilde{x}$ of (9) exponentially converges to the original state x of (4).*

Proof. Let $e = \tilde{x} - x, e_y = \tilde{y} - y = Fe$. From (4) and (9), we have

$$\begin{aligned} \dot{e} \;&\in\; (A - LF)e - B(\tilde{\omega} - \omega) + f_1(\tilde{x},u) - f_1(x,u) \\ &-\; \beta P^{-1}F^T\|h(\tilde{x},u)\|\mathrm{Sign}(e_y) - f_2(x,u)\theta. \end{aligned} \tag{10}$$

Consider the Lyapunov function $V(e) = \langle Pe, e\rangle$. Then the orbital derivative of V is

$$\begin{aligned} \dot{V}(e) \;&=\; 2\langle P\dot{e}, e\rangle = 2\langle P(A - LF)e - PB(\tilde{\omega} - \omega), e\rangle \\ &+\; 2\langle P(f_1(\tilde{x},u) - f_1(x,u)), e\rangle - 2\beta\|h(\tilde{x},u)\|\|e_y\| - 2\langle Pf_2(x,u)\theta, e\rangle. \end{aligned} \tag{11}$$

From (7) and the monotonicity of $\mathcal{F}$, we have

$$\langle PB(\tilde{\omega} - \omega), e\rangle = \langle \tilde{\omega} - \omega), B^T Pe\rangle = \langle \tilde{\omega} - \omega, (C - KF)e\rangle \leq 0. \tag{12}$$

On the other hand

$$2\langle P(f_1(\tilde{x},u) - f_1(x,u)), e\rangle \leq 2L_1\|e\|\|Pe\| \leq L_1(\|e\|^2 + \|Pe\|^2) \tag{13}$$

and

$$\begin{aligned} Y &:= -2\beta\|h(\tilde{x},u)\|\|e_y\| - 2\langle Pf_2(x,u)\theta, e\rangle \\ &= -2\beta\|h(\tilde{x},u)\|\|e_y\| - 2\langle P(f_2(x,u) - f_2(\tilde{x},u))\theta, e\rangle - 2\langle Pf_2(\tilde{x},u)\theta, e\rangle \\ &\leq -2\beta\|h(\tilde{x},u)\|\|e_y\| + 2L_2L_3\|e\|\|Pe\| - 2\langle \theta, h(\tilde{x},u)e_y\rangle \\ &\leq L_2L_3(\|e\|^2 + \|Pe\|^2). \end{aligned} \tag{14}$$

Let α_{max} and α_{min} be the largest eigenvalue and the smallest eigenvalues of P, respectively. From (6), (11), (12), (13) and (14), we have

$$\frac{dV}{dt} \leq -\epsilon\|e\|^2 \leq -\frac{\epsilon}{\alpha_{max}}V(t).$$

Using Gronwall's inequality, we obtain

$$\alpha_{min}\|e\|^2 \leq V(t) \leq \exp(\frac{-\epsilon t}{\alpha_{max}})V(t_0).$$

Hence, $\|e\| \leq \exp(\frac{-\epsilon t}{2\alpha_{max}})\sqrt{\frac{V(t_0)}{\alpha_{min}}}$ and the conclusion follows. □

Remark 3. (i) According to [15], under similar conditions, our implementation of a sliding mode observer leads to an exponential convergence of the observer state, whereas the adaptive observer in the same study only achieves convergence without a defined rate. Furthermore, our method avoids the need to solve a high-dimensional additional ODE, which can be computationally expensive, as seen in [15].
(ii) In Assumption 4, if h is bounded, then $\gamma = L_1 + L_2L_3$ can be replaced by just L_1, which is usually much smaller than γ. This makes it easier to satisfy (6)-(8) and enables us to solve a wider class of set-valued Lur'e systems that cannot be tackled using the method in [15]. Furthermore, our technique also ensures finite time convergence of the observation error e_y to zero. A bounded matrix function h can be guaranteed if f_2 is bounded and F is full row rank, as one can always choose a full row rank matrix F for the output $y = Fx$ without losing any information.

Lemma 3. *Suppose that $F \in \mathbb{R}^{p\times n}$ be a full row rank matrix. If f_2 is bounded then h is also bounded.*

Proof. From Remark 1, we imply that $h(x,u) = (FP^{-1}F^T)^{-1}Ff_2(x,u)$ and the condition follows. □

Theorem 4. *If $h(x,u)$ is bounded by some constant $L_4 > 0$ then γ in Assumption 4 can be reduced to L_1 to still obtain the exponential convergence of the original state by using the observer*

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B\tilde{\omega} - Le_y + f_1(\tilde{x},u) + \beta P^{-1}F^T\mathrm{Sign}(e_y), \\ \tilde{\omega} \in -\mathcal{F}(C\tilde{x} - Ke_y), \\ \tilde{y} = F\tilde{x} \end{cases} \tag{15}$$

where $\beta > L_3L_4$. In addition if $F \in \mathbb{R}^{p\times n}$ be a full row rank matrix and $\mathrm{im}(B) \subset \mathrm{im}(P^{-1}F)$ then the observation error e_y converges in explicitly finite time.

Proof. Under the new assumption, in (14) we have

$$-\beta\|e_y\| - \langle Pf_2(x,u)\theta, e\rangle = -\beta\|e_y\| - \langle \theta, h(x,u)e_y\rangle \leq -(\beta - L_3L_4)\|e_y\| \leq 0$$

and similarly one obtains the exponential convergence of the observer state.

To attain the finite time convergence of the observation error, we consider the new Lyapunov function $W = \frac{1}{2}\langle e_y, (FP^{-1}F^T)^{-1}e_y\rangle$. Then

$$\frac{dW}{dt} = \langle (FP^{-1}F^T)^{-1}\dot{e}_y, e_y\rangle.$$

From (10), we have

$$\begin{aligned} \dot{e}_y &= F(A-LF)e - FB(\tilde{\omega}-\omega) + F\Big(f_1(\tilde{x},u) - f_1(x,u)\Big) \\ &- \beta FP^{-1}F^T\text{Sign}(e_y) - Ff_2(x,u)\theta. \end{aligned} \tag{16}$$

Since $\text{im}(B) \subset \text{im}(P^{-1}F)$, using Lemma 1, we have

$$F^T(FP^{-1}F^T)^{-1}FB = PB$$

and hence

$$\begin{aligned} &\langle (FP^{-1}F^T)^{-1}FB(\tilde{\omega}-\omega), e_y\rangle = \langle F^T(FP^{-1}F^T)^{-1}FB(\tilde{\omega}-\omega), e\rangle \\ = \;& \langle PB(\tilde{\omega}-\omega), e\rangle = \langle \tilde{\omega}-\omega, B^TPe\rangle = \langle \tilde{\omega}-\omega, (C-KF)e\rangle \leq 0. \end{aligned}$$

On the other hand, from (8) we deduce that $f_2 = P^{-1}F^Th^T$ and thus

$$(FP^{-1}F^T)^{-1}Ff_2(x,u)\theta = (FP^{-1}F^T)^{-1}(FP^{-1}F^T)h^T(x,u)\theta = h^T(x,u)\theta. \tag{17}$$

Since $\|e\| \leq \exp(\frac{-\epsilon t}{2\alpha_{max}})\sqrt{\frac{V(t_0)}{\alpha_{min}}}$ as in the proof of Theorem 2, we can find some $t_1 > 0$ such that for all $t \geq t_1$, one has

$$\|(FP^{-1}F^T)^{-1}\Big(F(A-LF)e + F\big(f_1(\tilde{x},u) - f_1(x,u)\big)\Big)\| \leq \frac{\sigma}{2}$$

where $\sigma = \beta - L_3L_4 > 0$. Indeed, we can choose

$$t_1 := \frac{2\alpha_{max}}{\epsilon}\ln\Big(\frac{2\|(FP^{-1}F^T)^{-1}F\|(\|A-LF\| + L_1)}{\sigma}\sqrt{\frac{V(t_0)}{\alpha_{min}}}\Big).$$

Then for all $t \geq t_1$, we have

$$\begin{aligned} \frac{dW}{dt} &\leq \frac{\sigma}{2}\|e_y\| - (\beta - \|h^T(x,u)\theta\|)\|e_y\| \leq \frac{\sigma}{2}\|e_y\| - (\beta - L_3L_4)\|e_y\| \\ &= -\frac{\sigma}{2}\|e_y\| \leq -\kappa\sqrt{W(t)}, \end{aligned}$$

where $\kappa := \sigma/\sqrt{2\gamma_{max}}$ and γ_{max} is the largest eigenvalue of $(FP^{-1}F^T)^{-1}$ since $\|e_y\| \geq \sqrt{\frac{2W(t)}{\mu_{max}}}$. Suppose that $W(t) > 0$ for all $t \geq t_1$ then we have

$$\frac{W'}{2\sqrt{W}} \leq -\frac{\kappa}{2}$$

and thus

$$\sqrt{W(t)} - \sqrt{W(t_1)} \leq -\frac{\kappa}{2}(t - t_1) \to -\infty \text{ as } t \to \infty,$$

a contradiction. Let $t_f \geq t_1$ be the first time such that $W(t_f) = 0$, then we deduce that $W(t) = 0$ for all $t \geq t_f$ since W is non-negative and decreasing. It means that e_y converges to 0 in finite time. Similarly as above we have

$$-\sqrt{W(t_1)} = \sqrt{W(t_f)} - \sqrt{W(t_1)} \leq -\frac{\kappa}{2}(t_f - t_1)$$

and hence

$$t_f \leq t_1 + 2\sqrt{W(t_1)}/\kappa.$$

□

Remark 4. (i) Based on the observation (17), we can have a significantly better estimation for the gain β than in [20]. Indeed, in the current paper the gain β does not change to obtain the finite time convergence of the observation error from the exponential convergence of the observer state. In addition, we can also provide an explicit estimation for t_f.
(ii) If Assumption 1 is substituted by the maximal monotone property of the set-valued $\mathcal{F}$, similar results can still be obtained. The existence of solutions for the original system (4) remains guaranteed, as shown in references such as [5, 12]. Existence of solutions to the observer systems (9), (15) can be also obtained by approximating the Sign set-valued operator with smooth functions. It's worth mentioning that the normal cone operator N_C, which is associated to a nonempty closed convex set C, is a crucial maximal monotone set-valued mapping in mechanical and electrical engineering that does not satisfy the boundedness requirement in Assumption 1.

4 Reduced-order observer

Suppose that the given matrices and matrix-functions can be decomposed as follows

$$x = \begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} \\ \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ \\ B_2 \end{pmatrix}, C = (C_1 \;\; C_2), F = (F_q \;\; 0)$$

$$P = \begin{pmatrix} P_{11} & P_{12} \\ \\ P_{21} & P_{22} \end{pmatrix}, \quad f_1(x,u) = \begin{pmatrix} f_{11}(x,u) \\ \\ f_{12}(x,u) \end{pmatrix}, \quad f_2(x,u) = \begin{pmatrix} f_{21}(x,u) \\ \\ f_{22}(x,u) \end{pmatrix},$$

where $F_q \in R^{q\times q}$ is an invertible matrix and the following is satisfied:

Assumption 4': There exist $\epsilon > 0$, $Q \in \mathbb{R}^{(n-q)\times(n-q)} > 0$, $P_{22} \in \mathbb{R}^{(n-q)\times(n-q)}$ invertible, $P_{21} \in \mathbb{R}^{(n-q)\times q}$ such that

$$Q(A_{22} + KA_{12}) + (A_{22} + KA_{12})^T Q + L_1 Q(KK^T + I_{n-q})Q + (L_1 + \epsilon)I_{n-q} \leq 0, \tag{18}$$

$$(B_2 + KB_1)^T Q = C_2, \tag{19}$$

$$\begin{pmatrix} P_{21} & P_{22} \end{pmatrix} f_2(x,u) = 0, \tag{20}$$

where $K = P_{22}^{-1}P_{21}$.

Note that (4) can be rewritten as follows

$$\begin{cases} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1\omega + f_{11}\Big(\begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}, u\Big) + f_{21}\Big(\begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}, u\Big)\theta \\ \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2\omega + f_{12}\Big(\begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}, u\Big) + f_{22}\Big(\begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix}, u\Big)\theta \\ \omega \in -\mathcal{F}(C_1x_1 + C_2x_2) \\ y = F_qx_1. \end{cases} \tag{21}$$

Using (20), we have

$$\begin{cases} \dot{z} &= (A_{22} + KA_{12})z + (B_2 + KB_1)\omega + [(A_{21} + KA_{11}) - (A_{22} + KA_{12})K]x_1 \\ &+ (K \;\; I_{n-q})f_1\Big(\begin{pmatrix} x_1 \\ \\ z - Kx_1 \end{pmatrix}, u\Big) \\ \omega &\in -\mathcal{F}(C_2z + (C_1 - C_2K)x_1), \\ x_2 &= z - Kx_1. \end{cases} \tag{22}$$

The adaptive observer is

$$\begin{cases} \dot{\tilde{z}} &= (A_{22} + KA_{12})\tilde{z} + (B_2 + KB_1)\tilde{\omega} + [(A_{21} + KA_{11}) - (A_{22} + KA_{12})K]x_1 \\ &+ (K \;\; I_{n-q})f_1\Big(\begin{pmatrix} x_1 \\ \\ \tilde{z} - Kx_1 \end{pmatrix}, u\Big) \\ \tilde{\omega} &\in -\mathcal{F}(C_2\tilde{z} + (C_1 - C_2K)x_1), \\ \tilde{x}_2 &= \tilde{z} - Kx_1. \end{cases} \tag{23}$$

Theorem 5. *Let Assumptions 1, 2, 3, 4' hold. Then (23) is a reduced-order observer of (4), i.e.,* $\lim_{t\to\infty}(x_2(t) - \tilde{x}_2(t)) = 0$.

Proof. Let $e_z = \tilde{z} - z$. Then we have

$$\begin{cases} \dot{e}_z &= (A_{22} + KA_{12})e_z + (B_2 + KB_1)(\tilde{\omega} - \omega) \\ &+ (K \;\; I_{n-q})\Big[f_1\Big(\begin{pmatrix} x_1 \\ \\ \tilde{z} - Kx_1 \end{pmatrix}, u\Big) - f_1\Big(\begin{pmatrix} x_1 \\ \\ z - Kx_1 \end{pmatrix}, u\Big)\Big] \\ \omega &\in -\mathcal{F}(C_2z + (C_1 - C_2K)x_1). \\ \tilde{\omega} &\in -\mathcal{F}(C_2\tilde{z} + (C_1 - C_2K)x_1). \end{cases} \tag{24}$$

Let us consider the Lyapunov function $W(e_z) = \langle Qe_z, e_z\rangle$, then $\dot{W}(e_z) = 2\langle Q\dot{e}_z, e_z\rangle$. From (19) and the monotonicity of $\mathcal{F}$, we have

$$\langle Q(B_2 + KB_1)(\tilde{\omega} - \omega), e_z\rangle = \langle \tilde{\omega} - \omega, (B_2 + KB_1)^T Qe_z\rangle = \langle \tilde{\omega} - \omega, C_2 e_z\rangle \leq 0.$$

On the other hand

$$\begin{aligned} & \langle 2Q(K \;\; I_{n-q})\Big[f_1\Big(\begin{pmatrix} x_1 \\ \tilde{z} - Kx_1 \end{pmatrix}, u\Big) - f_1\Big(\begin{pmatrix} x_1 \\ z - Kx_1 \end{pmatrix}, u\Big)\Big], e_z\rangle \\ \leq \;& 2L_1\|Q(K \;\; I_{n-q})e_z\|\|e_z\| \leq L_1 e_z^T Q(K \;\; I_{n-q})(K \;\; I_{n-q})^T Qe_z + L_1 e_z^T e_z \\ \leq \;& L_1 e_z^T Q(KK^T + I_{n-q})Qe_z + L_1 e_z^T e_z. \end{aligned}$$

Combining with (18), similarly as in the proof of Theorem 2, we have $\dot{W}(e_z) \leq -\epsilon\|e_z\|^2$. It deduces that e_z converges to zero exponentially and the conclusion follows. □

Remark 5. (i) Note that if we have (8), then $Pf_2(x,u) = F^T h(x,u)$ and thus $\begin{pmatrix} P_{21} & P_{22}\end{pmatrix} f_2(x,u) = 0$. It would be interesting to improve or remove the condition (20).
(ii) When the matrices are decomposable, it is more effective to provide assumptions directly with the new lower-dimension matrices. Note that Assumption 4' is strictly weaker than Assumption 4 even when $Q = P_{22}$ [15]. It is remarkable that $Q > 0$ can be different from P and it is unnecessary to require that $P > 0$ but only invertible P_{22}.

5 A new continuous approximate of the sliding mode technique

Although the sliding mode method is effective, it has a persistent issue: the chattering effect caused by the discontinuity of the Sign function. To eliminate this issue, the Sign function in $\mathbb{R}$ is typically approximated by the "sigmoid function" (as noted in [22])

$$\text{Sign(x)} \approx \frac{x}{|x| + \epsilon}$$

or by [23]

$$\text{Sign(x)} \approx \frac{x}{\sqrt{x^2 + \epsilon}}$$

for some small fixed $\delta > 0$. Other continuous approximates can be also found in [23].

In this paper, we provide a new smooth approximate of the set-valued function Sign by using a time-dependent guiding function. Given a continuous guiding function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$, we define the function $\text{Sign}_\delta : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$\text{Sign}_\delta(t,x) = \begin{cases} \frac{x}{\|x\|} - \frac{1-\|x\|/\delta(t)}{(1+M\|x\|)^N}\frac{x}{\|x\|} & \text{if } \; x \neq 0 \text{ and } \|x\| \leq \delta(t), \\ \frac{x}{\|x\|} & \text{if} \quad \|x\| > \delta(t), \\ 0 & \text{if} \quad x = 0, \end{cases} \tag{25}$$

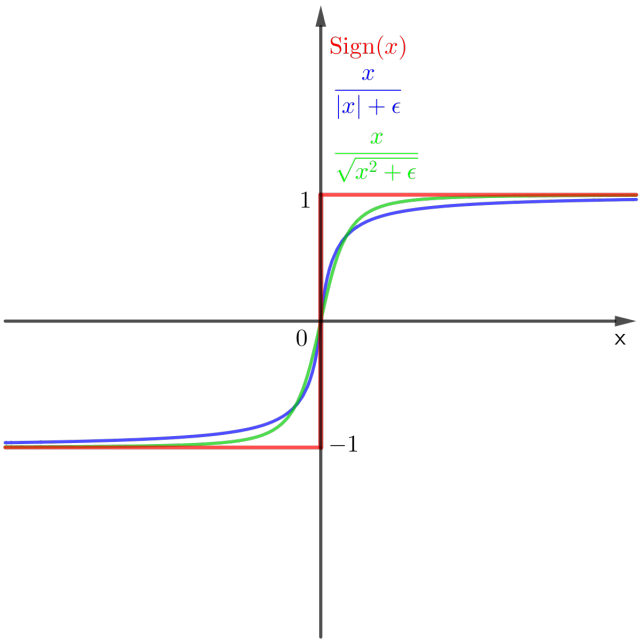

Figure 2: Sign function in $\mathbb{R}$ and its approximations

for some $M, N > 0$ and the function $\delta(\cdot)$ is decreasing which converges to zero when $t \to \infty$. For example, we can choose $\delta(t) = \exp(-k_1 t - k_2)$ for some $k_1 > 0, k_2 > 0$. It can be seen that the $\text{Sign}_\delta(t, x)$ function is a continuous and smooth function with respect to both time t and state x. When the magnitude of x is greater than $\delta(t)$, the function $\text{Sign}_\delta(t, x)$ becomes equal to $\text{Sign}(x)$. This property makes it effective in reducing the chattering effect while still handling uncertainty. In comparison, the norm of the sigmoid function $\frac{x}{|x|+\epsilon}$ is always less than 1 which does not deal with the uncertainty entirely. The sigmoid function only leads to convergence of the state to an approximate region of the sliding surface, as reported, e.g., in [22]. The suitable choice of the guiding function δ can completely remove the chattering effect. This can be seen in Example 1, Figure 3.

6 Numerical Examples

In this section we provide some numerical examples to show the effectiveness of our approach.

Example 1. To test the effectiveness of our new approximation method, we will analyze a simple one-dimensional system: $\dot{x} \in \mu - L\text{Sign}(x)$, where μ is an uncertainty with a constraint $|\mu| < L$. As an example, let us set $\mu = 3\sin x$ and $L = 4$. It is known that the state variable x will converge to zero in a finite time. To evaluate the numerical simulations, we use the explicit scheme with an initial value of $x_0 = 0.1$ (see Figure 3). The Sign function creates a chattering effect (as seen in the state x_1). However, by replacing the Sign function with $\text{Sign}_\delta(t, x)$ defined in equation (25), using a guiding function $\delta(t) = e^{-0.5t}$ and $M = 1, N = 3$, the chattering effect can be eliminated and the convergence to zero is achieved (as seen in the state x_3). The performance of this approximation

is better than using the sigmoid function with $\epsilon = 10^{-3}$ or $\epsilon = 10^{-6}$ (as seen in the state x_2). It is important to note that the implicit scheme can only remove the chattering effect numerically.

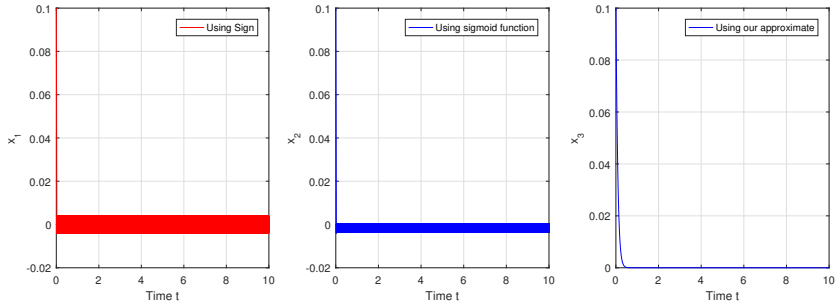

Figure 3: An effective continuous approximation of Sign function in $\mathbb{R}$

Example 2. Next we consider the system (4) with the following data

$$A = \begin{pmatrix} -1 & 5 & 0 \\ 9 & -0.9 & 0 \\ 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, f_1(x,u) = \begin{pmatrix} 3u + 0.8\sin x_2 \\ 2u + 0.9\cos x_1 \\ -u + 0.8\sin x_3 \end{pmatrix}$$

$$f_2(x,u) = \begin{pmatrix} 3\sin x_2 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 5 & -3 & 4 \end{pmatrix}, F = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

Suppose that the unknown $\theta = 3\sin t$, the control input $u = 8\cos t$ and

$$\mathcal{F}(x) = \begin{cases} \text{sign}(x)(2|x| + 5) & \text{if} \quad x \neq 0, \\ [-5, 5] & \text{if} \quad x = 0. \end{cases}$$

Then $L_1 = 0.8, L_2 = 3, L_3 = 3$ and $\gamma = 9.8$. We cannot find the matrices P, K, L such that the LMIs in Assumption 4 (and also the LMIs in [15]) are satisfied but Theorem 4 can be applied with

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ 14 \\ 0 \end{pmatrix}, \quad \epsilon = 0.2, \quad K = 3.$$

Using the sliding mode observer (15) with the gain $\beta = 10$ and initial point $x_0 = (3\ 2\ 1)^T$, $\tilde{x}_0 = (15\ 27\ 16)^T$, the convergence of the error $e = \tilde{x} - x$ to zero can be seen in the Figure 4.

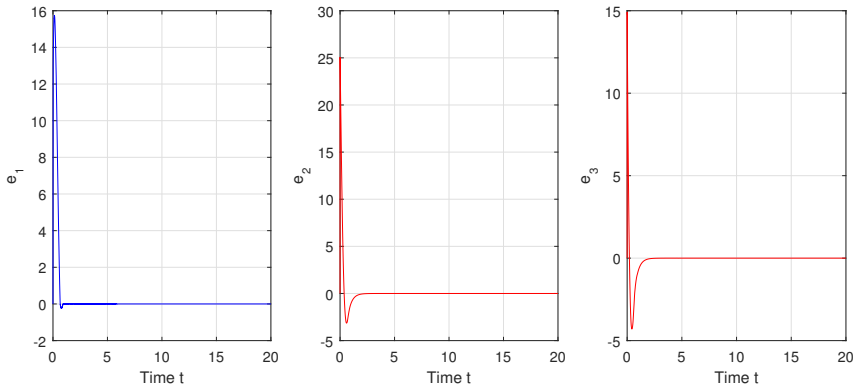

Figure 4: The convergence of the sliding mode observer (15)

7 Conclusion and perspectives

In this paper, the advantages of sliding mode observers are demonstrated for set-valued Lur'e dynamical systems that are faced with uncertainties. The robustness of this observer technique is a significant factor in the analysis and control of such systems. We also present a new and efficient continuous approximation of the sliding mode technique, which provides improved performance compared to conventional methods. However, the traditional condition (8) is a limiting factor in the applicability. Further research is needed to explore the possibility of relaxing or removing this condition to increase the range of systems that can be analyzed. This is an area that merits further investigations and has the potential to enhance the performance of sliding mode observers for set-valued Lur'e dynamical systems.

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