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Abstract: We investigate the range $\mathscr{R}_T$ of the diffusive biased walk $\mathbb{X}$ on a Galton-Watson tree $\mathbb{T}$ in random environment, that is to say the sub-tree of $\mathbb{T}$ of all distinct vertices visited by this walk up to the time T. We study the volume of the range with constraints and more precisely the number of k-tuples ($k \geq 2$) of distinct vertices in this sub-tree, in small generations and satisfying an hereditary condition. A special attention is paid to the vertices visited during distinct excursions of $\mathbb{X}$ above the root of the Galton-Watson tree as we observe they give the major contribution to this range. As an application, we study the genealogy of $k \geq 2$ distinct vertices of the tree $\mathscr{R}_T$ picked uniformly from those in small generations. It turns out that two or more vertices among them share a common ancestor for the last time in the remote past. We also point out an hereditary character in their genealogical tree due to the random environment.

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1. Introduction

1.1. Randomly biased random walk on trees

Let (N, A) be a random variable under a probability measure $\mathbf{P}$ taking values in $\mathbb{N} \times \mathbb{R}$ and consider the following Galton-Waston marked tree $(\mathbb{T}, (A_x; x \in \mathbb{T}))$ rooted at e: the generation 0 contains one marked individual $(e, A_e) = (e, 0)$. For any $n \in \mathbb{N}^*$, assume the generation $n-1$ has been built. If it is empty, then the generation n is also empty. Otherwise, any vertex x in the generation $n-1$ gives progeny to N_x marked children $(x^1, A_{x^1}), \dots, (x^{N_x}, A_{x^{N_x}})$ independently of other vertices in generation $n-1$ according to the law of (N, A), thus forming the generation n, denoted by $\mathbb{T}_n$. We assume $\mathbf{E}[N] > 1$ so that $\mathbb{T}$ is a super-critical Galton-Watson tree, that is $\mathbf{P}(\text{non-extinction of } \mathbb{T}) > 0$ and we define $\mathbf{P}^*(\cdot) := \mathbb{P}(\cdot | \text{non-extinction of } \mathbb{T})$.
For any vertex $x \in \mathbb{T}$, we denote by $|x|$ the generation of x, by x_i its ancestor in generation $i \in \{0, \dots, |x|\}$ and $x^* := x_{|x|-1}$ stands for the parent of x. In particular, $x_0 = e$ and $x_{|x|} = x$. For any $x, y \in \mathbb{T}$, we write $x \leq y$ if x is an ancestor of y (y is said to be a descendent of x) and $x < y$ if $x \leq y$ and $x \neq y$. We then write $[\![x_i, x]\!] := \{x_j; j \in \{i, \dots, |x|\}\}$. Finally, for any $x, y \in \mathbb{T}$, we denote by $x \wedge y$ the most recent common ancestor of x and y, that is the ancestor u of x and y such that $\max\{|z|;\ z \in [\![e, x]\!] \cap [\![e, y]\!]\} = |u|$.
Let us introduce the branching potential V: let $V(e) = A_e = 0$ and for any $x \in \mathbb{T} \setminus \{e\}$

$$V(x) := \sum_{e < z \leq x} A_z = \sum_{i=1}^{|x|} A_{x_i}.$$

Under $\mathbf{P}$, $\mathscr{E} := (\mathbb{T}, (V(x); x \in \mathbb{T}))$ is a real valued branching random walk such that $(V(x) - V(x^*))_{x \in \mathbb{T} \setminus \{e\}}$ is distributed as A. We will then refer to $\mathscr{E}$ as the random environment.
For convenience, we add a parent e^* to the root e and we introduce the $\mathbb{T} \cup \{e^*\}$-valued random

walk $\mathbb{X} := (X_j)_{j\in\mathbb{N}}$ reflected in e^* such that under the quenched probabilities $\{\mathbb{P}_z^{\mathscr{E}}; z \in \mathbb{T}\cup\{e^*\}\}$, the transition probabilities are given by: for any $x \in \mathbb{T}$

$$p^{\mathscr{E}}(x,x^*) = \frac{e^{-V(x)}}{e^{-V(x)} + \sum_{i=1}^{N_x} e^{V(x^i)}} \quad \text{and for all } 1 \le i \le N_x, \quad p^{\mathscr{E}}(x,x^i) = \frac{e^{-V(x^i)}}{e^{-V(x)} + \sum_{i=1}^{N_x} e^{-V(x^i)}}.$$

Otherwise, $p^{\mathscr{E}}(x,u) = 0$ and $p^{\mathscr{E}}(e^*,e) = 1$. Let $\mathbb{P}^{\mathscr{E}} := \mathbb{P}_e^{\mathscr{E}}$ and we finally define the following annealed probabilities

$$\mathbb{P}(\cdot) := \mathbf{E}[\mathbb{P}^{\mathscr{E}}(\cdot)] \quad \text{and} \quad \mathbb{P}^*(\cdot) := \mathbf{E}^*[\mathbb{P}^{\mathscr{E}}(\cdot)].$$

R. Lyons and R. Pemantle [LP92] initiated the study of the randomly biased random walk $\mathbb{X}$. When, for all $x \in \mathbb{T}$, $V(x) = \log\lambda$ for a some constant $\lambda > 0$, the walk $\mathbb{X}$ is known as the λ-biased random walk on $\mathbb{T}\cup\{e^*\}$ and was first introduced by R. Lyons (see [Lyo90] and [Lyo92]). The λ-biased random walk is transient unless the bias is strong enough: if $\lambda \ge \mathbf{E}[N]$ then, $\mathbf{P}^*$-almost surely, $\mathbb{X}$ is recurrent (positive recurrent if $\lambda > \mathbf{E}[N]$). It is known since Y. Peres and O. Zeitouni [PZ06] that when $\lambda = \mathbf{E}[N]$, $\mathbb{X}$ is diffusive: there exists $\boldsymbol{\sigma}^2 \in (0,\infty)$ such that $(|X_{\lfloor nt\rfloor}|/\sqrt{\boldsymbol{\sigma}^2 n})_{t\ge 0}$ converges in law to a standard reflected brownian motion. R. Lyons, R. Pemantle and Y. Peres (see [LPP96b] and [LPP96a]), later joined by E. Aïdékon [Aïd14] studied the transient case and showed that $\mathbb{X}$ has a deterministic and positive speed $v_\lambda := \lim_{n\to\infty} |X_n|/n$. Moreover, the expression of v_λ is explicit.

When the bias is random, the behavior of $\mathbb{X}$ depends on the fluctuations of the following log-Laplace transform $\psi(t) := \log \mathbf{E}[\sum_{|x|=1} e^{-tV(x)}]$ which we assume to be well defined on $[0,1]$: if $\inf_{t\in[0,1]}\psi(t) > 0$, then $\mathbf{P}^*$-almost surely, $\mathbb{X}$ is transient and we refer to the work of E. Aïdékon [Aïd08] for this case. Otherwise, it is recurrent. More specifically, G. Faraud [Far11] proved that the random walk $\mathbb{X}$ is $\mathbf{P}^*$-almost surely positive recurrent either if $\inf_{t\in[0,1]}\psi(t) < 0$ or if $\inf_{t\in[0,1]}\psi(t) = 0$ and $\psi'(1) > 0$. It is null recurrent if $\inf_{t\in[0,1]}\psi(t) = 0$ and $\psi'(1) \le 0$. When $\psi'(1) = 0$, the largest generation reached by the walk $\mathbb{X}$ is of order $(\log n)^3$ and the walk is in the slow regime (see [HS07a] and [FHS11])

In the present paper, we focus on the null recurrent randomly biased walk $\mathbb{X}$ and assume

Assumption 1.

$$\inf_{t\in[0,1]} \psi(t) = \psi(1) = 0 \quad \textit{and} \quad \psi'(1) < 0. \tag{1}$$

Let us introduce

$$\kappa := \inf\{t > 1;\ \psi(t) = 0\}, \tag{2}$$

and assume $\kappa \in (1,\infty)$. Under (1) and some integrability conditions, it has been proven that $|X_n|$ and $\max_{1\le j\le n}|X_j|$ is of order $n^{1-1/\min(\kappa,2)}$(see [HS07b], [Far11], [AdR17] and [dR22]). In other words, the random walk $\mathbb{X}$ is sub-diffusive for $\kappa \in (1,2]$ and diffusive for $\kappa > 2$. In this paper, we put ourselves in the latter case.

We now define the range of the random walk $\mathbb{X}$. Let $T \in \mathbb{N}^*$. The range $\mathscr{R}_T$ of the random walk $\mathbb{X}$ is the set of distinct vertices of $\mathbb{T}$ visited by $\mathbb{X}$ up to the time T: if $\mathscr{L}_u^T := \sum_{j=1}^T \mathbb{1}_{\{X_j=u\}}$ denotes the local time of a vertex $u \in \mathbb{T}$ at time T then

$$\mathscr{R}_T = \{u \in \mathbb{T};\ \mathscr{L}_u^T \ge 1\}, \tag{3}$$

its cardinal is denoted by R_T and we also called it range. It has been proved by E. Aïdékon and L. de Raphélis that R_n is of order n (see [AdR17]). Moreover, $\mathscr{R}_n$ is a finite sub-tree of $\mathbb{T}$ and properly renormalized, it converges in law to a random real tree when n goes to infinity.

Introduce T^j, the j-th return time to e^*: $T^0 = 0$ and for any $j \geq 1$, $T^j = \inf\{i > T^{j-1};\ X_i = e^*\}$. Thanks to (31), we know that $T^{n^{1/2}}$ is of order n. We will be focusing our attention on the range $\mathscr{R}_{T^{n^{1/2}}}$ and we shall finally present an extension of the range $\mathscr{R}_n$.

Let us now define an extension of the volume $R_{T^{n^{1/2}}}$: for any integer $k \geq 2$ and any subset $\mathfrak{D}$ of $\mathbb{T}$ with cardinal $|\mathfrak{D}| \geq k$, let $\mathfrak{D}^{\times k} := \mathfrak{D} \times \cdots \times \mathfrak{D}$, introduce $\Delta^k := \{\boldsymbol{x} = (x^{(1)}, \ldots, x^{(k)}) \in \mathbb{T}^{\times k};\ \forall i_1 \neq i_2, x^{(i_1)} \neq x^{(i_2)}\}$ which is the set of k-tuples of distinct vertices of $\mathbb{T}$. Also introduce the set (assumed to be nonempty) $\Delta^k(\mathfrak{D}) := \Delta^k \cap \mathfrak{D}^{\times k}$ of k-tuples of distinct vertices of $\mathfrak{D}$. For any $n \in \mathbb{N}^*$, any subset $\mathscr{D}_n$ of $\mathscr{R}_{T^{n^{1/2}}}$ with cardinal D_n and for any function $f : \Delta^k \longrightarrow \mathbb{R}^+$, if $D_n \geq k$, we define the range $\mathscr{A}^k(\mathscr{D}_n, f)$ by

$$\mathscr{A}^k(\mathscr{D}_n, f) := \sum_{\boldsymbol{x} \in \Delta^k(\mathscr{D}_n)} f(\boldsymbol{x}). \tag{4}$$

Otherwise, $\mathscr{A}^k(\mathscr{D}_n, f)$ is equal to 0. The aim of studying the range $\mathscr{A}^k(\mathscr{D}_n, f)$ is to understand the interactions between the vertices in the tree $\mathscr{R}_{T^{n^{1/2}}}$ and most of all to give a description of the genealogy of the vertices in $\mathscr{R}_{T^{n^{1/2}}}$. Note that the range we investigate here differs from the range studied in [AK23], where authors focus on the interactions between the trajectories of the random walk $\mathbb{X}$ and on the trajectories of the underlying branching potential V.

1.2. Genealogy of uniformly chosen vertices in the range

For a nonempty subset $\mathscr{D}_n$ of $\mathscr{R}_{T^{n^{1/2}}}$, introduce the random variable $\mathscr{X}^n = (\mathscr{X}^{(1,n)}, \ldots, \mathscr{X}^{(k,n)})$ taking values in Δ^k with law defined by: for any $\boldsymbol{x} = (x^{(1)}, \ldots, x^{(k)}) \in \Delta^k$, if $\mathbb{P}^*(D_n \geq k) > 0$, then

$$\mathbb{P}^*\big(\mathscr{X}^n = \boldsymbol{x}\big) = \frac{1}{\mathbb{P}^*(D_n \geq k)} \mathbb{E}^*\Big[\frac{\mathbb{1}_{\{\boldsymbol{x} \in \Delta^k(\mathscr{D}_n)\}}}{|\Delta^k(\mathscr{D}_n)|} \mathbb{1}_{\{D_n \geq k\}}\Big], \tag{5}$$

and $\mathbb{P}^*(\mathscr{X}^n = \boldsymbol{x}) = 0$ otherwise. Note that $|\Delta^k(\mathscr{D}_n)| = D_n(D_n - 1) \times \cdots \times (D_n - k + 1)$ so the vertices $\mathscr{X}^{(1,n)}, \ldots, \mathscr{X}^{(k,n)}$ are nothing but k vertices picked uniformly and without replacement in the set $\mathscr{D}_n$. For any bounded function $f : \Delta^k \longrightarrow \mathbb{R}^+$, we actually have

$$\mathbb{E}^*[f(\mathscr{X}^n)] = \frac{1}{\mathbb{P}^*(D_n \geq k)} \mathbb{E}^*\Big[\frac{\mathscr{A}^k(\mathscr{D}_n, f)}{\mathscr{A}^k(\mathscr{D}_n, 1)} \mathbb{1}_{\{D_n \geq k\}}\Big], \tag{6}$$

thus making a link between the generalized range $\mathscr{A}^k(\mathscr{D}_n, \cdot)$ and the law of $\mathscr{X}^n$. Recall that our main interest is the genealogy of the k vertices $\mathscr{X}^{(1,n)}, \ldots, \mathscr{X}^{(k,n)}$ so let us define the genealogical tree of these k vertices. First, introduce the largest generation $M_n := \max_{x \in \mathscr{D}_n} |x|$ of the set $\mathscr{D}_n$. Recall that in the diffusive regime (see (1) and (2) with $\kappa > 2$), $\max_{x \in \mathscr{R}_{T^{n^{1/2}}}} |x|$, the largest generation of the tree $\mathscr{R}_{T^{n^{1/2}}}$, is of order $n^{1/2}$ when $n \to \infty$.
If $D_n \geq k$, we then define for any $m \in \{0, \ldots, M_n\}$ the equivalence relation $\sim_m$ on $\{1, \ldots, k\}$ by: $i_1 \sim_m i_2$ if and only if $\mathscr{X}^{(i_1,n)}$ and $\mathscr{X}^{(i_2,n)}$ share a common ancestor in generation m. We denote by $\pi_m^{k,n}$ the partition of $\{1, \ldots, k\}$ whose blocks are given by equivalent classes of the relation $\sim_m$. The process $(\pi_m^{k,n})_{0 \leq m \leq M_n}$ is called the genealogical tree of $\mathscr{X}^{(1,n)}, \ldots, \mathscr{X}^{(k,n)}$. Let $\mathscr{G}^{(i,n)} = |\mathscr{X}^{(i,n)}|$ be the generation of $\mathscr{X}^{(i,n)}$. By definition,

$$\pi_0^{k,n} = \{\{1, \ldots, k\}\} \ \text{ and } \ \pi_m^{k,n} = \{\{1\}, \ldots, \{k\}\} \ \text{ for any } \ m \in \{\max_{1 \leq i \leq k} \mathscr{G}^{(i,n)}, \ldots, M_n\}.$$

Replacing $\mathscr{R}_{T^{n^{1/2}}}$ by a regular Galton-Watson tree $\mathfrak{T}$ and $\mathscr{D}_n$ by $\{x \in \mathfrak{T};\ |x| = T\}$ (the T-th generation of $\mathfrak{T}$), the genealogy of k vertices $\mathscr{X}_T^{(1)}, \ldots, \mathscr{X}_T^{(k)}$ uniformly chosen in $\{x \in \mathfrak{T};\ |x| = T\}$

has been deeply studied for fixed T as well as for $T \to \infty$. First, when $k = 2$, K.B. Athreya [Ath12b] proved that when $\mathfrak{T}$ is super-critical (the mean of the reproduction law in larger than 1) $\mathscr{X}_T^{(1)}$ and $\mathscr{X}_T^{(2)}$ share a common ancestor for the last time in the remote past: if $\mathscr{M}_T := |\mathscr{X}_T^{(1)} \wedge \mathscr{X}_T^{(2)}|$ denotes the generation of the most recent common ancestor of $\mathscr{X}_T^{(1)}$ and $\mathscr{X}_T^{(2)}$ then $(\mathscr{M}_T)$ converges in law to a non-negative random variable depending on the reproduction law N when T goes to ∞. However, when $\mathfrak{T}$ is critical (the mean of the reproduction law is equal to 1), $\mathscr{X}_T^{(1)}$ and $\mathscr{X}_T^{(2)}$ share a common ancestor for the last time in the recent past: $(\mathscr{M}_T/T)$ converges in law to a $[0, 1]$-valued random variable which doesn't depend on the reproduction law N when T goes to ∞, see [Ath12a]. K.B. Athreya also dealt with the sub-critical case (the mean of the reproduction law is smaller than 1) in the latter paper and it is quite similar to the critical case. More recently S. Harris, S. Johnston and M. Roberts gave a full description of the genealogy of the vertices $\mathscr{X}_T^{(1)}, \dots, \mathscr{X}_T^{(k)}$ for a given integer $k \geq 2$ for both fixed T and $T \to \infty$, when the underlying process is a continuous-time Galton-Watson process (see [HJR17] and [Joh19]). See also [AD21] for a study of the genealogy of randomly chosen individuals when the underlying process is a continuous-state branching process.

Let us return to the case of the random walk in random environment. The generations at which the vertices $\mathscr{X}^{(1,n)}, \dots, \mathscr{X}^{(k,n)}$ are chosen have a major influence on their genealogical structure. The next four subsections are dedicated to the three regimes we observe: the tiny generations, the small generations, on which we spend most of our time and the critical generations. For the second regime, we are able to give a quite full description of the genealogy of $\mathscr{X}^{(1,n)}, \dots, \mathscr{X}^{(k,n)}$, displaying five examples we believe to be relevant, see subsection 1.4, and a general result is proven in 1.5. Finally, we show that we can easily extend our results on $\mathscr{R}_{Tn^{1/2}}$ to the range up to the time n.

1.3. The tiny generations

Recall that $\psi(t) = \log \mathbf{E}[\sum_{|x|=1} e^{-tV(x)}]$ and introduce $\tilde{\gamma} := \sup\{a \in \mathbb{R};\ \inf_{t \geq 0}(\psi(-t) - at) > 0\}$. By tiny generations, we mean those of order ℓ_n where $\ell_n \to \infty$ when $n \to \infty$ and $\ell_n \leq G \log n$ with $G \in (0, (2\tilde{\gamma})^{-1})$. The fact is that for these generations, the random environment has a uniform impact. Indeed, P. Andreoletti and P. Debs proved in [AD14] that with high probability, $\mathscr{R}_n = \{x \in \mathbb{T};\ |x| \leq G \log n\}$ for all $G \in (0, (2\tilde{\gamma})^{-1})$. Moreover, the value $(2\tilde{\gamma})^{-1}$ is optimal: if G_n denotes the largest generation entirely visited by the random walk $\mathbb{X}$ up to the time n, then $\mathbb{P}^*$-almost surely

$$\frac{G_n}{\log n} \underset{n \to \infty}{\longrightarrow} \frac{1}{2\tilde{\gamma}}.$$

For this case, we are therefore capable of giving a description of the genealogy of $k \geq 2$ vertices uniformly chosen by adapting the works of S. Harris, S. Johnston and M. Roberts to discrete super-critical Galton-Watson trees.

1.4. The small generations: examples

In this subsection, we focus on the small generations of the tree $\mathscr{R}_{n^{1/2}}$ where we recall that T^j is the j-th return time to e^*: $T^0 = 0$ and for any $j \geq 1$, $T^j = \inf\{i > T^{j-1};\ X_i = e^*\}$. Let $(\mathfrak{L}_n)$ be a sequence of positive integers such that $\mathfrak{L}_n \geq \delta_0^{-1} \log n$ (see Lemma 3.4 for the definition of δ_0) but smaller than the typical generations of the diffusive random walk $\mathbb{X}$:

Assumption 2 (The small generations)**.** *Let $(\Lambda_i)_{i\in\mathbb{N}}$ be the sequence of functions defined recursively by: for all $t>0$, $\Lambda_0(t)=t$ and for any $i\in\mathbb{N}^*$, $\Lambda_{i-1}(t)=e^{\Lambda_i(t)}$. There exists $l_0\in\mathbb{N}$ such that*

$$\lim_{n\to\infty}\frac{\mathfrak{L}_n}{n^{1/2}}\Lambda_{l_0}(\mathfrak{L}_n)=0. \tag{7}$$

Assumption 2 ensures that $\mathfrak{L}_n/n^{1/2}$, renormalized by a sequence that grows very slowly, goes to 0 when n goes to ∞. Note that when $k=2$, it is enough to assume that $\mathfrak{L}_n=o(n^{1/2})$.
Let (ℓ_n) be a sequence of positive integers such that $\delta_0^{-1}\log n\le\ell_n\le\mathfrak{L}_n$ and introduce the set

$$\mathscr{D}_n:=\{x\in\mathscr{R}_{T^{n^{1/2}}};\ \ell_n\le|x|\le\mathfrak{L}_n\},$$

with high $\boldsymbol{L}_n-1$ where $\boldsymbol{L}_n:=\mathfrak{L}_n-\ell_n+1$. Recall that D_n is the cardinal of $\mathscr{D}_n$. Note that $\lim_{n\to\infty}\mathbb{P}^*(D_n\ge k)=1$ so we will refer to the set $\{D_n\ge k\}$ only if necessary.
For any $m\in\mathbb{N}$, recall that $\mathbb{T}_m=\{x\in\mathbb{T};\ |x|=m\}$ be the m-th generation of the tree $\mathbb{T}$ and let $\Delta_m^j:=\Delta^j(\mathbb{T}_m)$. In addition, we also require the following technical assumption.

Assumption 3. *There exists $\delta_1>0$ such that $\psi(t)<\infty$ for all $t\in[1-\delta_1,\lceil\kappa+\delta_1\rceil]$ and for all $1\le j\le\lceil\kappa+\delta_1\rceil$, for all $\boldsymbol{\beta}=(\beta_1,\dots,\beta_j)\in(\mathbb{N}^*)^{\times j}$ such that $\sum_{i=1}^j\beta_i\le\lceil\kappa+\delta_1\rceil$*

$$c_j(\boldsymbol{\beta}):=\mathbf{E}\Big[\sum_{\boldsymbol{x}\in\Delta_1^j}e^{-\langle\boldsymbol{\beta},V(\boldsymbol{x})\rangle_j}\Big]<\infty, \tag{8}$$

where $\langle\boldsymbol{\beta},V(\boldsymbol{x})\rangle_j:=\sum_{i=1}^j\beta_iV(x^{(i)})$.

The next assumption is an ellipticity condition.

Assumption 4. *There exists $\mathfrak{h}>0$ such that*

$$\mathbf{P}\big(\inf_{x\in\mathbb{T}}(V(x)-V(x^*))\ge-\mathfrak{h}\big)=1. \tag{9}$$

Remark 1. *Although we assumed $k\ge2$, the case $k=1$, that is to the say the volume $D_n=\sum_{\ell_n\le|z|\le\mathfrak{L}_n}\mathbb{1}_{\{z\in\mathscr{R}_{T^{n^{1/2}}}\}}$ of the regular range $\mathscr{D}_n$ is interesting. The convergence of $(D_n/(n^{1/2}\boldsymbol{L}_n))_n$ does not requires all the previous assumptions and holds for $\kappa>2$. However, since it is an easy consequence of Theorem 1.6 with $k=2$ and $f=1$, we state the following result:*

Theorem 1.1. *Let $\kappa>4$. Under the assumptions 1, 2, 3 and 4, in $\mathbb{P}^*$-probability*

$$\frac{1}{n^{1/2}\boldsymbol{L}_n}D_n\underset{n\to\infty}{\longrightarrow}c_\infty W_\infty,$$

where c_∞ is defined in (26), W_∞ is the limit of the $\mathscr{F}_n:=\sigma((\mathbb{T};V(x),|x|\le n))$ additive martingale $W_n=\sum_{|x|=n}e^{-V(x)}$ and according to [Big77], $\mathbf{P}^*(W_\infty>0)=0$. Moreover, $\lim_{t\to\infty}t^\kappa\mathbf{P}(W_\infty>t)$ exists (see [Liu00]).
In particular, if $R_n(\ell)=\sum_{|z|=\ell}\mathbb{1}_{\{z\in\mathscr{R}_{T^{n^{1/2}}}\}}$ denotes the volume of the ℓ-th generation of the range $\mathscr{R}_{T^{n^{1/2}}}$ and $\log n=o(\mathfrak{L}_n)$, then both $(R_n(\mathfrak{L}_n)/n^{1/2})$ and $(\sum_{\ell=\delta_0^{-1}\log n}^{\mathfrak{L}_n}R_n(\ell)/(n^{1/2}\mathfrak{L}_n))$ converge in $\mathbb{P}^*$-probability to $c_\infty W_\infty$.

In view of Theorem 1.1, we deduce that whenever $\mathfrak{L}_n$ is large enough but not to close to the largest generation of the tree $\mathscr{R}_{T^{n^{1/2}}}$, the range $R_n(\mathfrak{L}_n)$ is of order $n^{1/2}$. Moreover, $\boldsymbol{L}_n-1$ denotes the height of the set $\mathscr{D}_n$ in the tree $\mathscr{R}_{T^{n^{1/2}}}$ and the volume of $\mathscr{D}_n$ behaves like $\boldsymbol{L}_n\times R_n(\mathfrak{L}_n)$.

The following theorems are composed of two parts: the first part will be a convergence of the range $\mathscr{A}_n^k(f)$ for a given function f and the second part will be an application of this convergence to the genealogy of the vertices $\mathscr{X}^{(1,n)}, \dots, \mathscr{X}^{(k,n)}$.

In the second example, we present a range such that for a k-tuple $\boldsymbol{x} \in \Delta^k$, some of the vertices are free while others are obliged to interact with each other. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in (\mathbb{N}^*)^{\times k}$ and introduce

$$f_{\boldsymbol{\lambda}}(x^{(1)}, \dots, x^{(k)}) := \prod_{i=2}^{k} \mathbb{1}_{\{|x^{(i-1)} \wedge x^{(i)}| < \lambda_i\}}.$$

Note that there is no constraint between $x^{(i_1)}$ and $x^{(i_2)}$ if $i_1 \notin \{i_1 - 1, i_1 + 1\}$.

Theorem 1.2. *Let $k \geq 2$ and assume $\kappa > 2k$. Under the assumptions 1, 2, 3 and 4, in $\mathbb{P}^*$-probability*

$$\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k} \mathscr{A}^k(\mathscr{D}_n, f_{\boldsymbol{\lambda}}) \underset{n\to\infty}{\longrightarrow} (c_\infty)^k \mathscr{A}_\infty^k(f_{\boldsymbol{\lambda}}),$$

where $\mathscr{A}_\infty^k(f_{\boldsymbol{\lambda}}) = \lim_{l\to\infty} \sum_{\boldsymbol{x}\in\Delta_l^k} e^{-V(x^{(1)})} \prod_{i=2}^k e^{-V(x^{(i)})} \mathbb{1}_{\{|x^{((i-1))} \wedge x^{(i)}| < \lambda_i\}}$ and this limit holds in $L^2(\mathbf{P}^*)$.

In the next example, we are interested in the number of vertices of $\mathscr{D}_n$ sharing a common ancestor for the last time at the bottom of the tree $\mathscr{R}_{T^{n^{1/2}}}$. Let $k \geq 2$ be an integer. For any $\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$, let $\mathscr{S}^k(\boldsymbol{x})$ be the first generation at which none of $x^{(1)}, \dots, x^{(k)}$ share a common ancestor:

$$\mathscr{S}^k(\boldsymbol{x}) := \min\{m \geq 1;\ \forall\, i_1 \neq i_2,\ |x^{(i_1)} \wedge x^{(i_2)}| < m\}, \tag{10}$$

where we recall that $|x^{(i_1)} \wedge x^{(i_2)}|$ is the most recent common ancestor of $x^{(i_1)}$ and $x^{(i_2)}$. For any $m \in \mathbb{N}^*$, introduce $\mathscr{C}_m^k := \{\boldsymbol{x} \in \Delta^k;\ \mathscr{S}^k(\boldsymbol{x}) \leq m\}$ (see Figure 1).
Let us also introduce the coalescent times (or split times) of the vertices $\mathscr{X}^{(1,n)}, \dots, \mathscr{X}^{(k,n)}$, uniformly chosen in the set $\mathscr{D}_n$. For a given partition $\boldsymbol{\pi}$ of $\{1, \dots, k\}$, we denote by $|\boldsymbol{\pi}|$ the total number of blocks of $\boldsymbol{\pi}$. Define the coalescent times by: $\mathscr{S}_0^{k,n} := 0$ and for all $j \in \mathbb{N}^*$, $k \geq 2$

$$\mathscr{S}_j^{k,n} := \min\left\{m \geq \mathscr{S}_{j-1}^{k,n};\ |\pi_m^{k,n}| > |\pi_{\mathscr{S}_{j-1}^{k,n}}^{k,n}| \wedge (k-1)\right\}. \tag{11}$$

Note that there exists $\mathscr{J}^{k,n} \in \mathbb{N}$ such that for any $j \geq \mathscr{J}^{k,n}$, $\mathscr{S}_j^{k,n} = \mathscr{S}^k(\mathscr{X}^{(n)})$ and by definition, $2 \leq |\{\mathscr{S}_j^n;\ j \in \mathbb{N}\}| \leq k$. One can notice that seen backwards in time, each random time $\mathscr{S}_j^{k,n} - 1$ with $0 < j \leq \mathscr{J}^{k,n}$ corresponds to a generation at which two or more vertices among $\mathscr{X}^{(1,n)}, \dots, \mathscr{X}^{(k,n)}$ share a common ancestor for the first time. $\mathscr{J}_j^{k,n}$ is usually referred to as the j-th split time while $\mathscr{J}_{\mathscr{J}^{k,n}-j+1}^{k,n}$ is the j-th coalescent time.

It appears that the number of vertices visited by the random walk $\mathbb{X}$ belonging to $\mathscr{C}_m^k$ is large and as a consequence, the sequence of random times $(\mathscr{S}^k(\mathscr{X}^{(n)}) = \inf\{m \geq 1;\ \pi_m^{k,n} = \{\{1\}, \dots, \{k\}\}\})_n$ converges in law.

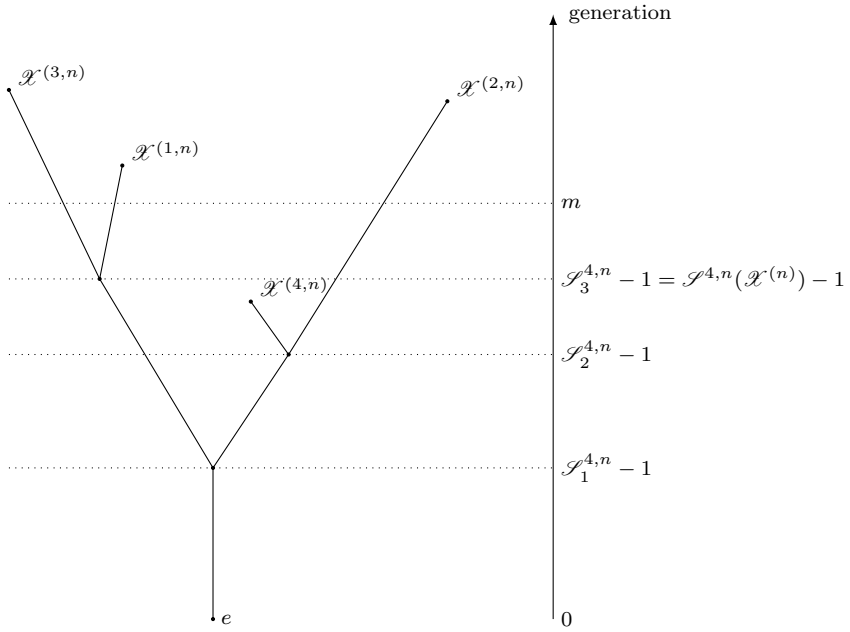

Figure 1: An example of four vertices belonging to $\mathscr{C}_m^4$ together with their three coalescent times.

Theorem 1.3. *Let $k \geq 2$. Assume that $\kappa > 2k$ and for any $m \in \mathbb{N}^*$, $\boldsymbol{x} \in \Delta^k$, $f_m(\boldsymbol{x}) = \mathbb{1}_{\mathscr{C}_m^k}(\boldsymbol{x})$. Recall that $\mathscr{A}^k(\mathscr{D}_n, f_m)$ is the number of k-tuples $\boldsymbol{x}$ of distinct vertices of $\mathscr{D}_n$ such that $\mathscr{S}^k(\boldsymbol{x}) \leq m$. Under the assumptions 1, 2, 3 and 4*

1. *in $\mathbb{P}^*$-probability*

$$\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_n, f_m) \underset{n\to\infty}{\longrightarrow} (c_\infty)^k \mathscr{A}_\infty^k(f_m),$$

   where $\mathscr{A}_\infty^k(f_m)$ is defined in (23). Note that $\lim_{m\to\infty} \mathscr{A}_\infty^k(f_m) = (W_\infty)^k$ in $L^2(\mathbf{P}^*)$.
2. *Moreover, the sequence of random times $(\mathscr{S}^k(\mathscr{X}^{(n)}))$ converges in law, under $\mathbb{P}^*$: for any $m \in \mathbb{N}^*$*

$$\mathbb{P}^*\big(\mathscr{S}^k(\mathscr{X}^{(n)}) \leq m\big) \underset{n\to\infty}{\longrightarrow} \mathbf{E}^*\Big[\frac{\mathscr{A}_\infty^k(f_m)}{(W_\infty)^k}\Big]. \tag{12}$$

The convergence in (12) is somewhat reminiscent of the result of K.B Athreya ([Ath12b], Theorem 2) for a super-critical Galton-Watson tree stated earlier: each coalescence occurs in a generation close to the root.

In the following result, we compute the law of $\pi^{k,n}$. Before that, we add, for convenience, a collection $\{e^{(i)};\ i \in \mathbb{N}^*\}$ of distinct leafs in the generation 0. Let $q \geq 2$ be an integer and $\boldsymbol{\pi}$ be a partition of $\{1, \ldots, q\}$. For any $m \in \mathbb{N}^*$, define the set $\Upsilon_{m,\boldsymbol{\pi}}$ by: $\boldsymbol{x} = (x^{(1)}, \ldots, x^{(q)}) \in \Upsilon_{m,\boldsymbol{\pi}}$ if and only if $\boldsymbol{x} \in \Delta^q$ and

$$\forall \boldsymbol{B} \in \boldsymbol{\pi}, \forall i_1, i_2 \in \boldsymbol{B} : (x^{(i_1)})_m = (x^{(i_2)})_m,$$

and if $|\boldsymbol{\pi}| \geq 2$

$$\forall \boldsymbol{B} \neq \tilde{\boldsymbol{B}} \in \boldsymbol{\pi}, \forall i_1 \in \boldsymbol{B}, i_2 \in \tilde{\boldsymbol{B}} : (x^{(i_1)})_m \neq (x^{(i_2)})_m,$$

where we recall that, when $|x^{(i)}| \geq m$, $(x^{(i)})_m$ denotes the ancestor of $x^{(i)}$ in generation m. Otherwise, if $|x^{(i)}| < m$, we set $(x^{(i)})_m := e^{(i)}$ so $\Upsilon_{m,\boldsymbol{\pi}}$ is well defined.

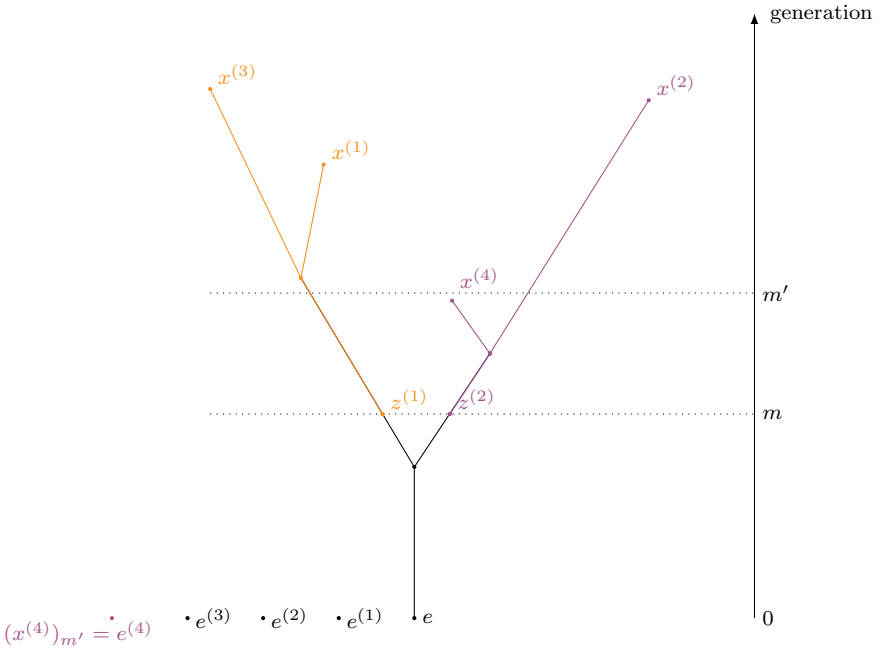

Figure 2: In the present illustration, the 4-tuple of vertices $(x^{(1)}, x^{(2)}, x^{(3)}x^{(4)})$ belongs to $\Upsilon_{m,\boldsymbol{\pi}}$ with $\boldsymbol{\pi} = \{\{1,3\},\{2,4\}\}$, since $z^{(1)} = (x^{(1)})_m = (x^{(3)})_m, z^{(2)} = (x^{(2)})_m = (x^{(4)})_m$ and $z^{(1)} \neq z^{(2)}$. However, it doesn't belong to $\Upsilon_{m',\boldsymbol{\pi}}$.

Now, let $1 \leq d < q$ be two integers. A collection $(\Xi_i)_{0\leq i\leq d}$ of partitions of $\{1,\dots,q\}$ is said to be increasing if it satisfies $\Xi_0 = \{\{1,\dots,q\}\}$, $\Xi_d = \{\{1\},\dots,\{q\}\}$ and for all $i \in \{1,\dots,d\}$, $|\Xi_{i-1}| < |\Xi_i|$, where we recall that $|\Xi_i|$ is the total number of blocks of the partition Ξ_i. For $p \in \{1,\dots,d\}$, the j-th block $\boldsymbol{B}_j^{p-1}$ of the partition Ξ_{p-1} (blocks are ordered by their least element) is the union of $b_{p-1}(\boldsymbol{B}_j^{p-1}) \geq 1$ (we will write $b_{p-1}(\boldsymbol{B}_j)$ instead) block(s) $\boldsymbol{B}_{l_1}^p,\dots,\boldsymbol{B}_{l_{b_{p-1}(\boldsymbol{B}_j)}}^p$, $1 \leq l_1 < \ldots < l_{b_{p-1}(\boldsymbol{B}_j)} \leq |\Xi_p|$, of the partition Ξ_p and for any $i \in \{1,\dots,l_{b_{p-1}(\boldsymbol{B}_j)}\}$, define

$$\beta_{j,i}^{p-1} := |B_{l_i}^p|, \tag{13}$$

be the cardinal of the block $B_{l_i}^p$.

Let $(\Xi_i)_{0\leq i\leq d}$ be an increasing collection of partitions of $\{1,\dots,q\}$ and let $\boldsymbol{t} = (t_1,\dots,t_d) \in \mathbb{N}^{\times d}$ such that $t_1 < \cdots < t_d$. Introduce the set $\Gamma_{\boldsymbol{t},\Xi}^i := \Upsilon_{t_i-1,\Xi_{i-1}} \cap \Upsilon_{t_i,\Xi_i}$. We then define the function $f_{\boldsymbol{t},\Xi}^d$ by: for all $\boldsymbol{x} \in \Delta^q$

$$f_{\boldsymbol{t},\Xi}^d(\boldsymbol{x}) = \prod_{i=1}^d \mathbb{1}_{\Gamma_{\boldsymbol{t},\Xi}^i}(\boldsymbol{x}). \tag{14}$$

The function defined in (14) plays a key role in our study: $f_{\boldsymbol{t},\Xi}^d(\boldsymbol{x})$ characterizes the genealogy of $\boldsymbol{x} := (x^{(1)},\dots,x^{(q)})$. Indeed, for any $i \in \{1,\dots,d\}$, the partition Ξ_i corresponds to the i-th

generation of the genealogical tree of $x^{(1)},\dots,x^{(q)}$ while t_i-1 denotes the i-th generation at which at least two branches of this genealogical tree split (t_i-1 therefore corresponds to a coalescent/split time, see Figure 3 for instance). We are now ready to state our result:

Theorem 1.4. *Let $k\geq 2$ and assume that $\kappa>2k$. Under the assumptions 1, 2, 3 and 4, for any $\ell\in\mathbb{N}^*$ such that $\ell<k$, any $\boldsymbol{s}=(s_1,\dots,s_\ell)\in\mathbb{N}^{\times\ell}$ such that $s_1<\cdots<s_\ell$ and any increasing collection $\Pi=(\boldsymbol{\pi}_i)_{0\leq i\leq\ell}$ of partitions of $\{1,\dots,k\}$*

1. *in $\mathbb{P}^*$-probability*

$$\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_n,f^\ell_{\boldsymbol{s},\Pi})\underset{n\to\infty}{\longrightarrow}(c_\infty)^k\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi}). \tag{15}$$

2. *Moreover, for any non-negative integers $m_0<m_1<\cdots<m_\ell$*

$$\mathbb{P}^*(\pi^{k,n}_{m_0}=\boldsymbol{\pi}_0,\dots,\pi^{k,n}_{m_\ell}=\boldsymbol{\pi}_\ell)\underset{n\to\infty}{\longrightarrow}\mathbf{E}^*\Big[\frac{1}{(W_\infty)^k}\sum_{\substack{\boldsymbol{s}=(s_1,\dots,s_\ell)\\ m_{i-1}<s_i\leq m_i}}\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})\Big], \tag{16}$$

where $\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})$ is the limit in $L^2(\mathbf{P}^*)$ of the martingale $(\mathscr{A}^k_l(f^\ell_{\boldsymbol{s},\Pi}))_l$, satisfying

$$\sum_{\Pi\ \textit{increasing}}\ \sum_{\substack{\boldsymbol{s}=(s_1,\dots,s_\ell)\\ m_{i-1}<s_i\leq m_i}}\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})=(W_\infty)^k,$$

and

$$\mathbf{E}^*\big[\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})\big]=e^{\psi(k)}\prod_{i=1}^{\ell}\prod_{j=1}^{|\boldsymbol{\pi}_{i-1}|}c_{b_{i-1}(\boldsymbol{B}_j)}(\boldsymbol{\beta}^{i-1}_j)\prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_i\\ |\mathfrak{B}|\geq 2}}e^{s^*_{i+1}\psi(|\mathfrak{B}|)}, \tag{17}$$

with $s^*_{i+1}=s_{i+1}-s_i-1$, $s^*_{\ell+1}=1$, $\boldsymbol{\beta}^p_j:=(\beta^p_{j,1},\dots,\beta^p_{j,b_p(\boldsymbol{B}_j)})$ (see (13)). We also use the convention $\prod_\varnothing=1$ and see the assumption 3 for the definition of $c_l(\boldsymbol{\beta})$.

Remark 2 (An hereditary character)**.** *There is an hereditary character hidden in the previous formula (17) due to the random environment. The fact is, unlike the case of regular super-critical Galton-Watson trees depending on $(b_i(\boldsymbol{B});\ \boldsymbol{B}\in\boldsymbol{\pi}_i,\ 0\leq i\leq\ell-1)$ (see [Joh19], Theorem 3.5), the limit law of the present genealogical tree depends on the collection $(\boldsymbol{\beta}^i_j;\ 1\leq i\leq\ell,\ 1\leq j\leq|\boldsymbol{\pi}_{i-1}|)$ and on $(|\mathfrak{B}|;\ \mathfrak{B}\in\boldsymbol{\pi}_i,\ 1\leq i\leq\ell)$, making a huge difference. Indeed, by definition, the latter take more account of the genealogical structure than $(b_i(\boldsymbol{B});\ \boldsymbol{B}\in\boldsymbol{\pi}_i,\ 0\leq i\leq\ell-1)$. For instance, let $k=4$, $\ell=3$ and define the increasing collection of partitions $\Pi=(\boldsymbol{\pi}_i)_{1\leq i\leq\ell}$ by $\boldsymbol{\pi}_3=\{\{1\},\{2\},\{3\},\{4\}\}$, $\boldsymbol{\pi}_2=\{\{1,3\},\{2\},\{4\}\}$, $\boldsymbol{\pi}_1=\{\{1,3\},\{2,4\}\}$ and $\boldsymbol{\pi}_0=\{1,2,3,4\}$. We have $\boldsymbol{\beta}^3_1=(1,1)$, $\boldsymbol{\beta}^3_2=1$, $\boldsymbol{\beta}^3_1=1$; $\boldsymbol{\beta}^2_1=2$, $\boldsymbol{\beta}^2_2=(1,1)$; $\boldsymbol{\beta}^1_1=(2,2)$ and thanks to (17), for any $\boldsymbol{t}=(t_1,t_2,t_3)\in\mathbb{N}^{\times 3}$ such that $t_1<t_2<t_3$*

$$\mathbf{E}^*[\mathscr{A}^4_\infty(f^3_{\boldsymbol{t},\Pi})]=\mathbf{E}\Big[\sum_{|x|=1}e^{-2V(x)}\Big]\mathbf{E}\Big[\sum_{\substack{x\neq y\\ |x|=|y|=1}}e^{-V(x)-V(y)}\Big]^2\mathbf{E}\Big[\sum_{\substack{x\neq y\\ |x|=|y|=1}}e^{-2V(x)-2V(y)}\Big]\times e^{t^*_3\psi(2)+2t^*_2\psi(2)+\psi(4)}.$$

Also introduce the increasing collection of partitions $\Pi'=(\boldsymbol{\pi}'_i)_{1\leq i\leq\ell}$ such that $\boldsymbol{\pi}'_3=\boldsymbol{\pi}_3$, $\boldsymbol{\pi}'_2=\boldsymbol{\pi}_2$, $\boldsymbol{\pi}'_1=\{\{1,3,4\},\{2\}\}$ and $\boldsymbol{\pi}'_0=\boldsymbol{\pi}_0$. We have $\boldsymbol{\beta}^3_1=(1,1)$, $\boldsymbol{\beta}^3_2=1$, $\boldsymbol{\beta}^3_3=1$; $\boldsymbol{\beta}^2_1=(2,1)$, $\boldsymbol{\beta}^2_2=1$;

and thanks to (17)*, for any* $\boldsymbol{t} = (t_1, t_2, t_3) \in \mathbb{N}^{\times 3}$ *such that* $t_1 < t_2 < t_3$

$$\mathbf{E}^*[\mathscr{A}_\infty^4(f^3_{\boldsymbol{s},\Pi'})] = \mathbf{E}\Big[\sum_{\substack{x\neq y\\ |x|=|y|=1}} e^{-V(x)-V(y)}\Big]\mathbf{E}\Big[\sum_{\substack{x\neq y\\ |x|=|y|=1}} e^{-2V(x)-V(y)}\Big]\mathbf{E}\Big[\sum_{\substack{x\neq y\\ |x|=|y|=1}} e^{-3V(x)-V(y)}\Big] \times e^{s_3^*\psi(2)+s_2^*\psi(3)+\psi(4)}.$$

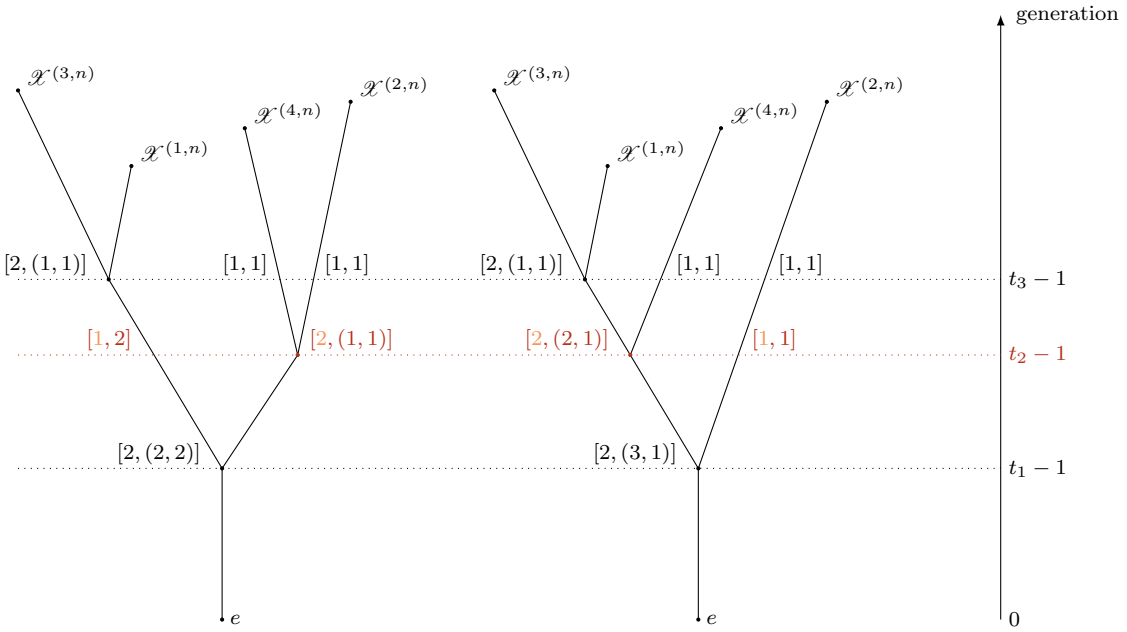

Figure 3: An example of a genealogical tree of the four vertices $\mathscr{X}^{(1,n)}$, $\mathscr{X}^{(2,n)}$, $\mathscr{X}^{(3,n)}$, $\mathscr{X}^{(4,n)}$ associated to Π (left) and associated to Π' (right). $[1,2]$ means that $b_{1,\Pi}(\{1,3\}) = 1$ and $\boldsymbol{\beta}_1^{2,\Pi} = 2$, $[2,(1,1)]$ means that $b_{2,\Pi}(\{2,4\}) = 2$ and $\boldsymbol{\beta}_2^{2,\Pi} = (1,1)$. In the same way, $[2,(2,1)]$ means that $b_{1,\Pi'}(\{1,3\}) = 2$ and $\boldsymbol{\beta}_1^{2,\Pi'} = (2,1)$, $[1,1]$ means that $b_{2,\Pi'}(\{2\}) = 1$ and $\boldsymbol{\beta}_2^{2,\Pi'} = 1$.

The difference between these two examples is that in the second one, we ask $(\mathscr{X}^{(4,n)})_{t_1-1}$ *(the ancestor of* $\mathscr{X}^{(4,n)}$ *of in generation* t_1-1*) to belong to both genealogical line* $[\![(\mathscr{X}^{(1,n)})_{t_1-1}, \mathscr{X}^{(1,n)}]\!]$ *and* $[\![(\mathscr{X}^{(3,n)})_{t_1-1}, \mathscr{X}^{(3,n)}]\!]$*. This constraint can be satisfied only if the vertex* $(\mathscr{X}^{(4,n)})_{t_1-1}$ *is often visited by the random walk* $\mathbb{X}$*, inducing more dependence in the trajectories of* $\mathbb{X}$ *thus giving the factor* $t_2^*\psi(3)$ *instead of* $2t_2^*\psi(2) = t_2^*\psi(2) + t_2^*\psi(2)$*.*
However, in the case of regular super-critical Galton-Watson trees, the events $\cap_{i=0}^3\{\boldsymbol{\pi}_i\}$ *and* $\cap_{i=0}^3\{\tilde{\boldsymbol{\pi}}_i\}$ *have the same probability under the limit law of the genealogical tree. Indeed, one can notice (see Figure 3) that for all* $i \in \{1,2,3\}$ *and all* $j \in \{1,\dots,|\boldsymbol{\pi}_i|\}$ *(*$|\boldsymbol{\pi}_i| = |\boldsymbol{\pi}'_i|$ *by definition),* $b_{i,\Pi}(\boldsymbol{B}_j) = b_{i,\Pi'}(\boldsymbol{B}_{\mathfrak{p}(j)})$ *for some permutation* $\mathfrak{p}$ *on* $[\![1,|\boldsymbol{\pi}_i|]\!]$*, but this not the case when replacing* $b_{i,\Pi}(\boldsymbol{B}_\cdot)$ *by* $\boldsymbol{\beta}_\cdot^{i,\Pi}$ *and* $b_{i,\Pi'}(\boldsymbol{B}_\cdot)$ *by* $\boldsymbol{\beta}_\cdot^{i,\Pi'}$*.*

Since all coalesences of the genealogical lines of $\mathscr{X}^{(1,n)},\dots,\mathscr{X}^{(k,n)}$ occur in the remote past with large probability, one could focus on this particular vertices of the tree $\mathscr{R}_{T^{n^{1/2}}}$. To do that, we pick a k-tuple $\mathscr{Y}^{(n)} = (\mathscr{Y}^{(1,n)},\dots,\mathscr{Y}^{(k,n)})$ uniformly in the set $\mathscr{D}_n^{\times k} \cap \mathscr{C}_{\mathfrak{s}}^k$ for $\mathfrak{s} \in \mathbb{N}^*$. In other words, the law of $\mathscr{Y}^{(n)}$ is given in (5) by replacing $\Delta^k(\mathscr{D}_n)$ with $\Delta^k(\mathscr{D}_n) \cap \mathscr{C}_{\mathfrak{s}}^k$. We keep the same

notations for $\mathscr{Y}^{(n)}$ as for $\mathscr{X}^{(n)}$.
The last example gives the law of the coalescent times $(\mathscr{S}^{k,n})_{1\le j\le \mathscr{J}^{k,n}}$ of $\mathscr{Y}^{(1,n)},\dots,\mathscr{Y}^{(k,n)}$:

Theorem 1.5. *Let $k\ge 2$ and assume that $\kappa>2k$. Let $1\le \ell<k$, $\mathfrak{s}\in\mathbb{N}^*$ be two integers, and $\boldsymbol{s}=(s_1,\dots,s_\ell)\in\mathbb{N}^{\times\ell}$ such that $s_1<\ldots<s_\ell\le\mathfrak{s}$. Assume that for all $\boldsymbol{x}\in\Delta^k$,*

$$F^\ell_{\boldsymbol{s}}(\boldsymbol{x})=\sum_{\Xi\ increasing} f^\ell_{\boldsymbol{s},\Xi}(\boldsymbol{x}),$$

where Ξ increasing means here that $\Xi=(\Xi_i)_{0\le i\le\ell}$ is an increasing collection of partitions of $\{1,\dots,k\}$. Under the assumptions 1, 2, 3 and 4,

1. *in $\mathbb{P}^*$-probability*

$$\frac{\mathscr{A}^k(\mathscr{D}_n,F^\ell_{\boldsymbol{s}})}{\mathscr{A}^k(\mathscr{D}_n,\mathbb{1}_{\mathscr{C}^k_{\mathfrak{s}}})}\mathbb{1}_{\{D_n\ge k\}}\underset{n\to\infty}{\longrightarrow}\frac{\mathscr{A}^k_\infty(F^\ell_{\boldsymbol{s}})}{\mathscr{A}^k_\infty(\mathbb{1}_{\mathscr{C}^k_{\mathfrak{s}}})}. \tag{18}$$

2. *Moreover*

$$\mathbb{P}^*(\mathscr{S}^{k,n}_1=s_1,\dots,\mathscr{S}^{k,n}_\ell=s_\ell,\mathscr{J}^{k,n}=\ell)\underset{n\to\infty}{\longrightarrow}\mathbf{E}^*\Big[\frac{\mathscr{A}^k_\infty(F^\ell_{\boldsymbol{s},\Xi})}{\mathscr{A}^k_\infty(\mathbb{1}_{\mathscr{C}^k_{\mathfrak{s}}})}\Big], \tag{19}$$

where $\mathscr{A}^k_\infty(F^\ell_{\boldsymbol{s},\Xi})$ is the limit in $L^2(\mathbb{P}^*)$ of the martingale $(\mathscr{A}^k_l(F^\ell_{\boldsymbol{s},\Xi}))_l$ and satisfying

$$\sum_{\ell=1}^{k-1}\sum_{\substack{\boldsymbol{s}=(s_1,\dots,s_\ell)\\ s_1<\cdots<s_\ell\le\mathfrak{s}}}\mathscr{A}^k_\infty(F^\ell_{\boldsymbol{s}})=\mathscr{A}^k_\infty(\mathbb{1}_{\mathscr{C}^k_{\mathfrak{s}}}).$$

1.5. The small generations: a general result

In this section, we present results for the range $\mathscr{A}^k(\mathscr{D}_n,f)$ with f satisfying a very natural heredity condition we will discuss later and including previous examples. First, recall that $\mathscr{D}_n=\{x\in\mathscr{R}_{T^{n^{1/2}}};\ \ell_n\le|x|\le\mathfrak{L}_n\}$ with $(\mathfrak{L}_n)$ a sequence of positive integers such that $\delta_0^{-1}\log n\le\mathfrak{L}_n\le n^{1/2}$ (see Lemma 3.4 for the definition of δ_0) and (ℓ_n) is a sequence of positive integers such that $\delta_0^{-1}\log n\le\ell_n\le\mathfrak{L}_n$. Then recall the definition of $\mathscr{A}^k(\mathscr{D}_n,f)$ defined in (4): if $D_n\ge k$

$$\mathscr{A}^k(\mathscr{D}_n,f):=\sum_{\boldsymbol{x}\in\Delta^k(\mathscr{D}_n)}f(\boldsymbol{x}),$$

with $\Delta^k(\mathscr{D}_n)=\{\boldsymbol{x}=(x^{(1)},\dots,x^{(k)})\in\mathscr{D}_n^{\times k};\ \forall i_1\ne i_2,x^{(i_1)}\ne x^{(i_2)}\}$ and equal to 0 otherwise. Although we obtain quite general results, we however require the following assumption on f: recall that for all $k\ge2$, $\boldsymbol{x}=(x^{(1)},\dots,x^{(k)})\in\Delta^k$, $\mathscr{C}^k_m=\{\boldsymbol{x}\in\Delta^k;\ \mathscr{S}^k(\boldsymbol{x})\le m\}$ where $\mathscr{S}^k(\boldsymbol{x})-1$ denotes the last generation at which two or more vertices among $x^{(1)},\dots,x^{(k)}$ share a common ancestor (see (10)). Assume

Assumption 5. *there exists $\mathfrak{g}\in\mathbb{N}^*$ such that for all integer $p\ge\mathfrak{g}$ and all $\boldsymbol{x}=(x^{(1)},\dots,x^{(k)})\in\Delta^k$, if $\min_{1\le i\le k}|x^{(i)}|\ge p$ and $\boldsymbol{x}\in\mathscr{C}^k_p$ then*

$$f((x^{(1)},\dots,x^{(k)}))=f\big((x^{(1)})_p,\dots,(x^{(k)})_p\big), \tag{20}$$

where we recall that $(x^{(i)})_p$ is the ancestor of $x^{(i)}$ in the generation p. In other words, we ask the constraint f to be hereditary from a given generation $\mathfrak{g}$.

Introduce the local time $\mathscr{L}^n := \sum_{j=1}^n \mathbb{1}_{\{X_j=e^*\}}$ of the parent e^* of the root e at time n. Recall that T^j is the j-th return time to e^*: $T^0 = 0$ and for any $j \geq 1$, $T^j = \inf\{i > T^{j-1};\ X_i = e^*\}$. Let $s \in \mathbb{N}^*$ and introduce $\mathscr{D}_{n,T^s} := \{x \in \mathscr{R}_{T^s};\ \ell_n \leq |x| \leq \mathfrak{L}_n\}$. We denote by $\mathfrak{E}^{k,s}$ the set defined by: for a given $\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$, $\boldsymbol{x} \in \mathfrak{E}^{k,s}$ if and only if the vertices of $x^{(1)}, \dots, x^{(k)}$ are visited during k distinct excursions before the instant T^s:

$$\mathfrak{E}^{k,s} := \bigcup_{\boldsymbol{j} \in [\![1,s]\!]_k} \bigcap_{i=1}^{k} \{\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k;\ \mathscr{L}^{T^{j_i}}_{x^{(i)}} - \mathscr{L}^{T^{j_i-1}}_{x^{(i)}} \geq 1\}, \tag{21}$$

where we denote by $[\![1,s]\!]_k$ the set of k-tuples $\boldsymbol{j}$ of $\{1, \dots, s\}$ such that for all $i_1 \neq i_2 \in \{1, \dots, s\}$, $j_{i_1} \neq j_{i_2}$. Our first proposition is a convergence of the range $\mathscr{A}^k(\mathscr{D}_{n,T^s}, f\mathbb{1}_{\mathfrak{E}^{k,s}})$ for any $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$, $\varepsilon_1 \in (0,1)$.

Proposition 1. *Let $k \geq 2$ and assume $\kappa > 2k$. Under the assumptions 1, 2, 3 and 4, if f satisfies the hereditary assumption 5 then for all $\varepsilon, \varepsilon_1 \in (0,1)$, $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$*

$$\mathbb{P}^*\Big(\Big|\frac{1}{(s\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_{n,T^s}, f\mathbb{1}_{\mathfrak{E}^{k,s}}) - (c_\infty)^k\mathscr{A}^k_\infty(f)\Big| > \varepsilon\Big) \underset{n\to\infty}{\longrightarrow} 0,$$

where $(S_i - S_{i-1})_{i\in\mathbb{N}^*}$ is a sequence of i.i.d real valued random variables such that $S_0 = 0$ and

$$\mathbf{E}[\boldsymbol{h}(S_1)] = \mathbf{E}\Big[\sum_{|x|=1} \boldsymbol{h}(V(x))e^{-V(x)}\Big], \tag{22}$$

and, in $L^2(\mathbf{P}^*)$

$$\mathscr{A}^k_\infty(f) := \lim_{l\to\infty} \mathscr{A}^k_l(f), \tag{23}$$

with $\mathscr{A}^k_l(f, \boldsymbol{\beta}) := \sum_{\boldsymbol{x}\in\Delta^k_l} f(\boldsymbol{x})e^{-\langle\boldsymbol{\beta}, V(\boldsymbol{x})\rangle_k}$, $\mathscr{A}_l(f) := \mathscr{A}_l(f, \mathbf{1})$ and $\mathbf{1} := (1, \dots, 1) \in \mathbb{N}^{\times k}$.

In the next proposition, we claim k-tuples in $\Delta^k \setminus \mathfrak{E}^{k,s}$ with $n^{1/2}/\varepsilon_1$ and $\varepsilon_1 \in (0,1)$, that is k-tuples of vertices such that at least two among them are visited during the same excursion above e^* and before T^s, have a minor contribution to the range $\mathscr{A}^k(\mathscr{D}_n, 1)$.

Proposition 2. *Let $\varepsilon \in (0,1)$, $k \geq 2$ and assume $\kappa > 2k$. Under the assumptions 1, 2, 3 and 4*

$$\mathbb{P}^*\Big(\sup_{s\leq n^{1/2}/\varepsilon_1} \mathscr{A}^k(\mathscr{D}_{n,T^s}, \mathbb{1}_{\Delta^k\setminus\mathfrak{E}^{k,s}}) > \varepsilon(n^{1/2}\boldsymbol{L}_n)^k\Big) \underset{n\to\infty}{\longrightarrow} 0 \tag{24}$$

We are now ready to state our main result:

Theorem 1.6. *Let $k \geq 2$ and assume $\kappa > 2k$. Under the assumptions 1, 2, 3 and 4, if f satisfies the hereditary assumption 5 then, in $\mathbb{P}^*$-probability*

$$\frac{\mathscr{A}^k(\mathscr{D}_n, f)}{(n^{1/2}\boldsymbol{L}_n)^k} \underset{n\to\infty}{\longrightarrow} (c_\infty)^k\mathscr{A}^k_\infty(f), \tag{25}$$

and if $g \not\equiv 0$ also satisfies assumption 5 then in $\mathbb{P}^*$-probability

$$\frac{\mathscr{A}^k(\mathscr{D}_n, f)}{\mathscr{A}^k(\mathscr{D}_n, g)}\mathbb{1}_{\{D_n\geq k\}} \underset{n\to\infty}{\longrightarrow} \frac{\mathscr{A}^k_\infty(f)}{\mathscr{A}^k_\infty(g)}, \tag{26}$$

where $\boldsymbol{L}_n = \mathfrak{L}_n - \ell_n + 1$, $c_\infty := \mathbf{E}[(\sum_{j\geq 0} e^{-S_j})^{-1}]$ *and* $\mathscr{A}^k_\infty(f)$ *is defined in* (23). *Note that a constraint satisfying assumption* 5 *doesn't have any influence on the normalization of the range. Moreover,* $\mathscr{A}^k(\mathscr{D}_n, f)$ *behaves like* $(\boldsymbol{L}_n \max_{x\in\mathscr{R}_{Tn^{1/2}}} |x|)^k$ *and the limiting value* $\mathscr{A}_\infty(f)$ *contains all the information about the interactions between the vertices of the tree.*
Since $\psi(2) < 0$, c_∞ *and* c_0 *are well defined in* $(0,\infty)$. *Indeed, the sequence* $((\sum_{j=0}^l e^{-S_j})^{-1})_{l\in\mathbb{N}}$ *is bounded and non-increasing and by Jensen inequality,* $1 \geq \mathbf{E}[(\sum_{j=0}^l e^{-S_j})^{-1}] \geq \mathbf{E}[(\sum_{j=0}^l e^{-S_j})]^{-1} = (\sum_{j=0}^l e^{j\psi(2)})^{-1} \geq 1 - e^{\psi(2)} > 0$.

We end this subsection by stating an extension of Theorem 1.6 to the range $\mathscr{R}_n$. Before that, introduce $\tilde{\mathscr{D}}_n := \{x \in \mathscr{R}_n;\ \ell_n \leq |x| \leq \mathfrak{L}_n\}$ with cardinal $\tilde{D}_n$.

Theorem 1.7. *Let* $k \geq 2$. *There exists a non-increasing sequence of positive integers* $(\mathfrak{q}_j)_j$, *satisfying* $\mathfrak{q}_j \in (0, 1/2)$ *and* $q_j \to 0$ *when* $j \to \infty$ *such that if* $\kappa > 2\xi k$ *for some integer* $\xi \geq 2$ *and* $\mathfrak{L}_n = o(n^{1/2-\mathfrak{q}_\xi})$, *then, in law, under* $\mathbb{P}^*$

$$\frac{\mathscr{A}^k(\tilde{\mathscr{D}}_n, f)}{(n^{1/2}\boldsymbol{L}_n)^k} \underset{n\to\infty}{\longrightarrow} \frac{\mathscr{A}^k_\infty(f)}{(W_\infty)^k}\big(c_\infty c_0^{1/2}|\mathscr{N}|\big)^k, \tag{27}$$

and if $g \not\equiv 0$ *also satisfies assumption* 5 *then in* $\mathbb{P}^*$*-probability*

$$\frac{\mathscr{A}^k(\tilde{\mathscr{D}}_n, f)}{\mathscr{A}^k(\tilde{\mathscr{D}}_n, g)} \mathbb{1}_{\{D_n \geq k\}} \underset{n\to\infty}{\longrightarrow} \frac{\mathscr{A}^k_\infty(f)}{\mathscr{A}^k_\infty(g)}, \tag{28}$$

where $c_0 := \mathbf{E}[\sum_{x\neq y;|x|=|y|=1} e^{-V(x)-V(y)}]/(1 - e^{\psi(2)})$ *and* $\mathscr{N}$ *is a standard Gaussian random variable.*
In particular, all the previous results on $\mathscr{D}_n$ *hold for* $\tilde{\mathscr{D}}_n$ *with* $\mathfrak{L}_n = o(n^{1/2-\mathfrak{q}_\xi})$.

1.6. Further discussion: the critical generations

The critical generations, that is to say of order $n^{1/2}$, correspond to the typical generations but also to the largest reached by the diffusive random walk $\mathbb{X}$ up to the time n. E. Aïdékon and L. de Raphélis [AdR17] showed that $n^{1/2}$ is also the right normalisation for the tree $\mathscr{R}_n$: in law, under $\mathbb{P}^*$

$$\frac{c_0^{1/2}}{n^{1/2}}\mathscr{R}_n \underset{n\to\infty}{\longrightarrow} \mathscr{T}_{|B|},$$

where for any $\mathfrak{c} > 0$, $\mathfrak{c}\mathscr{R}_T$ is tree $\mathscr{R}_T$ with edge lengths equal to $\mathfrak{c}$ and $\mathscr{T}_{|B|}$ is the real tree coded by the standard reflected Brownian motion $|B| = (|B_t|)_{t\in[0,1]}$ on $[0,1]$ (see [Gal06]). $\mathscr{T}_{|B|}$ is what we can call a Brownian forest thus suggesting that two vertices $\mathscr{X}^{(1,n)}$ and $\mathscr{X}^{(2,n)}$ chosen uniformly in the range $\mathscr{R}_n$ at a generation of order $n^{1/2}$ can share a common ancestor in both remote past and recent past. That is actually what is happening when considering two vertices $\tilde{\mathscr{X}}^{(1,n)}$ and $\tilde{\mathscr{X}}^{(2,n)}$ picked uniformly at generation $n^{1/2}$ in the tree $\mathscr{R}_{\mathfrak{t}_n}$, where $\mathfrak{t}_n$ is the $n^{1/2}$-th return time of $\mathbb{X}$ to e^* (which is quite similar to $\mathscr{R}_n$): let $\tilde{\mathscr{M}}_n$ be the most recent common ancestor of $\tilde{\mathscr{X}}^{(1,n)}$ and $\tilde{\mathscr{X}}^{(2,n)}$. First observe that

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty} \mathbb{P}^*(\tilde{\mathscr{M}}_n < 1/\varepsilon) > 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\limsup_{n\to\infty} \mathbb{P}^*(\varepsilon n^{1/2} \leq \tilde{\mathscr{M}}_n < n^{1/2}) > 0. \tag{29}$$

Moreover, coalescence can't occur anywhere else:

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} \mathbb{E}^*\Big[\frac{1}{n} \sum_{\substack{x\neq y\\ |x|=|y|=n^{1/2}}} \mathbb{1}_{\{x,y\in\mathscr{R}_{\mathfrak{t}_n},\ 1/\varepsilon\leq|x\wedge y|<\varepsilon n^{1/2}\}}\Big] = 0.$$

Although $\mathbb{T}$ is a super-critical Galton-Watson tree, the genealogy of $\mathscr{R}_{\mathfrak{t}_n}$ (or $\mathscr{R}_n$) is a mix of the super-critical case and the critical case for a regular Galton-Watson trees (see subsection 1.2).
The fact is using standard techniques for randomly biased random walks and branching random walks, we are able to deal with the quenched mean of $(D_{\mathfrak{t}_n})^{p_1}$ for $p_1 \leq \lfloor \kappa \rfloor$ and $(\mathscr{A}^2(\mathscr{D}_{\mathfrak{t}_n}, f))^{p_2}$ with $p_2 \leq \lfloor \kappa/2 \rfloor$ but not with the actual random variables.
The computation for any $m > 0$ and any $0 < a < b < 1$ of $\mathbb{P}^*(\tilde{\mathscr{M}}_n < m)$ and $\mathbb{P}^*(an^{1/2} \leq \tilde{\mathscr{M}}_n < bn^{1/2})$ is part of an ongoing work with P. Andreoletti and L. de Raphélis.
The present paper aims in some way to describe the interaction between the vertices of the tree $\mathscr{R}_{T^{n^{1/2}}}$ in the set of generations «squashed» when rescaling the tree by $n^{1/2}$.

Remark 3. *The curiosity here is the fact that critical generations and small generations equally contributed to the range. Indeed, whether $\mathfrak{L}_n$ is negligible with respect to $n^{1/2}$ (with $\mathfrak{L}_n \geq \delta_0^{-1} \log n$) or not, $\sum_{|u|=\mathfrak{L}_n} \mathbb{1}_{\{u \in \mathscr{R}_{\mathfrak{t}_n}\}}$ is of order $n^{1/2}$. This fact makes a deep difference with the slow regime in which only the critical generations (that is typical generations, of order $(\log n)^2$) contribute significantly to the range (see [AC18], Theorem 1.2 and Proposition 1.4).*

2. Proofs of the theorems

In this section, we prove theorems presented as examples and end it with the proof of Theorem 1.6.

2.1. Proofs of Theorems 1.2 to 1.5

In this subsection, we give a proof of each example stated above except for the Theorem 1.1 which is the simplest application of Theorem 1.6, taking $f = 1$. For each example, the procedure is as follows: we first prove the function f we consider satisfies the hereditary assumption 5 and we then give useful precisions on $\mathscr{A}^k_\infty(f)$ for the description of the genealogy of the vertices $\mathscr{X}^{(1,n)}, \dots, \mathscr{X}^{(k,n)}$.

Proof of Theorem 1.2. Recall that for $\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in (\mathbb{N}^*)^{\times k}$

$$f_{\boldsymbol{\lambda}}(x^{(1)}, \dots, x^{(k)}) := \prod_{i=2}^{k} \mathbb{1}_{\{|x^{(i-1)} \wedge x^{(i)}| < \lambda_i\}}.$$

Let us prove that the hereditary assumption 5 is satisfied by $f_{\boldsymbol{\lambda}}$. Recall that for $\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$, $\mathscr{S}^k(\boldsymbol{x}) - 1$ denotes the last generation at which two or more vertices among $x^{(1)}, \dots, x^{(k)}$ share a common ancestor. If $p \geq \max_{2 \leq i \leq k} \lambda_i$ and $\boldsymbol{x} \in \Delta^k$ such that $p \leq \min_{1 \leq i \leq k} |x^{(i)}|$, then $\mathscr{S}^k(\boldsymbol{x}) \leq p$ implies that for any $\boldsymbol{z} \in [\![(x^{(1)})_p, x^{(1)}]\!] \times \cdots \times [\![(x^{(k)})_p, x^{(k)}]\!]$, we have $\mathscr{S}^k(\boldsymbol{z}) = \mathscr{S}^k(\boldsymbol{x})$ since, by definition of $f_{\boldsymbol{\lambda}}$, the highest obliged coalescence between at least two vertices among $(x^{(1)}, \dots, x^{(k)})$ must append at generation $\max_{2 \leq i \leq k} \lambda_i$. Thus, $\mathscr{S}^k((x^{(1)})_p, \dots, (x^{(k)})_p) \leq m$. Moreover, by definition, $\mathscr{S}^k((x^{(1)})_p, \dots, (x^{(k)})_p) \leq \max_{2 \leq i \leq k} \lambda_i$ implies $\mathscr{S}^k(\boldsymbol{x}) \leq \max_{2 \leq i \leq k} \lambda_i$. Consequently, assumption 5 holds for $\mathfrak{g} = \max_{2 \leq i \leq k} \lambda_i$. We conclude using Theorem 1.6. □

We now prove Theorem 1.3:

Proof of Theorem 1.3. Recall that for $\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$, $\mathscr{S}^k(\boldsymbol{x}) - 1$ denotes the last generation at which two or more vertices among $x^{(1)}, \dots, x^{(k)}$ share a common ancestor and for $m \in \mathbb{N}^*$, recall that

$$f_m(\boldsymbol{x}) = \mathbb{1}_{\{\mathscr{S}^k(\boldsymbol{x}) \leq m\}}.$$

First, note that the hereditary assumption 5 is satisfied by f_m. Indeed, if $p \geq m$ and $\boldsymbol{x} \in \Delta^k$ such that $p \leq \min_{1\leq i\leq k} |x^{(i)}|$, then $\mathscr{S}^k(\boldsymbol{x}) \leq p$ implies that for any $\boldsymbol{z} \in [\![(x^{(1)})_p, x^{(1)}]\!] \times \cdots \times [\![(x^{(k)})_p, x^{(k)}]\!]$, we have $\mathscr{S}^k(\boldsymbol{z}) = \mathscr{S}^k(\boldsymbol{x})$. Thus, $\mathscr{S}^k((x^{(1)})_p, \dots, (x^{(k)})_p) \leq m$. Moreover, by definition, $\mathscr{S}^k((x^{(1)})_p, \dots, (x^{(k)})_p) \leq m$ implies $\mathscr{S}^k(\boldsymbol{x}) \leq m$. Consequently, assumption 5 holds for $\mathfrak{g} = m$.
We then deduce the converge of the trace in (15) by using Theorem 1.6.
We now move to the limit law of $(\mathscr{S}^k(\mathscr{X}^{(n)}))$ in (12). Note, by definition, that

$$\mathbb{P}^*\big(\mathscr{S}^k(\mathscr{X}^n) \leq m\big) = \frac{1}{\mathbb{P}^*(D_n \geq k)} \mathbb{E}^*\Big[\frac{\mathscr{A}^k(\mathscr{D}_n, f_m)}{\mathscr{A}^k(\mathscr{D}_n, 1)} \mathbb{1}_{\{D_n \geq k\}}\Big],$$

so $\mathbb{P}^*(\mathscr{S}^k(\mathscr{X}^n) \leq m)$ goes to $\mathbf{E}^*[\mathscr{A}^k_\infty(f_m)/(W_\infty)^k]$ when n goes to ∞ thanks to Theorem 1.6 with $f = f_m$ and $g = 1$ together with the fact that $\lim_{n\to\infty} \mathbb{P}^*(D_n \geq k) = 1$. It is left to show that $\lim_{m\to\infty} \mathscr{A}^k_\infty(f_m) = (W_\infty)^k$. For that, we use Lemma 3.9 with $f = 1$ and $\boldsymbol{p} = (l, \dots, l) \in (\mathbb{N}^*)^{\times k}$

$$\sup_{l>m} \mathbf{E}^*\big[\big|\mathscr{A}^k_l(f_m) - \mathscr{A}^k_l(1)\big|^2\big] \underset{m\to\infty}{\longrightarrow} 0.$$

Moreover, $\lim_{l\to\infty} \mathscr{A}^k_l(1) = (W_\infty)^k$ and $\lim_{l\to\infty} \mathscr{A}^k_l(f_m) = \mathscr{A}^k_\infty(f_m)$ so $(\mathscr{A}^k_\infty(f_m))_m$ converges to $(W_\infty)^k$ in $L^2(\mathbf{P}^*)$, which allows to end the proof. □

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. Recall that for any $1 \leq d < q \in \mathbb{N}^*$, for an increasing collection $\Xi = (\Xi_i)_{0\leq i\leq d}$ of partitions of $\{1, \dots, q\}$, for all $\boldsymbol{x} = (x^{(1)}, \dots, x^{(q)}) \in \Delta^q$ and all $\boldsymbol{t} = (t_1, \dots, t_d) \in \mathbb{N}^{\times d}$ such that $t_1 < t_2 < \cdots < t_d$,

$$f^d_{\boldsymbol{t},\Xi}(\boldsymbol{x}) = \prod_{i=1}^d \mathbb{1}_{\Gamma^i_{\boldsymbol{t},\Xi}}(\boldsymbol{x}),$$

where $\Gamma^i_{\boldsymbol{t},\Xi} = \Upsilon_{t_i-1,\Xi_{i-1}} \cap \Upsilon_{t_i,\Xi_i}$ and for any $r \in \{1, \dots, d\}$ and any $m \in \mathbb{N}^*$, $\boldsymbol{x}$ belongs to Υ_{m,Ξ_r} if and only if

$$\forall \boldsymbol{B} \in \boldsymbol{\Xi}_r, \forall i_1, i_2 \in \boldsymbol{B} : (x^{(i_1)})_m = (x^{(i_2)})_m,$$

and for $r \neq 0$

$$\forall \boldsymbol{B} \neq \tilde{\boldsymbol{B}} \in \boldsymbol{\Xi}_r, \forall i_1 \in \boldsymbol{B}, i_2 \in \tilde{\boldsymbol{B}} : (x^{(i_1)})_m \neq (x^{(i_2)})_m,$$

where we recall that $(x^{(i)})_m$ denotes the ancestor of $x^{(i)}$ in generation m if exists, $(x^{(i)})_m = e^{(i)}$ otherwise. Recall that $\mathscr{C}^k_{\mathfrak{g}} = \{\boldsymbol{y} \in \Delta^q;\ \mathscr{S}^q(\boldsymbol{y}) \leq \mathfrak{g}\}$ where $\mathscr{S}^q(\boldsymbol{y}) - 1$ is the last generation at which two or more vertices among $y^{(1)}, \dots, y^{(q)}$ share a common ancestor. Let $p \geq t_d$ such that $\min_{1\leq i\leq q} x^{(i)} \geq p$ and $\boldsymbol{x} \in \mathscr{C}^k_p$. If $\boldsymbol{x} \in \cap_{j=1}^d \Gamma^j_{\boldsymbol{t},\Xi}$, then $(z^{(i)})_t = (x^{(i)})_t$ for all $\boldsymbol{z} \in [\![(x^{(1)})_p, x^{(1)}]\!] \times \cdots \times [\![(x^{(q)})_p, x^{(q)}]\!]$, $1 \leq i \leq q$ and $t \in \{0, \dots, p\}$ thus giving $((x^{(1)})_p, \dots, (x^{(q)})_p) \in \cap_{j=1}^d \Gamma^j_{\boldsymbol{t},\Xi}$. Moreover, by definition, $((x^{(1)})_p, \dots, (x^{(q)})_p)) \in \cap_{j=1}^d \Gamma^j_{\boldsymbol{t},\Xi}$ implies $\boldsymbol{x} \in \cap_{j=1}^d \Gamma^j_{\boldsymbol{t},\Xi}$, then $(z^{(i)})_t = (x^{(i)})_t$. Consequently, $f^d_{\boldsymbol{t},\Xi}$ satisfies assumption 5 with $\mathfrak{g} = t_d$ and this prove that the convergence in (15) holds.
We move to the limit law of $(\pi^{k,n})$ in (16). Recall the definition of $\mathscr{S}^{k,n}_i$ in (11). First, note that

$$\mathbb{P}^*(\pi^{k,n}_{m_0} = \boldsymbol{\pi}_0, \dots, \pi^{k,n}_{m_\ell} = \boldsymbol{\pi}_\ell) = \mathbb{P}^*\Big(\bigcap_{i=1}^\ell \big\{\pi^{k,n}_{m_{i-1}} = \boldsymbol{\pi}_{i-1}, \pi^{k,n}_{m_i} = \boldsymbol{\pi}_i, m_{i-1} < \mathscr{S}^{k,n}_i \leq m_i\big\}\Big).$$

Indeed, for all $1 \leq i \leq \ell$, $|\boldsymbol{\pi}_{i-1}| < |\boldsymbol{\pi}_i|$ so the interval $(m_{i-1}, m_i]$ necessarily contains at least one coalescent time. But since $\boldsymbol{\pi}_0 = \{\{1, \dots, k\}\}$ and $\boldsymbol{\pi}_\ell = \{\{1\}, \dots, \{k\}\}$, $\cup_{i=1}^{\ell}(m_{i-1}, m_i]$ can't contain more than ℓ coalescent times so $\mathscr{S}_i^{k,n}$ is the only one belonging to $(m_{i-1}, m_i]$. We now write

$$\begin{aligned}\mathbb{P}^*\Big(\bigcap_{i=1}^{\ell}\big\{\pi^{k,n}_{m_{i-1}} = \boldsymbol{\pi}_{i-1}, m_{i-1} < \mathscr{S}_i^{k,n} \leq m_i\big\}\Big) &= \sum_{s_1=m_0+1}^{m_1} \cdots \sum_{s_\ell=m_{\ell-1}+1}^{m_\ell} \mathbb{P}^*\Big(\bigcap_{i=1}^{\ell}\big\{\pi^{k,n}_{m_{i-1}} = \boldsymbol{\pi}_{i-1}, \pi^{k,n}_{m_i} = \boldsymbol{\pi}_i, \mathscr{S}_i^{k,n} = s_i\big\}\Big)\\ &= \sum_{s_1=m_0+1}^{m_1} \cdots \sum_{s_\ell=m_{\ell-1}+1}^{m_\ell} \mathbb{P}^*\Big(\bigcap_{i=1}^{\ell}\big\{\pi^{k,n}_{s_i-1} = \boldsymbol{\pi}_{i-1}, \pi^{k,n}_{s_i} = \boldsymbol{\pi}_i\big\}\Big),\end{aligned}$$

Moreover, $\pi^{k,n}_{s_i-1} = \boldsymbol{\pi}_{i-1}, \pi^{k,n}_{s_i} = \boldsymbol{\pi}_i$ means nothing but $\mathscr{X}^{(n)} \in \Gamma^i_{\boldsymbol{s},\Pi}$ and it follows that

$$\begin{aligned}\mathbb{P}^*\Big(\bigcap_{i=1}^{\ell}\big\{\pi^{k,n}_{m_{i-1}} = \boldsymbol{\pi}_{i-1}, \pi^{k,n}_{m_i} = \boldsymbol{\pi}_i, \mathscr{S}_i^{k,n} = s_i\big\}\Big) &= \mathbb{E}^*\big[f^\ell_{\boldsymbol{s},\Pi}(\mathscr{X}^{(n)})\big]\\ &= \mathbb{E}^*\Big[\frac{\mathscr{A}^k(\mathscr{D}_n, f^\ell_{\boldsymbol{s},\Pi})}{\mathscr{A}^k(\mathscr{D}_n, 1)}\mathbb{1}_{\{D_n \geq k\}}\Big],\end{aligned}$$

where we have used the definition of $\mathscr{X}^{(n)}$ (see (6)) in the last equation. Since $f^\ell_{\boldsymbol{s},\Pi}$ satisfies the hereditary assumption 5, we finally get (15) from (25) with $f = f^\ell_{\boldsymbol{s},\Pi}$ and by (26) with $g = 1$

$$\lim_{n\to\infty} \mathbb{P}^*(\pi^{k,n}_{m_0} = \boldsymbol{\pi}_0, \dots, \pi^{k,n}_{m_\ell} = \boldsymbol{\pi}_\ell) = \sum_{s_1=m_0+1}^{m_1} \cdots \sum_{s_\ell=m_{\ell-1}+1}^{m_\ell} \mathbf{E}^*\Big[\frac{\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})}{(W_\infty)^k}\Big].$$

We now compute the conditional expectation of $\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})$ conditionally given the sigma-algebra $\mathscr{F}_{s_p-1} = \sigma(\mathbb{T}; (V(x); |x| < s_p))$. Start with $p = \ell$. Let $s_i \in \{m_{i-1}+1, \dots, m_i\}$ for all $i \in \{1, \dots, \ell\}$. Using the definition of $\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})$ and the fact that $\boldsymbol{x} \in \Delta^k_l \cap \Gamma^\ell_{\boldsymbol{s},\Pi}$ for $l > s_\ell$ implies $\mathscr{S}^k(\boldsymbol{x}) \leq s_\ell$, we obtain, on the set of non-extinction

$$\mathbf{E}^*\big[\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})|\mathscr{F}_{s_\ell}\big] = \lim_{l\to\infty} \mathbf{E}^*\Big[\sum_{\boldsymbol{x}\in\Delta^k_l} f^\ell_{\boldsymbol{s},\Pi}(\boldsymbol{x})e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k}|\mathscr{F}_{s_\ell}\Big] = \sum_{\boldsymbol{x}\in\Delta^k_{s_\ell}} f^\ell_{\boldsymbol{s},\Pi}(\boldsymbol{x})e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k},$$

since $s_\ell - 1$ corresponds to the last generation at which two or more vertices among $x^{(1)}, \dots, x^{(l)}$ share a common ancestor and we recall that $\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k = \sum_{i=1}^k V(x^{(i)})$. In particular, these vertices don't share any common ancestor in generation s_ℓ and last inequality comes from independence of the increments of the branching random walk $(\mathbb{T}, (V(x), x \in \mathbb{T}))$ together with the fact that $\psi(1) = 0$. Before going any further, let us define a transformation of the collection of partitions Π. We build from Π, which is a collection of partitions of the set $\{1, \dots, k\}$, a new collection $\Pi^{\ell-1} = (\tilde{\boldsymbol{\pi}}_i)_{0\leq i\leq \ell-1}$ of partitions of the set $\{1, \dots, |\boldsymbol{\pi}_{\ell-1}|\}$ as follows:

- $\tilde{\boldsymbol{\pi}}_{\ell-1} = \{\{1\}, \dots, \{|\boldsymbol{\pi}_{\ell-1}|\}\}$;
- for any $1 \leq i \leq \ell - 2$ and any $1 \leq j \leq |\boldsymbol{\pi}_i|$, the j-th block $\boldsymbol{B}^i_j$ of the partition $\boldsymbol{\pi}_i$ is the union of $\mathfrak{b}_{\ell-1}(\boldsymbol{B}^i_j) \geq 1$ block(s) of the partition $\boldsymbol{\pi}_{\ell-1}$. We then denote by $\tilde{\boldsymbol{B}}^i_j$ the subset of $\{1, \dots, |\boldsymbol{\pi}_{\ell-1}|\}$ composed of all indices of these $\mathfrak{b}_{\ell-1}(\boldsymbol{B}^i_j)$ block(s) and let $\tilde{\boldsymbol{\pi}}_i = \{\tilde{\boldsymbol{B}}^i_1, \dots, \tilde{\boldsymbol{B}}^i_{|\boldsymbol{\pi}_i|}\}$. By definition, $\tilde{\boldsymbol{\pi}}_0$ remains a one-block partition: $\tilde{\boldsymbol{\pi}}_0 = \{\{1, \dots, |\boldsymbol{\pi}_{\ell-1}|\}\}$.

Note that for any $0 \le i \le \ell-1$, $|\tilde{\boldsymbol{\pi}}_i| = |\boldsymbol{\pi}_i|$ and for any $0 \le i \le \ell-2$, $1 \le j \le |\boldsymbol{\pi}_i|$, $b_i(\boldsymbol{B}_j) = \tilde{b}_i(\tilde{\boldsymbol{B}}_j)$, where $\tilde{\boldsymbol{B}}_j \in \tilde{\boldsymbol{\pi}}_i$ is the union of $\tilde{b}_i(\tilde{\boldsymbol{B}}_j) \ge 1$ block(s) of $\tilde{\pi}_{i+1}$.

Example 4. *If* Π *is defined by* $\boldsymbol{\pi}_4 = \{\{1\},\{2\},\{3\},\{4\},\{5\}\}$,$\boldsymbol{\pi}_3 = \{\{1,3\},\{2\},\{4\},\{5\}\}$, $\boldsymbol{\pi}_2 = \{\{1,3\},\{2,5\},\{4\}\}$, $\boldsymbol{\pi}_1 = \{\{1,3,4\},\{2,5\}\}$ *and* $\boldsymbol{\pi}_0 = \{\{1,2,3,4,5\}\}$ *then we have:*
$\tilde{\boldsymbol{\pi}}_3 = \{\{1\},\{2\},\{3\},\{4\}\}$, $\tilde{\boldsymbol{\pi}}_2 = \{\{1\},\{2,4\},\{3\}\}$,$\tilde{\boldsymbol{\pi}}_1 = \{\{1,3\},\{2,4\}\}$, *and* $\tilde{\boldsymbol{\pi}}_0 = \{\{1,2,3,4\}\}$.

If $\Pi^\ell := \Pi$, then for any $i \in \{0,\dots,\ell-1\}$, let Π^i be the collection of partitions of $\{1,\dots,|\boldsymbol{\pi}_i|\}$ resulting from the previous procedure applied to Π^{i+1}. Note that Π^i is an increasing collection of partitions of $\{1,\dots,|\boldsymbol{\pi}_i|\}$. This construction is a way of preserving the genealogical information through the generations.
Let $\boldsymbol{s}^{\ell-1} = (s_1,\dots,s_{\ell-1})$ and recall the definitions regarding partitions in (13). One can now notice that, since that the number of vertices of the k-tuple $\boldsymbol{x} \in \Delta^k_{s_\ell}$ sharing the same parent $u^{(j)}$ is $b_{\ell-1}(\boldsymbol{B}_j)$ (where we recall that $b_{\ell-1}(\boldsymbol{B}_j)$ stands for $b_{\ell-1}(\boldsymbol{B}_j^{\ell-1})$), we have

$$\sum_{\boldsymbol{x}\in\Delta^k_{s_\ell}} f^\ell_{\boldsymbol{s},\Pi}(\boldsymbol{x}) e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k} = \sum_{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}_{\ell-1}|}_{s_{\ell-1}}} f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}(\boldsymbol{u}) \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} \sum_{\boldsymbol{x}^{(j)}\in\Delta^{b_{\ell-1}(\boldsymbol{B}_j)}_{s_\ell}} \prod_{i=1}^{b_{\ell-1}(\boldsymbol{B}_j)} \mathbb{1}_{\{(x^{(j,i)})^* = u^{(j)}\}} \times e^{-V(x^{(j,i)})},$$

where $\boldsymbol{x}^{(j)} = (x^{(j,1)},\dots,x^{(j,b_{\ell-1}(\boldsymbol{B}_j))})$ and $(x^{(j,i)})^*$ is the parent of $x^{(j,i)}$. Moreover, by definition, $b_{\ell-1}(\boldsymbol{B}_j) = |\boldsymbol{B}_j^{\ell-1}|$ (it comes from the fact that $\boldsymbol{\pi}_\ell = \{\{1\},\dots,\{k\}\}$) so

$$\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} \sum_{\boldsymbol{x}^{(j)}\in\Delta^{b_{\ell-1}(\boldsymbol{B}_j)}_{s_\ell}} \prod_{i=1}^{b_{\ell-1}(\boldsymbol{B}_j)} \mathbb{1}_{\{(x^{(j,i)})^* = u^{(j)}\}} e^{-V(x^{(j,i)})} = e^{-\langle \boldsymbol{\beta}^{\ell-1}, V(\boldsymbol{u})\rangle_{|\boldsymbol{\pi}_{\ell-1}|}} \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} \sum_{\boldsymbol{x}^{(j)}\in\Delta^{b_{\ell-1}(\boldsymbol{B}_j)}_{s_\ell}} \times \prod_{i=1}^{b_{\ell-1}(\boldsymbol{B}_j)} \mathbb{1}_{\{(x^{(j,i)})^* = u^{(j)}\}} e^{-V_{u^{(j)}}(x^{(j,i)})},$$

where $\boldsymbol{\beta}^{\ell-1} = (|\boldsymbol{B}_1^{\ell-1}|,\dots,|\boldsymbol{B}^{\ell-1}_{|\boldsymbol{\pi}_{\ell-1}|}|)$ and $V_{u^{(j)}}(x^{(j,i)}) = V(x^{(j,i)}) - V(u^{(j)})$. By independence of the increments of the branching random walk $(\mathbb{T}, (V(x), x\in\mathbb{T}))$, since $\psi(1) = 0$

$$\begin{aligned}\mathbf{E}\Big[\sum_{\boldsymbol{x}\in\Delta^k_{s_\ell}} f^\ell_{\boldsymbol{s},\Pi}(\boldsymbol{x}) e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k} \big| \mathscr{F}_{s_\ell - 1}\Big] &= \mathscr{A}^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}, \boldsymbol{\beta}^{\ell-1}\big) \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} c_{b_{\ell-1}(\boldsymbol{B}_j)}(\mathbf{1}) \prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_\ell \\ |\mathfrak{B}|\ge 2}} e^{\psi(|\mathfrak{B}|)} \\ &= \mathscr{A}^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}, \boldsymbol{\beta}^{\ell-1}\big) \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} c_{b_{\ell-1}(\boldsymbol{B}_j)}(\boldsymbol{\beta}_j^{\ell-1}),\end{aligned}$$

where $\boldsymbol{\beta}_j^{\ell-1} := (\beta^{\ell-1}_{j,1},\dots,\beta^{\ell-1}_{j,b_{\ell-1}(\boldsymbol{B}_j)}) = (1,\dots,1)$, see (13). We also recall that $\mathscr{A}^m_l(g,\boldsymbol{\beta}) = \sum_{\boldsymbol{x}\in\Delta^m_l} g(\boldsymbol{x}) e^{-\langle \boldsymbol{\beta}, V(\boldsymbol{x})\rangle_m}$ and see assumption 3 for the definition of $c_l(\boldsymbol{\beta})$. Now recall that $\Pi^{\ell-2}$ is the collection of partitions of $\{1,\dots,|\boldsymbol{\pi}_{\ell-2}|\}$ obtain from $\Pi^{\ell-1}$ with the same procedure as above (see Example 4). Let $\boldsymbol{s}^{\ell-2} = (s_1,\dots,s_{\ell-2})$. Again, exactly $b_{\ell-2}(\boldsymbol{B}_j)$ vertices in generation

$s_{\ell-1}$ are sharing the same parent $z^{(j)}$ so

$$\mathscr{A}_{s_{\ell-1}}^{|\boldsymbol{\pi}_{\ell-1}|}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\boldsymbol{\beta}^{\ell-1}\big)=\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\ell-2}|}_{s_{\ell-1}-1}} f^{\ell-2}_{\boldsymbol{s}^{\ell-2},\Pi^{\ell-2}}(\boldsymbol{z})\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|}\sum_{\boldsymbol{u}^{(j)}\in\Delta^{b_{\ell-2}(\boldsymbol{B}_j)}_{s^{\ell-1}}}\prod_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}\mathbb{1}_{\{(u^{(j,i)})^*=z^{(j)}\}}$$
$$\times\, e^{-\beta^{\ell-2}_{j,i}V(u^{(j,i)})}\sum_{\boldsymbol{x}^{(j)}\in\Delta^{b_{\ell-2}(\boldsymbol{B}_j)}_{s^{\ell-1}}}\mathbb{1}_{\{x^{(j,i)}\geq u^{(j,i)}\}}e^{-\beta^{\ell-2}_{j,i}V_{u^{(j,i)}}(x^{(j,i)})},$$

where $\boldsymbol{u}^{(j)}=(u^{(j,1)},\dots,u^{(j,b_{\ell-2}(\boldsymbol{B}_j))})$, $\boldsymbol{x}^{(j)}=(x^{(j,1)},\dots,x^{(j,b_{\ell-2}(\boldsymbol{B}_j))})$ and $V_{u^{(j,i)}}(x^{(j,i)})$ is the increment $V(x^{(j,i)})-V(u^{(j,i)})$. Then, by independence of the increments of the branching random walk $(\mathbb{T},(V(x),x\in\mathbb{T}))$,

$$\mathbf{E}\Big[\mathscr{A}_{s_{\ell-1}}^{|\boldsymbol{\pi}_{\ell-1}|}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\boldsymbol{\beta}^{\ell-1}\big)|\mathscr{F}_{s_{\ell-1}}\Big]=\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\ell-2}|}_{s_{\ell-1}-1}} f^{\ell-2}_{\boldsymbol{s}^{\ell-2},\Pi^{\ell-2}}(\boldsymbol{z})\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|}\sum_{\boldsymbol{u}^{(j)}\in\Delta^{b_{\ell-2}(\boldsymbol{B}_j)}_{s^{\ell-1}}}\prod_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}$$
$$\times\,\mathbb{1}_{\{(u^{(j,i)})^*=z^{(j)}\}}e^{-\beta^{\ell-2}_{j,i}V(u^{(j,i)})}e^{s^*_\ell\psi(\beta^{\ell-2}_{j,i})},$$

with $s^*_\ell=s_\ell-s_{\ell-1}-1$. Moreover, since $\sum_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}\beta^{\ell-2}_{j,i}=|\boldsymbol{B}^{\ell-2}_j|$ (see (13)), we have

$$\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|}\sum_{\boldsymbol{u}^{(j)}\in\Delta^{b_{\ell-2}(\boldsymbol{B}_j)}_{s^{\ell-1}}}\prod_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}\mathbb{1}_{\{(u^{(j,i)})^*=z^{(j)}\}}e^{-\beta^{\ell-2}_{j,i}V(u^{(j,i)})}$$
$$=e^{-\langle\boldsymbol{\beta}^{\boldsymbol{\ell-2}},V(\boldsymbol{z})\rangle_{|\boldsymbol{\pi}_{\ell-2}|}}\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|}\sum_{\boldsymbol{u}^{(j)}\in\Delta^{b_{\ell-2}(\boldsymbol{B}_j)}_{s^{\ell-1}}}\prod_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}\mathbb{1}_{\{(u^{(j,i)})^*=z^{(j)}\}}e^{-\beta^{\ell-2}_{j,i}V_{z^{(j)}}(u^{(j,i)})},$$

with $\boldsymbol{\beta}^{\ell-2}=(|\boldsymbol{B}^{\ell-1}_1|,\dots,|\boldsymbol{B}^{\ell-2}_{|\boldsymbol{\pi}_{\ell-2}|}|)$ and again, by independence of the increments of the branching random walk $(\mathbb{T},(V(x),x\in\mathbb{T}))$, using again that $\sum_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}\beta^{\ell-2}_{j,i}=|\boldsymbol{B}^{\ell-2}_j|$

$$\mathbf{E}\Big[\mathscr{A}_{s_\ell-1}^{|\boldsymbol{\pi}_{\ell-1}|}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\boldsymbol{\beta}^{\ell-1}\big)|\mathscr{F}_{s_{\ell-1}-1}\Big]=\mathscr{A}_{s_{\ell-1}-1}^{|\boldsymbol{\pi}_{\ell-2}|}\big(f^{\ell-2}_{\boldsymbol{s}^{\ell-2},\Pi^{\ell-2}},\boldsymbol{\beta}^{\ell-2}\big)\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|}c_{b_{\ell-2}(\boldsymbol{B}_j)}(\boldsymbol{\beta}^{\ell-2}_j)$$
$$\times\prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_{\ell-1}\\|\mathfrak{B}|\geq 2}}e^{s^*_\ell\psi(|\mathfrak{B}|)},$$

where $\boldsymbol{\beta}^{\ell-2}_j=(\beta^{\ell-2}_{j,1},\dots,\beta^{\ell-2}_{j,b_{\ell-2}(\boldsymbol{B}_j)})$. Thus, we obtain

$$\mathbf{E}\Big[\sum_{\boldsymbol{x}\in\Delta^k_{s_\ell}}f^\ell_{\boldsymbol{s},\Pi}(\boldsymbol{x})e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_k}|\mathscr{F}_{s_{\ell-1}-1}\Big]=\mathscr{A}_{s_{\ell-1}-1}^{|\boldsymbol{\pi}_{\ell-2}|}\big(f^{\ell-2}_{\boldsymbol{s}^{\ell-2},\Pi^{\ell-2}},\boldsymbol{\beta}^{\ell-2}\big)\prod_{i=\ell-1}^{\ell}\prod_{j=1}^{|\boldsymbol{\pi}_{i-1}|}c_{b_{i-1}(\boldsymbol{B}_j)}(\boldsymbol{\beta}^{i-1}_j)$$
$$\times\prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_i\\|\mathfrak{B}|\geq 2}}e^{s^*_\ell\psi(|\mathfrak{B}|)}.$$

By induction on $2\leq p\leq\ell$, we finally get, on the set of non-extinction

$$\mathbf{E}^*\big[\mathscr{A}^k_\infty(f^\ell_{\boldsymbol{s},\Pi})|\mathscr{F}_{s_p-1}\big]=\mathscr{A}_{s_p-1}^{|\boldsymbol{\pi}_{p-1}|}\big(f^{p-1}_{\boldsymbol{s}^{p-1},\Pi^{p-1}},\boldsymbol{\beta}^{p-1}\big)\prod_{i=p}^{\ell}\prod_{j=1}^{|\boldsymbol{\pi}_{i-1}|}c_{b_{i-1}(\boldsymbol{B}_j)}(\boldsymbol{\beta}^{i-1}_j)\prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_i\\|\mathfrak{B}|\geq 2}}e^{s^*_{i+1}\psi(|\mathfrak{B}|)}.$$

Taking $p = 2$ in the above formula, we have, on the set of non-extinction

$$\mathbf{E}^*\big[\mathscr{A}_\infty^k(f^\ell_{\boldsymbol{s},\Pi})|\mathscr{F}_{s_2-1}\big] = \mathscr{A}_{s_2-1}^{|\boldsymbol{\pi}_1|}\big(f^1_{\boldsymbol{s}^1,\Pi^1},\boldsymbol{\beta}^1\big)\prod_{i=2}^{\ell}\prod_{j=1}^{|\boldsymbol{\pi}_{i-1}|} c_{b_{i-1}(\boldsymbol{B}_j)}(\boldsymbol{\beta}_j^{i-1})\prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_i\\|\mathfrak{B}|\geq 2}} e^{s^*_{i+1}\psi(|\mathfrak{B}|)},$$

where for any $i \in \{2,\dots,\ell\}$, $s_i^* = s_i - s_{i-1} - 1$ and $s^*_{\ell+1} = 1$. Since $\sum_{j=1}^{b_0(\boldsymbol{B}_1)}|\boldsymbol{B}_j^1| = k$ (it comes from the fact that $\boldsymbol{\pi}_0 = \{\{1,\dots,k\}\}$), we have

$$\begin{aligned}\mathbf{E}^*\big[\mathscr{A}_{s_2-1}^{|\boldsymbol{\pi}_1|}\big(f^1_{\boldsymbol{s}^1,\Pi^1},\boldsymbol{\beta}^1\big)|\mathscr{F}_{s_1-1}\big] &= \sum_{|z|=s_1-1} e^{-kV(z)}c_{b_0(\boldsymbol{B}_1)}(\boldsymbol{\beta}^1)\prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_1\\|\mathfrak{B}|\geq 2}} e^{s_2^*\psi(|\mathfrak{B}|)}\\ &= \sum_{|z|=s_1-1} e^{-kV(z)}\prod_{j=1}^{|\boldsymbol{\pi}_0|} c_{b_0(\boldsymbol{B}_j)}(\boldsymbol{\beta}_j^0)\prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_1\\|\mathfrak{B}|\geq 2}} e^{s_2^*\psi(|\mathfrak{B}|)},\end{aligned}$$

the last equality coming from the fact $\boldsymbol{\beta}_j^0 = \boldsymbol{\beta}^1 = (|\boldsymbol{B}_1^1|,\dots,|\boldsymbol{B}^1_{|\boldsymbol{\pi}_1|}|)$. Finally,

$$\mathbf{E}^*\big[\mathscr{A}_\infty^k(f^\ell_{\boldsymbol{s},\Pi})\big] = e^{\psi(k)}\prod_{i=1}^{\ell}\prod_{j=1}^{|\boldsymbol{\pi}_{i-1}|} c_{b_{i-1}(\boldsymbol{B}_j)}(\boldsymbol{\beta}_j^{i-1})\prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_i\\|\mathfrak{B}|\geq 2}} e^{s^*_{i+1}\psi(|\mathfrak{B}|)},$$

thus completing to proof. □

We end this subsection with the proof of Theorem 1.5.

Proof of Theorem 1.5. First recall that for $1 \leq \ell < k$, $\mathfrak{s} \in \mathbb{N}^*$ and $\boldsymbol{s} = (s_1,\dots,s_\ell) \in \mathbb{N}^{\times\ell}$ such that $s_1 < \cdots < s_\ell \leq \mathfrak{s}$, for all $\boldsymbol{x} \in \Delta^k$ such that $\min_{1\leq j\leq k}|x^{(j)}| \geq \mathfrak{s}$,

$$F^\ell_{\boldsymbol{s}}(\boldsymbol{x}) = \sum_{\Xi \text{ increasing}} f^\ell_{\boldsymbol{s},\Xi}(\boldsymbol{x}).$$

By Ξ increasing, we mean here that $\Xi = (\Xi_i)_{0\leq i\leq \ell}$ is an increasing collection of partitions of $\{1,\dots,k\}$. Since $f^\ell_{\boldsymbol{s},\Xi}$ satisfies the hereditary assumption 5, the same goes for $F^\ell_{\boldsymbol{s}}$ by taking $\mathfrak{g} = \mathfrak{s}$. Using the linearity of $g \mapsto \mathscr{A}_l^k(g)$, we get (18) thanks to Theorem 1.6.
First note that

$$\{\mathscr{J}^{k,n} = \ell,\ \mathscr{S}^k(\mathscr{Y}^{(n)}) \leq \mathfrak{s}\} = \bigcup_{\boldsymbol{m};m_1<\cdots<m_\ell\leq\mathfrak{s}}\ \bigcup_{\Pi \text{ increasing}}\ \bigcap_{i=1}^{\ell}\{\boldsymbol{\pi}^{k,n}_{m_i-1} = \boldsymbol{\pi}_{i-1}, \boldsymbol{\pi}^{k,n}_{m_i} = \boldsymbol{\pi}_i\},$$

where Π increasing means here that $\Pi = (\boldsymbol{\pi}_i)_{0\leq i\leq\ell}$ is an increasing collection of partitions of $\{1,\dots,k\}$. It follows that

$$\mathbb{P}^*\big(\mathscr{S}_1^{k,n} = s_1,\dots,\mathscr{S}_\ell^{k,n} = s_\ell,\ \mathscr{J}^{k,n} = \ell\big) = \mathbb{E}^*\Big[\frac{\mathscr{A}^k(\mathscr{D}_n,F^\ell_{\boldsymbol{s}})}{\mathscr{A}^k(\mathscr{D}_n,\mathbb{1}_{\mathscr{C}^k_{\mathfrak{s}}})}\mathbb{1}_{\{D_n\geq k\}}\Big],$$

and we conclude using Theorem 1.6. □

2.2. Proof of Theorem 1.6 and Theorem 1.7

Proof of Theorem 1.6. First, $\mathscr{A}^k(\mathscr{D}_{n,T^{n^{1/2}}},f)=\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathfrak{E}^{k,n^{1/2}}})+\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\Delta^k\setminus\mathfrak{E}^{k,n^{1/2}}})$ and then for any $\varepsilon\in(0,1)$

$$\begin{aligned}&\mathbb{P}^*\Big(\Big|\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_n,f)-(c_\infty)^k\mathscr{A}^k_\infty(f)\Big|>\varepsilon\Big)\\&\le\mathbb{P}^*\Big(\Big|\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_n,f)-(c_\infty)^k\mathscr{A}^k_\infty(f)\Big|>\frac{\varepsilon}{2}\Big)+\mathbb{P}^*\big(\mathscr{A}^k(\mathscr{D}_n,\mathbb{1}_{\Delta^k\setminus\mathfrak{E}^{k,n^{1/2}}})>\frac{\varepsilon}{2}(n^{1/2}\boldsymbol{L}_n)^k\big).\end{aligned}$$

Noticing that $\mathscr{D}_n=\mathscr{D}_{n,T^{n^{1/2}}}$, the first probability in this sum goes to 0 when $n\to\infty$ thanks to Proposition 1 with $s=n^{1/2}$ and the second one also goes to 0 thanks to Proposition 2 thus giving (25). For the convergence in $\mathbb{P}^*$-probability (26), note that

$$\begin{aligned}&\mathbb{P}^*\Big(\Big|\frac{\mathscr{A}^k(\mathscr{D}_n,f)}{\mathscr{A}^k(\mathscr{D}_n,f)}\mathbb{1}_{\{D_n\ge k\}}-\frac{\mathscr{A}^k_\infty(f)}{\mathscr{A}^k_\infty(g)}\Big|>\varepsilon\Big)\\&\le\mathbb{P}^*\Big(\Big|\frac{\mathscr{A}^k(\mathscr{D}_n,f)}{\mathscr{A}^k(\mathscr{D}_n,f)}-\frac{\mathscr{A}^k_\infty(f)}{\mathscr{A}^k_\infty(g)}\Big|>\varepsilon,D_n\ge k\Big)+\mathbb{P}^*(D_n<k),\end{aligned}$$

these two probabilities go to 0 when $n\to\infty$ and the proof is completed. □

We now prove Theorem 1.7. Recall that $\tilde{\mathscr{D}}_n=\{x\in\mathscr{R}_n;\ \ell_n\le|x|\le\mathfrak{L}_n\}$. The main idea of the proof is to show that, when $\kappa>2\xi k$, $\xi\ge2$, and $\mathfrak{L}_n=o(n^{1/2-\mathfrak{q}_\xi})$ for some non-increasing sequence $\mathfrak{q}$ such that $\mathfrak{q}_j\to0$ when $j\to\infty$, the volume $\tilde{D}_n$ of the range $\tilde{\mathscr{D}}_n$ behaves like the volume of the range up to the last complete excursion of $(X)_{j\le n}$ above the parent e^* of the root e.
For that, one can notice that for this choice of κ, Proposition 1 holds uniformly in s: there exists a non-increasing sequence of positive integers $(\mathfrak{q}_j)_j$, satisfying $\mathfrak{q}_j\in(0,1/2)$ and $q_j\to0$ when $j\to\infty$ such that if $\kappa>2\xi k$ for some integer $\xi\ge2$ and $\mathfrak{L}_n=o(n^{1/2-\mathfrak{q}_\xi})$ then, for any $\varepsilon_1\in(0,1)$

$$\mathbb{P}^*\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\Big|\frac{1}{(s\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}})-(c_\infty)^k\mathscr{A}^k_\infty(f)\Big|>\varepsilon\Big)\underset{n\to\infty}{\longrightarrow}0.\tag{30}$$

The proof of (30) is the same as the proof of Proposition 1 but for any $\varepsilon,\varepsilon_1\in(0,1)$, by Markov inequality

$$\begin{aligned}&\mathbb{P}\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\Big\{\Big|\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})-\mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big]\Big|>\varepsilon(s\boldsymbol{L}_n)^k/16\Big\}\Big)\\&\le\sum_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\frac{16^{2\xi k}}{\varepsilon^{2\xi k}(s\boldsymbol{L}_n)^{2\xi k}}\mathbb{E}\Big[\Big(\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})-\mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big]\Big)^{2\xi k}\Big]\\&\le16^{2\xi k}\mathfrak{C}_{3.7}\sum_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\Big(\frac{\mathfrak{L}_n}{s}\Big)^{\tilde{\mathfrak{q}}_\xi}\le\mathfrak{C}_1\frac{(\mathfrak{L}_n)^{\tilde{\mathfrak{q}}_\xi}}{n^{(\tilde{\mathfrak{q}}_\xi-1)/2}},\end{aligned}$$

where we have used Lemma 3.7 with $\mathfrak{a}=\xi$ for second inequality. Note that $\tilde{\mathfrak{q}}_\xi\ge2$ since $\xi\ge2$ so, as in the proof of Proposition 1, we obtain (30) by taking $\mathfrak{q}_j:=(2\tilde{\mathfrak{q}}_j)^{-1}$.

Proof of Theorem 1.7. First, let us state the following fact, proved by Y. Hu ([Hu17], Corollary 1.2): in law, under $\mathbb{P}^*$

$$\frac{1}{n^{1/2}}\sum_{j=1}^{n}\mathbb{1}_{\{X_k=e\}} \underset{n\to\infty}{\longrightarrow} \frac{1}{p^{\mathscr{E}}(e,e^*)}\frac{c_0^{1/2}}{W_\infty}|\mathscr{N}|.$$

We can actually adapt this result to the local time $\mathscr{L}^n$ of the parent e^* of the root e: in law, under $\mathbb{P}^*$

$$\frac{1}{n^{1/2}}\mathscr{L}^n \underset{n\to\infty}{\longrightarrow} \frac{c_0^{1/2}}{W_\infty}|\mathscr{N}|, \tag{31}$$

where c_0 is defined in (26). Moreover, recall that $\mathscr{N}$ denotes, under $\mathbb{P}^{\mathscr{E}}$, a standard Gaussian variable. Then, we show that $\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f)$ and $\mathscr{A}^k(\mathscr{D}_n,f)$ are close in the following:

$$\mathbb{P}^*\Big(\frac{1}{(\mathscr{L}^n\boldsymbol{L}_n)^k}\big|\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f)-\mathscr{A}^k(\mathscr{D}_n,f)\big|>\varepsilon\Big)\underset{n\to\infty}{\longrightarrow}0. \tag{32}$$

Indeed, by (31), $\lim_{\varepsilon_1\to 0}\limsup_{n\to\infty}\mathbb{P}^*(\varepsilon_1 n^{1/2}\le\mathscr{L}^n\le n^{1/2}/\varepsilon_1)=1$ so thanks to Proposition 2, together with Lemma 3.6 taking $\mathfrak{R}=\Delta^k\setminus\mathscr{C}^k_{a_n}$ and Lemma 3.10, $(\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}})+\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}}))/(\mathscr{L}^n\boldsymbol{L}_n)^k\to 0$ when $n\to\infty$, in $\mathbb{P}^*$-probability where we recall that $a_n=(2\delta_0)^{-1}\log n$. Therefore, it is enough to show that

$$\mathbb{P}^*\Big(\frac{1}{(\mathscr{L}^n\boldsymbol{L}_n)^k}\big|\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f\mathbb{1}_{\mathscr{C}^k_{a_n}})-\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})\big|>\varepsilon\Big)\underset{n\to\infty}{\longrightarrow}0.$$

For that, assume $\mathscr{L}^n=s\in\{\varepsilon_1 n^{1/2},\dots,n^{1/2}/\varepsilon_1\}$. By definition, $\mathscr{L}^n=\sup\{j\ge 1;\ T^j\le n\}$ so on the set $\{D_{n,T^s}\ge k\}$, where D_{n,T^s} is the cardinal of $\mathscr{D}_{n,T^s}$, both $\mathscr{D}_{n,T^s}$ and $\mathscr{D}_n$ are nonempty and note that

$$\begin{aligned}\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})-\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathscr{C}^k_{a_n}}) &= \sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\boldsymbol{x}\in\Delta^k}\mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{1}_{\{T^s<T_{\boldsymbol{x}}\le n\}}\\ &\le \sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\boldsymbol{x}\in\Delta^k}\mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{1}_{\{T^s<T_{\boldsymbol{x}}<T^{s+1}\}},\end{aligned}$$

where $|\boldsymbol{x}|=|\boldsymbol{p}|$ means that for all $1\le i\le k$, $x^{(i)}=p_i$. Using the strong Markov property at time T^s first and Markov inequality then, we have

$$\begin{aligned}&\mathbb{P}^{\mathscr{E}}\Big(\frac{1}{(s\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})-\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathscr{C}^k_{a_n}})>\varepsilon, D_{n,T^s}\ge k,\mathscr{L}^n=s\Big)\\ &\le \mathbb{P}^{\mathscr{E}}\Big(\frac{1}{((s\boldsymbol{L}_n)^k}\sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\boldsymbol{x}\in\Delta^k}\mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{1}_{\{T_{\boldsymbol{x}}<T^1\}}>\varepsilon\Big)\\ &\le \frac{1}{(\varepsilon s\boldsymbol{L}_n)^{2k}}\mathbb{E}^{\mathscr{E}}\Big[\Big(\sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\boldsymbol{x}\in\Delta^k}\mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{1}_{\{T_{\boldsymbol{x}}<T^1\}}\Big)^2\Big].\end{aligned}$$

Moreover $\mathbb{E}^{\mathscr{E}}[(\sum_{\boldsymbol{x}\in\Delta^k}\mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{1}_{\{T_{\boldsymbol{x}}<T^1\}})^2]$ is equal to

$$\sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Delta^k\\ \boldsymbol{x}\neq\boldsymbol{y}}}\mathbb{1}_{\{|\boldsymbol{x}|=|\boldsymbol{y}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\vee\mathscr{S}^k(\boldsymbol{y})\le a_n\}}\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}}\vee T_{\boldsymbol{y}}<T^1)+\sum_{\boldsymbol{x}\in\Delta^k}\mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}}<T^1),$$

where $t \vee s = \max(t,s)$ and thanks to Lemma 3.2

$$\mathbb{E}^{\mathscr{E}}\Big[\Big(\sum_{\boldsymbol{x}\in\Delta^k} \mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{1}_{\{T_{\boldsymbol{x}}<T^1\}}\Big)^2\Big] \le \mathfrak{C}'(a_n\boldsymbol{L}_n)^{2k},$$

for some constant $\mathfrak{C}'>0$ that doesn't depend on $\boldsymbol{p}$ thus giving

$$\mathbb{P}^*\Big(\frac{1}{(s\boldsymbol{L}_n)^k}\sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\boldsymbol{x}\in\Delta^k}\mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{1}_{\{T^s<T_{\boldsymbol{x}}\le n\}}>\varepsilon, \mathscr{L}^n=s\Big) \le \frac{\tilde{\mathfrak{C}}(a_n)^{2k}}{(\varepsilon s)^2},$$

for some constant $\tilde{\mathfrak{C}}>0$. Finally, for all $\varepsilon_1\in(0,1)$

$$\begin{aligned}
&\mathbb{P}^*\Big(\Big\{\frac{1}{(\mathscr{L}^n\boldsymbol{L}_n)^k}\Big|\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f\mathbb{1}_{\mathscr{C}^k_{a_n}})-\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big|>\varepsilon\Big\}\Big)\le\mathbb{P}^*(D_{n,T^{\mathscr{L}^n}}<k)\\
&+\sum_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\mathbb{P}^*\Big(\frac{1}{(s\boldsymbol{L}_n)^k}\sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\mathscr{S}^k(\boldsymbol{x})\le a_n\}}\mathbb{1}_{\{T^s<T_{\boldsymbol{x}}\le n\}}>\varepsilon, D_{n,T^s}\ge k,\mathscr{L}^n=s\Big)\\
&+\mathbb{P}^*(\mathscr{L}^n<\varepsilon_1 n^{1/2})+\mathbb{P}^*(\mathscr{L}^n>n^{1/2}/\varepsilon_1)\le\frac{\tilde{\mathfrak{C}}(a_n)^{2k}}{\varepsilon^2\varepsilon_1 n^{1/2}}+\mathbb{P}^*(\mathscr{L}^n<\varepsilon_1 n^{1/2})+\mathbb{P}^*(\mathscr{L}^n>n^{1/2}/\varepsilon_1).
\end{aligned}$$

Note that $\lim_{n\to\infty}\mathbb{P}^*(D_{n,T^{\mathscr{L}^n}}<k)=0$ and using (31) with the definition of $a_n=(2\delta_0)^{-1}\log n$, we have $\lim_{\varepsilon_1\to0}\limsup_{n\to\infty}(\tilde{\mathfrak{C}}(a_n)^{2k}/(\varepsilon^2\varepsilon_1 n^{1/2})+\mathbb{P}^*(\mathscr{L}^n<\varepsilon_1 n^{1/2})+\mathbb{P}^*(\mathscr{L}^n>n^{1/2}/\varepsilon_1))=0$, which yields (32).
Now, since $\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f\mathbb{1}_{\Delta^k\setminus\mathfrak{E}^{k,\mathscr{L}^n}})/(\mathscr{L}^n\boldsymbol{L}_n)^k\to0$ when $n\to\infty$, in $\mathbb{P}^*$-probability and thanks to (32), we can focus our attention on $\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f\mathbb{1}_{\mathfrak{E}^{k,\mathscr{L}^n}})/(\mathscr{L}^n\boldsymbol{L}_n)^k$.
Note the $\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f\mathbb{1}_{\mathfrak{E}^{k,\mathscr{L}^n}})$ concentrates around $(c_\infty)^k\mathscr{A}^k_\infty(f)$. Indeed, for any $\varepsilon,\varepsilon_1\in(0,1)$

$$\begin{aligned}
&\mathbb{P}^*\Big(\Big|\frac{1}{(\mathscr{L}^n\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f\mathbb{1}_{\mathfrak{E}^{k,\mathscr{L}^n}})-(c_\infty)^k\mathscr{A}^k_\infty(f)\Big|>\varepsilon\Big)\\
&\le\mathbb{P}^*\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\Big\{\Big|\frac{1}{(s\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}})-(c_\infty)^k\mathscr{A}^k_\infty(f)\Big|>\varepsilon\Big\}\Big)+\mathbb{P}^*(\mathscr{L}^n<\varepsilon_1 n^{1/2})\\
&+\mathbb{P}^*(\mathscr{L}^n>n^{1/2}/\varepsilon_1).
\end{aligned}$$

Thanks to equation (30), the first probability above goes to 0 when n goes to ∞ and by (31), $\lim_{\varepsilon_1\to0}\lim_{n\to\infty}(\mathbb{P}^*(\mathscr{L}^n<\varepsilon_1 n^{1/2})+\mathbb{P}^*(\mathscr{L}^n>n^{1/2}/\varepsilon_1))=0$ thus giving

$$\lim_{n\to\infty}\mathbb{P}^*\Big(\Big|\frac{1}{(\mathscr{L}^n\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_{n,T^{\mathscr{L}^n}},f\mathbb{1}_{\mathfrak{E}^{k,\mathscr{L}^n}})-(c_\infty)^k\mathscr{A}^k_\infty(f)\Big|>\varepsilon\Big)=0. \tag{33}$$

We obtain from (33), together with (32) that

$$\lim_{n\to\infty}\mathbb{P}^*\Big(\Big|\frac{1}{(\mathscr{L}^n\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})-(c_\infty)^k\mathscr{A}^k_\infty(f)\Big|>\varepsilon\Big)=0,$$

which gives (26). We also deduce (25) using (31). For the convergence in $\mathbb{P}^*$-probability (26), note that

$$\begin{aligned}
&\mathbb{P}^*\Big(\Big|\frac{\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})}{\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})}\mathbb{1}_{\{D_n\ge k\}}-\frac{\mathscr{A}^k_\infty(f)}{\mathscr{A}^k_\infty(g)}\Big|>\varepsilon\Big)\\
&\le\mathbb{P}^*\Big(\Big|\frac{\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})}{\mathscr{A}^k(\mathscr{D}_n,f\mathbb{1}_{\mathscr{C}^k_{a_n}})}-\frac{\mathscr{A}^k_\infty(f)}{\mathscr{A}^k_\infty(g)}\Big|>\varepsilon, D_n\ge k\Big)+\mathbb{P}^*(D_n<k),
\end{aligned}$$

which goes to 0 when n goes to ∞ and the proof is completed. □

3. Proofs of Propositions 1 and 2

This section is devoted to the proofs of our two propositions. We show that relevant k-tuples of visited vertices are those in the set $\mathfrak{E}^{k,\cdot}$.

Let us recall the well-known many-to-one lemma:

Lemma 3.1 (many-to-one)**.** *For any $p \in \mathbb{N}^*$ and any bounded function $\boldsymbol{h}$*

$$\mathbf{E}[\boldsymbol{h}(S_1,\dots,S_p)] = \mathbf{E}\Big[\sum_{|x|=p} e^{-V(x)}\boldsymbol{h}(V(x_1),\dots,V(x_p))\Big],$$

where $(S_i)_{i\in\mathbb{N}}$ is the real valued random walk defined in (22).

We now state and prove a lemma that will be useful all along this section. For any vertex $x \in \mathbb{T}$, introduce $T_x := \inf\{i \geq 1\ X_i = x\}$, the hitting time of x and for any $\boldsymbol{x} = (x^{(1)},\dots,x^{(q)}) \in \Delta^q$, $T_{\boldsymbol{x}} := \max_{1\leq i\leq q} T_{x^{(i)}}$. Recall that for any $j \in \mathbb{N}^*$, T^j denotes the j-th return time to the parent e^* of the root e. For $1 \leq \ell < q$ two integers, $\boldsymbol{m} = (m_1,\dots,m_\ell) \in \mathbb{N}^{\times\ell}$ such that $m_1 < \cdots < m_\ell$ and $\Pi = (\boldsymbol{\pi})_{0\leq i\leq \ell}$ an increasing collection of partitions of $\{1,\dots,q\}$ that is to say $|\boldsymbol{\pi}_{i-1}| < |\boldsymbol{\pi}_i|$ with $\pi_0 = \{\{1,\dots,q\}\}$ and $\boldsymbol{\pi}_\ell = \{\{1\},\dots,\{q\}\}$, recall the definition of $f^\ell_{\boldsymbol{m},\Pi}$ in (14).

Lemma 3.2. *Let $k \geq 2$ and $\mathfrak{a} \geq 1$ be two integers and assume $\kappa > 2\mathfrak{a}k$. Let $q \in \{k,\dots,2\mathfrak{a}k\}$ and $\boldsymbol{p} = (p_1,\dots,p_q) \in \mathbb{N}^{\times q}$. Under the assumptions 1 and 3, there exists a constant $\mathfrak{C} > 0$ doesn't depending neither on $\boldsymbol{p}$, nor on $\boldsymbol{m}$ such that*

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\ |\boldsymbol{x}|=\boldsymbol{p}}} f^\ell_{\boldsymbol{m},\Pi}(\boldsymbol{x})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}} < T^1)\Big] \leq \mathfrak{C},$$

where $|\boldsymbol{x}| = \boldsymbol{p}$ means that $|x^{(i)}| = p_i$ for any $i \in \{1,\dots,q\}$. In particular, for any integer $m \in \mathbb{N}^*$, there exists a constant $C_{3.2} > 0$ doesn't depending on $\boldsymbol{p}$ such that

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\ |\boldsymbol{x}|=\boldsymbol{p}}} \mathbb{1}_{\mathscr{C}^k_m}(\boldsymbol{x})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}} < T^1)\Big] \leq C_{3.2}m^{q-1}. \tag{34}$$

Proof in the case $\cap_{j=1}^\ell \Gamma^j_{\boldsymbol{m},\Pi} \subset \{\boldsymbol{x} \in \Delta^q;\ \mathscr{C}^q(\boldsymbol{x}) < \min_{1\leq i\leq q} p_i\}$. First recall that Π^i is the partition of $\{1,\dots,|\boldsymbol{\pi}_i|\}$ obtained via the procedure defined above Example 4 and for any $i \in \{1,\dots,\ell\}$, any $j \in \{1,\dots,|\boldsymbol{\pi}_{i-1}|\}$, the j-th block $\boldsymbol{B}^i_j$ of the partition $\boldsymbol{\pi}_{i-1}$ is the union of $b_{i-1}(\boldsymbol{B}_j) \geq 1$ block(s) of the partition $\boldsymbol{\pi}_i$. Note (see the proof of Theorem 1.4) that

$$\sum_{\substack{\boldsymbol{x}\in\Delta^q\\ |\boldsymbol{x}|=\boldsymbol{p}}} f^\ell_{\boldsymbol{m},\Pi}(\boldsymbol{x})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}} < T^1) = \sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\ell-1}|}_{m_{\ell-1}}} f^{\ell-1}_{\boldsymbol{m}^{\ell-1},\Pi^{\ell-1}}(\boldsymbol{z}) \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} \sum_{\boldsymbol{u}^{(j)}\in\Delta^{b_{\ell-1}(\boldsymbol{B}_j)}_{m_\ell}} \prod_{i=1}^{b_{\ell-1}(\boldsymbol{B}_j)} \mathbb{1}_{\{(u^{(j,i)})^*=z^{(j)}\}}$$
$$\times \sum_{\boldsymbol{x}\in\Delta^q} \mathbb{1}_{\{|\boldsymbol{x}|=\boldsymbol{p},\ \boldsymbol{x}\geq\mathfrak{u}\}}\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}} < T^1),$$

where $\boldsymbol{m}^{\ell-1} = (m_1,\dots,m_{\ell-1})$, $\mathfrak{u}$ is the concatenation of $\boldsymbol{u}^{(1)},\dots,\boldsymbol{u}^{(|\boldsymbol{\pi}_{\ell-1}|)}$ and $\boldsymbol{x} \geq \mathfrak{u}$ means that $x^{(p)} \geq \mathfrak{u}^{(p)}$. Thanks to the strong Markov property at time $T_{z^{(i)}}$, there exists a constant $C_q \geq 1$

such that

$$\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}} f^{\ell}_{\boldsymbol{m},\Pi}(\boldsymbol{x})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}}<T^1) \le C_q \sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\ell-1}|}_{m_\ell-1}} f^{\ell-1}_{\boldsymbol{m}^{\ell-1},\Pi^{\ell-1}}(\boldsymbol{z})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{z}}<T^1)\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|}\sum_{\boldsymbol{u}^{(j)}\in\Delta^{b_{\ell-1}(\boldsymbol{B}_j)}_{m_\ell}}\prod_{i=1}^{b_{\ell-1}(\boldsymbol{B}_j)} \times \mathbb{1}_{\{(u^{(j,i)})^*=z^{(j)}\}}\sum_{\substack{\boldsymbol{x}^{(j)}\in\Delta^{b_{\ell-1}(\boldsymbol{B}_j)}\\|\boldsymbol{x}^{(j)}|=\boldsymbol{p}^{(j)}}}\mathbb{1}_{\{x^{(j,i)}\ge u^{(j,i)}\}}\mathbb{P}^{\mathscr{E}}_{z^{(j)}}(T_{x^{(j,i)}}<T^1),$$

where $\boldsymbol{p}$ is now seen as the concatenation of $\boldsymbol{p}^{(1)},\dots,\boldsymbol{p}^{(|\boldsymbol{\pi}_{\ell-1}|)}$. Moreover, it is known that for all $z\le x$ in $\mathbb{T}$,

$$\mathbb{P}^{\mathscr{E}}_z(T_x<T^1)=\frac{\sum_{e\le w\le z}e^{V(w)}}{\sum_{e\le w\le x}e^{V(w)}}\quad\text{if } z\neq e,\quad \mathbb{P}^{\mathscr{E}}(T_x<T^1)=\frac{1}{\sum_{e\le w\le x}e^{V(w)}}\quad\text{else,}\tag{35}$$

so $\mathbb{P}^{\mathscr{E}}_z(T_x<T^1)\le e^{-V(x)}\sum_{e\le w\le z}e^{V(w)}$. By independence of the increments of the branching random walk $(\mathbb{T},(V(x),x\in\mathbb{T}))$, using that $b_{\ell-1}(\boldsymbol{B}_j)=|\boldsymbol{B}_j^{\ell-1}|$ and $\psi(1)=0$

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}} f^{\ell}_{\boldsymbol{m},\Pi}(\boldsymbol{x})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{z}}<T^1)\Big]\le \mathfrak{C}_{\ell-1}\mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\ell-1}|}_{m_\ell-1}} f^{\ell-1}_{\boldsymbol{m}^{\ell-1},\Pi^{\ell-1}}(\boldsymbol{z})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{z}}<T^1)\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|}(H_{z^{(j)}})^{|\boldsymbol{B}_j^{\ell-1}|}\Big] \times C_q\prod_{\boldsymbol{B}\in\boldsymbol{\pi}_{\ell-1}}c_{|\boldsymbol{B}|}(\mathbf{1}),$$

with $H_z=\sum_{e\le w\le z}e^{V(w)-V(z)}$ and $\mathfrak{C}_{\ell-1}=C_q\prod_{\boldsymbol{B}\in\boldsymbol{\pi}_{\ell-1}}c_{|\boldsymbol{B}|}(\mathbf{1})\in(0,\infty)$ thanks to assumption 3 since for any $\boldsymbol{B}\in\pi_{\ell-1}$, $|\boldsymbol{B}|<q\le 4k<\kappa$. Again, thanks to the strong Markov property at time $T_{w^{(i)}}$

$$\begin{aligned}&\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\ell-1}|}_{m_\ell-1}} f^{\ell-1}_{\boldsymbol{m}^{\ell-1},\Pi^{\ell-1}}(\boldsymbol{z})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{z}}<T^1)\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|}(H_{z^{(j)}})^{|\boldsymbol{B}_j^{\ell-1}|}\\ &\le C_{\ell-1}\sum_{\boldsymbol{w}\in\Delta^{|\boldsymbol{\pi}_{\ell-2}|}_{m_{\ell-1}-1}} f^{\ell-2}_{\boldsymbol{m}^{\ell-2},\Pi^{\ell-2}}(\boldsymbol{w})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{w}}<T^1)\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|}\sum_{\boldsymbol{v}^{(j)}\in\Delta^{b_{\ell-2}(\boldsymbol{B}_j)}_{m_{\ell-1}}}\prod_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}\mathbb{1}_{\{(v^{(j,i)})^*=w^{(j)}\}}\\ &\times\sum_{\boldsymbol{z}^{(j)}\in\Delta^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}}\mathbb{1}_{\{z^{(j,i)}\ge v^{(j,i)}\}}(H_{z^{(j,i)}})^{\beta^{\ell-2}_{j,i}}\mathbb{P}^{\mathscr{E}}_{w^{(j)}}(T_{z^{(j,i)}}<T^1),\end{aligned}$$

for some constant $C_{\ell-1}\ge 1$, where $\boldsymbol{v}^{(j)}=(u^{(j,1)},\dots,u^{(j,b_{\ell-2}(\boldsymbol{B}_j))})$ and recall the definition of $\beta^{\ell-2}_{j,i}$ in (13). Thanks to (35)

$$(H_{z^{(j,i)}})^{\beta^{\ell-2}_{j,i}}\mathbb{P}^{\mathscr{E}}_{w^{(j)}}(T_{z^{(j,i)}}<T^1)=H_{w^{(j)}}e^{-V_{w^{(j)}}(z^{(j,i)})}(H_{z^{(j,i)}})^{\beta^{\ell-2}_{j,i}-1},$$

and $H_{z^{(j,i)}}=H_{v^{(j,i)}}e^{-V_{v^{(j,i)}}(z^{(j,i)})}+H_{v^{(j,i)},z^{(j,i)}}$ where, for any $u<x$, $H_{u,x}:=\sum_{u<w\le x}e^{V(w)-V(x)}$. Since $H_u\ge 1$ for all $u\in\mathbb{T}$, we have

$$H_{z^{(j,i)}}\le H_{w^{(j)}}\big(e^{-V_{w^{(i)}}(v^{(j,i)})}+1\big)\big(e^{-V_{v^{(j,i)}}(z^{(j,i)})}+H_{v^{(j,i)},z^{(j,i)}}\big),$$

thus giving that $(H_{z^{(j,i)}})^{\beta_{j,i}^{\ell-2}}\mathbb{P}^{\mathscr{E}}_{w^{(j)}}(T_{z^{(j,i)}}<T^1)$ is smaller than

$$(H_{w^{(j)}})^{\beta_{j,i}^{\ell-2}}e^{-V_{w^{(j)}}(v^{(j,i)})}\big(e^{-V_{w^{(i)}}(v^{(j,i)})}+1\big)^{\beta_{j,i}^{\ell-2}-1}e^{-V_{v^{(j,i)}}(z^{(j,i)})}\\ \times\big(e^{-V_{v^{(j,i)}}(z^{(j,i)})}+H_{v^{(j,i)},z^{(j,i)}}\big)^{\beta_{j,i}^{\ell-2}-1}.$$

By independence of the increments of the branching random walk $(\mathbb{T},(V(x),x\in\mathbb{T}))$, using that $\sum_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}\beta_{j,i}^{\ell-2}=|\boldsymbol{B}_j^{\ell-2}|$

$$\mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta_{m_{\ell-1}}^{|\boldsymbol{\pi}_{\ell-1}|}}f^{\ell-1}_{\boldsymbol{m}^{\ell-1},\Pi^{\ell-1}}(\boldsymbol{z})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{z}}<T^1)\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|}(H_{z^{(j)}})^{|\boldsymbol{B}_j^{\ell-1}|}\Big]$$

$$\le\mathfrak{C}_{\ell-2}\mathbf{E}\Big[\sum_{\boldsymbol{w}\in\Delta_{m_{\ell-1}-1}^{|\boldsymbol{\pi}_{\ell-2}|}}f^{\ell-2}_{\boldsymbol{m}^{\ell-2},\Pi^{\ell-2}}(\boldsymbol{w})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{w}}<T^1)\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|}(H_{w^{(j)}})^{|\boldsymbol{B}_j^{\ell-2}|}\Big],$$

where, thanks to the many-to-one Lemma 3.1

$$\mathfrak{C}_{\ell-2}=\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|}\mathbf{E}\Big[\sum_{\boldsymbol{v}\in\Delta_1^{b_{\ell-2}(\boldsymbol{B}_j)}}\prod_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}e^{-V(v^{(i)})}(e^{-V(v^{(i)})}+1)^{\beta_{j,i}^{\ell-2}}\Big]\prod_{\boldsymbol{B}\in\pi_{\ell-1}}\mathbf{E}\big[(e^{-S_{m_\ell^*}}+H^S_{m_\ell^*})^{|\boldsymbol{B}|-1}\big],$$

$m_\ell^*=m_\ell-m_{\ell-1}-1$, $H_m^S:=\sum_{p=1}^m e^{S_p-S_m}$ (the random walk (S_p) is defined in (22)). Note that $\mathfrak{C}_{\ell-2}\in(0,\infty)$. Indeed, the first mean in the definition of $\mathfrak{C}_{\ell-2}$ belongs to $(0,\infty)$ thanks to assumption 3 since for any $1\le j\le|\boldsymbol{\pi}_{\ell-2}|$, $b_{\ell-2}(\boldsymbol{B}_j)<q\le 2\mathfrak{a}k<\kappa$ and $\sum_{i=1}^{b_{\ell-2}(\boldsymbol{B}_j)}\beta_{j,i}^{\ell-2}=|\boldsymbol{B}_j^{\ell-2}|<q$. The second one also belongs to $(0,\infty)$ since for all $\boldsymbol{B}\in\pi_{\ell-1}$, $|\boldsymbol{B}|-1\le q-2<\kappa-2$ and as it is proved in [AD20] that $\sup_{m\in\mathbb{N}^*}\mathbf{E}[(H_m)^{\kappa-1-\varepsilon}]<\infty$ for any $\varepsilon>0$. We also deduce from this, together with the fact that $\psi'(1)<0$ and $m_\ell^*\ge 0$ that $\mathfrak{C}_{\ell-2}$ is bounded by a positive constant doesn't depending on $\boldsymbol{m}$. By induction, there exists a constant $\mathfrak{C}_2\in(0,\infty)$ (still not depending on $\boldsymbol{m}$) such that

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}}f^{\ell}_{\boldsymbol{m},\Pi}(\boldsymbol{x})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}}<T^1)\Big]\le\mathfrak{C}_2\mathbf{E}\Big[\sum_{|z|=m_1-1}\sum_{\boldsymbol{u}\in\Delta_{m_2-1}^{|\boldsymbol{\pi}_1|}}\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{u}}<T^1)\prod_{i=1}^{|\boldsymbol{\pi}_1|}(H_{u^{(i)}})^{|\boldsymbol{B}_i^1|}\mathbb{1}_{\{u^{(i)}>z\}}\Big].$$

Thanks to the strong Markov property, $\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{u}}<T^1)\le C_{|\boldsymbol{\pi}_1|}\mathbb{P}^{\mathscr{E}}(T_z<T^1)\prod_{i=1}^{|\boldsymbol{\pi}_1|}\mathbb{P}^{\mathscr{E}}_z(T_{u^{(i)}}<T^1)=C_{|\boldsymbol{\pi}_1|}\mathbb{P}^{\mathscr{E}}(T_z<T^1)\prod_{i=1}^{|\boldsymbol{\pi}_1|}e^{-V(u^{(i)})}/H_{u^{(i)}}$ for some constant $C_{|\boldsymbol{\pi}_1|}\ge 1$ and then, using (35) together with the many-to-one Lemma 3.1

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}}f^{\ell}_{\boldsymbol{m},\Pi}(\boldsymbol{x})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}}<T^1)\Big]\le\mathfrak{C}_1\mathbf{E}\Big[\sum_{|z|=m_1-1}e^{-V(z)}(H_z)^{|\boldsymbol{\pi}_1|-1}\Big]=\mathfrak{C}_1\mathbf{E}\big[(H^S_{m_1-1})^{|\boldsymbol{\pi}_1|-1}\big].$$

Again, $|\boldsymbol{\pi}_1|-1\le q-1\le 2\mathfrak{a}k-1<\kappa-1$ so $\mathbf{E}[(H^S_{m_1-1})^{|\boldsymbol{\pi}_1|-1}]\le\sup_{m\in\mathbb{N}^*}\mathbf{E}[(H^S_{m-1})^{|\boldsymbol{\pi}_1|-1}]\in(0,\infty)$ which ends the proof. □

3.1. The range on $\mathfrak{E}^{k,\cdot}$

This section is dedicated to the proof of Proposition 1 in which the range is restricted to the k-tuples of vertices belonging to the set $\mathfrak{E}^{k,\cdot}$, that is such that the vertices are visited during k distinct excursions, see (21) for the definition of $\mathfrak{E}^{k,\cdot}$.

3.1.1. The relevant vertices: the set $\mathscr{C}^k_{a_n}$

First recall that $\mathscr{C}^k_m = \{\boldsymbol{x} \in \Delta^k;\ \mathscr{S}^k(\boldsymbol{x}) \le m\}$ where, for any $\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)}) \in \Delta^k$ and $\mathscr{S}^k(\boldsymbol{x}) - 1$ is the last generation at which two or more vertices among $x^{(1)}, \dots, x^{(k)}$ share a common ancestor (see (10)). In this subsection, we focus on the range on $\mathfrak{C}^{k,\cdot} \cap \mathscr{C}^k_{a_n}$ with $a_n = (2\delta_0)^{-1} \log n$, which is the set of relevant k-tuples of vertices in the case of small generations. Before going any further, let us state and prove the following lemma. Recall that $H_u = \sum_{e \le z \le u} e^{V(z)-V(u)}$.

Lemma 3.3. *Let $k \ge 2$ and $\mathfrak{a} \ge 1$ be two integers and assume $\kappa > 2\mathfrak{a}k$. Under the assumptions 1, 3 and 4*

(i) for any integer $q \in \{k, \dots, 2\mathfrak{a}k\}$ and any $\boldsymbol{\beta} = (\beta_1, \dots, \beta_q) \in (\mathbb{N}^*)^{\times q}$ such that $\sum_{j=1}^q \beta_j \le 2\mathfrak{a}k$, there exists a constant $\mathfrak{C}_{3.3,1} > 0$ such that

$$\sup_{\boldsymbol{p} \in (\mathbb{N}^*)^{\times q}} \mathbf{E}\Big[\sum_{\substack{\boldsymbol{x} \in \Delta^q \\ |\boldsymbol{x}| = \boldsymbol{p}}} e^{-\langle \boldsymbol{\beta}, V(\boldsymbol{x}) \rangle_q} \Big] \le \mathfrak{C}_{3.3,1};$$

(ii) for any integer $q \in \{k, \dots, 2\mathfrak{a}k\}$ there exists a constant $\mathfrak{C}_{3.3,2} > 0$ such that for n large enough and any $h > 0$

$$\mathbf{E}\Big[\sum_{\boldsymbol{x} \in \Delta^q_{a_n}} \mathbb{1}_{\{\max_{1 \le i \le q} H_{x^{(i)}} > h\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x}) \rangle_q} \Big] \le \frac{\mathfrak{C}_{3.3,2}}{h^{\kappa - 1}} + o(1).$$

Proof in the case $\cap_{j=1}^{\ell} \Gamma^j_{\boldsymbol{m},\Pi} \subset \{\boldsymbol{x} \in \Delta^q;\ \mathscr{C}^q(\boldsymbol{x}) < \min_{1 \le i \le q} p_i\}$. Not that, since $H_u \ge 1$, we have $\mathbf{E}[\sum_{\boldsymbol{x} \in \Delta^q_{a_n}} e^{-\langle \mathbf{1}, V(\boldsymbol{x}) \rangle_q}] = \mathbf{E}[\sum_{\boldsymbol{x} \in \Delta^q_{a_n}} \mathbb{1}_{\{\max_{1 \le i \le q} H_{x^{(i)}} > h\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x}) \rangle_q}]$ for all $h \le 1$. The proof of *(i)* is similar to the proof of Theorem 1.4 and Lemma 3.2 so we focus on *(ii)*. In order to avoid unnecessary technical difficulties, we prove it for any $\mathfrak{a} \ge 2$. Recall the definition of $f^{\ell}_{\boldsymbol{s},\Pi}$ in (14) for $\ell \in \{1, \dots, q-1\}$, $\boldsymbol{s} = (s_1, \dots, s_\ell) \in \mathbb{N}^{\times \ell}$ such that $s_1 < \dots < s_\ell$ and $\Pi = (\boldsymbol{\pi}_i)_{0 \le i \le \ell}$ an increasing collection of partitions of $\{1, \dots, q\}$. Note that

$$\sum_{\boldsymbol{x} \in \Delta^q_{a_n}} \sum_{j=1}^{q} \mathbb{1}_{\{\max_{1 \le i \le q} H_{x^{(i)}} > h\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x}) \rangle_q} = \sum_{\ell=1}^{q-1} \sum_{\boldsymbol{s}; s_1 < \dots < s_\ell \le a_n} \sum_{\Pi \text{ increasing}} \sum_{\boldsymbol{x} \in \Delta^q_{a_n}} f^{\ell}_{\boldsymbol{s},\Pi}(\boldsymbol{x}) e^{-\langle \mathbf{1}, V(\boldsymbol{x}) \rangle_q} \times \mathbb{1}_{\{\max_{1 \le i \le q} H_{x^{(i)}} > h\}},$$

and $\sum_{\boldsymbol{x} \in \Delta^q_{a_n}} f^{\ell}_{\boldsymbol{s},\Pi}(\boldsymbol{x}) \mathbb{1}_{\{\max_{1 \le i \le q} H_{x^{(j)}} > h\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x}) \rangle_q}$ is equal to

$$\sum_{\boldsymbol{z} \in \Delta^{|\boldsymbol{\pi}_{\ell-1}|}_{s_{\ell-1}}} f^{\ell-1}_{\boldsymbol{s}^{\ell-1}, \Pi^{\ell-1}}(\boldsymbol{z}) \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} \sum_{\boldsymbol{u}^{(j)} \in \Delta^{b_{\ell-1}(\boldsymbol{B}_j)}_{s_\ell}} \prod_{i=1}^{b_{\ell-1}(\boldsymbol{B}_j)} \mathbb{1}_{\{(u^{(j,i)})^* = z^{(j)}\}} \sum_{\boldsymbol{x}^{(j)} \in \Delta^{b_{\ell-1}(\boldsymbol{B}_j)}_{a_n}} \mathbb{1}_{\{x^{(j,i)} \ge u^{(j,i)}\}} \times e^{-V(x^{(j,i)})} \mathbb{1}_{\{\max_{1 \le j' \le |\boldsymbol{\pi}_{\ell-1}|} \max_{1 \le i' \le b_{\ell-1}(\boldsymbol{B}_{j'})} H_{x^{(j',i')}} > h\}}.$$

Recall that for any $u < x$, $H_{u,x} = \sum_{u < z \le x} e^{V(z)-V(x)}$, thanks to assumption 4 together with the fact that $H_{z^{(j')}} \ge 1$

$$H_{x^{(j',i')}} \le H_{z^{(j')}}(1 + e^{\mathfrak{h}}) e^{-V_{u^{(j',i')}}(x^{(j',i')})} + H_{u^{(j',i')}, x^{(j',i')}},$$

so $H_{x^{(j',i')}} > h$ implies that $H_{z^{(j')}}(1+e^{\mathfrak{h}})e^{-V_{u^{(j',i')}}(x^{(j',i')})} > h/2$ or $H_{u^{(j',i')},x^{(j',i')}} > h/2$. We also decompose according to the values of $H_{z^{(j')}}$:

$$\begin{aligned}&\mathbb{1}_{\left\{\max_{1\le j'\le|\boldsymbol{\pi}_{\ell-1}|}\max_{1\le i'\le b_{\ell-1}(\boldsymbol{B}_{j'})}H_{z^{(j')}}(1+e^{\mathfrak{h}})e^{-V_{u^{(j',i')}}(x^{(j',i')})}>h/2\right\}}\\&\le \mathbb{1}_{\{\max_{1\le j'\le|\boldsymbol{\pi}_{\ell-1}|}H_{z^{(j')}}>h\}}+\mathbb{1}_{\left\{\max_{1\le j'\le|\boldsymbol{\pi}_{\ell-1}|}\max_{1\le i'\le b_{\ell-1}(\boldsymbol{B}_{j'})}2(1+e^{\mathfrak{h}})e^{-V_{u^{(j',i')}}(x^{(j',i')})}>1\right\}}.\end{aligned}$$

We therefore deduce that $\mathbb{1}_{\{\max_{1\le j'\le|\boldsymbol{\pi}_{\ell-1}|}\max_{1\le i'\le b_{\ell-1}(\boldsymbol{B}_{j'})}H_{x^{(j',i')}}>h\}}$ is smaller than

$$\mathbb{1}_{\{\max_{1\le j'\le|\boldsymbol{\pi}_{\ell-1}|}H_{z^{(j')}}>h\}}+\sum_{j'=1}^{|\boldsymbol{\pi}_{\ell-1}|}\sum_{i'=1}^{b_{\ell-1}(\boldsymbol{B}_{j'})}\Big(\mathbb{1}_{\{H_{u^{(j',i')},x^{(j',i')}}>h/2\}}+\mathbb{1}_{\left\{2(1+e^{\mathfrak{h}})e^{-V_{u^{(j',i')}}(x^{(j',i')})}>1\right\}}\Big).$$

By independence of the increments of the branching random walk $(\mathbb{T},(V(x),x\in\mathbb{T}))$, since $\psi(1)=0$

$$\begin{aligned}&\mathbf{E}\Big[\sum_{\boldsymbol{x}\in\Delta^q_{a_n}}f^{\ell}_{\boldsymbol{s},\Pi}(\boldsymbol{x})\mathbb{1}_{\{\max_{1\le i\le q}H_{x^{(i)}}>h\}}e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_q}\Big]\\&\le\mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}}f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}(\boldsymbol{z})\mathbb{1}_{\{\max_{1\le j'\le|\boldsymbol{\pi}_{\ell-1}|}H_{z^{(j')}}>h\}}e^{-\langle\boldsymbol{\beta}^{\ell-1},V(\boldsymbol{z})\rangle_{|\boldsymbol{\pi}_{\ell-1}|}}\Big]\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|}c_{b_{\ell-1}(\boldsymbol{B}_j)}(\mathbf{1})\\&+q(\mathfrak{z}_{1,n}+\mathfrak{z}_{2,n})\prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|}c_{b_{\ell-1}(\boldsymbol{B}_j)}(\mathbf{1})\mathbf{E}\Big[\mathscr{A}^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\boldsymbol{\beta}^{\ell-1}\big)\Big],\end{aligned}$$

where we recall that $\mathscr{A}^q_l(g,\beta)=\sum_{\boldsymbol{x}\in\Delta^q_l}e^{-\langle\beta,V(\boldsymbol{x})\rangle_q}$,

$$\mathfrak{z}_{1,n}=\mathbf{E}[\sum_{|x|=a_n-s_\ell}e^{-V(x)}\mathbb{1}_{\{H_x>h/2\}}]\quad\text{and}\quad\mathfrak{z}_{2,n}=\mathbf{E}[\sum_{|x|=a_n-s_\ell}e^{-V(x)}\mathbb{1}_{\{2(1+e^{\mathfrak{h}})e^{-V(x)}>1\}}].$$

Thanks to the many-to-one Lemma 3.1

$$\mathfrak{z}_{1,n}=\mathbf{P}(H^S_{a_n-s_\ell}>h/2)\le\mathfrak{C}_{3.3,3}/h^{\kappa-1},$$

for some constant $\mathfrak{C}_{3.3,3}>0$, the last inequality coming from ([AD20], Lemma 2.2). We now turn to $\mathfrak{z}_{n,2}$. If $s_\ell\le a_n/2$ then, for any $\rho\in(0,\kappa-1)$

$$\mathfrak{z}_{2,n}\le 2^{\rho}(1+e^{\mathfrak{h}})^{\rho}e^{a_n\psi(1+\rho)/2}.$$

Otherwise $s_\ell-1\ge a_n/2$ and thanks to the Cauchy–Schwarz inequality

$$\begin{aligned}\mathbf{E}\Big[\mathscr{A}^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\boldsymbol{\beta}^{\ell-1}\big)\Big]\le&\mathbf{E}\Big[\mathscr{A}^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\boldsymbol{\beta}^{\ell-1}\big)\mathbb{1}_{\tilde{\mathscr{V}}_n}\Big]+\\&+\big(1-\mathbf{P}(\tilde{\mathscr{V}}_n)\big)^{1/2}\mathbf{E}\Big[\Big(\mathscr{A}^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\boldsymbol{\beta}^{\ell-1}\big)\Big)^2\Big]^{1/2},\end{aligned}$$

where $\tilde{\mathscr{V}}_n:=\{\min_{a_n/2\le|z|\le a_n}V(z)>3/2\log n\}$ (recall that $a_n=(2\delta_0)^{-1}\log n$). On the one hand, by definition, there exists $i_\alpha\in\{1,\dots,|\boldsymbol{\pi}_{\ell-1}|\}$ such that $|\boldsymbol{B}^{\ell-1}_{i_\alpha}|\ge 2$. It follows that

$$\mathbf{E}\Big[\mathscr{A}^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\boldsymbol{\beta}^{\ell-1}\big)\mathbb{1}_{\tilde{\mathscr{V}}_n}\Big]\le n^{-3/2}\mathbf{E}\Big[\mathscr{A}^{|\boldsymbol{\pi}_{\ell-1}|}_{s_\ell-1}\big(f^{\ell-1}_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}},\tilde{\boldsymbol{\beta}}^{\ell-1}\big)\Big],$$

where $\tilde{\boldsymbol{\beta}}_j^{\ell-1} = |\boldsymbol{B}_j^{\ell-1}|$ for all $j \neq i_\alpha$ and $\tilde{\boldsymbol{\beta}}_{i_\alpha}^{\ell-1} = |\boldsymbol{B}_{i_\alpha}^{\ell-1}| - 1 \geq 1$. One the other hand, $1 - \mathbf{P}(\tilde{\mathscr{Y}}_n) \leq n^{-\rho_5}$ with $\rho_5 > 0$ thanks to Lemma 3.4. Moreover, both $\sum_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} \boldsymbol{\beta}_j^{\ell-1}$ and $\sum_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} \tilde{\boldsymbol{\beta}}_j^{\ell-1}$ are smaller than $2k$ since $q \leq 2k$. Hence, thanks to *(i)*

$$\mathbf{E}\Big[\mathscr{A}_{s_\ell-1}^{|\boldsymbol{\pi}_{\ell-1}|}\big(f_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}^{\ell-1}, \tilde{\boldsymbol{\beta}}^{\ell-1}\big)\Big] + \mathbf{E}\Big[\Big(\mathscr{A}_{s_\ell-1}^{|\boldsymbol{\pi}_{\ell-1}|}\big(f_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}^{\ell-1}, \boldsymbol{\beta}^{\ell-1}\big)\Big)^2\Big]^{1/2} \leq \mathfrak{C}_{3.3,4},$$

for some constant $\mathfrak{C}_{3.3,4} > 0$. We obtain

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{\boldsymbol{x}\in\Delta_{a_n}^q} f_{\boldsymbol{s},\Pi}^{\ell}(\boldsymbol{x})\mathbb{1}_{\{\max_{1\leq i\leq q} H_{x(i)}>h\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_q}\Big] \\
&\leq \mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta_{s_\ell-1}^{|\boldsymbol{\pi}_{\ell-1}|}} f_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}^{\ell-1}(\boldsymbol{z})\mathbb{1}_{\{\max_{1\leq j'\leq|\boldsymbol{\pi}_{\ell-1}|} H_{z(j')}>h\}} e^{-\langle \boldsymbol{\beta}^{\ell-1}, V(\boldsymbol{z})\rangle_{|\boldsymbol{\pi}_{\ell-1}|}}\Big] \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} c_{b_{\ell-1}(\boldsymbol{B}_j)}(\mathbf{1}) \\
&+ q_{\mathfrak{Z}1,n} \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-1}|} c_{b_{\ell-1}(\boldsymbol{B}_j)}(\mathbf{1}) \mathbf{E}\Big[\mathscr{A}_{s_\ell-1}^{|\boldsymbol{\pi}_{\ell-1}|}\big(f_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}^{\ell-1}, \boldsymbol{\beta}^{\ell-1}\big)\Big] + n^{-\rho_6},
\end{aligned}$$

thanks to the assumption 3 and for $\rho_6 > 0$. Note (see the proof of Theorem 1.4) that

$$\mathbf{E}\Big[\mathscr{A}_{s_\ell-1}^{|\boldsymbol{\pi}_{\ell-1}|}\big(f_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}^{\ell-1}, \boldsymbol{\beta}^{\ell-1}\big)\Big] = \mathscr{A}_{s_{\ell-1}-1}^{|\boldsymbol{\pi}_{\ell-2}|}\big(f_{\boldsymbol{s}^{\ell-2},\Pi^{\ell-2}}^{\ell-2}, \boldsymbol{\beta}^{\ell-2}\big) \prod_{j=1}^{|\boldsymbol{\pi}_{\ell-2}|} c_{b_{\ell-2}(\boldsymbol{B}_j)}(\boldsymbol{\beta}_j^{\ell-2}) \prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_{\ell-1}\\ |\mathfrak{B}|\geq 2}} e^{s_\ell^* \psi(|\mathfrak{B}|)},$$

with $\boldsymbol{\beta}_j^{\ell-2} = (\beta_{j,1}^{\ell-2}, \ldots, \beta_{j,b_{\ell-2}(\boldsymbol{B}_j)}^{\ell-2})$ and $s_\ell^* = s_\ell - s_{\ell-1} - 1$. Since for any $\mathfrak{B} \in \boldsymbol{\pi}_{\ell-1}$ such that $|\mathfrak{B}| \geq 2$, $\psi(|\mathfrak{B}|) < 0$, we have

$$\begin{aligned}
\sum_{s_\ell=s_{\ell-1}+1}^{a_n} \mathbf{E}\Big[\mathscr{A}_{s_\ell-1}^{|\boldsymbol{\pi}_{\ell-1}|}\big(f_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}^{\ell-1}, \boldsymbol{\beta}^{\ell-1}\big)\Big] &\leq \mathbf{E}\Big[\mathscr{A}_{s_{\ell-1}-1}^{|\boldsymbol{\pi}_{\ell-2}|}\big(f_{\boldsymbol{s}^{\ell-2},\Pi^{\ell-2}}^{\ell-2}, \boldsymbol{\beta}^{\ell-2}\big)\Big] \prod_{\boldsymbol{B}\in\boldsymbol{\pi}_{\ell-2}} c_{b_{\ell-2}(\boldsymbol{B})}(\boldsymbol{\beta}^{\ell-1}) \\
&\quad \Big(1 - \prod_{\substack{\mathfrak{B}\in\boldsymbol{\pi}_{\ell-1}\\ |\mathfrak{B}|\geq 2}} e^{\psi(|\mathfrak{B}|)}\Big)^{-1}.
\end{aligned}$$

Doing the same for $\mathbf{E}[\sum_{\boldsymbol{z}\in\Delta_{s_\ell-1}^{|\boldsymbol{\pi}_{\ell-1}|}} f_{\boldsymbol{s}^{\ell-1},\Pi^{\ell-1}}^{\ell-1}(\boldsymbol{z})\mathbb{1}_{\{\max_{1\leq j'\leq|\boldsymbol{\pi}_{\ell-1}|} H_{z(j')}>h\}} e^{-\langle \boldsymbol{\beta}^{\ell-1}, V(\boldsymbol{z})\rangle_{|\boldsymbol{\pi}_{\ell-1}|}}]$, we obtain, thanks to assumption 3

$$\begin{aligned}
&\sum_{\boldsymbol{s};s_1<\ldots<s_\ell\leq a_n} \mathbf{E}\Big[\sum_{\boldsymbol{x}\in\Delta_{a_n}^q} f_{\boldsymbol{s},\Pi}^{\ell}(\boldsymbol{x})\mathbb{1}_{\{\max_{1\leq i\leq q} H_{x(i)}>h\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_q}\Big] \\
&\leq \sum_{s_1<\ldots<s_{\ell-1}\leq a_n} \mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta_{s_{\ell-1}-1}^{|\boldsymbol{\pi}_{\ell-2}|}} f_{\boldsymbol{s}^{\ell-2},\Pi^{\ell-2}}^{\ell-2}(\boldsymbol{z})\mathbb{1}_{\{\max_{1\leq j'\leq|\boldsymbol{\pi}_{\ell-2}|} H_{z(j')}>h\}} e^{-\langle \boldsymbol{\beta}^{\ell-2}, V(\boldsymbol{z})\rangle_{|\boldsymbol{\pi}_{\ell-2}|}}\Big] \\
&+ \frac{\mathfrak{C}_{3.3,5}}{h^{\kappa-1}} + n^{-\rho_7},
\end{aligned}$$

for some constant $\mathfrak{C}_{3.3,5} > 0$ and $\rho_7 > 0$. We conclude by induction together with assumption 3. □

We remind the definition of the range $\mathscr{A}^k(\mathscr{D}_{n,T^s}, g)$

$$\mathscr{A}^k(\mathscr{D}_{n,T^s}, g) = \sum_{\substack{\boldsymbol{x}\in\Delta^k \\ \ell_n\le|\boldsymbol{x}|\le\mathfrak{L}_n}} g(\boldsymbol{x})\mathbb{1}_{\{T_{\boldsymbol{x}}<T^s\}},$$

where $T_{\boldsymbol{x}} = \max_{1\le i\le k} T_{x^{(i)}}$ and $\ell_n \le |\boldsymbol{x}| \le \mathfrak{L}_n$ means that $\ell_n \le |x^{(i)}| \le \mathfrak{L}_n$ for all $i \in \{1,\dots,k\}$. Vertices with high potential have a major contribution to the range. One can note that under the assumption 1, the potential $V(u)$ of the vertex $u \in \mathbb{T}$ behaves like $|u|$ when $|u|$ is large (see [Big76] and [HS09] for instance). It allows to say that

Fact 1. For all $\varepsilon \in (0,1)$, there exists $a_\varepsilon > 0$ such that

$$\mathbf{P}^*\big(\inf_{z\in\mathbb{T}} V(z) \ge -a_\varepsilon\big) \ge 1-\varepsilon. \tag{36}$$

Moreover,

Lemma 3.4. *Under the assumption 1, there exists $\delta_0 > 0$ and $\rho_1 > 1/2$ such that for any positive integer ζ*

$$\mathbf{P}\big(\min_{|z|=\delta_0^{-1}\zeta} V(z) \ge 3\zeta\big) \ge 1-e^{-\rho_1\zeta},$$

Using Lemma 3.4, we are able to prove that any vertex $x \in \mathbb{T}$ in a generation between $\delta_0^{-1}\log n$ and $n^{1/2}$ is visited during a single excursion above the parent e^* of the root e. For that, let us define the edge local time $N_u^T := \sum_{j=1}^T \mathbb{1}_{\{X_{j-1}=u^*, X_j=u\}}$ of the vertex $u \in \mathbb{T}$ and introduce

$$E_u^s := \sum_{j=1}^s \mathbb{1}_{\{N_u^{T^j} - N_u^{T^{j-1}} \ge 1\}},$$

the number of excursions during which the vertex x is visited by the random walk $\mathbb{X}$.

Lemma 3.5. *Under the assumption 1, for all $\varepsilon_1 \in (0,1)$, there exists $\rho_2 := \rho_2(\varepsilon_1) > 0$ such that for n large enough*

$$\mathbb{P}^*\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \bigcup_{|z|=\delta_0^{-1}\log n} \{E_z^s \ge 2\}\Big) \le n^{-\rho_2}.$$

The proof of Lemma 3.5 is similar to the one of Lemma 3.5 in [AD20].
Introduce the set $\mathfrak{S}^{k,s}$ of k-tuples of vertices visited during a single excursion:

$$\mathfrak{S}^{k,s} := \{\boldsymbol{x} = (x^{(1)},\dots,x^{(k)}) \in \Delta^k;\ \forall\, 1\le i\le k,\ E^s_{x^{(i)}} = 1\}. \tag{37}$$

In other words, Lemma 3.5 says that we can restrict the study of the range $\mathscr{A}^k(\mathscr{D}_{n,T^s}, f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathscr{C}^k_{a_n}})$ to the set $\mathfrak{S}^{k,s}$. This restriction allows to get quasi-independence in the trajectory of the random walk $\mathbb{X}$ and the resulting quasi-independent version of the range $\mathscr{A}^k(\mathscr{D}_{n,T^s}, f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathscr{C}^k_{a_n}})$ is easier to deal with. A similar idea is developed in [AD20] and [AK23]. Let $\boldsymbol{j} \in [\![1,s]\!]_k$, $\boldsymbol{p} \in \{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}$ and define

$$\mathscr{A}^{k,n}_{\boldsymbol{p}}(\boldsymbol{j}, g) := \sum_{\substack{\boldsymbol{x}\in\Delta^k \\ |\boldsymbol{x}|=\boldsymbol{p}}} g(\boldsymbol{x}) \prod_{i=1}^k \mathbb{1}_{\{N^{T^{j_i}}_{x^{(i)}} - N^{T^{j_i-1}}_{x^{(i)}} \ge 1\}} \quad \text{and} \quad \mathscr{A}^{k,n}(\boldsymbol{j}, g) := \sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}} \mathscr{A}^{k,n}_{\boldsymbol{p}}(\boldsymbol{j}, g), \tag{38}$$

where for any $\boldsymbol{x} = (x^{(1)}, \dots, x^{(k)})$, $|\boldsymbol{x}| = \boldsymbol{p}$ means nothing but $|x^{(i)}| = p_i$ for all $i \in \{1, \dots, k\}$. In the next lemma, we show that $\mathscr{A}^k(\mathscr{D}_{n,T^s}, f\mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathscr{C}^k})$ and $\sum_{\boldsymbol{j} \in [\![1,s]\!]_k} \mathscr{A}^{k,n}(\boldsymbol{j}, f\mathbb{1}_{\mathscr{C}^k})$ have the same behavior

Lemma 3.6. *Let $k \geq 2$ be an integer and assume $\kappa > 2k$. Under the assumptions 1 and 3, for all bounded function g, any $\varepsilon, \varepsilon_1 \in (0,1)$, there exists $\rho_4 := \rho_4(\varepsilon, \varepsilon_1) > 0$ such that for n large enough and any subset $\mathfrak{R}$ of $\mathbb{T}^{\times k}$*

$$\mathbb{P}^*\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1} \Big\{ \Big| \mathscr{A}^k(\mathscr{D}_{n,T^s}, g\mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{R}}) - \sum_{\boldsymbol{j} \in [\![1,s]\!]_k} \mathscr{A}^{k,n}(\boldsymbol{j}, g\mathbb{1}_{\mathfrak{R}}) \Big| > \varepsilon (s\boldsymbol{L}_n)^k \Big\} \Big) \leq n^{-\rho_4}.$$

Proof. We first decompose as follows

$$\mathscr{A}^k(\mathscr{D}_{n,T^s}, g\mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{R}}) = \mathscr{A}^k(\mathscr{D}_{n,T^s}, g\mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{S}^{k,s} \cap \mathfrak{R}}) + \mathscr{A}^k(\mathscr{D}_{n,T^s}, g\mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{R} \cap \Delta^k \setminus \mathfrak{S}^{k,s}}).$$

We first deal with $\mathscr{A}^k(\mathscr{D}_{n,T^s}, g\mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{S}^{k,s} \cap \mathfrak{R}})$. Note that $\boldsymbol{x} \in \mathfrak{E}^{k,s} \cap \mathfrak{S}^{k,s}$ means nothing but there exists $\boldsymbol{j} \in [\![1,s]\!]_k$ such that for any $i \in \{1, \dots, k\}$, $N^{T^{j_i}}_{x^{(i)}} - N^{T^{j_i - 1}}_{x^{(i)}} \geq 1$ and for all $\mathfrak{j} \neq j_i$, $N^{T^{\mathfrak{j}}}_{x^{(i)}} - N^{T^{\mathfrak{j}-1}}_{x^{(i)}} = 0$, thus giving that $\mathscr{A}^k_{\boldsymbol{p}}(\mathscr{D}_{n,T^s}, g\mathbb{1}_{\mathfrak{E}^{k,s} \cap \mathfrak{S}^{k,s} \cap \mathfrak{R}})$ is equal to

$$\sum_{\boldsymbol{p} \in \{\ell_n, \dots, \mathfrak{L}_n\}^{\times k}} \sum_{\boldsymbol{j} \in [\![1,s]\!]_k} \sum_{\substack{\boldsymbol{x} \in \Delta^k \\ |\boldsymbol{x}| = \boldsymbol{p}}} g\mathbb{1}_{\mathfrak{R}}(\boldsymbol{x}) \prod_{i=1}^k \mathbb{1}_{\{N^{T^{j_i}}_{x^{(i)}} - N^{T^{j_i-1}}_{x^{(i)}} \geq 1;\ \forall \mathfrak{j} \neq j_i, N^{T^{\mathfrak{p}}}_{x^{(i)}} - N^{T^{\mathfrak{p}-1}}_{x^{(i)}} = 0\}},$$

Thanks to the strong Markov property, the random variables $N^{T^l}_z - N^{T^{l-1}}_z$ are i.i.d under $\mathbb{P}^{\mathscr{E}}$ and distributed as $N^{T^1}_z$. It follows that

$$\begin{aligned}
&\mathbb{E}^{\mathscr{E}}\Big[\sum_{\substack{\boldsymbol{x} \in \Delta^k \\ |\boldsymbol{x}| = \boldsymbol{p}}} g\mathbb{1}_{\mathfrak{R}}(\boldsymbol{x}) \prod_{i=1}^k \mathbb{1}_{\{N^{T^{j_i}}_{x^{(i)}} - N^{T^{j_i-1}}_{x^{(i)}} \geq 1;\ \forall \mathfrak{j} \neq j_i, N^{T^{\mathfrak{p}}}_{x^{(i)}} - N^{T^{\mathfrak{p}-1}}_{x^{(i)}} = 0\}} \Big] \\
&= \sum_{\substack{\boldsymbol{x} \in \Delta^k \\ |\boldsymbol{x}| = \boldsymbol{p}}} g\mathbb{1}_{\mathfrak{R}}(\boldsymbol{x}) \prod_{i=1}^k \mathbb{P}^{\mathscr{E}}(T_{x^{(i)}} < T^1)(1 - \mathbb{P}^{\mathscr{E}}(T_{x^{(i)}} < T^1))^{s-1}.
\end{aligned}$$

Using (35), $\mathbb{P}^{\mathscr{E}}(T_{x^{(i)}} < T^1) \leq e^{-V(x^{(i)})}$ and on $\mathscr{V}_n = \{\min_{\delta_0^{-1} \log n \leq |x| \leq n^{1/2}} V(z) \geq 3 \log n\}$

$$(1 - \mathbb{P}^{\mathscr{E}}(T_{x^{(i)}} < T^1))^{s-1} \geq (1 - e^{-V(x^{(i)})})^s \geq (1 - n^{-3})^s \geq (1 - n^{-3})^{n^{1/2}/\varepsilon_1},$$

so

$$\begin{aligned}
&\mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j} \in [\![1,s]\!]_k} \sum_{\substack{\boldsymbol{x} \in \Delta^k \\ |\boldsymbol{x}| = \boldsymbol{p}}} g\mathbb{1}_{\mathfrak{R}}(\boldsymbol{x}) \prod_{i=1}^k \mathbb{1}_{\{N^{T^{j_i}}_{x^{(i)}} - N^{T^{j_i-1}}_{x^{(i)}} \geq 1\}} \Big] \\
&- \mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j} \in [\![1,s]\!]_k} \sum_{\substack{\boldsymbol{x} \in \Delta^k \\ |\boldsymbol{x}| = \boldsymbol{p}}} g\mathbb{1}_{\mathfrak{R}}(\boldsymbol{x}) \prod_{i=1}^k \mathbb{1}_{\{N^{T^{j_i}}_{x^{(i)}} - N^{T^{j_i-1}}_{x^{(i)}} \geq 1;\ \forall p \neq j_i, N^{T^{p}}_{x^{(i)}} - N^{T^{p-1}}_{x^{(i)}} = 0\}} \Big] \\
&\leq s(s-1) \times \cdots \times (s-k+1) \sum_{\substack{\boldsymbol{x} \in \Delta^k \\ |\boldsymbol{x}| = \boldsymbol{p}}} g(\boldsymbol{x}) e^{-\langle \mathbf{1}, V(\boldsymbol{x}) \rangle_k} \big(1 - (1 - n^{-3})^{kn^{1/2}/\varepsilon_1}\big).
\end{aligned}$$

The next step is to show that the mean of the previous sum goes to 0 when n goes to ∞, uniformly in $\boldsymbol{p}$. For that, one can notice that

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}} g(\boldsymbol{x})e^{-\langle \mathbf{1},V(\boldsymbol{x})\rangle_k}\big(1-(1-n^{-3})^{\frac{kn^{1/2}}{\varepsilon_1}}\big)\Big] \le \frac{2kn^{-5/2}}{\varepsilon_1}\|g\|_\infty \mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}} e^{-\langle \mathbf{1},V(\boldsymbol{x})\rangle_k}\Big],$$

thus giving, thanks to Lemma 3.3 (i) with $\boldsymbol{\beta}=\mathbf{1}$

$$\begin{aligned}\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}} g\mathbb{1}_{\mathfrak{R}}(\boldsymbol{x})e^{-\langle \mathbf{1},V(\boldsymbol{x})\rangle_k}\big(1-(1-n^{-3})^{\frac{n^{1/2}}{\varepsilon_1}}\big)\Big] &\le \frac{2kn^{-5/2}}{\varepsilon_1}\|g\|_\infty \sup_{\boldsymbol{p}\in(\mathbb{N}^*)^{\times k}} \mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta_p^k\\|\boldsymbol{x}|=\boldsymbol{p}}} e^{-\langle \boldsymbol{\beta},V(\boldsymbol{x})\rangle_k}\Big]\\ &\le \mathfrak{C}_{3.6}n^{-5/2}, \end{aligned} \tag{39}$$

for some constant $\mathfrak{C}_{3.6}>0$ doesn't depending on $\boldsymbol{p}$. We then use (39) to conclude. Note that

$$\begin{aligned}&\mathbb{P}^*\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\Big\{\Big|\mathscr{A}^k(\mathscr{D}_{n,T^s},g\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathfrak{R}})-\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},g\mathbb{1}_{\mathfrak{R}})\Big|>\varepsilon(s\boldsymbol{L}_n)^k\Big\}\Big)\\
&\le \mathbb{P}^*\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\Big\{\Big|\mathscr{A}^k(\mathscr{D}_{n,T^s},g\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathfrak{R}})-\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},g\mathbb{1}_{\mathfrak{R}})\Big|>\varepsilon(s\boldsymbol{L}_n)^k\Big\},\mathscr{V}_n\Big)\\
&+1-\mathbf{P}^*(\mathscr{V}_n).\end{aligned}$$

By Lemma 3.4, for n large enough $1-\mathbf{P}^*(\mathscr{V}_n)\le n^{-\rho_1'}$ for some $\rho_1'>0$ and thanks to Markov inequality

$$\begin{aligned}&\mathbb{P}^*\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\Big\{\Big|\mathscr{A}^k(\mathscr{D}_{n,T^s},g\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathfrak{R}})-\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},g\mathbb{1}_{\mathfrak{R}})\Big|>\varepsilon(s\boldsymbol{L}_n)^k\Big\},\mathscr{V}_n\Big)\\
&\le \mathbb{P}^*\Big(\bigcup_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\{\mathscr{A}^k(\mathscr{D}_{n,T^s},g\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathfrak{R}\cap\Delta^k\setminus\mathfrak{S}^{k,s}})>\varepsilon(s\boldsymbol{L}_n)^k/2\}\Big)\\
&+\sum_{s=\varepsilon_1 n^{1/2}}^{n^{1/2}/\varepsilon_1}\sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\frac{2}{\varepsilon(s\boldsymbol{L}_n)^k}\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}} g\mathbb{1}_{\mathfrak{R}}(\boldsymbol{x})e^{-\langle \mathbf{1},V(\boldsymbol{x})\rangle_k}\big(1-(1-n^{-3})^{\frac{kn^{1/2}}{\varepsilon_1}}\big)\Big],\end{aligned}$$

which, thanks to Lemma 3.5 and (39), is smaller than $n^{-\rho_2}+\frac{2\mathfrak{C}_{3.6}}{n^2\varepsilon_1\varepsilon}$ for n large enough and the proof is completed. □

The next lemma relates $\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})$ with its quenched mean and illustrates why this quasi-independent version of the range is easier to deal with.

Lemma 3.7. *Let $k\ge 2$ and $\mathfrak{a}\ge 1$ be two integers and assume $\kappa>2\mathfrak{a}k$. Under the assumptions 1, 2 and 3, there exits a constant $\mathfrak{C}_{3.7}>0$ and a non-decreasing sequence of positive numbers $(\tilde{\mathfrak{q}}_j)_{j\ge 2}$ satisfying $\tilde{\mathfrak{q}}_2=1$ and $\tilde{\mathfrak{q}}_j\to\infty$ when $j\to\infty$ such that for n large enough and any $\varepsilon_1 n^{1/2}\le s\le n^{1/2}/\varepsilon_1$*

$$\mathbb{E}\Big[\Big(\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})-\mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big]\Big)^{2\mathfrak{a}}\Big]\le \mathfrak{C}_{3.7}(\boldsymbol{L}_n)^{2\mathfrak{a}k}(\mathfrak{L}_n)^{\tilde{\mathfrak{q}}_{\mathfrak{a}}}s^{2\mathfrak{a}k-\tilde{\mathfrak{q}}_{\mathfrak{a}}}.$$

Proof. Recall the definition of $\mathscr{A}^{k,n}(\boldsymbol{j}, f\mathbb{1}_{\mathscr{C}^k_{a_n}})$ in (38). Thanks to the strong Markov property, the random variables $N_z^{T^i} - N_z^{T^{i-1}}$ are i.i.d under $\mathbb{P}^{\mathscr{E}}$ and distributed as $N_z^{T^1}$ for any $z \in \mathbb{T}$. It follows that

$$\begin{aligned}&\mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k} \mathscr{A}^{k,n}(\boldsymbol{j}, f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big]\\ &= s(s-1)\cdots(s-k+1)\sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\substack{\boldsymbol{x}\in\Delta^k\\ |\boldsymbol{x}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x})\prod_{i=1}^k \mathbb{P}^{\mathscr{E}}(N^{T^1}_{x^{(i)}}\geq 1).\end{aligned}$$

and for $\mathfrak{a} = 1$, the term s^{2k} in the above mean is equal to 0 and using again the strong Markov property leads to

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{\boldsymbol{j}\in[\![1,s]\!]_k} \mathscr{A}^{k,n}(\boldsymbol{j}, f\mathbb{1}_{\mathscr{C}^k_{a_n}}) - \mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k} \mathscr{A}^{k,n}(\boldsymbol{j}, f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big]\Big)^2\Big]\\ &\leq \mathfrak{C}_{3.7,1}(\boldsymbol{L}_n)^{2k}\big((\mathfrak{L}_n)^2 s^{2k-2} + \mathfrak{L}_n s^{2k-1}\big) \leq 2\mathfrak{C}_{3.7,1}(\boldsymbol{L}_n)^{2k}\mathfrak{L}_n s^{2k-1},\end{aligned}$$

where the constant $\mathfrak{C}_{3.7,1} > 0$ comes from Lemma 3.2 and the last inequality comes the fact that $\mathfrak{L}_n \leq s$ for n large enough.
When $\mathfrak{a} \geq 2$, using similar arguments we have

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{\boldsymbol{j}\in[\![1,s]\!]_k} \mathscr{A}^{k,n}(\boldsymbol{j}, f\mathbb{1}_{\mathscr{C}^k_{a_n}}) - \mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k} \mathscr{A}^{k,n}(\boldsymbol{j}, f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big]\Big)^{2\mathfrak{a}}\Big]\\ &\leq \mathfrak{C}_{3.7,2} \leq (\boldsymbol{L}_n)^{2\mathfrak{a}k}(\mathfrak{L}_n)^{2\lfloor \mathfrak{a}/2\rfloor} s^{2\mathfrak{a}k - 2\lfloor \mathfrak{a}/2\rfloor}.\end{aligned}$$

We finally obtain the result by taking $\mathfrak{q}_{\mathfrak{a}} := \mathfrak{a}\mathbb{1}_{\mathfrak{a}=1} + 2\lfloor \mathfrak{a}/2\rfloor \mathbb{1}_{\mathfrak{a}\geq 2}$. □

3.1.2. Convergence of the quenched mean of the range on $\mathscr{C}^k_{a_n}$

We prove that the quenched mean of the quasi-independent version $\sum_{\boldsymbol{j}\in[\![1,s]\!]_k} \mathscr{A}^k(\boldsymbol{j}, f\mathscr{C}^k_{a_n})$ of the range on the set $\mathscr{C}^k_{a_n}$ converges in $\mathbf{P}^*$-probability by using the hereditary assumption 5.

Lemma 3.8. *Let $k \geq 2$ be an integer and assume $\kappa > 2k$. Under the assumptions 1, 3 and 4, if f satisfies the hereditary assumption 5 then*

$$\lim_{n\to\infty} \mathbf{E}^*\Big[\Big|\frac{1}{(\boldsymbol{L}_n)^k}\sum_{\substack{\boldsymbol{x}\in\Delta^k\\ \ell_n\leq|\boldsymbol{x}|\leq\mathfrak{L}_n}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x})\prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} - (c_\infty)^k \mathscr{A}^k_{a_n}(f)\Big|\Big] = 0$$

Proof. The first step is to decompose $\sum_{\boldsymbol{x}\in\Delta^k;\ell_n\leq|\boldsymbol{x}|\leq\mathfrak{L}_n} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x})\prod_{i=1}^k e^{-V(x^{(i)})}/H_{x^{(i)}}$:

$$\begin{aligned}\sum_{\substack{\boldsymbol{x}\in\Delta^k\\ \ell_n\leq|\boldsymbol{x}|\leq\mathfrak{L}_n}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x})\prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} &= \sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\boldsymbol{z}\in\Delta^k_{a_n}}\sum_{\substack{\boldsymbol{x}\in\Delta^k\\ |\boldsymbol{x}|=\boldsymbol{p};\ x^{(i)}>z^{(i)}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x})\prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}}\\ &= \sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\boldsymbol{z}\in\Delta^k_{a_n}} f(\boldsymbol{z})\sum_{\substack{\boldsymbol{x}\in\Delta^k\\ |\boldsymbol{x}|=\boldsymbol{p};\ x^{(i)}>z^{(i)}}}\prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}},\end{aligned}$$

where the last equality comes from the hereditary assumption 5. As we did above, we decompose $H_{x^{(i)}}$: $H_{x^{(i)}} = H_{z^{(i)}} e^{-V_{z^{(i)}}(x^{(i)})} + H_{z^{(i)},x^{(i)}}$. By independence of the increments of the branching random walk $(\mathbb{T}, (V(x), x \in \mathbb{T}))$

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k \\ |\boldsymbol{x}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \Big| \mathscr{F}_{a_n} \Big] = \sum_{\boldsymbol{z}\in\Delta^k_{a_n}} f(\boldsymbol{z}) \prod_{i=1}^k e^{-V(z^{(i)})} \varphi_{n,p_i}(H_{z^{(i)}}), \tag{40}$$

where $\mathscr{F}_{a_n} = \sigma(\mathbb{T}, (V(x); |x| \le a_n))$ and $\varphi_{n,p_i}(r) = \mathbf{E}[\sum_{|x|=p_i-a_n} e^{-V(x)}/(re^{-V(x)} + H_x)]$. Let us prove that

$$\lim_{n\to\infty} \sup_{\ell_n \le \boldsymbol{p} \le \mathfrak{L}_n} \mathbf{E}\Big[\Big(\sum_{\substack{\boldsymbol{x}\in\Delta^k \\ |\boldsymbol{x}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} - \sum_{\boldsymbol{x}\in\Delta^k_{a_n}} f(\boldsymbol{x}) \prod_{i=1}^k e^{-V(x^{(i)})} \varphi_{n,p_i}(H_{x^{(i)}}) \Big)^2 \Big] = 0. \tag{41}$$

Thanks to (40)

$$\begin{aligned}
&\mathbf{E}\Big[\Big(\sum_{\substack{\boldsymbol{x}\in\Delta^k \\ |\boldsymbol{x}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} - \sum_{\boldsymbol{x}\in\Delta^k_{a_n}} f(\boldsymbol{x}) \prod_{i=1}^k e^{-V(x^{(i)})} \varphi_{n,p_i}(H_{x^{(i)}}) \Big)^2 \Big] \\
&= \mathbf{E}\Big[\Big(\sum_{\substack{\boldsymbol{x}\in\Delta^k \\ |\boldsymbol{x}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \Big)^2 \Big] - \mathbf{E}\Big[\Big(\sum_{\boldsymbol{x}\in\Delta^k_{a_n}} f(\boldsymbol{x}) \prod_{i=1}^k e^{-V(x^{(i)})} \varphi_{n,p_i}(H_{x^{(i)}}) \Big)^2 \Big].
\end{aligned}$$

For $\boldsymbol{x}, \boldsymbol{y} \in \Delta^k$, denote by $\boldsymbol{xy} = (x^{(1)}, \dots, x^{(k)}, y^{(1)}, \dots, y^{(k)})$ the concatenation of $\boldsymbol{x}$ and $\boldsymbol{y}$. Note that

$$\begin{aligned}
\Big(\sum_{\substack{\boldsymbol{x}\in\Delta^k \\ |\boldsymbol{x}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \Big)^2 &= \sum_{\substack{\boldsymbol{x},\boldsymbol{y};\ \boldsymbol{xy}\in\Delta^{2k} \\ |\boldsymbol{x}|=|\boldsymbol{y}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{y}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \frac{e^{-V(y^{(i)})}}{H_{y^{(i)}}} \\
&\quad + \sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Delta^k;\ \boldsymbol{xy}\notin\Delta^{2k} \\ |\boldsymbol{x}|=|\boldsymbol{y}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{y}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \frac{e^{-V(y^{(i)})}}{H_{y^{(i)}}},
\end{aligned}$$

where for any $\boldsymbol{x}, \boldsymbol{y} \in \Delta^k$, $\boldsymbol{xy} \notin \Delta^{2k}$ means that there exists $\alpha \in \{1, \dots, k\}$ and $i_1, \dots, i_\alpha \in \{1, \dots, k\}$ distinct such that $x^{(i_j)} = y^{(i_j)}$ for all $j \in \{1, \dots, \alpha\}$. It follows

$$\lim_{n\to\infty} \sup_{\ell_n \le \boldsymbol{p} \le \mathfrak{L}_n} \mathbf{E}\Big[\sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Delta^k;\ \boldsymbol{xy}\notin\Delta^{2k} \\ |\boldsymbol{x}|=|\boldsymbol{y}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{y}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \frac{e^{-V(y^{(i)})}}{H_{y^{(i)}}} \Big] = 0,$$

Indeed, by independence of the increments of the branching random walk $(\mathbb{T}, (V(x), x \in \mathbb{T}))$, since

$H_x \geq 1$

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x},\boldsymbol{y}\in\Delta^k;\ \boldsymbol{x}\boldsymbol{y}\notin\Delta^{2k}\\ |\boldsymbol{x}|=|\boldsymbol{y}|=\boldsymbol{p}}} f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x}) f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{y}) \prod_{i=1}^k \frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}} \frac{e^{-V(y^{(i)})}}{H_{y^{(i)}}}\Big]$$
$$\leq \|f\|_\infty^2 \sum_{\alpha=1}^k \sum_{i_1\neq i_2\ldots\neq i_\alpha=1}^k \prod_{j=1}^\alpha e^{(p_{i_j}-a_n)\psi(2)} \mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta^{2k-\alpha}_{a_n}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_{2k-\alpha}}\Big].$$

By Lemma 3.3, $\sum_{n\in\mathbb{N}} \mathbf{E}[\sum_{\boldsymbol{z}\in\Delta^{2k-\alpha}_{a_n}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_{2k-\alpha}}] < \infty$ and for any $p_i \in \{\ell_n, \ldots, \mathfrak{L}_n\}$, $i \in \{1,\ldots,k\}$, $e^{(p_{i_j}-a_n)\psi(2)} \leq e^{(\ell_n-a_n)\psi(2)} \to 0$ when $n\to\infty$ since $\ell_n \geq \delta_0^{-1}\log n$ and $a_n = (2\delta_0)^{-1}\log n$, which gives (41). We now prove that

$$\lim_{n\to\infty} \mathbf{E}\Big[\Big|(c_\infty)^k \sum_{\boldsymbol{z}\in\Delta^k_{a_n}} f(\boldsymbol{z}) \prod_{i=1}^k e^{-V(z^{(i)})} - \sum_{\boldsymbol{z}\in\Delta^k_{a_n}} f(\boldsymbol{z}) \prod_{i=1}^k e^{-V(z^{(i)})} \varphi_{n,p_i}(H_{z^{(i)}})\Big|\Big] = 0. \tag{42}$$

Let $h_n = \log n$ (the choice of h_n is almost arbitrary, $h_n\to\infty$ with $h_n = o(n^\theta)$ for all $\theta > 0$ should be enough). Note that $|c_\infty - \varphi_{n,p_i}(H_{z^{(i)}})| \leq 2$ so

$$\mathbf{E}\Big[\Big|(c_\infty)^k \sum_{\boldsymbol{z}\in\Delta^k_{a_n}} f(\boldsymbol{z}) \prod_{i=1}^k e^{-V(z^{(i)})} - \sum_{\boldsymbol{z}\in\Delta^k_{a_n}} f(\boldsymbol{z}) \prod_{i=1}^k e^{-V(z^{(i)})} \varphi_{n,p_i}(H_{z^{(i)}})\Big|\Big]$$
$$\leq \|f\|_\infty \mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta^k_{a_n}} \prod_{i=1}^k e^{-V(z^{(i)})} |c_\infty - \varphi_{n,p_i}(H_{z^{(i)}})| \mathbb{1}_{\{H_{z^{(i)}}\leq h_n\}}\Big]$$
$$+ 2\|f\|_\infty \mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta^k_{a_n}} \mathbb{1}_{\{\max_{1\leq i\leq k} H_{z^{(i)}} > h_n\}} e^{-\langle \mathbf{1}, V(\boldsymbol{z})\rangle_k}\Big].$$

We show that $\lim_{n\to\infty} \sup_{\{\ell_n\leq p_i\leq \mathfrak{L}_n,\ 1\leq r\leq h_n\}} |\varphi_{n,p_i}(r) - c_\infty| = 0$. For that, on the first hand, $\varphi_{n,p_i}(r) \leq \mathbf{E}[1/H^S_{\ell_n - a_n}]$ where we recall that $H^S_m = \sum_{j=1}^m e^{S_j - S_m}$ (see (22) for the definition of the random walk S). On the other, for any $\ell_n \leq p_i \leq \mathfrak{L}_n$ and $1\leq r\leq h_n$, $\varphi_{n,p_i}(r)$ is larger, for any $r_1 > 0$, than

$$\mathbf{E}\Big[\sum_{|x|=p_i-a_n} \frac{e^{-V(x)}}{h_n e^{-V(x)} + H_x} \mathbb{1}_{\{V(x)\geq r_1\log n\}}\Big] \geq \mathbf{E}\Big[\frac{1}{h_n n^{-r_1} + H^S_{\mathfrak{L}_n - a_n}}\Big] - \mathbf{P}(S_{p_i-a_n} < r_1\log n).$$

where we have used the many-to-one Lemma 3.1.
Note that $\mathbf{P}(S_{p_i-a_n} < 3\log n) \leq \mathbf{P}(\min_{(2\delta_0)^{-1}\log n\leq j\leq \mathfrak{L}_n} S_j < r_1\log n) \to 0$ when $n\to\infty$ for some $r_1 > 0$ since $a_n = (2\delta_0)^{-1}\log n$ and $\psi'(1) < 0$. Moreover, by definition, both ($\mathbf{E}[1/H^S_{\ell_n-a_n}]$) and ($\mathbf{E}[1/(h_n n^{-r_1} + H^S_{\mathfrak{L}_n-a_n})]$) goes to c_∞ when n goes to ∞ and we obtain the convergence. Then

$$\mathbf{E}\Big[\Big|(c_\infty)^k \sum_{\boldsymbol{z}\in\Delta^k_{a_n}} f(\boldsymbol{z}) \prod_{i=1}^k e^{-V(z^{(i)})} - \sum_{\boldsymbol{z}\in\Delta^k_{a_n}} f(\boldsymbol{z}) \prod_{i=1}^k e^{-V(z^{(i)})} \varphi_{n,p_i}(H_{z^{(i)}})\Big|\Big]$$
$$\leq \|f\|_\infty \mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta^k_{a_n}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k}\Big] \max_{1\leq i\leq k} \sup_{\ell_n\leq p_i\leq \mathfrak{L}_n,\ 1\leq r\leq h_n} |\varphi_{n,p_i}(r) - c_\infty|^k$$
$$+ 2\|f\|_\infty \mathbf{E}\Big[\sum_{\boldsymbol{z}\in\Delta^k_{a_n}} \mathbb{1}_{\{\max_{1\leq i\leq k} H_{z^{(i)}} > h_n\}} e^{-\langle \mathbf{1}, V(\boldsymbol{z})\rangle_k}\Big].$$

Using Lemma 3.3, first *(i)*, then *(ii)* with $h = h_n$, $\sup_{n\in\mathbb{N}} \mathbf{E}[\sum_{\boldsymbol{z}\in\Delta^k_{a_n}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k}] < \infty$ and $\lim_{n\to\infty} \mathbf{E}[\sum_{\boldsymbol{z}\in\Delta^k_{a_n}} \mathbb{1}_{\{\max_{1\le i\le k} H_{z(i)} > h_n\}} e^{-\langle \mathbf{1}, V(\boldsymbol{z})\rangle_k}] = 0$ thus giving (42).
Finally, putting together (41) and (42), we obtain the result. □

3.1.3. Convergence of the quasi-martingale $\mathscr{A}^k_l(f)$

Recall that

$$\mathscr{A}^k_l(f,\boldsymbol{\beta}) = \sum_{\boldsymbol{x}\in\Delta^k_l} f(\boldsymbol{x}) e^{-\langle \boldsymbol{\beta}, V(\boldsymbol{x})\rangle_k} = \sum_{\boldsymbol{x}\in\Delta^k_l} f(\boldsymbol{x}) \prod_{i=1}^k e^{-\beta_i V(x^{(i)})} \quad \text{and} \quad \mathscr{A}^k_l(f) = \mathscr{A}^k_l(f,\mathbf{1}).$$

The aim of this subsection is to prove that $\mathscr{A}^k_\infty := \lim_{l\to\infty} \mathscr{A}^k_l(f)$ exists when f satisfies our hereditary assumption 5. For that, let us define for any $\boldsymbol{p} \in (\mathbb{N}^*)^{\times k}$

$$\mathscr{A}^k_{\boldsymbol{p}}(f) := \sum_{\substack{\boldsymbol{x}\in\Delta^k \\ |\boldsymbol{x}|=\boldsymbol{p}}} f(\boldsymbol{x}) e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k},$$

where we recall that for any $\boldsymbol{x} = (x^{(1)},\dots,x^{(k)}) \in \Delta^k$, $|\boldsymbol{x}| = \boldsymbol{p}$ if and only if $|x^{(i)}| = p_i$ for all $i \in \{1,\dots,k\}$. One can notice that when $\boldsymbol{p} = (l,\dots,l) \in (\mathbb{N}^*)^{\times k}$, we have $\mathscr{A}^k_l(f) = \mathscr{A}^k_{\boldsymbol{p}}(f)$.

Lemma 3.9. *Let $k \ge 2$ be an integer and assume $\kappa > 2k$. Under the assumptions 1, 3 and 4, for any bounded function $f : \Delta^k \to \mathbb{R}^+$, there exists two constants $\mathfrak{C}_{3.9} > 0$ and $\mathfrak{b} \in (0,1)$ such that for any $\boldsymbol{p} \in (\mathbb{N}^*)^{\times k}$ and any integer $m \ge 1$ such that $m \le \max \boldsymbol{p} := \max_{1\le i\le} p_i$*

$$\mathbf{E}^*\big[\big|\mathscr{A}^k_{\boldsymbol{p}}(f\mathbb{1}_{\mathscr{C}^k_m}) - \mathscr{A}^k_{\boldsymbol{p}}(f)\big|^2\big] \le \mathfrak{C}_{3.9} e^{-\mathfrak{b}m}.$$

Proof. In order to avoid unnecessary technical difficulties, we prove it for any $\kappa > 4$. First note that $\mathscr{A}^k_{\boldsymbol{p}}(f) - \mathscr{A}^k_{\boldsymbol{p}}(f\mathbb{1}_{\mathscr{C}^k_m}) = \sum_{\boldsymbol{x}\in\Delta^k;\ |\boldsymbol{x}|=\boldsymbol{p}} f(\boldsymbol{x}) \mathbb{1}_{\{\mathscr{S}^k(\boldsymbol{x})>m\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k}$ which is smaller than $\|f\|_\infty \sum_{\boldsymbol{x}\in\Delta^k;\ |\boldsymbol{x}|=\boldsymbol{p}} \mathbb{1}_{\{\mathscr{S}^k(\boldsymbol{x})>m\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_k}$. Using a similar argument as we developed in the proof of Lemma 3.8, it is enough to show the following estimation:

$$\mathbf{E}^*\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q \\ |\boldsymbol{x}|=\boldsymbol{p}}} \mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{x})>m\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_q}\Big] \le \mathfrak{C}_{3.9,1} e^{-\mathfrak{b}m}, \tag{43}$$

for any $q \in \{k,\dots,2k\}$ and some constant $\mathfrak{C}_{3.9,1} > 0$. Assume that $\min \boldsymbol{p} < \max \boldsymbol{p}$ (the proof is similar when $\min \boldsymbol{p} = \max \boldsymbol{p}$). Note that if $m < \min \boldsymbol{p}$, then

$$\begin{aligned}\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q \\ |\boldsymbol{x}|=\boldsymbol{p}}} \mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{x})>m\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_q}\Big] =& \mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q \\ |\boldsymbol{x}|=\boldsymbol{p}}} \mathbb{1}_{\{m<\mathscr{S}^q(\boldsymbol{x})\le \min \boldsymbol{p}\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_q}\Big] \\ &+ \mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q \\ |\boldsymbol{x}|=\boldsymbol{p}}} \mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{x})>\min \boldsymbol{p}\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_q}\Big].\end{aligned}$$

One can notice that, if $|\boldsymbol{x}| = \boldsymbol{p}$ and $\mathscr{S}^q(x) \le \min \boldsymbol{p}$, then $\mathscr{S}^q(x) = \mathscr{S}^q(u)$ for any $\boldsymbol{u} \in \Delta^q$ such that $\max|\boldsymbol{u}| = \min|\boldsymbol{u}| = \min \boldsymbol{p}$. Hence, as usual

$$\begin{aligned}\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q \\ |\boldsymbol{x}|=\boldsymbol{p}}} \mathbb{1}_{\{m<\mathscr{S}^q(\boldsymbol{x})\le \min \boldsymbol{p}\}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_q}\Big] &= \mathbf{E}\Big[\sum_{\boldsymbol{u}\in\Delta^q_{\min \boldsymbol{p}}} \mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{u})>m\}} \sum_{\boldsymbol{x}\in\Delta^q;\ \boldsymbol{x}\ge\boldsymbol{u}} e^{-\langle \mathbf{1}, V(\boldsymbol{x})\rangle_q}\Big] \\ &= \mathbf{E}\Big[\sum_{\boldsymbol{u}\in\Delta^q_{\min \boldsymbol{p}}} \mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{u})>m\}} e^{-\langle \mathbf{1}, V(\boldsymbol{u})\rangle_q}\Big],\end{aligned}$$

thus giving

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{x})>m\}}e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_q}\Big]=\mathbf{E}\Big[\sum_{\boldsymbol{u}\in\Delta^q_{\min\boldsymbol{p}}}\mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{u})>m\}}e^{-\langle\mathbf{1},V(\boldsymbol{u})\rangle_q}\Big]+\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{x})>\min\boldsymbol{p}\}}e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_q}\Big].$$

We deduce from this equality that it is enough to prove (43) for any $m\le\min\boldsymbol{p}$ with $q\ge3$. Again, we focus on the case $\min\boldsymbol{p}<\max\boldsymbol{p}$.
Assume $m\le\min\boldsymbol{p}$. Let $\boldsymbol{x}\in\Delta^q$ such that $|\boldsymbol{x}|=\boldsymbol{p}$ and $\mathscr{C}^q(\boldsymbol{x})>m$. There exists an integer $\mathfrak{f}\in\{m+1,\dots,\max\boldsymbol{p}\}$ such that, seen backwards in time, at least two vertices among $x^{(1)},\dots,x^{(q)}$ share a common ancestor for the first times in the generation $\mathfrak{f}-1$ and there exits at least one vertex among these vertices in a generation smaller or equal to $\mathfrak{f}-1$. Then, one can notice that

$$\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{x})>m\}}e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_q}=\sum_{\mathfrak{f}=m+1}^{\max\boldsymbol{p}}\sum_{\substack{\boldsymbol{\pi}\text{ partition of}\\\{1,\dots,q\},\,|\boldsymbol{\pi}|<q}}\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\Upsilon_{\mathfrak{f}-1,\boldsymbol{\pi}}\cap\Upsilon_{\mathfrak{f},\boldsymbol{\eta}}}(\boldsymbol{x})e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_q},$$

where $\boldsymbol{\eta}=\{\{1\},\dots,\{q\}\}$ (recall the definition of $\Upsilon_{p-1,\boldsymbol{\pi}}\cap\Upsilon_{p,\boldsymbol{\eta}}$ in (14)).
By definition, there exists $\mathfrak{y}\in\{1,\dots,q-2\}$ and $(i_1,\dots,i_{\mathfrak{y}},i_{\mathfrak{y}+1},\dots,i_q)\in[\![1,q]\!]_q$ such that $\max_{1\le\mathfrak{l}\le\mathfrak{y}}p_{i_\mathfrak{l}}\le\mathfrak{f}-1$ and $\min_{\mathfrak{y}+1\le\mathfrak{l}\le q}p_{i_\mathfrak{l}}\ge\mathfrak{f}-1$. By definition of the set $\Upsilon_{\cdot,\cdot}$, for all $\mathfrak{l}\in\{1,\dots,\mathfrak{y}\}$, if $i_\mathfrak{l}$ belongs to the block $\boldsymbol{B}$ of the partition $\boldsymbol{\pi}$, then $\boldsymbol{B}=\{i_\mathfrak{l}\}$. Let $\bar{\boldsymbol{\pi}}:=\boldsymbol{\pi}\setminus\{\{i_1\},\dots,\{i_\mathfrak{y}\}\}$ and for all $j\in\{1,\dots,|\boldsymbol{\pi}|-\mathfrak{y}\}$, denote by $\bar{\boldsymbol{B}}_j$ the j-th block (ordered by their least element) of the partition $\bar{\boldsymbol{\pi}}$ of the set $\{i_{\mathfrak{y}+1},\dots,i_q\}=\{1,\dots,q\}\setminus\{i_1,\dots,i_\mathfrak{y}\}$. We have

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\Upsilon_{\mathfrak{f}-1,\boldsymbol{\pi}}\cap\Upsilon_{\mathfrak{f},\boldsymbol{\eta}}}(\boldsymbol{x})e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_q}\Big|\mathscr{F}_\mathfrak{f}\Big]=\sum_{\substack{\boldsymbol{u}\in\Delta^\mathfrak{y}\\|\boldsymbol{u}|=\boldsymbol{p}_\cdot}}e^{-\langle\mathbf{1},V(\boldsymbol{u})\rangle_\mathfrak{y}}\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}|-\mathfrak{y}}_{\mathfrak{f}-1}}\prod_{j=1}^{|\boldsymbol{\pi}|-\mathfrak{y}}\sum_{\boldsymbol{v}^{(j)}\in\Delta^{|\bar{\boldsymbol{B}}_j|}_\mathfrak{f}}\prod_{i=1}^{|\bar{\boldsymbol{B}}_j|}\times\mathbb{1}_{\{(v^{(j,i)})^*=z^{(j)}\}}e^{-V(v^{(j,i)})},$$

where $|\boldsymbol{u}|=\boldsymbol{p}_\cdot$ means that $u^{(\mathfrak{l})}=p_{i_\mathfrak{l}}$ for all $\mathfrak{l}\in\{1,\dots,\mathfrak{y}\}$, $\boldsymbol{v}^{(j)}=(v^{(j,1)},\dots,v^{(j,|\bar{\boldsymbol{B}}_j|)})$. Thus

$$\begin{aligned}\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\Upsilon_{\mathfrak{f}-1,\boldsymbol{\pi}}\cap\Upsilon_{\mathfrak{f},\boldsymbol{\eta}}}(\boldsymbol{x})e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_q}\Big|\mathscr{F}_{\mathfrak{f}-1}\Big]&=\sum_{\substack{\boldsymbol{u}\in\Delta^\mathfrak{y}\\|\boldsymbol{u}|=\boldsymbol{p}_\cdot}}e^{-\langle\mathbf{1},V(\boldsymbol{u})\rangle_\mathfrak{y}}\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}|-\mathfrak{y}}_{\mathfrak{f}-1}}e^{-\langle\bar{\boldsymbol{\beta}},V(\boldsymbol{z})\rangle_{|\boldsymbol{\pi}|-\mathfrak{y}}}\prod_{\bar{\boldsymbol{B}}\in\bar{\boldsymbol{\pi}}}c_{\bar{\boldsymbol{B}}}(\mathbf{1})\\&=\prod_{\bar{\boldsymbol{B}}\in\bar{\boldsymbol{\pi}}}c_{\bar{\boldsymbol{B}}}(\mathbf{1})\sum_{\substack{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}|}\\|\boldsymbol{u}|=\tilde{\boldsymbol{p}}}}e^{-\langle\tilde{\boldsymbol{\beta}},V(\boldsymbol{u})\rangle_{|\boldsymbol{\pi}|}},\end{aligned}$$

where $\tilde{\boldsymbol{p}}=(p_1,\dots,p_\mathfrak{y},\mathfrak{f}-1,\dots,\mathfrak{f}-1)\in(\mathbb{N}^*)^{\times|\boldsymbol{\pi}|}$ and $\tilde{\boldsymbol{\beta}}=(1,\dots,1,\bar{\boldsymbol{B}}_1,\dots,\bar{\boldsymbol{B}}_{|\boldsymbol{\pi}|-\mathfrak{y}})\in(\mathbb{N}^*)^{\times|\boldsymbol{\pi}|}$.
One can notice that there exists $r_0>0$ such that

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}|}\\|\boldsymbol{u}|=\tilde{\boldsymbol{p}}}}e^{-\langle\tilde{\boldsymbol{\beta}},V(\boldsymbol{u})\rangle_{|\boldsymbol{\pi}|}}\mathbb{1}_{\{\min_{|w|=\mathfrak{f}-1}V(w)<r_0(\mathfrak{f}-1)\}}\Big]\le\mathfrak{C}_{3.9,2}e^{-(\mathfrak{f}-1)},\tag{44}$$

for some constant $\mathfrak{C}_{3.9,2} > 0$. Indeed, By the Cauchy–Schwarz inequality,

$$\mathbf{E}\Big[\Big(\sum_{\substack{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}|}\\ |\boldsymbol{u}|=\tilde{\boldsymbol{p}}}} e^{-\langle \mathbf{1},V(\boldsymbol{u})\rangle_{|\boldsymbol{\pi}|}}\mathbb{1}_{\{\min_{|w|=\mathfrak{f}-1}V(w)<r_0(\mathfrak{f}-1)\}}\Big] \le \mathbf{E}\Big[\sum_{\substack{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}|}\\ |\boldsymbol{u}|=\bar{\boldsymbol{p}}}} e^{-\langle \mathbf{1},V(\boldsymbol{u})\rangle_{|\boldsymbol{\pi}|}}\Big)^2\Big]^{1/2} \times \mathbf{P}\Big(\min_{|w|=\mathfrak{f}-1} V(w) < r_0(\mathfrak{f}-1)\Big)^{1/2},$$

and thanks to Lemma 3.3 *(i)*, $\mathbf{E}[(\sum_{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}|};\ |\boldsymbol{u}|=\bar{\boldsymbol{p}}} e^{-\langle \mathbf{1},V(\boldsymbol{u})\rangle_{|\boldsymbol{\pi}|}})^2] \le \mathfrak{C}_{3.3,1}$, where we recall that $\mathfrak{C}_{3.3,1} > 0$ is a constant doesn't depending on $\boldsymbol{p}$ (or $\tilde{\boldsymbol{p}}$) since $|\boldsymbol{\pi}| < q \le 2k$. Moreover, since $\psi'(1) < 0$, we can find $r_0 > 0$ and a constant $\mathfrak{C}_{3.9,3} > 0$ such that $\mathbf{P}(\min_{|w|=\mathfrak{f}-1} V(w) < r_0(\mathfrak{f}-1)) \le \mathfrak{C}_{3.9,3}e^{2(\mathfrak{f}-1)}$. This yields (44).

Now, note that, since $|\boldsymbol{\pi}| < q$, there is at least one block of the partition $\boldsymbol{\pi}$ with cardinal larger or equal to 2 so $\langle \tilde{\boldsymbol{\beta}}, V(\boldsymbol{z})\rangle_{|\boldsymbol{\pi}|} \ge \langle \mathbf{1}, V(\boldsymbol{z})\rangle_{|\boldsymbol{\pi}|} + \min_{|w|=\mathfrak{f}-1} V(w)$ thus giving that the mean $\mathbf{E}[\sum_{\boldsymbol{x}\in\Delta^q;\ |\boldsymbol{x}|=\boldsymbol{p}} \mathbb{1}_{\Upsilon_{\mathfrak{f}-1,\boldsymbol{\pi}}\cap\Upsilon_{\mathfrak{f},\boldsymbol{\eta}}}(\boldsymbol{x})e^{-\langle \mathbf{1},V(\boldsymbol{x})\rangle_q}]$ is smaller than

$$\prod_{\bar{\boldsymbol{B}}\in\bar{\boldsymbol{\pi}}} c_{\bar{\boldsymbol{B}}}(\mathbf{1})\Big(\mathbf{E}\Big[\Big(\sum_{\substack{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}|}\\ |\boldsymbol{u}|=\tilde{\boldsymbol{p}}}} e^{-\langle \tilde{\boldsymbol{\beta}},V(\boldsymbol{u})\rangle_{|\boldsymbol{\pi}|}}\mathbb{1}_{\{\min_{|w|=\mathfrak{f}-1}V(w)<r_0(\mathfrak{f}-1)\}}\Big] + \mathbf{E}\Big[e^{-\min_{|w|=\mathfrak{f}-1}V(w)}\sum_{\substack{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}|}\\ |\boldsymbol{u}|=\bar{\boldsymbol{p}}}} e^{-\langle \mathbf{1},V(\boldsymbol{u})\rangle_{|\boldsymbol{\pi}|}}\mathbb{1}_{\{\min_{|w|=\mathfrak{f}-1}V(w)\ge r_0(\mathfrak{f}-1)\}}\Big]\Big),$$

which, thanks to Lemma 3.3 *(i)* and (44), is smaller than $\mathfrak{C}_{3.9,4}e^{-(1\wedge r_0)(\mathfrak{f}-1)}$ for some constant $\mathfrak{C}_{3.9,4} > 0$. Finally

$$\mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^q\\ |\boldsymbol{x}|=\boldsymbol{p}}} \mathbb{1}_{\{\mathscr{S}^q(\boldsymbol{x})>m\}}e^{-\langle \mathbf{1},V(\boldsymbol{x})\rangle_q}\Big] \le \mathfrak{C}_{3.9,5}\sum_{\mathfrak{f}=m+1}^{\max \boldsymbol{p}} e^{-(1\wedge r_0)(\mathfrak{f}-1)} \le \mathfrak{C}_{3.9,1}e^{-(1\wedge r_0)m},$$

for some constant $\mathfrak{C}_{3.9,5} > 0$ and (43) is proved. □

The convergence of the sequence of random variables $(\mathscr{A}_l^k(f))_l$ directly follows from Lemma 3.9. Indeed, let f be a bounded function satisfying the hereditary assumption 5. For any $l > l' > m > \mathfrak{g}$

$$\mathscr{A}_l^k(f\mathbb{1}_{\mathscr{C}_m^k}) = \sum_{x\in\Delta_l^k} f\mathbb{1}_{\mathscr{C}_m^k}(\boldsymbol{x})e^{-\langle \mathbf{1},V(\boldsymbol{x})\rangle_k} = \sum_{u\in\Delta_{l'}^k} f\mathbb{1}_{\mathscr{C}_m^k}(\boldsymbol{u})\sum_{\substack{x\in\Delta_l^k\\ \boldsymbol{x}>\boldsymbol{u}}} e^{-\langle \mathbf{1},V(\boldsymbol{x})\rangle_k},$$

so $\mathbf{E}[\mathscr{A}_l^k(f\mathbb{1}_{\mathscr{C}_m^k})|\mathscr{F}_{l'}] = \mathscr{A}_{l'}^k(f\mathbb{1}_{\mathscr{C}_m^k})$ where $\mathscr{F}_m = \sigma(\mathbb{T}, (V(x); |x| \le m))$ and $(\mathscr{A}_l^k(f\mathbb{1}_{\mathscr{C}_m^k}))_{l>m}$ is a martingale bounded in $L^2(\mathbf{P})$. In particular, for any integer $m > \mathfrak{g}$, $(\mathscr{A}_l^k(f\mathbb{1}_{\mathscr{C}_m^k}))_{l>m}$ converges in $L^2(\mathbf{P}^*)$ and $\mathbf{P}^*$-almost surely. Hence, thanks to Lemma 3.9, $(\mathscr{A}_l(f))_l$ is a Cauchy sequence in $L^2(\mathbf{P}^*)$ and therefore, $\mathscr{A}_\infty^k(f)$ exists.

3.1.4. k-tuples in the set $\Delta^k \setminus \mathscr{C}_{a_n}^k$

Before proving Proposition 1, let us show that the contribution of the k-tuples in the set $\mathfrak{E}^{k,\cdot} \cap \Delta^k \setminus \mathscr{C}_{a_n}^k$ is not significant. To do that, the following lemma provides an estimation for the quasi-independent version (38) of the range on the set $\mathscr{C}_{a_n}^k$:

Lemma 3.10. *Let $\varepsilon_1 \in (0,1)$, $k \geq 2$ and assume $\kappa > 2k$. Under the assumptions 1, 3 and 4, there exist two constants $C_{3.10} > 0$ and $\rho_8 > 0$ such that*

$$\mathbb{E}^*\Big[\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k}\sum_{\boldsymbol{j}\in[\![1,\mathfrak{s}_n]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}})\Big] \leq C_{3.10}n^{-\rho_8}, \tag{45}$$

with $\mathfrak{s}_n = n^{1/2}/\varepsilon_1$.

Proof. Recall that, thanks to the strong Markov property together with (35)

$$\mathbb{E}^{\mathscr{E}}\Big[\mathscr{A}^{k,n}(\boldsymbol{j},\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}})\Big] = \sum_{\substack{\boldsymbol{x}\in\Delta^k\\ \ell_n\leq|\boldsymbol{x}|\leq\mathfrak{L}_n}}\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}}(\boldsymbol{x})\prod_{i=1}^k\frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}},$$

and since $H_{x^{(i)}} \geq 1$

$$\begin{aligned}\mathbb{E}^*\Big[\sum_{\boldsymbol{j}\in[\![1,\mathfrak{s}_n]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}})\Big] &\leq (\mathfrak{s}_n)^k\sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}}\mathbf{E}^*\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\mathscr{S}^k(\boldsymbol{x})>a_n\}}e^{-\langle\mathbf{1},V(\boldsymbol{x})\rangle_k}\Big]\\ &\leq (\mathfrak{s}_n\boldsymbol{L}_n)^k\mathfrak{C}_{3.9}e^{-\mathfrak{b}a_n} = (\mathfrak{s}_n\boldsymbol{L}_n)^k\mathfrak{C}_{3.9}n^{-\mathfrak{b}(2\delta_0)^{-1}},\end{aligned}$$

which ends the proof. □

We are now ready to prove Proposition 1.

Proof of Proposition 1. We have to prove that for any $\varepsilon_1 n^{1/2} \leq s \leq n^{1/2}/\varepsilon_1$, $\varepsilon_1 \in (0,1)$

$$\mathbb{P}^*\Big(\Big|\frac{1}{(s\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathscr{C}^k_{a_n}}) - (c_\infty)^k\mathscr{A}^k_\infty(f)\Big| > \varepsilon/2\Big) \underset{n\to\infty}{\longrightarrow} 0. \tag{46}$$

We deduce from Lemma 3.7 with $\mathfrak{a} = 1$ that the range $\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})$ concentrates around its quenched mean. Indeed, for any $\varepsilon \in (0,1)$, by Markov inequality

$$\begin{aligned}&\mathbb{P}\Big(\Big|\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}}) - \mathbb{E}^{\mathscr{E}}\big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})\big]\Big| > \varepsilon(s\boldsymbol{L}_n)^k/16\Big)\\ &\leq \frac{16^2}{\varepsilon^1(s\boldsymbol{L}_n)^{2k}}\mathbb{E}\Big[\Big(\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}}) - \mathbb{E}^{\mathscr{E}}\big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})\big]\Big)^2\Big]\\ &\leq 16^2\mathfrak{C}_{3.7}\frac{\mathfrak{L}_n}{s} \leq \mathfrak{C}_1\frac{\mathfrak{L}_n}{n^{1/2}} \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

where the last inequality comes from the fact that $\mathfrak{L}_n = o(n^{1/2})$. Then, we know, thanks to Lemma 3.6 with $g = f$ and $\mathfrak{R} = \mathscr{C}^k_{a_n}$, that $\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathscr{C}^k_{a_n}})$ behaves like its quasi-independent version $\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})$: for n large enough

$$\mathbb{P}^*\Big(\Big|\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathscr{C}^k_{a_n}}) - \sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big| > \varepsilon(s\boldsymbol{L}_n)^k/16\Big) \leq n^{-\rho_4},$$

hence

$$\lim_{n\to\infty}\mathbb{P}^*\Big(\Big|\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathscr{C}^k_{a_n}}) - \mathbb{E}^{\mathscr{E}}\big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})\big]\Big| > \varepsilon(s\boldsymbol{L}_n)^k/8\Big) = 0.$$

One can notice that

$$\mathbb{E}^{\mathscr{E}}\Big[\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\mathscr{C}^k_{a_n}})\Big]=s(s-1)\cdots(s-k+1)\sum_{\substack{\boldsymbol{x}\in\Delta^k\\ \ell_n\le|\boldsymbol{x}|\le\mathfrak{L}_n}}f\mathbb{1}_{\mathscr{C}^k_{a_n}}(\boldsymbol{x})\prod_{i=1}^k\frac{e^{-V(x^{(i)})}}{H_{x^{(i)}}}.$$

Finally, Lemma 3.8 yields

$$\mathbb{P}^*\Big(\Big|\frac{1}{(s\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\mathscr{C}^k_{a_n}})-(c_\infty)^k\mathscr{A}^k_{a_n}(f)\Big|>\varepsilon/4\Big)\underset{n\to\infty}{\longrightarrow}0,$$

and the result of the subsection 3.1.3 leads to the convergence in (46).
Now using Lemma 3.10, we show that

$$\mathbb{P}^*\Big(\frac{1}{(s\boldsymbol{L}_n)^k}\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\Delta^k\setminus\mathscr{C}^k_{a_n}})>\varepsilon/2\Big)\underset{n\to\infty}{\longrightarrow}0. \tag{47}$$

Indeed, by Lemma 3.6 with $g=f$ and $\mathfrak{R}=\Delta^k\setminus\mathscr{C}^k_{a_n}$

$$\mathbb{P}^*\Big(\Big|\mathscr{A}^k(\mathscr{D}_{n,T^s},f\mathbb{1}_{\mathfrak{E}^{k,s}\cap\Delta^k\setminus\mathscr{C}^k_{a_n}})-\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}})\Big|>\varepsilon(s\boldsymbol{L}_n)^k/4\Big)\le n^{-\rho_4},$$

with $\rho_4=\rho_4(\varepsilon/4,\varepsilon_1)$ instead of $\rho_4(\varepsilon,\varepsilon_1)$. Finally, note that for any $s\le n^{1/2}/\varepsilon_1$, the sum $\sum_{\boldsymbol{j}\in[\![1,s]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}})$ is smaller than $\sum_{\boldsymbol{j}\in[\![1,n^{1/2}/\varepsilon_1]\!]_k}\mathscr{A}^{k,n}(\boldsymbol{j},f\mathbb{1}_{\Delta^k\setminus\mathscr{C}^k_{a_n}})$ so Lemma 3.10, together with Markov inequality leads to (47). We end the proof putting together (46) and (47). □

3.2. The range on $\Delta^k\setminus\mathfrak{E}^{k,\cdot}$

Recall

$$\mathscr{A}^k(\mathscr{D}_{n,T^s},g)=\sum_{\substack{\boldsymbol{x}\in\Delta^k\\ \ell_n\le|\boldsymbol{x}|\le\mathfrak{L}_n}}g(\boldsymbol{x})\mathbb{1}_{\{T_{\boldsymbol{x}}<T^s\}},$$

where $T_{\boldsymbol{x}}=\max_{1\le i\le k}T_{x^{(i)}}$, $T_z=\min\{j\ge0;\ X_j=z\}$, $T^0=0$ and $T^s=\min\{j>T^{s-1};\ X_j=e\}$ for $s\in\mathbb{N}^*$. Also recall that (ℓ_n) and $(\mathfrak{L}_n)$ are two sequences of positive integers such that $\delta_0^{-1}\log n\le\ell_n\le\mathfrak{L}_n\le n^{1/2}$.
The last step of our study is to show that the contribution of the k-tuples of vertices in small generations (see (7)) and such that at least two of these vertices are visited during the same excursion is not significant. This section is thus devoted to the proof of Proposition 2, claiming that

$$\mathbb{P}^*\Big(\sup_{s\le n^{1/2}/\varepsilon_1}\mathscr{A}^k(\mathscr{D}_{n,T^s},\mathbb{1}_{\Delta^k\setminus\mathfrak{E}^{k,s}})>\varepsilon(n^{1/2}\boldsymbol{L}_n)^k\Big)\underset{n\to\infty}{\longrightarrow}0$$

Lemma 3.11. *Let $\varepsilon_1\in(0,1)$, $k\ge2$, let $\mathfrak{s}_n=n^{1/2}/\varepsilon_1$ and assume $\kappa>2k$. Assume that the assumptions 1, 3, 4 hold and that $\mathfrak{L}_n=o(n^{1/2})$.*

(i) If

$$\mathfrak{E}_1^{k,s}:=\bigcup_{j=1}^s\bigcap_{i=1}^k\{\boldsymbol{x}=(x^{(1)},\dots,x^{(k)})\in\Delta^k;\ \mathscr{L}^{T^j}_{x^{(i)}}-\mathscr{L}^{T^{j-1}}_{x^{(i)}}\}$$

denotes the set of k-tuples of vertices visited during the same excursion before the instant T^s, then

$$\lim_{n\to\infty} \mathbb{E}^*\Big[\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k} \sup_{s\le \mathfrak{s}_n} \mathscr{A}^k(\mathscr{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s}\cap \mathfrak{E}_1^{k,s}})\Big] = 0.$$

(ii) Let $\mathfrak{E}_2^{k,s} := \Delta^k \setminus (\mathfrak{E}^{k,s} \cup \mathfrak{E}_1^{k,s})$. If $k \ge 3$ and the assumption 2 hold, then, for all $B > 0$

$$\lim_{n\to\infty} \mathbb{E}^*\Big[\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k} \sup_{s\le \mathfrak{s}_n} \mathscr{A}^k(\mathscr{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s}\cap \mathfrak{E}_2^{k,s}} \mathbb{1}_{\{\underline{V}(\cdot)\ge -B\}})\Big] = 0,$$

with $\underline{V}(\boldsymbol{x}) \ge -B$ if and only if $\underline{V}(x^{(i)}) \ge -B$ for all $i \in \{1,\dots,k\}$.

Proof. In order to avoid unnecessary technical difficulties, we prove it for any $\kappa > 4$. Let us start with the proof of *(i)*. By definition, $\boldsymbol{x} \in \mathfrak{S}^{k,s} \cap \mathfrak{E}_1^{k,s}$ if and only if there exists $j \in \{1,\dots,s\}$ such that for all $1 \le i \le k$, $N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} \ge 1$ and for all $p \ne j$, $N_{x^{(i)}}^{T^p} - N_{x^{(i)}}^{T^{p-1}} = 0$. Thus, using again the strong Markov property

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\le \mathfrak{s}_n} \mathscr{A}^k(\mathscr{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s}\cap \mathfrak{E}_1^{k,s}})\Big] &= \mathbf{E}\Big[\sup_{s\le \mathfrak{s}_n} \sum_{j=1}^{s} \sum_{\substack{\boldsymbol{x}\in\Delta^k \\ \ell_n\le|\boldsymbol{x}|\le \mathfrak{L}_n}} \mathbb{1}_{\cap_{i=1}^k \cap_{p\ne j}\{N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} \ge 1, N_{x^{(i)}}^{T^p} - N_{x^{(i)}}^{T^{p-1}} = 0\}}\Big] \\
&\le \sum_{j=1}^{\mathfrak{s}_n} \mathbb{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k \\ \ell_n\le|\boldsymbol{x}|\le \mathfrak{L}_n}} \prod_{i=1}^{k} \mathbb{1}_{\{N_{x^{(i)}}^{T^j} - N_{x^{(i)}}^{T^{j-1}} \ge 1\}}\Big] \\
&\le \mathfrak{s}_n \sum_{\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}} \mathbf{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k \\ |\boldsymbol{x}|=\boldsymbol{p}}} \mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{x}} < T^1)\Big] \\
&\le \mathfrak{C}_{2,1}\mathfrak{s}_n(\boldsymbol{L}_n)^k(\mathfrak{L}_n)^{k-1},
\end{aligned}$$

where we have used Lemma 3.2 (34) with $m = \mathfrak{L}_n$ for the last inequality, recalling that the constant $\mathfrak{C}'_{2,1} > 0$ doesn't depend on $\boldsymbol{p}$. By definition of $\mathfrak{s}_n$

$$\mathbb{E}\Big[\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k} \sup_{s\le \mathfrak{s}_n} \mathscr{A}^k(\mathscr{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s}\cap \mathfrak{E}_1^{k,s}})\Big] \le \frac{\mathfrak{C}_{2,1}}{\varepsilon_1}\Big(\frac{\mathfrak{L}_n}{n^{1/2}}\Big)^{k-1},$$

which goes to 0 when n goes to ∞ since $\mathfrak{L}_n = o(n^{1/2})$ and this yields *(i)*.

We now focus on *(ii)*. Since $k \ge 3$, $\mathfrak{E}_2^{k,s}$ is nothing but the set of k-tuples of distinct vertices of $\mathbb{T}$ neither visited during k distinct excursions, nor during the same excursion. Therefore, there exists $\mathfrak{e} \in \{2,\dots,k-1\}$ and $\mathfrak{e}$ disjoint subsets $I_1,\dots,I_{\mathfrak{e}}$ of $\{1,\dots,k\}$ such that $\{1,\dots,k\} = I_1 \cup \dots \cup I_{\mathfrak{e}}$ and for any $j \in \{1,\dots,\mathfrak{e}\}$, $i, i' \in I_j$ if and only if $x^{(i)}$ and $x^{(i')}$ are visited during the same excursion before the instant T^s:

$$\exists\, \mathfrak{j} \in \{1,\dots,s\} :\ \big(\mathscr{L}_{x^{(i)}}^{T^{\mathfrak{j}}} - \mathscr{L}_{x^{(i)}}^{T^{\mathfrak{j}-1}}\big) \wedge \big(\mathscr{L}_{x^{(i')}}^{T^{\mathfrak{j}}} - \mathscr{L}_{x^{(i')}}^{T^{\mathfrak{j}-1}}\big) \ge 1.$$

Let $m \in \mathbb{N}^*$ and introduce the following subset of Δ^k

$$\Upsilon_m^{k,s} := \{\boldsymbol{x} = (x^{(1)},\dots,x^{(k)}) \in \Delta^k;\ \forall j \ne j' \in \{1,\dots,\mathfrak{e}\}, \forall\, i \in I_j, \forall\, i' \in I_{j'} :\ |x^{(i)} \wedge x^{(i')}| < m\},$$

where we recall that $u \wedge v$ is the most recent common ancestor (MRCA) of u and v. Υ_m^k is the set of k-tuples of vertices such that the MRCA of two vertices visited during two distinct excursions

before the instant T^s has to be in a generation smaller than m. Note that the MRCA of two vertices visited during the same excursion can be in a generation larger or equal to m.
Recall that $(\Lambda_l)_{l\in\mathbb{N}}$ is the sequence of functions such that for all $t>0$, $\Lambda_0(t)=t$ and for any $l\in\{1,\dots,l_0\}$, $\Lambda_{l-1}(t)=e^{\Lambda_l(t)}$ (see the assumption 2). Introduce $\mathfrak{g}_{l,n}:=4k\delta_0^{-1}\Lambda_l(\mathfrak{L}_n)$. Note that $\mathfrak{g}_{0,n}>\mathfrak{L}_n$ so

$$\mathbb{E}\Big[\sup_{s\le\mathfrak{s}_n}\mathscr{A}^k(\mathscr{D}_{n,T^s},\mathbb{1}_{\mathfrak{S}^{k,s}\cap\mathfrak{E}_2^{k,s}}\mathbb{1}_{\{\underline{V}(\cdot)\ge -B\}})\Big]=\mathbb{E}\Big[\sup_{s\le\mathfrak{s}_n}\mathscr{A}^k(\mathscr{D}_{n,T^s},\mathbb{1}_{\mathfrak{S}^{k,s}\cap\mathfrak{E}_2^{k,s}}\mathbb{1}_{\{\underline{V}(\cdot)\ge -B\}\cap\mathscr{C}^k_{\mathfrak{g}_{0,n}}})\Big].$$

Recall that for any $\boldsymbol{x}=(x^{(1)},\dots,x^{(k)})\in\Delta^k$, it belongs to $\mathfrak{S}^{k,s}$ if and only if $x^{(i)}$ is visited during a single excursion before the instant T^s for all $i\in\{1,\dots,k\}$. Using what we previously said, we have, for any $s\le\mathfrak{s}_n$

$$\begin{aligned}\mathbb{1}_{\mathfrak{S}^{k,s}\cap\mathfrak{E}_2^{k,s}}(\boldsymbol{x})&\le\sum_{\mathfrak{e}=2}^{k-1}\sum_{\boldsymbol{j}\in[\![1,s]\!]_{\mathfrak{e}}}\sum_{\substack{I_1,\dots,I_{\mathfrak{e}}\text{ sets}\\ \cup_{\mathfrak{l}=1}^{\mathfrak{e}}I_{\mathfrak{l}}=\{1,\dots,k\}}}\prod_{p=1}^{\mathfrak{e}}Y_p\\&\le\sum_{\mathfrak{e}=2}^{k-1}\sum_{\boldsymbol{j}\in[\![1,\mathfrak{s}_n]\!]_{\mathfrak{e}}}\sum_{\substack{I_1,\dots,I_{\mathfrak{e}}\text{ sets}\\ \cup_{\mathfrak{l}=1}^{\mathfrak{e}}I_{\mathfrak{l}}=\{1,\dots,k\}}}\prod_{p=1}^{\mathfrak{e}}Y_p,\end{aligned}$$

where, for any $p\in\{1,\dots,\mathfrak{e}\}$, $Y_p:=\mathbb{1}_{\cap_{i\in I_p}\{\mathscr{L}^{T^{j_p}}_{x^{(i)}}-\mathscr{L}^{T^{j_p-1}}_{x^{(i)}}\ge 1\}}$. It follows that

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\le\mathfrak{s}_n}\mathscr{A}^k(\mathscr{D}_{n,T^s},\mathbb{1}_{\mathfrak{S}^{k,s}\cap\mathfrak{E}_2^{k,s}}\mathbb{1}_{\{\underline{V}(\cdot)\ge -B\}\cap\mathscr{C}^k_{\mathfrak{g}_{0,n}}})\Big]\\&\le\sum_{\boldsymbol{p}\in\{\ell,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{l=1}^{l_0}\sum_{\mathfrak{e}=2}^{k-1}\sum_{\boldsymbol{j}\in[\![1,\mathfrak{s}_n]\!]_{\mathfrak{e}}}\sum_{\substack{I_1,\dots,I_{\mathfrak{e}}\text{ sets}\\ \cup_{\mathfrak{l}=1}^{\mathfrak{e}}I_{\mathfrak{l}}=\{1,\dots,k\}}}\mathbb{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\ge -B\}}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{e}}Y_p\Big]\\&+\sum_{\boldsymbol{p}\in\{\ell,\dots,\mathfrak{L}_n\}^{\times k}}\sum_{\mathfrak{e}=2}^{k-1}\sum_{\boldsymbol{j}\in[\![1,\mathfrak{s}_n]\!]_{\mathfrak{e}}}\sum_{\substack{I_1,\dots,I_{\mathfrak{e}}\text{ sets}\\ \cup_{\mathfrak{l}=1}^{\mathfrak{e}}I_{\mathfrak{l}}=\{1,\dots,k\}}}\mathbb{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\ge -B\}}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l_0,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{e}}Y_p\Big].\end{aligned}\tag{48}$$

First, let us prove that for any $\boldsymbol{p}\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}$,

$$\mathbb{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\ge -B\}}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{e}}Y_p\Big]\le\mathfrak{C}_{3.11,1}(\mathfrak{L}_n)^{k-\mathfrak{e}}.\tag{49}$$

The proof of (49) is quite technical so in order to keep it as clear as possible, as one can notice in the proof of Lemmas 3.3 *(i)* and 3.2 (34) with $m=\mathfrak{L}_n$, we can and shall restrict to the case $\boldsymbol{p}=(m,\dots,m)\in\{\ell_n,\dots,\mathfrak{L}_n\}^{\times k}$.

Thanks to the strong Markov property, the random variables $Y_1,\dots,Y_{\mathfrak{e}}$ are i.i.d under $\mathbb{P}^{\mathscr{E}}$ and

$$\begin{aligned}&\mathbb{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\ge -B\}}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{e}}Y_p\Big]\\&=\sum_{\boldsymbol{x}\in\Delta^k_m}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\ge -B\}}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{e}}\mathbb{P}^{\mathscr{E}}\big(\max_{i\in I_p}T_{x^{(i)}}<T^1\big).\end{aligned}$$

As usual, $\sum_{\boldsymbol{x}\in\Delta_m^k} \mathbb{1}_{\{\underline{V}(\boldsymbol{x})\geq -B\}} \mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x}) \prod_{p=1}^{\mathfrak{c}} \mathbb{P}^{\mathscr{E}}(\max_{i\in I_p} T_{x^{(i)}} < T^1)$ is equal to

$$\sum_{\ell=1}^{k-1} \sum_{\Pi \text{ increasing}} \sum_{\boldsymbol{t};t_1<\ldots<t_\ell<m} \sum_{\boldsymbol{x}\in\Delta_m^k} \mathbb{1}_{\{\underline{V}(\boldsymbol{x})\geq -B\}} f^{\ell}_{\boldsymbol{t},\Pi} \mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x}) \prod_{p=1}^{\mathfrak{c}} \mathbb{P}^{\mathscr{E}}\big(\max_{i\in I_p} T_{x^{(i)}} < T^1\big), \tag{50}$$

where the genealogical tree function $f^{\ell}_{\boldsymbol{t},\Pi}$ is defined in (14). Recall that $t_1-1,\ldots,t_\ell-1$ correspond to the consecutive coalescent/split times. We then define

$$\tau^{\ell} := \max\{\mathfrak{j}\in\{1,\ldots,\ell\};\ \exists\, p\neq p'\in\{2,\ldots,\mathfrak{c}\},\ \exists\, \boldsymbol{B}\in\pi_{\mathfrak{j}-1} : \boldsymbol{B}\cap I_p\neq\varnothing \text{ and } \boldsymbol{B}\cap I_{p'}\neq\varnothing\},$$

and the $\boldsymbol{x}$-version $\tau^{\ell}(\boldsymbol{x})$ of τ^{ℓ}:

$$\tau^{\ell}(\boldsymbol{x}) := \max\{\mathfrak{j}\in\{1,\ldots,\ell\};\ \exists\, p\neq p'\in\{2,\ldots,\mathfrak{c}\},\ \exists\, i\in I_p,\ i'\in I_{p'} :\ |x^{(i)}\wedge x^{(i')}| = t_{\mathfrak{j}}-1\}.$$

In other words, if the genealogical tree of $\boldsymbol{x}\in\Delta^k$ is given by $f^{\ell}_{\boldsymbol{t},\Pi}$, then $\tau^{\ell}=\tau^{\ell}(\boldsymbol{x})$ and $t_{\tau^{\ell}}-1$ is the last generation at which two or more vertices visited during two distinct excursions share a common ancestor.

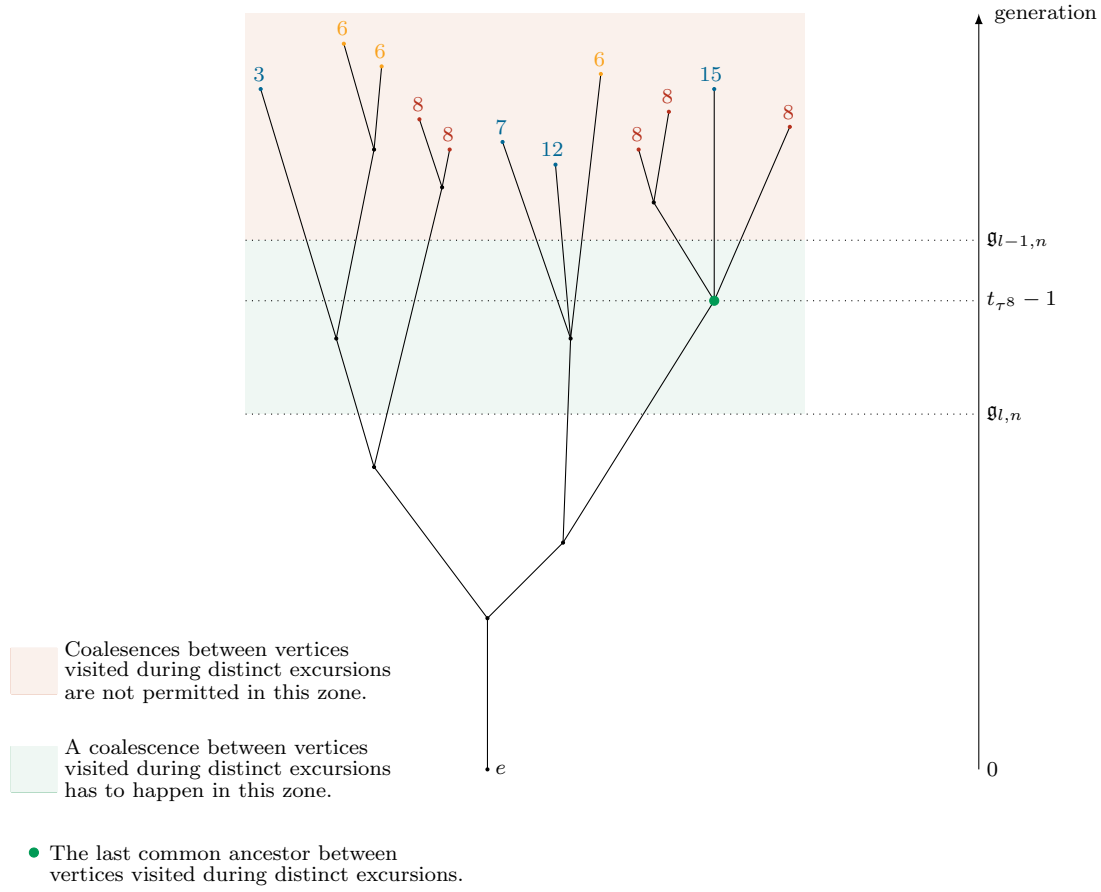

Figure 4: An example of a 12-tuple belonging to $\Upsilon^{12,\cdot}_{\mathfrak{g}_{l-1,n}} \setminus \Upsilon^{12,\cdot}_{\mathfrak{g}_{l,n}}$ whose genealogical tree is given by $f^{\ell}_{\boldsymbol{t},\Pi}$. 6 means that the corresponding vertex is visited during the 6-th excursion above e^*. In the present example, $\ell = 8$ and $\tau^8 = 4$.

By definition of τ^ℓ, for all $\mathfrak{j} \geq \tau^\ell$, if $\boldsymbol{B} \in \pi_{\mathfrak{j}}$, then $\boldsymbol{B}$ is necessarily a subset of $I_{p'}$ for some $p' \in \{1,\dots,\mathfrak{e}\}$. In other words, each coalescence that occurs between $t_{\tau^\ell+1}$ and t_ℓ involves exclusively two or more vertices visited during the same excursion. As a consequence, for any $i \in \{\tau^\ell,\dots,\ell\}$ and $p \in \{1,\dots,\mathfrak{e}\}$, we can defined the set I^i_p as follows: we first set $I^\ell_p := I_p$ so $I^\ell_1,\dots,I^\ell_{\mathfrak{e}}$ form a partition of $\{1,\dots,k\}$. As we said before, by definition of τ^ℓ, coalescences can only happen between two or more vertices which indexes belong to the same I^ℓ_p. Thus, for any $p \in \{1,\dots,\mathfrak{e}\}$, there exists an integer $\mathfrak{e}^{\ell-1}_p \geq 1$ and $\mathfrak{e}^{\ell-1}_p$ distinct integers $k^{\ell-1}_{p,1},\dots,k^{\ell-1}_{p,\mathfrak{e}^{\ell-1}_p}$ in $\{1,\dots,|\boldsymbol{\pi}_{\ell-1}|\}$ such that for any $j \in \{k^{\ell-1}_{p,1},\dots,k^{\ell-1}_{p,\mathfrak{e}^{\ell-1}_p}\}$, the block $\boldsymbol{B}^{\ell-1}_j$ of the partition $|\boldsymbol{\pi}_{\ell-1}|$ is the union of $b_{\ell-1}(\boldsymbol{B}_j)$ block(s) of the partition $\boldsymbol{\pi}_\ell$ of elements of F^ℓ_p. We set $F^{\ell-1}_p := \{k^{\ell-1}_{p,1},\dots,k^{\ell-1}_{p,\mathfrak{e}^{\ell-1}_p}\}$ so $I^{\ell-1}_1,\dots,I^{\ell-1}_{\mathfrak{e}}$ form a partition of $\{1,\dots,|\boldsymbol{\pi}_{\ell-1}|\}$. Now, let $i \in \{\tau^\ell+1,\dots,\ell\}$ and assume that F^i_p has been built. By definition of τ^ℓ, for any $p \in \{1,\dots,\mathfrak{e}\}$, there exists an integer $\mathfrak{e}^{i-1}_p \geq 1$ and $\mathfrak{e}^{i-1}_p$ distinct integer $k^{i-1}_{p,1},\dots,k^{i-1}_{p,\mathfrak{e}^{i-1}_p}$ in $\{1,\dots,|\boldsymbol{\pi}_{i-1}|\}$ such that for any $j \in \{k^{i-1}_{p,1},\dots,k^{i-1}_{p,\mathfrak{e}^{i-1}_p}\}$, the block $\boldsymbol{B}^{i-1}_j$ of the partition $|\boldsymbol{\pi}_{i-1}|$ is the union of $b_{i-1}(\boldsymbol{B}_j)$ block(s) of the partition $\boldsymbol{\pi}_i$ of elements of I^i_p. We set $I^{i-1}_p := \{k^{i-1}_{p,1},\dots,k^{\ell-1}_{p,\mathfrak{e}^{i-1}_p}\}$ so $I^{i-1}_1,\dots,I^{i-1}_{\mathfrak{e}}$ form a partition of $\{1,\dots,|\boldsymbol{\pi}_{i-1}|\}$.
Hence, noticing that

$$f^{\ell}_{\boldsymbol{t},\Pi}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x}) \leq f^{\ell}_{\boldsymbol{t},\Pi}(\boldsymbol{x})\mathbb{1}_{\{\mathfrak{g}_{l,n}\leq t_{\tau^\ell(\boldsymbol{x})}-1<\mathfrak{g}_{l-1,n}\}} = f^{\ell}_{\boldsymbol{t},\Pi}(\boldsymbol{x})\mathbb{1}_{\{\mathfrak{g}_{l,n}\leq t_{\tau^\ell}-1<\mathfrak{g}_{l-1,n}\}},$$

it is enough to show (49) for $\mathfrak{g}_{l,n} \leq t_{\tau^\ell} - 1 < \mathfrak{g}_{l-1,n}$. We then have

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{\boldsymbol{x}\in\Delta^k_m}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\geq -B\}}f^{\ell}_{\boldsymbol{t},\Pi}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{e}}\mathbb{P}^{\mathscr{E}}\big(\max_{i\in I_p}T_{x^{(i)}}<T^1\big)\big|\mathscr{F}_{t_{\tau^\ell}}\Big]\\
&\leq \mathfrak{C}_{3.11,2}\sum_{\boldsymbol{u}\in\Delta^{|\boldsymbol{\pi}_{\tau^\ell}|}_{t_{\tau^\ell}}}\mathbb{1}_{\{\underline{V}(\boldsymbol{u})\geq -B\}}f^{\tau^\ell}_{\boldsymbol{t}^{\tau^\ell},\Pi^{\tau^\ell}}(\boldsymbol{u})\prod_{p=1}^{\mathfrak{e}}\mathbb{P}^{\mathscr{E}}\big(\max_{i\in I^{\tau^\ell}_p}T_{u^{(i)}}<T^1\big)\prod_{j=1}^{|\boldsymbol{\pi}_{\tau^\ell}|}(H_{u^{(j)}})^{|\boldsymbol{B}^{\tau^\ell}_j|},
\end{aligned}$$

for some constant $\mathfrak{C}_{3.11,2} > 0$ where $\boldsymbol{t}^{\tau^\ell}$ and Π^{τ^ℓ} are defined in Example 4.
Note that $t_{\tau^\ell} - 1$ is the first generation (backwards in time) at which a coalescence between two or more vertices visited during distinct excursions occurs so there exists a subset J_ℓ of $\{1,\dots,|\boldsymbol{\pi}_{t_{\tau^\ell-1}}|\}$ and a collection $\{\alpha_i;\ i \in J_\ell\}$ of $|J_\ell|$ integers satisfying $\alpha_i \geq 1$ for all $i \in J_\ell$ and $\sum_{i\in J_\ell}\alpha_i \leq k$ such that

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{\boldsymbol{x}\in\Delta^k_m}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\geq -B\}}f^{\ell}_{\boldsymbol{t},\Pi}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{e}}\mathbb{P}^{\mathscr{E}}\big(\max_{i\in I_p}T_{x^{(i)}}<T^1\big)\big|\mathscr{F}_{t_{\tau^\ell}-1}\Big]\\
&\leq \mathfrak{C}_{3.11,3}\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\tau^\ell-1}|}_{t_{\tau^\ell}-1}}f^{\tau^\ell-1}_{\boldsymbol{t}^{\tau^\ell-1},\Pi^{\tau^\ell-1}}(\boldsymbol{z})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{z}}<T^1)\prod_{j=1}^{|\boldsymbol{\pi}_{\tau^\ell-1}|}(H_{u^{(j)}})^{|\boldsymbol{B}^{\tau^\ell-1}_j|}\prod_{i\in J_\ell}e^{-\alpha_i V(z^{(i)})}\mathbb{1}_{\{\underline{V}(\boldsymbol{z})\geq -B\}}.
\end{aligned}$$

Note that

$$\prod_{i\in J_\ell} e^{-\alpha_i V(z^{(i)})}\mathbb{1}_{\{\underline{V}(\boldsymbol{z})\geq -B\}} \leq \prod_{i\in J_\ell} e^{-\alpha_i V(z^{(i)})}\mathbb{1}_{\{\min_{i\in J_\ell} V(z^{(i)})\geq -B,\, \min_{|z|=t_{\tau^\ell-1}} V(z)<\delta_0(t_{\tau_\ell}-1)\}} + e^{-\min_{|z|=t_{\tau^\ell-1}} V(z)}\mathbb{1}_{\{\min_{|z|=t_{\tau^\ell-1}} V(z)\geq\delta_0(t_{\tau_\ell}-1)\}},$$

so $\mathbf{E}[\sum_{\boldsymbol{x}\in\Delta_m^k}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\geq -B\}} f^\ell_{\boldsymbol{t},\Pi}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{c}}\mathbb{P}^{\mathscr{E}}(\max_{i\in I_p} T_{x^{(i)}}<T^1)]$ is smaller than

$$\mathbf{E}\Big[\mathfrak{C}_{3.11,3}\sum_{\boldsymbol{z}\in\Delta^{|\boldsymbol{\pi}_{\tau^\ell-1}|}_{t_{\tau^\ell-1}}} f^{\tau^\ell-1}_{\boldsymbol{t}^{\tau^\ell-1},\Pi^{\tau^\ell-1}}(\boldsymbol{z})\mathbb{P}^{\mathscr{E}}(T_{\boldsymbol{z}}<T^1)\prod_{j=1}^{|\boldsymbol{\pi}_{\tau^\ell-1}|}(H_{u^{(j)}})^{|\boldsymbol{B}^{\tau^\ell-1}_j|} \times\Big(e^{kB}\mathbb{1}_{\{\min_{|z|=t_{\tau^\ell-1}} V(z)<\delta_0(t_{\tau_\ell}-1)\}}+e^{-3\delta_0(t_{\tau_\ell}-1)}\Big)\Big].$$

Using the same argument as the one we used in the proof of Lemma 3.2 together with the Cauchy–Schwarz inequality, we obtain that the previous mean is smaller than

$$\begin{aligned}&\mathfrak{C}_{3.11,3}\sup_{d\in\mathbb{N}^*}\mathbf{E}[(H^S_{d-1})^{4k-1}]\Big(e^{kB}\mathbb{P}\big(\min_{|z|=t_{\tau^\ell-1}} V(z)<\delta_0(t_{\tau_\ell}-1)\big)^{1/2}+e^{-3\delta_0(t_{\tau_\ell}-1)}\Big)\\ &\leq \mathfrak{C}_{3.11,3}\sup_{d\in\mathbb{N}^*}\mathbf{E}[(H^S_{d-1})^{4k-1}](e^{kB}+1)e^{-k\Lambda_l(\mathfrak{L}_n)},\end{aligned}$$

where we have used Lemma 3.4 with $\zeta=\delta_0 t_{\tau_\ell}$ and the fact that $t_{\tau_\ell}-1\geq\mathfrak{g}_{l,n}$.
Back to (50) together with what we have just obtained and the fact that for all $\mathfrak{j}\in\{1,\dots,\tau_\ell\}$, $t_{\mathfrak{j}}\leq\mathfrak{g}_{l-1,n}$, $\mathbb{E}[\sum_{\boldsymbol{x}\in\Delta^k,\,|\boldsymbol{x}|=\boldsymbol{p}}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\geq -B\}}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\mathbb{1}_{\mathfrak{S}^{k,\mathfrak{s}_n}\cap\mathfrak{E}^{k,\mathfrak{s}_n}_2}(\boldsymbol{x})]$ is smaller than

$$\mathfrak{C}_{3.11,3}\sup_{d\in\mathbb{N}^*}\mathbf{E}[(H^S_{d-1})^{4k-1}](e^{kB}+1)e^{-k\Lambda_l(\mathfrak{L}_n)}\sum_{\ell=1}^{k-1}\sum_{\Pi\text{ increasing}}(\mathfrak{g}_{l-1,n})^{\tau_\ell}(\mathfrak{L}_n)^{\ell-\tau_\ell}.$$

Note that $\tau_\ell\leq\ell<k$. Moreover, by definition, $\ell-\tau_\ell$ is smaller than the total number of coalescences occurring between two or more vertices which indexes belong to the same set I^ℓ_p and this number is smaller than $\sum_{p=1}^{\mathfrak{c}}(|I^\ell_p|-1)=k-\mathfrak{c}$ thus giving

$$\mathbb{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\geq -B\}}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l-1,n}}\setminus\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{c}}Y_p\Big]\leq\mathfrak{C}_{3.11,1}\big(\Lambda_{l-1}(\mathfrak{L}_n)e^{-\Lambda_l(\mathfrak{L}_n)}\big)^k(\mathfrak{L}_n)^{k-\mathfrak{c}},$$

which, by definition of $\Lambda_l(\mathfrak{L}_n)$, is equal to $\mathfrak{C}_{3.11,1}(\mathfrak{L}_n)^{k-\mathfrak{c}}$ and it yields (49).
In the same way, we can prove that

$$\mathbb{E}\Big[\sum_{\substack{\boldsymbol{x}\in\Delta^k\\|\boldsymbol{x}|=\boldsymbol{p}}}\mathbb{1}_{\{\underline{V}(\boldsymbol{x})\geq -B\}}\mathbb{1}_{\Upsilon^{k,\mathfrak{s}_n}_{\mathfrak{g}_{l,n}}}(\boldsymbol{x})\prod_{p=1}^{\mathfrak{c}}Y_p\Big]\leq\mathfrak{C}'_{3.11,1}\big(1+\Lambda_{l+1}(\mathfrak{L}_n)^k\big)(\mathfrak{L}_n)^{k-\mathfrak{c}},\tag{51}$$

for some constant $\mathfrak{C}'_{3.11,1}>0$. Putting together (48), (49) and (51), we obtain, for some constant $\mathfrak{C}_{3.11,4}>0$

$$\begin{aligned}&\mathbb{E}\Big[\frac{1}{(n^{1/2}\boldsymbol{L}_n)^k}\sup_{s\leq\mathfrak{s}_n}\mathscr{A}^k(\mathscr{D}_{n,T^s},\mathbb{1}_{\mathfrak{S}^{k,s}\cap\mathfrak{E}^{k,s}_2}\mathbb{1}_{\{\underline{V}(\cdot)\geq -B\}\cap\mathscr{C}^k_{\mathfrak{g}_{0,n}}})\Big]\\ &\leq\mathfrak{C}_{3.11,4}\sum_{\mathfrak{c}=2}^{k-1}\Big(\frac{\mathfrak{L}_n}{n^{1/2}}\Big)^{k-\mathfrak{c}}\big(2+\Lambda_{l_0+1}(\mathfrak{L}_n)^k\big).\end{aligned}$$

Using the fact that $\Lambda_{l_0+1}(\mathfrak{L}_n)^k=(\log\Lambda_{l_0}(\mathfrak{L}_n))^k$, we obtain *(ii)* thanks to the assumption 2. □

We are now ready to prove Proposition 2:

Proof of Proposition 2. Let $\varepsilon' > 0$. First, note that thanks to Lemma 3.5 and **Fact 1** (36) there exists $a_{\varepsilon'} > 0$ such that we can restrict our study to the k-tuples of vertices in the set $\mathfrak{S}^{k,s} \cap \{\underline{V}(\cdot) \geq -a_{\varepsilon'}\}$

$$\lim_{\varepsilon' \to 0} \limsup_{n \to \infty} \mathbb{P}^* \Big(\sup_{s \leq \mathfrak{s}_n} \mathscr{A}^k \big(\mathscr{D}_{n,T^s}, \mathbb{1}_{\Delta^k \setminus \mathfrak{E}^{k,s}} (1 - \mathbb{1}_{\mathfrak{S}^{k,s} \cap \{\underline{V}(\cdot) \geq -a_{\varepsilon'}\}}) \big) > \varepsilon (n^{1/2} \boldsymbol{L}_n)^k \Big) = 0,$$

where we recall that $\mathfrak{s}_n = n^{1/2}/\varepsilon_1$. Then, note that $\mathscr{A}^k(\mathscr{D}_{n,T^s}, \mathbb{1}_{\Delta^k \setminus \mathfrak{E}^{k,s}} \mathbb{1}_{\mathfrak{S}^{k,s} \cap \{\underline{V}(\cdot) \geq -a_{\varepsilon'}\}})$ is smaller than

$$\mathscr{A}^k(\mathscr{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s} \cap \mathfrak{E}_1^{k,s}}) + \mathscr{A}^k(\mathscr{D}_{n,T^s}, \mathbb{1}_{\mathfrak{S}^{k,s} \cap \mathfrak{E}_2^{k,s}} \mathbb{1}_{\{\underline{V}(\cdot) \geq -a_{\varepsilon'}\}}).$$

Hence, by Markov inequality, the result follows using Lemma 3.11 with $B = a_{\varepsilon'}$. □

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