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Polyhedral Investigation of k Edge-Connected L-Hop-Constrained Network Design Problem

I. Diarrassouba[1], A. R. Mahjoub[2]

1. Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, 76600 Le Havre, France
diarrasi@univ-lehavre.fr
2. Laboratoire LAMSADE, CNRS UMR 7243, Université Paris-Dauphine, PSL, Place du Maréchal De Lattre de Tassigny, 75775, Paris Cedex 16, France
ridha.mahjoub@lamsade.dauphine.fr

Abstract. In this paper, we study the k edge-connected L-hop-constrained network design problem. Given a weighted graph $G = (V, E)$, a set D of pairs of nodes, two integers $L \geq 2$ and $k \geq 2$, the problem consists in finding a minimum weight subgraph of G containing at least k edge-disjoint paths of length at most L between every pair $\{s, t\}$ of D. The problem has several applications in telecommunications network design. It also has applications in reliable container transportation network design. Even if the problem has been studied for several decades, it appears, to the best of our knowledge, that the associated polytope is not well known, even when $L \in \{2, 3\}$.
In this paper, we investigate the problem from a polyhedral point of view. In particular, we consider the case where $L \in \{2, 3\}$ and investigate the polytopes induced by the integer programming formulations introduced in [24]. We first present these latter formulation, and introduce several new classes of valid inequalities. Then, we study the conditions for these inequalities to define facets and present separation algorithms for these inequalities.

Keywords. Hop-constrained survivable network, edge-disjoint paths, hop-constrained path, valid inequalities, Branch-and-Cut algorithm.

1 Introduction

Let $G = (V, E)$ be an undirected graph with node set V and edge set E and $D \subseteq V \times V$ a set of pairs of nodes, called *demands* with $|D| = d$. If a pair $\{s, t\}$ is a demand in D, we call s and t *demand nodes* or *terminal nodes*. Let $L \geq 2$ be a fixed integer. If s and t are two nodes of V, an L-st-path in G is a path between s and t of length at most L, where the length is the number of edges (also called *hops*).
Given a weight function $c : E \to \mathbb{R}$, which associates the weight $c(e)$ to each edge $e \in E$ and an integer $k \geq 2$, the *k-edge-connected L-hop-constrained network design problem* (kHNDP for short) consists in finding a minimum cost subgraph of G having at least k edge-disjoint L-st-paths between each demand $\{s, t\} \in D$.

The kHNDP has applications in the design of survivable telecommunication networks where bounded-length paths are required. Survivable networks must satisfy some connectivity requirements; that is, the networks should still be functional after the failure of certain links. As pointed out in [53] (see also [50]), the topology that seems to be very efficient (and needed in practice) is the uniform topology, that is to say that corresponding to networks that survive after the failure of $k - 1$ or fewer edges, for some $k \geq 2$. However, this requirement is often insufficient regarding the reliability of a telecommunication network. In fact, the alternative paths could be too long to guarantee an effective routing. In data networks, such as the Internet, the elongation of the route of the information could cause a major loss in the transfer speed.

For other networks, the signal itself could be degraded by a longer routing. In such cases, the L-path requirement guarantees exactly the needed quality of the alternative routes. Moreover, in a telecommunication network, usually several commodities have to be routed in the network between pairs of terminals. In order to ensure an effective routing, there must exist a sufficient number of hop-constrained paths between each pair of terminals.

One can also find application of the kHNDP when designing reliable a container shipping service in martitime transportation. Indeed, one of the key issue for container liner shipping companies is to limit as much as possible the number of transshipments for each container, and this, for reducing the chance of losing the container (see Balakrishnan and Vad Karsten [6]). Moreover, ensuring edge- or node-disjoint maritime routes guaranties that the liner shipping network is less vulnerable to disruption (See Lhomme [52]).

The kHNDP has been extensively investigated when there is only one demand in the network ($|D| = 1$). In particular, the associated polytope has received special attention. In [48], Huygens et al. study the kHNDP for $k = 2$ and $L = 2, 3$. They give an integer programming formulation for the problem and show that the linear programming relaxation of this formulation completely describes the associated polytope. From this, they obtain a minimal linear description of that polytope. They also show that this formulation is no longer valid when $L \geq 4$. In [22], Dahl et al. study the kHNDP when $L = 2$ and $k \geq 2$. They give a complete description of the associated polytope in this case and show that it can be solved in polynomial time using linear programming. Bendali et al. [9] generalize the results of [48] and [22] and give a complete description of the polytope of the kHNDP, with $|D| = 1$, for any $k \geq 2$ and $L \in \{2, 3\}$
In [18], Dahl considers the kHNDP for $k = 1$ and $L = 3$. He gives a complete description of the dominant of the associated polytope. Dahl and Gouveia [19] consider the directed hop-constrained path problem. They describe valid inequalities and characterize the associated polytope when $L \leq 3$. Huygens and Mahjoub [46] study the kHNDP when $k = 2$ and $L \geq 4$. They also study the variant of the problem where k node-disjoint paths of length at most L are required between two terminals. They give an integer programming formulation for these two problems when $L = 4$.

Diarrassouba et al. [24] consider the problem when $L \in \{2, 3\}$ and $k \geq 2$. They introduce four extended integer programming formulations and compare these formulations to the so-called natural formulation. They show that for $L \in \{2, 3\}$, the LP-bound provided by these extended formulations has the same value as that provided by the natural formulation. They also solve, for $L = 2, 3$ and $k = 3, 4, 5$, some instances of the problem using CPLEX and compare the formulations in terms of efficiency.

Botton et al. [10] propose an extended formulation for the kHNDP for every $L \geq 2$. They also develop an exact algorithm based on a Benders decomposition method, and report computational results for $L \in \{3, 4, 5\}$, $k \in \{1, 2, 3\}$, and with graphs having up to 21 nodes and $|D| \in \{5, 10\}$.

More recently, Arslan et al. [57] investigate a variant of the kHNDP in which the designed network is such that there exists an L-st-path after the removal of at most $k - 1$ edges from the network, for any $k \geq 2$. They present an integer programming formulation relying on the design variables and present some computational results for this problem.

In [12], Coullard et al. investigate the structure of the polyhedron associated with the st-walks of length L of a graph, where a walk is a path that may go through the same node more than once. They present an extended formulation of the problem, and, using projection, they give a linear description of the associated polyhedron. They also discuss classes of facets of that polyhedron.

The kHNDP has also been studied when $|D| \geq 2$. In [21], Dahl and Johannessen consider the case where $k = 1$ and $L = 2$. They introduce valid inequalities and develop a Branch-and-Cut algorithm. The problem of finding a minimum cost spanning tree with hop-constraints is also considered in [37, 38, 40]. Here, the hop-constraints limit to a positive integer H the number of links between the root and any terminal in the network. Dahl [17] studies the problem when $H = 2$ from a polyhedral point of view, and gives a complete description of the associated polytope when the graph is a wheel. Finally, Huygens et al. [49] consider the problem of finding a minimum cost subgraph with at least two edge-disjoint L-hop-constrained paths between each given pair of terminal nodes. They give an integer programming formulation of that problem for $L = 2, 3$ and present several classes of valid inequalities. They also devise separation routines. Using these, they propose a Branch-and-Cut algorithm and discuss some computational results.

Besides hop-constraints, another reliability condition, which is used in order to limit the length of the routing, requires that each link of the network belongs to a ring (cycle) of bounded length. In [33], Fortz et al. consider the 2-node connected subgraph problem with bounded rings. This problem consists in finding a minimum cost 2-node connected subgraph (V, F) such that each edge of F belongs to a cycle of length at most L. They describe several classes of facet-defining inequalities for the associated polytope and devise a Branch-and-Cut algorithm for the problem. In [32], Fortz et al. study the edge version of that problem. They give an integer programming formulation for the problem in the space of the natural design variables and describe different classes of valid inequalities. They study the separation problem for these inequalities and discuss Branch-and-Cut algorithms.

The related k-edge-connected subgraph problem and its associated polytope have also been the subject of extensive research in the past years. Grötschel and Monma [41] and Grötschel et al. [42, 43] study the k-edge-connected subgraph problem within the framework of a general survivable model. They discuss polyhedral aspects and devise cutting plane algorithms. Didi Biha and Mahjoub [26] study that problem and give a complete description of the associated polytope when the graph is series-parallel. In [27], Didi Biha and Mahjoub study the Steiner version of that problem and characterize the polytope when k is even. Chopra in [11] studies the dominant of that problem and introduces a class of valid inequalities for its polyhedron. Barahona and Mahjoub [5] characterize the polytope for the class of Halin graphs. In [31], Fonlupt and Mahjoub study the fractional extreme points of the linear programming relaxation of the 2-edge-connected subgraph polytope. They introduce an ordering on these extreme points and characterize the minimal extreme points with respect to that ordering. As a consequence, they obtain a characterization of the graphs for which the linear programming relaxation of that problem is integral. Didi Biha and Mahjoub [25] extend the results of Fonlupt and Mahjoub [31] to the case $k \geq 3$ and introduce some graph reduction operations. Kerivin et al. [51] study that problem in the more general case where each node of the graph has a specific connectivity requirement. They present different classes of facets of the associated polytope when the connectivity requirement of each node is at most 2 and devise a Branch-and-Cut algorithm for the problem in this case. In [8], Bendali et al. study the k-edge-connected subgraph problem for

the case $k \geq 3$. They introduce several classes of valid inequalities and discuss the separation algorithm for these inequalities. They devise a Branch-and-Cut algorithm using the reduction operations of [25] and give some computational results for $k = 3, 4, 5$. A complete survey on the k-edge-connected subgraph problem can be found in [50].

In this work, we are mainly interested in the polyhedral description of the kHNDP polytope when $L \in \{2, 3\}$. We first present the integer programming formulations for the kHNDP introduced by Diarrassouba et al. [24] when $L = 2, 3$ and $k \geq 2$. We then introduce several classes of valid inequalities for the polytope associated with the so-called natural formulation as well as valid inequalities associated with some extended formulations. We also investigate the conditions under which these inequalities define facets. Then, we discuss the separation problem associated with these inequalities.

Notice that in this work, we consider two types of demand sets: rooted demands and disjoint demands. A set of rooted demands is composed of demands which have the same node as source node, while a set of disjoint demands is composed of demands in which each source node is associated with only one destination node and vice-versa.

The paper is organized as follows. In Section 2, we present the integer programming formulations introduced by [24], and in Section 3 we discuss the basic properties of the polytope associated with these formulations. Then, in Sections 4, 5, 6 and 7, we introduce new valid inequalities for the problem. In Section 8, we discuss some conditions under which they define facets and Section **??** is dedicated to the separation algorithms for each class of inequality.
The rest of this section is devoted to more definitions and notation. An edge $e \in E$ with endnodes u and v is denoted by uv. Given two node subsets W and W', we denote by $[W, W']$ the set of edges having one endnode in W and the other in W'. If $W = \{u\}$, we then write $[u, W']$ for $[\{u\}, W']$. We also denote by $\overline{W}$ the node set $V \setminus W$. The set of edges having only one node in W is called a *cut* and denoted by $\delta(W)$. We will write $\delta(u)$ for $\delta(\{u\})$. Given two nodes $s, t \in V$, a cut $\delta(W)$ such that $s \in W$ and $t \in \overline{W}$ is called an *st-cut*.
We will also denote by $H = (U, A)$ a directed graph where U is the set of nodes and A is the set of arcs. An arc a with origin u and destination v will be denoted by (u, v). Given two node subsets W and W' of U, we will denote by $[W, W']$ the set of arcs whose origin is in W and whose destination is in W'. As before, we will write $[u, W']$ for $[\{u\}, W']$ and $\overline{W}$ will denote the node set $U \setminus W$. The set of arcs having their origin in W and their destination in $\overline{W}$ is called a *cut* or *dicut* in H and is denoted by $\delta^+(W)$. We will also write $\delta^+(u)$ for $\delta^+(\{u\})$ with $u \in U$. If s and t are two nodes of H such that $s \in W$ and $t \in \overline{W}$, then $\delta^+(W)$ will be called an *st-cut* or *st-dicut* in H. If W and W' are two node subsets of H, then $[W, W']^+$ will denote the set of arcs of H whose origins are in W and destinations are in W'. As for undirected graphs, we will write $[u, W']^+$ for $[\{u\}, W']^+$.
Given an undirected graph $G = (V, E)$ (resp. a directed graph $H = (U, A)$) and an edge subset $F \subseteq E$ (resp. an arc subset $B \subseteq A$), we let $x^F \in \mathbb{R}^E$ (resp. $y^B \in \mathbb{R}^A$) be the *incidence vector* of F (resp. B), that is the $0-1$ vector such that $x^F(e) = 1$ if $e \in F$ (resp. $y^B(a) = 1$ if $a \in B$) and 0 otherwise. Given F a subset of E (resp. A) and a vector $x \in \mathbb{R}^E$ (resp. $y \in \mathbb{R}^A$), $x(F)$ (resp. $y(F)$) will represent the term $\sum_{e \in F} x(e)$ (resp. $\sum_{e \in F} y(e)$).

2 Integer programming formulations

In this section, we present five integer programming formulations, presented by Diarrassouba et al. [24], for the kHNDP when $L = 2, 3$. The first formulation is the so-called natural formulation whose variables set corresponds to the set of edges of the input graph while the other formulations are extended formulations based on auxilliary graphs.

2.1 The natural formulation

Let $G = (V, E)$ be an undirected graph, $D \subseteq V \times V$ be a demand set, and two integers $k \geq 2$ and $L \in \{2, 3\}$. If an edge subset $F \subseteq E$ induces a solution of the kHNDP, that is a subgraph (V, F) contains k-edge-disjoint L-st-paths for every $\{s, t\} \in D$, then its incidence vector x satisfies the following inequalities

$$x(\delta(W)) \geq k \text{ for all } st\text{-cuts } \delta(W),\ W \subset V, \{s, t\} \in D, \tag{2.1}$$

$$x(e) \geq 0 \qquad \text{for all } e \in E, \tag{2.2}$$

$$x(e) \leq 1 \qquad \text{for all } e \in E. \tag{2.3}$$

Inequalities (2.1) are the so-called *st-cut inequalities* while (2.2) and (2.3) are the *trivial inequalities*.

Now, for a demand $\{s, t\} \in D$, we consider the partition $\pi = (V_0, V_1, ..., V_{L+1})$ of V such that $s \in V_0$, $t \in V_{L+1}$ and $V_i \neq \emptyset$, for all $i \in \{1, ..., L\}$. Let T be the set of edges $e = uv$, where $u \in V_i$, $v \in V_j$, and $|i - j| > 1$. The edge set T is called an *L-st-path-cut*. Figure 1 below gives an example of L-st-path-cut with $L = 3$ and, $V_0 = \{s\}$ and $V_{L+1} = \{t\}$.

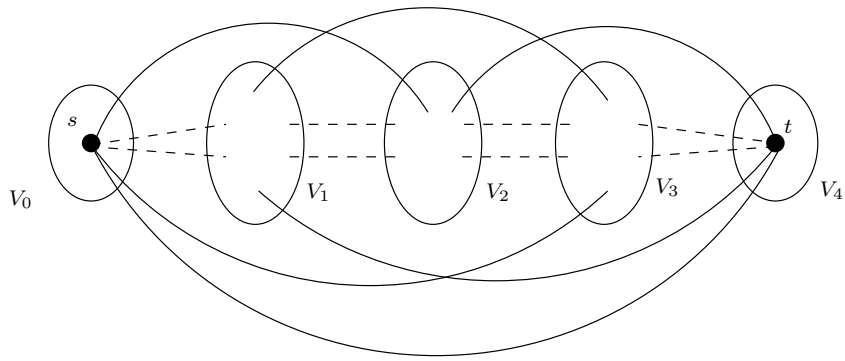

Figure 1: Support graph of a L-st-path-cut with $L = 3$, $V_0 = \{s\}$, $V_{L+1} = \{t\}$ and T formed by the solid edges.

The following inequality

$$x(T) \geq k \tag{2.4}$$

is the so-called *L-st-path-cut inequality* induced by π. Dahl [18] showed that the L-st-path-cut (2.4) inequalities are valid for the kHNDP polytope when $k = 1$ and $|D| = 1$. It is not hard to see that L-st-path-cut inequalities (2.4) are valid for the kHNDP polytope for any $k \geq 1$, $L \geq 2$ and $|D| \geq 2$.

It can be shown (see [24] and [49]) that L-st-path-cut inequalities together with the st-cut inequalities (2.1), (2.2) and (2.3) provide an integer programming formulation for the kHNDP when $L = 2, 3$.

Theorem 2.1 [49] *Let $G = (V, E)$ be a graph, $k \geq 2$ and $L \in \{2, 3\}$. Then, the kHNDP is equivalent to the following integer program*

$$min\{cx;\ subject\ to\ (2.1) - (2.4),\ x \in \mathbb{Z}^E\}. \tag{2.5}$$

Formulation (2.5) is called *the natural formulation* and is denoted by $k\text{HNDP}_{Nat}$. In [49], Huygens et al. studied the polytope associated with this formulation and introduce some facet-defining inequalities for the problem. They also develop a Branch-and-Cut algorithm for the kHNDP when $k = 2$ and $L = 2, 3$.

In [9], Bendali et al. showed that L-st-path-cut inequalities define facets of the kHNDP if and only if an L-st-path-cut inequality induced by a partition $(V_0, ..., V_{L+1})$, with $s \in V_0$ and $t \in V_{L+1}$, is facet-defining only if $|V_0| = |V_{L+1}| = 1$. Therefore, in the remainder of the paper, the only L-st-path-cuts that we will consider are those induced by partitions of the form $(\{s\}, V_1, ..., V_L, \{t\})$.

2.2 Demand decomposition based formulations

In this section, we present three integer programming formulations for the kHNDP for $L = 2, 3$ where we use a directed layered graph to model each hop-constrained subproblem. These formulations are called *separated formulations*. The graph transformation supporting these formulations is given below.

2.2.1 Graph transformation

Given $G = (V, E)$ and $\{s, t\} \in D$, let $\widetilde{G}_{st} = (\widetilde{V}_{st}, \widetilde{A}_{st})$ be the layered digraph obtained from G as follows:

- $\widetilde{V}_{st} = N_{st} \cup N'_{st} \cup \{s, t\}$ with $N_{st} = V \setminus \{s, t\}$ and N'_{st} is a copy of N_{st} (each node $u \in N_{st}$ corresponds to a node u' of N'_{st}),
- $\widetilde{A}_{st}$ is composed of four kinds of arcs:
  - for all $su \in E$, $(s, u) \in \widetilde{A}_{st}$,
  - for all $vt \in E$, $(v', t) \in \widetilde{A}_{st}$,
  - for all $u \in N_{st}$, we introduce $\min\{|[s, u]|, |[u, t]|, k\}$ arcs of the form $(u, u') \in \widetilde{A}_{st}$,
  - if $L = 3$, for all $uv \in E \setminus \{st\}$ with $u, v \in N_{st}$, $\{(u, v'), (v, u')\} \in \widetilde{A}_{st}$ (see Figure 2 for an illustration with $L = 3$).

For an edge $e = uv \in E$, we denote by $\widetilde{A}_{st}(e)$ the set of arcs of $\widetilde{G}_{st}$ corresponding to the edge e:

- when $u = s$ (resp. $v = t$), $\widetilde{A}_{st}(e)$ contains (s, v) (resp. (u', t)),
- when $u \neq s$ and $v \neq t$, if $L = 3$, $\widetilde{A}_{st}(e) = \{(u, v'), (v, u')\}$ and, if $L = 2$, $\widetilde{A}_{st}(e)$ is empty.

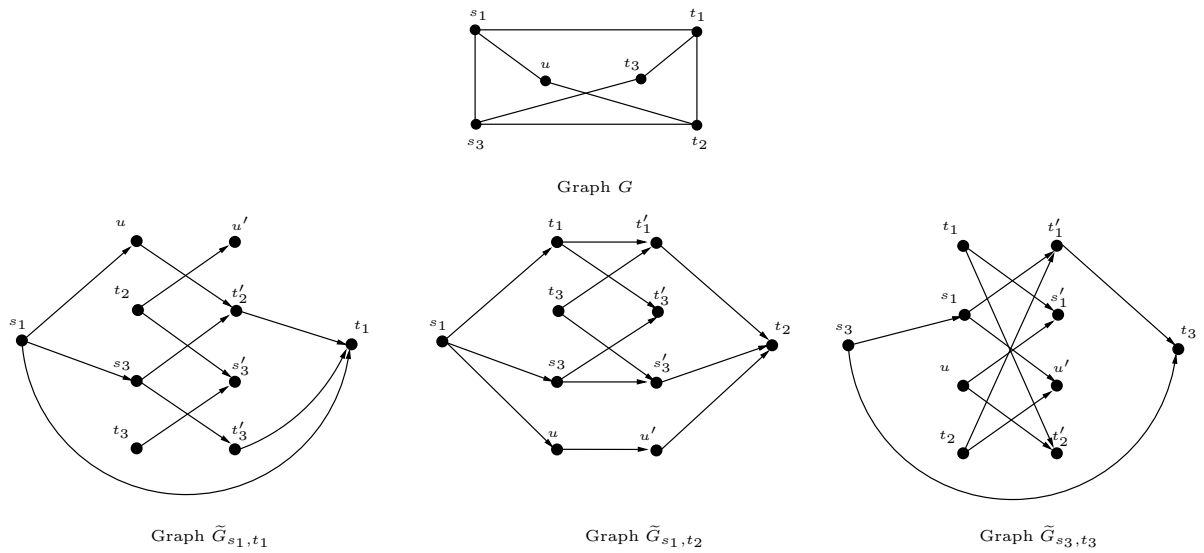

Figure 2: Construction of graphs $\widetilde{G}_{st}$ with $D = \{\{s_1, t_1\}, \{s_1, t_2\}, \{s_3, t_3\}\}$ for $L = 3$ and $k = 1$.

Note that $\widetilde{G}_{st}$ may have nodes different from $u \in N_{st} \cup N'_{st}$ with indegree or outdegree equal to zero. These nodes can be removed from $\widetilde{G}_{st}$ after its construction.

$\widetilde{G}_{st}$ contains four layers: $\{s\}, N_{st}, N'_{st}, \{t\}$ and no circuit. Also, any st-dipath in $\widetilde{G}_{st}$ is of length no more than 3:

- the length is equal to 1 if the st-dipath is composed of the single arc (s, t),
- the length is equal to 3 for both st-dipaths of the form (s, u, u', t) corresponding to path (s, u, t) of length 2 in G, and st-dipaths of the form (s, u, v', t) corresponding to path (s, u, v, t), with $u \neq v$, of length 3 in G.

Moreover, notice that in $\widetilde{G}_{st}$ there exist exactly $\min\{|[s, u]|, |[u, t]|, k\}$ arcs between two vertices (u, u'), for every $u \in V \setminus \{s, t\}$. If G is simple, that is does not contain parallel edges, then $\min\{|[s, u]|, |[u, t]|, k\} \leq 1$, for every $u \in V \setminus \{s, t\}$, and $\min\{|[s, u]|, |[u, t]|, k\} \leq k$ for general graphs. If G is simple and complete, then $\min\{|[s, u]|, |[u, t]|, k\} = 1$, for every $u \in V \setminus \{s, t\}$.

Note that each graph $\widetilde{G}_{st}$ contains $|\widetilde{V}_{st}| = 2|V| - 2$ $(= |N_{st} \cup N'_{st} \cup \{s, t\}|)$ nodes and $|\widetilde{A}_{st}| \leq |\delta(s)| + |\delta(t)| - |[s, t]| + k(|V| - 2)$ arcs if $L = 2$ and $|\widetilde{A}_{st}| \leq 2|E| - |\delta(s)| - |\delta(t)| + |[s, t]| + k(|V| - 2)$ arcs if $L = 3$, for all $\{s, t\} \in D$.

Diarrassouba et al. [24] (see also [9]) pointed out that a set of k arc-disjoint st-dipaths of $\widetilde{G}_{st}$ corresponds to a set of k edge-disjoint 3-st-paths in G, and vice-versa. This is summarized by the following Corollary.

Corollary 2.1 [24]
Let H be a subgraph of G and let $\widetilde{H}_{st}$, $\{s, t\} \in D$, be the subgraph of $\widetilde{G}_{st}$ obtained by considering all the arcs of $\widetilde{G}_{st}$ corresponding to an edge of H, plus the arcs of the form (u, u'), for all $u \in V\{s, t\}$. Then H induces a solution of the kHNDP if $\widetilde{H}_{st}$ contains k arc-disjoint st-dipaths, for every $\{s, t\} \in D$. Conversely, given a set of subgraphs $\widetilde{H}_{st}$ of $\widetilde{G}_{st}$, $\{s, t\} \in D$, if H is the

subgraph of G obtained by considering all the edges of G associated with at least one arc in a subgraph $\widetilde{H}_{st}$, then H induces a solution of the kHNDP only if $\widetilde{H}_{st}$ contains k arc-disjoint st-dipaths, for every $\{s,t\} \in D$.

Also, Bendali et al. [9] showed that st-cuts and 3-st-path-cuts of G correspond to a set st-discuts of $\widetilde{G}_{st}$ which do not contain any arc of the form (u,u'), and vice-versa.

Lemma 2.1 [9] *An edge set C which induces an st-cut or a 3-st-path-cut of G corresponds to an st-dicut of $\widetilde{G}_{st}$ which do not contain any arc of the form (u,u'), $u \in V \setminus \{s,t\}$, and vice-versa.*

2.2.2 Separated Flow Formulation

Given a demand $\{s,t\}$, we let $f^{st} \in \mathbb{R}^{\widetilde{A}_{st}}$ be a flow vector on $\widetilde{G}_{st}$ of value k between s and t. Then f^{st} satisfies the *flow conservation constraints* (2.6), given by

$$\sum_{a\in\delta^+(u)} f_a^{st} - \sum_{a\in\delta^-(u)} f_a^{st} = \left\{ \begin{array}{ll} k & \text{if } u = s, \\ 0 & \text{if } u \in \widetilde{V}_{st} \setminus \{s,t\}, \\ -k & \text{if } u = t, \end{array} \right\}, \quad \text{for all } u \in \widetilde{V}_{st}, \{s,t\} \in D \tag{2.6}$$

and

$$f_a^{st} \leq x(e), \text{ for all } a \in \widetilde{A}_{st}(e), e \in E, \{s,t\} \in D. \tag{2.7}$$
$$f_a^{st} \leq 1, \text{ for all } a = (u,u'),\ u \in V \setminus \{s,t\},\ \{s,t\} \in D. \tag{2.8}$$
$$f_a^{st} \geq 0, \text{ for all } a \in \widetilde{A}_{st},\ \{s,t\} \in D. \tag{2.9}$$
$$x(e) \leq 1, \text{ for all } e \in E. \tag{2.10}$$

Inequalities (2.7) are also called *linking inequalities*. They indicate that if an edge $e \in E$ is not in the solution, then the flow on every arc corresponding to e is 0. Inequalities (2.9)-(2.10) are the *trivial inequalities*.

Thus, we have the following theorem.

Theorem 2.2 [24] *The kHNDP for $L = 2,3$ is equivalent to the following integer program*

$$\begin{aligned} min\{cx;\ \textit{subject to } (2.6)-(2.10),\ x \in \mathbb{Z}_+^E,\ f^{st} \in \mathbb{R}_+^{\widetilde{A}_{st}}, \\ \textit{for all } \{s,t\} \in D\}. \end{aligned} \tag{2.11}$$

Formulation (2.11) will be called *the separated flow formulation* and will be denoted by $k\text{HNDP}_{Flow}^{Sep}$.

2.2.3 Separated Path Formulation

For each demand $\{s,t\} \in D$, let $\widetilde{\mathcal{P}}_{st}$ be the set of st-dipaths in $\widetilde{G}_{st}$ and, for each $P \in \widetilde{\mathcal{P}}_{st}$, let $\mu^{st}(P)$ be a binary variable whose value is 1 if P is used in a solution and 0 if not.

If an edge subset $F \subseteq E$ induces a solution of the kHNDP, then x^F and $(\mu^{st}(P),\ P \in \widetilde{\mathcal{P}}_{st}, \{s,t\} \in D)$ satisfy the following inequalities.

$$\sum_{P \in \widetilde{\mathcal{P}}_{st}} \mu^{st}(P) \geq k, \text{ for all } \{s,t\} \in D, \tag{2.12}$$

$$\sum_{P \in \widetilde{\mathcal{P}}_{st}, a \in P} \mu^{st}(P) \leq x(e), \text{ for all } a \in \widetilde{A}_{st}(e),\ e \in E, \{s,t\} \in D, \tag{2.13}$$

$$\sum_{P \in \widetilde{P}_{st}, a \in P} \mu^{st}(P) \leq 1, \text{ for all } a = (u,u'), u \in V \setminus \{s,t\},\ \{s,t\} \in D, \tag{2.14}$$

$$x(e) \leq 1, \text{ for all edge } e \in E, \tag{2.15}$$

$$\mu^{st}(P) \geq 0, \text{ for all } P \in \widetilde{\mathcal{P}}_{st}, \{s,t\} \in D. \tag{2.16}$$

Inequalities (2.12) express the fact that the solution must contain at least k st-dipaths. Inequalities (2.13) and (2.14) indicate that these st-dipaths are arc-disjoint.

The following theorem gives an integer programming formulation for the kHNDP using the path-based model described above.

Theorem 2.3 [24] *The kHNDP for $L = 2, 3$ is equivalent to the following integer program*

$$\begin{aligned} min\{cx;\ \textit{subject to } (2.12) - (2.16),\ x \in \mathbb{Z}_+^E, \mu^{st} \in \mathbb{Z}_+^{\widetilde{\mathcal{P}}_{st}}, \\ \textit{for all } \{s,t\} \in D\}. \end{aligned} \tag{2.17}$$

Formulation (2.17) is called *the separated path formulation* and is denoted by $k\text{HNDP}_{Path}^{Sep}$. Note that for each demand $\{s,t\} \in D$, the number of st-paths in the graph $\widetilde{G}_{st}$ is bounded by $|V|^{L-1}$, which is polynomial for $L = 2, 3$. Thus, this formulation contains a polynomial number of variables while the number of nontrivial inequalities is at most

$$d + \sum_{\{s,t\} \in D} (|\delta(s)| + |\delta(t)| - |[s,t]|) + dk(|V| - 2) \qquad \text{if } L = 2,$$

$$d + 2d|E| - \sum_{\{s,t\} \in D} (|\delta(s)| + |\delta(t)| - |[s,t]|) + dk(|V| - 2) \quad \text{if } L = 3,$$

which is polynomial.

2.2.4 Separated Cut Formulation

The previous two models include constraints guaranteeing that for each demand $\{s,t\} \in D$, there exists a flow of value k under the arc capacities given by x. By the Max flow-Min cut theorem, such a flow exists if and only if the capacity of any st-dicut, in each graph $\widetilde{G}_{st}$, is at least k. This observation leads at once to the following formulation which provides the same LP bound as the previous separated flow and path formulations.

Let $H \subseteq E$ be an edge subset which induces a solution of the kHNDP in G and let $\widetilde{H}_{st}$ be the arc subset of $\widetilde{G}_{st}$, $\{s,t\} \in D$, corresponding to H. Then, the incidence vector x^H of H and the vectors $y_{st}^{\widetilde{H}_{st}}$, $\{s,t\} \in D$, satisfy the following inequalities.

$$y_{st}(\delta^+(\widetilde{W})) \geq k, \text{ for all } st\text{-dicut } \delta^+(\widetilde{W}) \text{ of } \widetilde{G}_{st}, \text{ for all } \{s,t\} \in D, \tag{2.18}$$

$$y_{st}(a) \leq x(e), \text{ for all } a \in \widetilde{A}(e), \ e \in E, \ \{s,t\} \in D, \tag{2.19}$$

$$y_{st}(a) \leq 1, \text{ for all } a = (u,u'), \text{ for all } u \in V \setminus \{s,t\}, \ \{s,t\} \in D, \tag{2.20}$$

$$y_{st}(a) \geq 0, \text{ for all } a \in \widetilde{A}_{st}, \ \{s,t\} \in D, \tag{2.21}$$

$$x(e) \leq 1, \text{ for all } e \in E. \tag{2.22}$$

Inequalities (2.18) will be called *directed st-cut inequalities* or *st-dicut inequalities* and inequalities (2.19) *linking inequalities*. Inequalities (2.19) indicate that an arc $a \in \widetilde{A}_{st}$ corresponding to an edge e is not in $\widetilde{H}_{st}$ if e is not taken in H. Inequalities (2.20)-(2.22) are the *trivial inequalities*.

Theorem 2.4 [24] *The kHNDP for $L = 2, 3$ is equivalent to the following integer program*

$$\begin{aligned} min\{cx;\ \textit{subject to } (2.18) - (2.22),\ x \in \mathbb{Z}_+^E,\ y_{st} \in \mathbb{Z}_+^{\widetilde{A}_{st}}, \\ \textit{for all } \{s,t\} \in D\}. \end{aligned} \tag{2.23}$$

This formulation is called *the separated cut formulation* and is denoted by $k\text{HNDP}_{Cut}^{Sep}$. It contains a polynomial number of variables. Indeed, for $L = 2$, the number of variables is

$$|E| + \sum_{\{s,t\} \in D} |\widetilde{A}_{st}| \leq |E| + \sum_{\{s,t\} \in D} (|\delta(s)| + |\delta(t)| - |[s,t]|) + dk(|V| - 2),$$

and for $L = 3$, it is

$$|E| + \sum_{\{s,t\} \in D} |\widetilde{A}_{st}| \leq |E| + 2d|E| - \sum_{\{s,t\} \in D} (|\delta(s)| + |\delta(t)| - |[s,t]|) + dk(|V| - 2)$$

(recall that $d = |D|$).

However, the number of constraints is exponential since the number of directed st-cuts is exponential in the size of $\widetilde{G}_{st}$, for all $\{s,t\} \in D$. As we will see in Section **??** linear programming relaxation can be solved in polynomial time using a cutting plane algorithm.

In the next section, we present a further formulation for the kHNDP also based on directed graphs. However, unlike the separated formulations, this formulation is supported by only one directed graph.

2.3 Aggregated Formulation for the kHNDP

We denote by S_D and T_D respectively the sets of source and destination nodes of D. In the case where two demands $\{s_1, t_1\}$ and $\{s_2, t_2\}$ are such that $s_1 = t_2 = s$, we keep a copy of s in both S_D and T_D.
In this section, we will introduce a new formulation for the kHNDP which is supported by a single directed graph $\widetilde{G} = (\widetilde{V}, \widetilde{A})$ obtained from G as follows:

- $\widetilde{V} = S_D \cup N' \cup N'' \cup T_D$ with N' and N'' two copies of V; we denote by u' and u'' the nodes of N' and N'' corresponding to a node $u \in V$.

- $\widetilde{A}$ contains 6 kinds of arcs:

  1. for each demand $\{s,t\} \in D$
     - if $st \in E$, we add in $\widetilde{A}$ the arc (s,t'),
     - if $su \in E$ with $u \in V \setminus \{s,t\}$, we add an arc (s,u'),
     - if $vt \in E$ with $v \in V \setminus \{s,t\}$, we add an arc (v'',t),
  2. for each node $u \in V$, we add $\max\limits_{\{s,t\}\in D}\{\min\{|[s,u]|,|[u,t]|,k\}\}$ arcs of the form (u',u''),
  3. for each $t \in T_D$, we add $\min\{k, \max\{|[s,t]|,\ s \in S_D \text{ with } \{s,t\} \in D\}\}$ arcs of the form (t',t),
  4. if $L = 3$, for each edge $e = uv \in E$, we add two arcs (u',v'') and (v',u'').

Figures 3 and 4 show examples for $L = 2$ and $L = 3$ with $k = 1$, respectively.

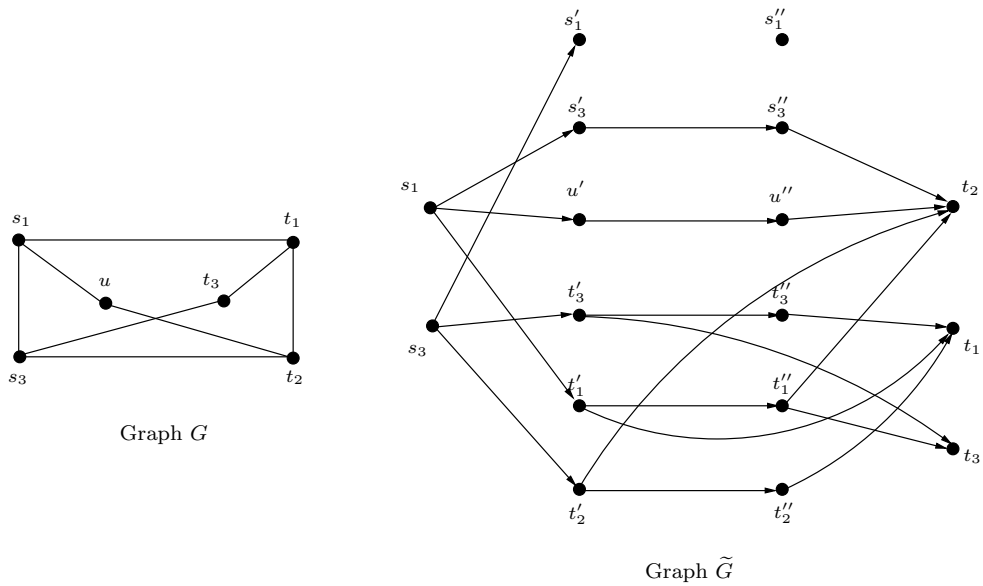

Figure 3: Construction of graph $\widetilde{G}$ with $D = \{\{s_1,t_1\},\{s_1,t_2\},\{s_3,t_3\}\}$, $L = 2$ and $k = 1$.

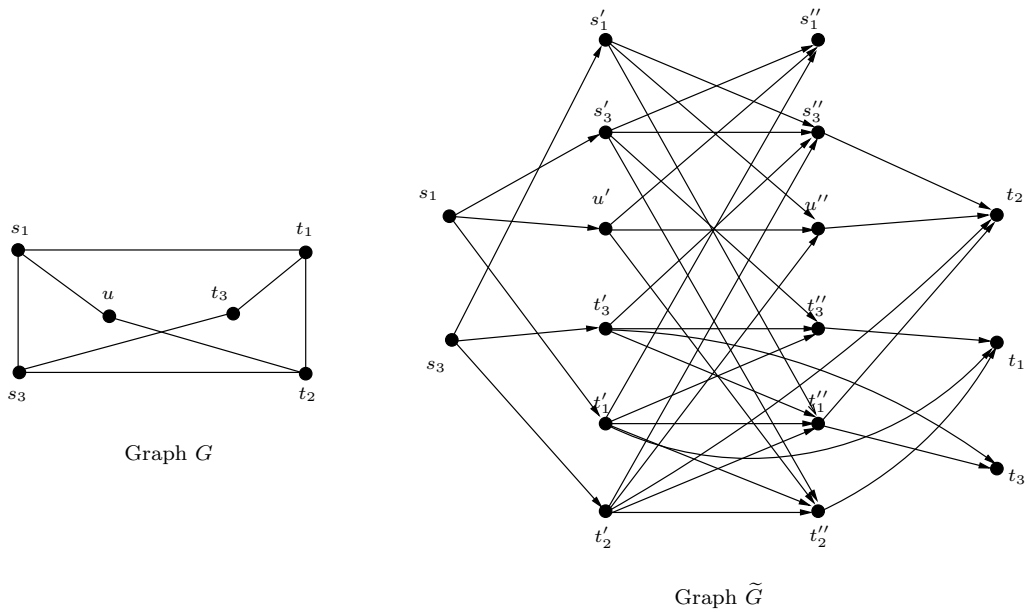

Figure 4: Construction of graph $\widetilde{G}$ with $D = \{\{s_1, t_1\}, \{s_1, t_2\}, \{s_3, t_3\}\}$, $L = 3$ and $k = 1$.

Notice that the digraph $\widetilde{G}$ may have nodes $u \in N \cup N'$ with indegree or outdegree equal to zero. These nodes can be removed from $\widetilde{G}$ after its construction. Also, note that when G is simple (that is with no parallel edges), $|[u', u'']^+| = |[t', t]^+| \leq 1$, for every $u \in V$ and every $t \in T_D$, and $|[u', u'']^+| = |[t', t]^+| = 1$ if G is complete.

Remark that $\widetilde{G}$ contains $|\widetilde{V}| = 2|V| + |S_D| + |T_D|$ nodes and $|\widetilde{A}| \leq k|V| + \sum_{s \in S} |\delta(s)| + \sum_{t \in T} |\delta(t)|$ arcs if $L = 2$, and $|\widetilde{A}| \leq 2|E| + k|V| + \sum_{s \in S} |\delta(s)| + \sum_{t \in T} |\delta(t)|$ arcs if $L = 3$.
If $\widetilde{G} = (\widetilde{V}, \widetilde{A})$ is the digraph associated with G, then for an edge $e \in E$, we denote by $\widetilde{A}(e)$ the set of arcs of $\widetilde{G}$ corresponding to e.

Observe that $\widetilde{G}$ is acyclic. Also note that for a given demand $\{s, t\} \in D$, every st-dipath in $\widetilde{G}$ contains at most 3 arcs. An L-st-path $P = (s, u, v, t)$ of G, where u and v may be the same, corresponds to an st-dipath $\widetilde{P} = (s, u', v'', t)$ in $\widetilde{G}$. Conversely, every st-dipath $\widetilde{P} = (s, u', v'', t)$ of $\widetilde{G}$, where u' and v'' may correspond to the same node of V, corresponds to an L-st-path $P = (s, u, v, t)$, where u and v may be the same. Moreover $\widetilde{G}$ does not contain any arc of the form (s, s') and (t'', t), for every $s \in S_D$ and $t \in T_D$. If a node $t \in T_D$ appears in exactly one demand $\{s, t\}$, then $[s'', t] = \emptyset$. In the remainder of this section we will suppose w.l.o.g. that each node of T_D is involved, as destination, in only one demand. In fact, in general, if a node $t \in T_D$ is involved, as destination, in more than one demand, say $\{s_1, t\}, ..., \{s_p, t\}$, with $p \geq 2$, then one may replace in T_D t by p nodes $t_1, ..., t_p$ and in D each demand $\{s_i, t\}$ by $\{s_i, t_i\}$, $i = 1, ..., p$.

Diarrassouba et al. [24] showed that looking for a solution of the kHNDP in G reduces to finding k arc-disjoint st-dipaths in $\widetilde{G}$, for all $\{s, t\} \in D$. Thus, if $F \subseteq E$ is an edge subset of G that

induces a solution of the kHNDP, then $x \in \mathbb{R}^E$ and $y \in \mathbb{R}_+^{\widetilde{A}}$ satisfy the following inequalities

$$y(\delta^+(\widetilde{W})) \geq k, \quad \text{for all } st\text{-dicut } \delta^+(\widetilde{W}), \{s,t\} \in D, \tag{2.24}$$

$$y(a) \leq x(e), \quad \text{for all } a \in \widetilde{A}(e),\ e \in E, \tag{2.25}$$

$$y(a) \geq 0, \quad \text{for all } a \in \widetilde{A}, \tag{2.26}$$

$$x(e) \leq 1, \quad \text{for all } e \in E. \tag{2.27}$$

They give the following theorem.

Theorem 2.5 [24] *The kHNDP for $L = 2, 3$ is equivalent to the following integer program*

$$min\{cx;\ subject\ to\ (2.24) - (2.27),\ x \in \mathbb{Z}_+^E, y \in \mathbb{Z}_+^{\widetilde{A}}\}. \tag{2.28}$$

Formulation (2.28) will be called *the aggregated formulation* and is denoted by $k\text{HNDP}_{Ag}$. Inequalities (2.24) will be called *directed st-cut inequalities* or *st-dicut inequalities* and (2.25) will be called *linking inequalities*. The latter indicate that an arc a, corresponding to an edge e, is not chosen in the solution of kSNDP if e is not chosen in the solution of kHNDP.
This formulation contains $|E| + |\widetilde{A}| \leq |E| + k|V| + \sum_{s \in S_D} |\delta(s)| + \sum_{t \in T_D} |\delta(t)|$ variables if $L = 2$ and $|E| + |\widetilde{A}| \leq 3|E| + k|V| + \sum_{s \in S_D} |\delta(s)| + \sum_{t \in T_D} |\delta(t)|$ variables if $L = 3$. The number of constraints is exponential since the st-dicuts are exponential in number. But, as it will turn out, the separation problem for inequalities (2.24) can be solved in polynomial time and hence, the linear programming relaxation of (2.28) so is.

3 The kHNDP polytopes

In the present section, we give some basic results on the polytope associated with the different formulations we have presented before. These polytopes are named as follows

- $k\text{HNDP}_{Nat}(G, D)$: polytope associated with the natural formulation,
- $k\text{HNDP}_{Ag}(G, D)$: polytope associated with the aggregated formulation,
- $k\text{HNDP}_{Cu}^{Sep}(G, D)$: polytope associated with the separated cut formulation,
- $k\text{HNDP}_{Flo}^{Sep}(G, D)$: polytope associated with the separed flow formulation,
- $k\text{HNDP}_{Pa}^{Sep}(G, D)$: polytope associated with the separated path formulation.

3.1 Polytope $k\text{HNDP}_{Ag}(G, D)$

We first consider the polytope $k\text{HNDP}_{Ag}(G, D)$. Let $\widetilde{G} = (\widetilde{V}, \widetilde{A})$ be the directed graph associated with G and D in the case of the Aggregated formulation. Let E^* be the set of edges $e \in E$ such that there exists a demand $\{s,t\} \in D$ such that $G \setminus \{e\}$ does not contain k edge-disjoint L-st-paths. Such an edge is said to be *L-st-essential*. Also consider an arc $a \in \widetilde{A}$ such that there exists a demand $\{s,t\} \in D$ such that the graph $\widetilde{G} \setminus \{a\}$ does not contain k arc-disjoint st-dipaths. Such an arc a is said to be *st-essential*. We will denote by $\widetilde{A}^*$ the set of st-essential arcs of $\widetilde{G}$. The following theorem characterizes the dimension of $k\text{HNDP}_{Ag}(G, D)$.

Theorem 3.1 *$dim(kHNDP_{Ag}(G,D)) = |E| + |\widetilde{A}| - |E^*| - |\widetilde{A}^*|$.*

Proof. Obviously, we have that $dim(k\text{HNDP}_{Ag}(G,D)) \leq |E| + |\widetilde{A}| - |E^*| - |\widetilde{A}^*|$. Now we show that $dim(k\text{HNDP}_{Ag}(G,D)) \geq |E| + |\widetilde{A}| - |E^*| - |\widetilde{A}^*|$. Recall that a solution of $k\text{HNDP}_{Ag}(G,D)$ is described by a pair $(\widetilde{F}, F)$ where $\widetilde{F} \subseteq \widetilde{A}$ and $F \subseteq E$ is the associated edge set. Also note that an edge set F induces a solution of the kHNDP if and only if the associated arc set $\widetilde{F}$ induces a subgraph of $\widetilde{G}$ containing k arc-disjoint st-dipaths for every $\{s,t\} \in D$.
Consider the pairs $(\widetilde{A} \setminus \{a\}, E)$, for all $a \in \widetilde{A} \setminus \widetilde{A}^*$. As $a \notin \widetilde{A}^*$, these pairs induce solutions of $k\text{HNDP}_{Ag}(G,D)$.
For every edge $e \in E \setminus E^*$, consider the pair $(\widetilde{A} \setminus \widetilde{A}(e), E \setminus \{e\})$. Recall that, for all $e \in E$, $\widetilde{A}(e)$ is the set of arcs of $\widetilde{A}$ corresponding to e. As $e \in E \setminus E^*$, the subgraph induced by $E \setminus \{e\}$ contains k edge-disjoint L-st-paths for every $\{s,t\} \in D$ and the subgraph of $\widetilde{G}$ induced by $\widetilde{A} \setminus \widetilde{A}(e)$ also contains k arc-disjoint st-dipaths for every $\{s,t\} \in D$. Hence this pair induces a solution of $k\text{HNDP}_{Ag}(G,D)$.
One can easily observe that these solutions, together with the solution given by the pair $(\widetilde{A}, E)$, form a family of $|E \setminus E^*| + |\widetilde{A} \setminus \widetilde{A}^*| + 1$ solutions of the $k\text{HNDP}_{Ag}$ whose incident vectors are affinely independant. Therefore, $dim(k\text{HNDP}_{Ag}(G,D)) \geq |E| + |\widetilde{A}| - |E^*| - |\widetilde{A}^*|$, which ends the proof of the theorem. □

Consequently, $k\text{HNDP}_{Ag}(G,D)$ is full dimensional if and only if $E^* = \emptyset = \widetilde{A}^*$. The next result shows that if G is complete and $|V| \geq k+2$, then $E^* = \emptyset = \widetilde{A}^*$, implying that $k\text{HNDP}_{Ag}(G,D)$ is full dimensional. But before, we give the following lemma.

Lemma 3.1 *If G is complete, then for every $\{s,t\} \in D$, there exist at least $|V|-1$ edge-disjoint 3-st-paths in G, and $|V|-1$ arc-disjoint st-dipaths in $\widetilde{G}$.*

Proof. Suppose that G is complete. Consider a demand $\{s,t\} \in D$ and the arc set $\widetilde{H} = [s,N'] \cup [N',N''] \cup [N'',t] \cup [t',t]$. Clearly, since G is complete, $|[s,N']| = |V|-1$, $|[N'',t]| = |V|-2$. Moreover, by the construction of $\widetilde{G}$, there are $|V|-2$ arcs of the form (u',u'') and the arc (t',t). The graph H thus contains st-dipaths (s,u',u'',t), for all $u \in V \setminus \{s,t\}$, and (s,t',t). These paths clearly forms $|V|-1$ arc-disjoint st-dipath of H. Finally, these paths correspond in G to paths (s,u,t), for all $u \in V \setminus \{s,t\}$, and path (s,t), which clearly form $|V|-1$ edge-disjoint 3-st-paths of G. □

A consequence of Lemma 3.1 is that for a complete graph G with $|V| \geq k+2$, the graph $\widetilde{G}$ contains at least $k+1$ arc-disjoint st-dipaths for every $\{s,t\} \in D$. This implies that $E^* = \emptyset = \widetilde{A}^*$. We thus have the following.

Corollary 3.1 *If G is complete and $|V| \geq k+2$, then $k\text{HNDP}_{Ag}(G,D)$ is full dimensional.*

From here, and until the end of this section, we will assume that G is complete and $|V| \geq k+2$, and hence $k\text{HNDP}_{Ag}(G,D)$ is full dimensional.

In what follows, we give necessary and sufficient conditions for the trivial inequalities to define facets of $k\text{HNDP}_{Ag}(G,D)$. Remark that the inequalities $y(a) \leq 1$, for all $a \in \widetilde{A}$, and $x(e) \geq 0$, for all $e \in E$, are redundant with respect to the inequalities

$$\begin{aligned} y(a) &\geq 0 \quad \text{for all } a \in \widetilde{A}, \\ x(e) &\leq 1 \quad \text{for all } e \in E, \\ y(a) &\leq x(e) \text{ for all arc } a \in \widetilde{A}(e), \end{aligned}$$

and hence, do not define facets.

Theorem 3.2 *We have the following results.*

i) Every inequality $x(e) \leq 1$ defines a facet of kHNDP$_{Ag}(G, D)$;

ii) An inequality $y(a) \geq 0$ defines a facet of kHNDP$_{Ag}(G, D)$ if and only if either $|V| \geq k+3$ or $|V| = k+2$ and a does not belong to an st-dicut of $\widetilde{G}$ of cardinality $k+1$.

Proof. i) Let $a \in \widetilde{A}$. Since G is complete and $|V| \geq k+2$, by Lemma 3.1, the subgraph induced by $\widetilde{A} \setminus \{a\}$ contains k arc-disjoint st-dipaths for every $\{s,t\} \in D$. Thus, the pair $(\widetilde{A} \setminus \{a\}, E)$ induces a solution of kHNDP$_{Ag}(G, D)$. Moreover, its incidence vector satisfies $x(e) = 1$.
Now let $f \in E \setminus \{e\}$. As before, the subgraph induced by $E \setminus \{f\}$ contains k edge-disjoint L-st-paths, for every $\{s,t\} \in D$. Thus, the pair $(\widetilde{A} \setminus \widetilde{A}(f), E \setminus \{f\})$ induces a solution of kHNDP$_{Ag}(G, D)$, whose incidence vector satisfies $x(e) = 1$.
It is not hard to see that these two families of solutions, together with the solution induced by the pair $(\widetilde{A}, E)$, form $|E| + |\widetilde{A}|$ solutions whose incidence vectors satisfy $x(e) = 1$ and are affinely independant. This yields $x(e) \leq 1$ defines a facet of kHNDP$_{Ag}(G, D)$.
ii) Consider an arc $a \in \widetilde{A}$ and suppose that $|V| \geq k+3$. By Lemma 3.1, $\widetilde{G}$ contains at least $k+2$ arc-disjoint st-dipaths for every $\{s,t\} \in D$, and G contains at least $k+2$ edge-disjoint L-st-paths. Thus for an edge $e \in E$, the pair $(\widetilde{A} \setminus (\{a\} \cup \widetilde{A}(e)), E \setminus \{e\})$ induces a solution of kHNDP$_{Ag}(G, D)$. Also, for an arc $a' \in \widetilde{A} \setminus \{a\}$, the pair $(\widetilde{A} \setminus \{a, a'\}, E)$ induces a solution of kHNDP$_{Ag}(G, D)$. These solutions together with the solution $(\widetilde{A} \setminus \{a\}, E)$ form a family of $|\widetilde{A}| + |E|$ solutions whose incidence vectors satisfy $y(a) = 0$ and are affinely independant. Thus, $y(a) \geq 0$ defines a facet.
Now suppose that $|V| = k+2$. If a belongs to an st-dicut $\delta^+(\widetilde{W})$ of $k+1$ arcs, then $y(a) \geq 0$ is redundant with respect to the inequalities

$$\begin{aligned} & y(\delta^+(\widetilde{W})) \geq k, \\ & -y(a') \geq -1, \text{ for all } a' \in \delta^+(\widetilde{W}) \setminus \{a\}, \end{aligned}$$

and hence cannot define a facet. If a does not belong to an st-dicut of $k+1$ arcs, then, the pairs $(\widetilde{A} \setminus (\{a\} \cup \widetilde{A}(e)), E \setminus \{e\})$, for all $e \in E$, and $(\widetilde{A} \setminus \{a, a'\}, E)$, for all $a' \in \widetilde{A} \setminus \{a\}$ induce solutions of kHNDP$_{Ag}(G, D)$. These solutions together with the solution $(\widetilde{A} \setminus \{a\}, E)$ form a family of $|\widetilde{A}| + |E|$ solutions whose incidence vectors satisfy $y(a) = 0$ and are affinely independant. Thus $y(a) \geq 0$ defines a facet of kHNDP$_{Ag}(G, D)$. □ □

The next theorem gives necessary conditions for the directed st-cut inequalities to define facets of kHNDP$_{Ag}(G, D)$.

Theorem 3.3 *Let $\widetilde{W} \subseteq \widetilde{V}$ be a node set such that there is a demand $\{s,t\} \in D$ with $s \in \widetilde{W}$ and $t \in \widetilde{V} \setminus \widetilde{W}$. Then the st-dicut inequality $y(\delta^+(\widetilde{W})) \geq k$ defines a facet of kHNDP$_{Ag}(G, D)$ only if the following conditions hold*

i) $\widetilde{W} \cap S_D = \{s\}$ and $(\widetilde{V} \setminus \widetilde{W}) \cap T_D = \{t\}$ (Recall that S_D (resp. T_D) is the set of terminals of G that are source (resp. destination) in at least one demand);

ii) $s' \in \widetilde{V} \setminus \widetilde{W}$, $s'' \in \widetilde{W}$ and $t'' \in \widetilde{W}$.

Proof. We will only show i). The proof for ii) follows the same lines. Suppose on the contrary that there exists another node $s_1 \neq s$ in $\widetilde{W} \cap S_D$. This implies that $[s, s_1] = \emptyset$, and thus, $\delta^+(\widetilde{W} \setminus \{s_1\}) = \delta^+(\widetilde{W}) \setminus \delta^+(s_1)$. Note that the edges of G associated with those of $\delta^+(s_1)$ are those of $\delta(s_1)$. As G is complete, $\delta^+(s_1) \neq \emptyset$. Therefore, the st-dicut inequality induced by $\widetilde{W}$ is redundant with respect to the inequalities

$$\begin{aligned} & y(\delta^+(\widetilde{W} \setminus \{s_1\})) \geq k, \\ & y(a) \geq 0 \qquad\qquad \text{for all } a \in \delta^+(s_1), \end{aligned}$$

and hence, cannot define a facet. □

3.2 Polytope kHNDP$^{Sep}_{Cu}(G, D)$

Now we consider the Cut formulation. The results of this section will be given without proof. In fact their proofs are similar to those of the previous section.
As before, we denote by E^* the set of L-st-essential edges of G and $\widetilde{A}^*_{st}$ the set of st-essential arcs of $\widetilde{G}_{st}$, for every $\{s, t\} \in D$. The following theorem gives the dimension of kHNDP$^{Sep}_{Cu}(G, D)$.

Theorem 3.4 *$dim(kHNDP^{Sep}_{Cu}(G, D)) = |E| + \sum_{\{s,t\} \in D} |\widetilde{A}_{st}| - |E^*| - \sum_{\{s,t\} \in D} |\widetilde{A}^*_{st}|$.*

Proof. Similar to the proof of Theorem 3.1. □

Lemma 3.2 *If G is complete, then for every demand $\{s, t\} \in D$, there exist at least $|V| - 1$ arc-disjoint st-dipaths in $\widetilde{G}_{st}$.*

Proof. Similar to the proof of Lemma 3.1. □

As a consequence, we have the following corollary.

Corollary 3.2 *If G is complete and $|V| \geq k + 2$, then $kHNDP^{Sep}_{Cu}(G, D)$ is full dimensional.*

Note that the inequalities $y_{st}(a) \leq 1$ and $x(e) \geq 0$ are redundant with respect to $y_{st}(a) \geq 0$, $x(e) \leq 1$ and $y_{st}(a) \leq x(e)$. The next theorem gives necessary and sufficient conditions for inequalities (2.21) and (2.22) to define facets of kHNDP$^{Sep}_{Cu}(G, D)$.

Theorem 3.5 *If G is complete and $|V| \geq k + 2$, then the following hold.*

i) Every inequality $x(e) \leq 1$ defines a facet of $kHNDP^{Sep}_{Cu}(G, D)$.

ii) An inequality $y(a) \geq 0$ defines a facet of $kHNDP^{Sep}_{Cu}(G, D)$ if and only if either $|V| \geq k + 3$ or $|V| = k + 2$ and a does not belong to an st-cut of cardinality $k + 1$.

Proof. Similar to the proof of Theorem 3.2. □

Now we address the dimension of the polytope associated with the natural formulation. The following theorem gives the dimension of kHNDP$_{Nat}(G, D)$. Its proof is omitted as it uses similar arguments as Theorems 3.1. Also, the condition for kHNDP$_{Ag}(G, D)$ to be full dimensional is the same for kHNDP$_{Nat}(G, D)$.

Theorem 3.6 *$dim(kHNDP_{Nat}(G,D)) = |E| - |E^*|$. Moreover, if G is complete and $|V| \geq k+2$, then $kHNDP_{Nat}(G,D)$ is full dimensional.*

Proof. Similar to the proof of Theorem 3.1. □

In the following sections, we present several classes of valid inequalities for the kHNDP polytopes. From Section 4 to Section 6, the presented inequalities are valid for the polytope associated with the natural formulation of the problem, $k\text{HNDP}_{Nat}(G,D)$. The inequalities presented in Section 7, on the other hand, are valid for the aggregated and separated cut formulations $k\text{HNDP}_{Ag}(G,D)$ and $k\text{HNDP}^{Sep}_{Flo}(G,D)$.

Recall that all the classes of inequalities we will introduce for the $k\text{HNDP}_{Nat}(G,D)$ (*i.e.* for the natural formulation) are also valid for the polytopes associated with the extended formulations presented in the paper.

We start with the so-called Hop-Constrained partition inequalities for $k \geq 2$ and $L = 2, 3$. To the best of our knowledge, these inequalities have never been presented before in the literature.

4 Hop-Constrained Partition Inequalities for $k\text{HNDP}_{Nat}(G,D)$

Our new inequalities are defined for any $k \geq 2$ and for $L \in \{2,3\}$. Also we distinguish the case where the demands are rooted or disjoint. We recall that S_D and T_D denote the set of origins and the set of destinations of the demands, respectively. Also, remember that when the demands are rooted, the set S_D is reduced to one node s, that is $S_D = \{s\}$, while for disjoint demands, each source node is associated with only one destination node and vice-versa.

The main results of the present section are given in Theorems 4.1 and 4.2. Their proofs are established later in subsections 4.3-4.6.

4.1 Hop-Constrained Partition inequalities for rooted demands

We first consider the case of rooted demands, that is $S_D = \{s\}$. Let $\pi = \{V_0, V_1, ..., V_p\}$, $p \geq 2$, be a partition of V such that $s \in V_0$ and $V_i \cap T_D \neq \emptyset$, $i = 1, ..., p$. The inequalities

$$x(\delta(V_0, V_1, ..., V_p)) \geq \begin{cases} \left\lceil \dfrac{(k+1)p}{2} \right\rceil, & \text{if } L = 2, \quad (4.1) \\ \left\lceil \dfrac{(k+\frac{2}{k+1})p}{2} \right\rceil, & \text{if } L = 3, \quad (4.2) \end{cases}$$

are called *Hop-Constrained Partition inequality.* The following theorem states that these inequalities are valid for the kHNDP when the demands are rooted.

Theorem 4.1 *Consider the kHNDP with rooted demands and let $\pi = \{V_0, V_1, ..., V_p\}$, $p \geq 2$, be a partition of V such that $s \in V_0$ and $V_i \cap T_D \neq \emptyset$, $i = 1, ..., p$. The Hop-Constrained partition inequality (4.1) (resp. (4.2)), induced by π, is valid for the kHNDP for any $k \geq 2$ and when $L = 2$ (resp. $L = 3$).*

Figures 5 and 6 present examples of partition supporting Hop-Constrained partition inequalities in cases $L = 2$ and $L = 3$, respectively.

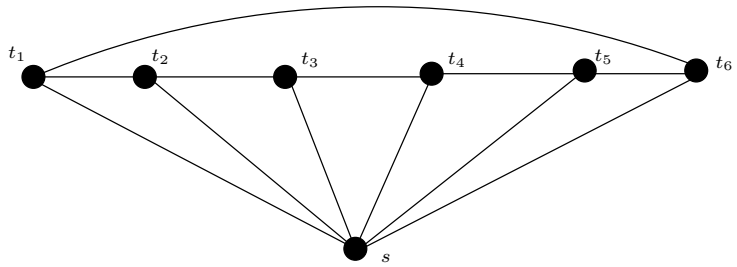

Figure 5: Support graph of a Hop-Constrained partition inequality for rooted demands, $L = 2$ and $k = 3$, $p = 6$, $V_0 = \{s\}$ and $V_i = \{t_i\}$, $i = 1, ..., p$.

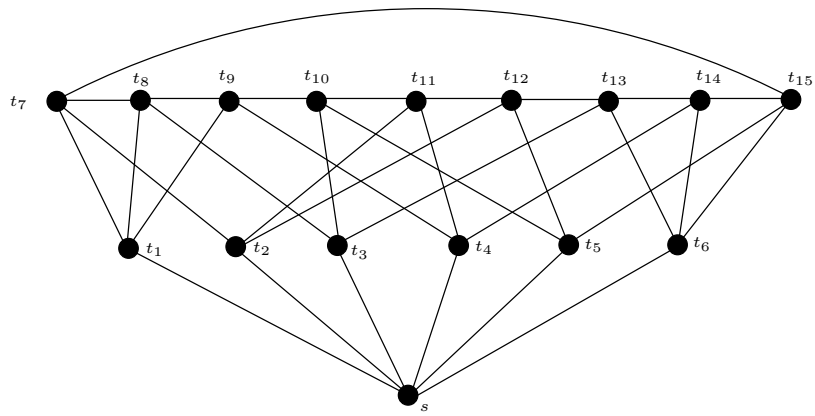

Figure 6: Support graph of a Hop-Constrained partition inequality for rooted demands, $L = 3$ and $k = 4$, $p = 15$, $V_0 = \{s\}$ and $V_i = \{t_i\}$, $i = 1, ..., p$.

Figures 5 and 6 present partitions with $p = 6$ and $p = 15$, respectively, and where $|V_i| = 1$, for every $i \in \{0, ..., p\}$. We can see that the solution depicted in Figure 5 is feasible for the kHNDP for $L = 2$ and $k = 3$. Also, the number of edges in $\delta(\pi)$ is $12 = \left\lceil \frac{(3+1)\cdot 6}{2} \right\rceil$. For Figure 6, the solution is feasible for the kHNDP with $L = 3$ and $k = 4$, and the number of edges in $\delta(\pi)$ is $33 = \left\lceil \frac{(4+2/5)\cdot 15}{2} \right\rceil$.

The proof of Theorem 4.1 will be established in Sections 4.3 and 4.4. Next we address the case where the demands are disjoint.

4.2 Partition inequalities for disjoint demands

Consider the kHNDP with disjoint demands and let $\pi = \{V_0, V_1, ..., V_p\}$, $p \geq 2$, be a partition of V such that $V_0 \supseteq S_D$ and $V_i \cap T_D \neq \emptyset$, $i = 1, ..., p$. The inequalities

$$x(\delta(V_0, V_1, ..., V_p)) \geq \begin{cases} \left\lceil \dfrac{(k + \lceil \frac{k}{2} \rceil)p}{2} \right\rceil, & \text{if } L = 2, \quad (4.3) \\ \left\lceil \dfrac{(k + \frac{k}{k+1})p}{2} \right\rceil, & \text{if } L = 3, \quad (4.4) \end{cases}$$

are called *Hop-Constrained partition inequalities.*

Theorem 4.2 *Consider the kHNDP with disjoint demands and let $\pi = \{V_0, V_1, ..., V_p\}$, $p \geq 2$, be a partition such that $V_0 \supseteq S_D$ and $V_i \cap T_D \neq \emptyset$, $i = 1, ..., p$. The Hop-Constrained partition inequality (4.3) (resp. (4.4)), induced by π, is valid for the kHNDP for any $k \geq 2$ and when $L = 2$ (resp. $L = 3$).*

Figures 7 and 8 illustrate the support graphs of Hop-Constrained partition inequalities in cases $L = 2$ and $L = 3$, respectively, and disjoint demands.

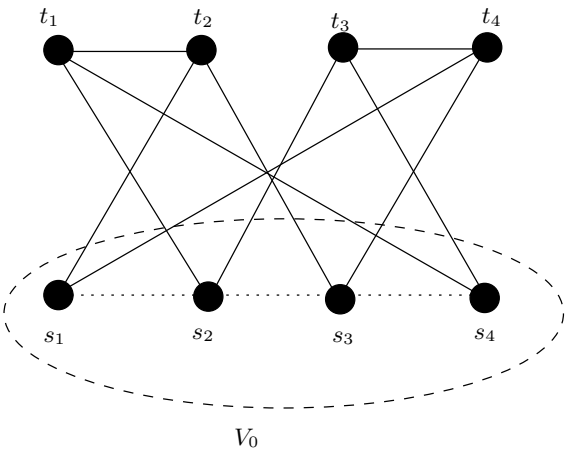

Figure 7: Support graph of a Hop-Constrained partition inequality for disjoint demands, $L = 2$ and $k = 3$. Here $p = 4$, $V_i = \{t_i\}$, $i = 1, ..., 4$, and $V_0 = \{s_1, ..., s_4\}$.

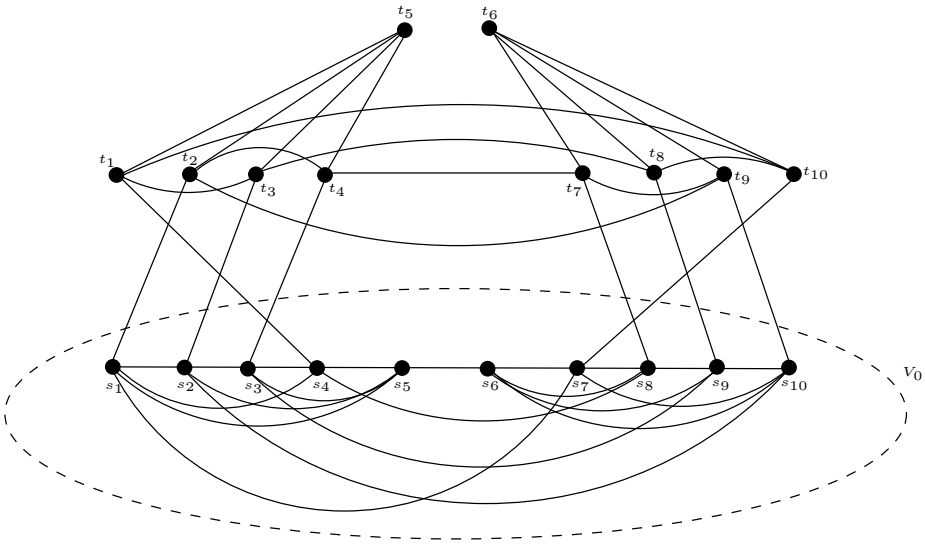

Figure 8: Support graph of a Hop-Constrained partition inequality for disjoint demands, $L = 3$ and $k = 4$, $p = 10$, $V_i = \{t_i\}$, $i = 1, ..., p$, and $V_0 = \{s_1, ..., s_p\}$.

In Figure 7, the demand set is composed of pairs (s_i, t_i), $i = 1, ..., 4$, and the considered partition is $\pi = \{V_0, V_1, ..., V_p\}$, with $p = 4$, and $V_1 = \{t_i\}$, $i = 1, ..., 4$, and $V_0 = \{s_1, s_2, s_3, s_4\}$. It is easy to see that the solution presented here is feasible for the kHNDP with $L = 2$ and $k = 3$. Also,

the number of edges involved in $\delta(\pi)$ is $10 = \lceil \frac{5 \cdot 4}{2} \rceil$.

With Figure 8, the demand set is composed of pairs (s_i, t_i), $i = 1, ..., 10$. The partition is obtained for $p = 10$ and $V_i = \{t_i\}$, $i = 1, ..., 10$ and $V_0 = \{s_1, ..., s_{10}\}$. Here also, the solution is feasible for the kHNDP with $k = 4$ and $L = 3$ and the number of edges in $\delta(\pi)$ is $24 = \left\lceil \frac{(4+4/5) \cdot 10}{2} \right\rceil$.

Remark that the solutions presented in Figures 5, 6, 7 and 8 satisfy with equality the Hop-Constrained partition inequality associated with each situation. This remark suggests that, under suitable conditions, the Hop-Constrained partition inequalities (4.1)-(4.4) may define facets of the kHNDP polytope.

In the next sections, we establish the proofs of Theorems 4.1 and 4.2. Notice that for each proof and for sake of simplicity, we assume that the demand set is composed of pairs (s_i, t_i), $i = 1, ..., p$, $V_i = \{t_i\}$, and $s_i \in V_0$, for every $i \in \{1, ..., p\}$. Also, notice that for rooted demands, $s_i = s$, for every $i \in \{1, ..., p\}$. Indeed, such an assumption does not change the number of edges in $\delta(V_0, V_1, ..., V_p)$ nor the right-hand-side of each inequality.

4.3 Proof of Theorem 4.1 for rooted demands and $L = 2$

Consider a feasible solution for the kHNDP and let H be the subgraph of G induced by this solution. Also, let Z_1 (resp. Z_2) be the set of nodes t_i which are at distance 1 (resp. 2) from s in H (see Figure 9 for illustration).

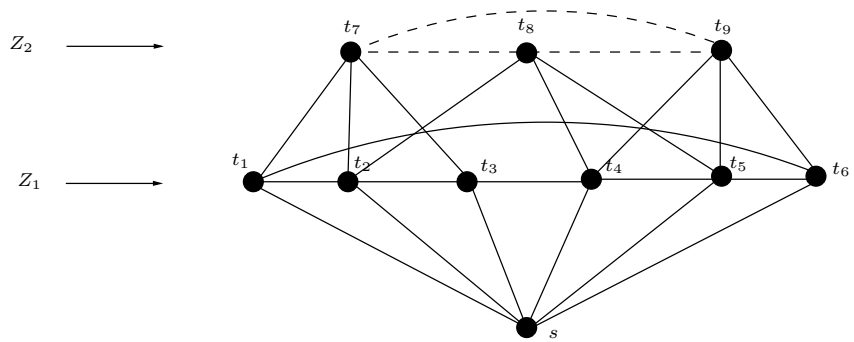

Figure 9: Support graph of a Hop-Constrained partition inequality for rooted demands, $L = 2$ and $k = 3$. Dashed edges are never used in a 2-st_i-path.

Note that $p = |Z_1| + |Z_2|$. Now remark that, for every node $t_i \in Z_2$, a 2-st_i-path of H cannot use an edge passing through another node of Z_2. Similarly, for every node $t_i \in Z_1$, a 2-st_i-path cannot use an edge passing through another node of Z_2. Thus,

$$
\begin{aligned}
|\delta_H(\pi)| &\geq k|Z_2| + \left\lceil \frac{k|Z_1| + |Z_1|}{2} \right\rceil, \\
&= \left\lceil \frac{k(|Z_1| + |Z_2|) + |Z_1| + k|Z_2|}{2} \right\rceil, \\
&= \left\lceil \frac{kp + |Z_1| + k|Z_2|}{2} \right\rceil, \\
&= \left\lceil \frac{(k+1)p + (k-1)|Z_2|}{2} \right\rceil, \\
&\geq \left\lceil \frac{(k+1)p}{2} \right\rceil,
\end{aligned}
$$

which proves the theorem.

4.4 Proof of Theorem 4.1 for rooted demands and $L = 3$

Consider a feasible solution of the kHNDP with $k \geq 2$ and $L = 3$ and let H be the subgraph of G induced by this solution. As before, we consider the graph G_π induced by π and let Z_1 (resp. Z_2) (resp. Z_3) be the set of nodes at distance 1 (resp. 2) (resp. 3) from s. Also let Z_1^r be the set of nodes $t_i \in Z_1$ such that $|[s, t_i]| + |[t_i, Z_1 \setminus \{t_i\}]| = r \leq k - 1$, and Z_1^k the set of nodes of Z_1 such that $|[s, t_i]| + |[t_i, Z_1 \setminus \{t_i\}]| \geq k$. In a similar way, let Z_2^r be the set of nodes $t_i \in Z_2$ such that $|[t_i, Z_1]| = r \leq k - 1$ and Z_2^k be the set of nodes $t_i \in Z_2$ such that $|[t_i, Z_1]| \geq k$.
Figure 10 gives an illustration for $L = 3$ and $k = 3$.

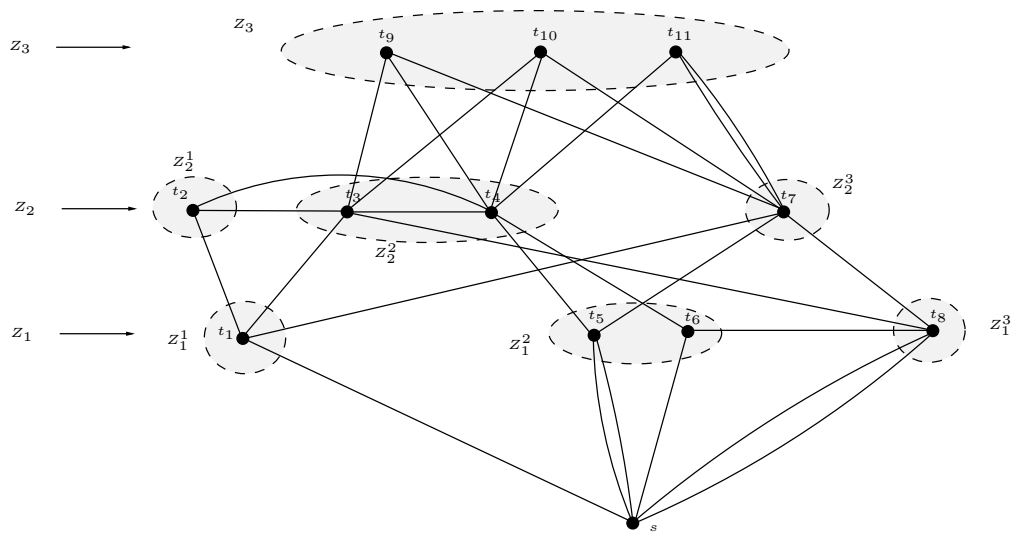

Figure 10: Support graph of a Hop-Constrained partition inequality for rooted demands, $L = 3$, $k = 3$ and $p = 11$.

Observe that

1. for every node $t_i \in Z_1 \cup Z_2 \cup Z_3$, a 3-$st_i$-path never uses an edge passing through another node of Z_3.
2. for every node $t_i \in Z_1$, a 3-st_i-path never uses an edge passing through a node of Z_2^1.

Observation 2) above implies that for every $t_i \in Z_1$,

$$|\delta_H(t_i)| = |[t_i, Z_2^1]| + \sum_{r=2}^{k} |[t_i, Z_2^r]| + |[t_i, Z_1 \setminus \{t_i\}]| + |[s, t_i]| \geq k + |[t_i, Z_2^1]|.$$

Also, recall that by the definition of Z_2^1, we have that $|[Z_2^1, Z_1]| = |Z_2^1|$.

By adding the inequalities

$$\begin{aligned} &|\delta_H(t_i)| \geq k, && \text{for every } t_i \in Z_2 \cup Z_3, \\ &|\delta_H(t_i)| \geq k + |[t_i, Z_2^1]|, && \text{for every } t_i \in Z_1, \\ &|\delta_H(s)| \geq \sum_{r=1}^{k} |Z_1^r|, \end{aligned}$$

we obtain

$$2|\delta_H(\pi)| \geq kp + |[Z_2^1, Z_1]| + \sum_{r=1}^{k} |Z_1^r| = kp + |Z_2^1| + \sum_{r=1}^{k} |Z_1^r|.$$

By divinding this inequality by 2 and rounding up the right-hand-side, we get

$$|\delta_H(\pi)| \geq \left\lceil \frac{kp + |Z_2^1| + \sum_{r=1}^{k} |Z_1^r|}{2} \right\rceil. \tag{4.5}$$

Now, if $|Z_2^1| + \sum_{r=1}^{k} |Z_1^r| \geq \frac{2p}{k+1}$, then inequality (4.5) implies that

$$|\delta(\pi)| \geq \left\lceil \frac{kp + \frac{2p}{k+1}}{2} \right\rceil,$$

and the result of the theorem holds.

Thus, assume now that $|Z_2^1| + \sum_{r=1}^{k} |Z_1^r| < \frac{2p}{k+1}$. This implies that

$$\sum_{r=2}^{k} |Z_2^r| + |Z_3| > \frac{(k-1)p}{k+1}.$$

Also, remember that

$$|[s, Z_1]| \geq \sum_{r=1}^{k} |Z_1^r|, \tag{4.6}$$

and that, by the definition of node sets Z_1^r, $r = 1, ..., k$, we have

$$|[s, t_i]| + |[t_i, Z_1 \setminus \{t_i\}]| = r, \text{for every } t_i \in Z_1^r, \text{if } r \leq k-1,$$
$$|[s, t_i]| + |[t_i, Z_1 \setminus \{t_i\}]| \geq k, \text{for every } t_i \in Z_1^k.$$

This also implies that

$$|[s, Z_1]| + \sum_{u \in Z_1} |[u, Z_1 \setminus \{u\}]| = \sum_{r=1}^{k} \sum_{u \in Z_1^r} (|[s, u]| + |[u, Z_1 \setminus \{u\}]|) \geq \sum_{r=1}^{k} r|Z_1^r|. \tag{4.7}$$

Moreover, the definition of nodes sets Z_2^r, $r = 1, ..., k$, implies that

$$|[Z_2, Z_1]| \geq \sum_{r=1}^{k} r|Z_2^r|, \tag{4.8}$$

$$\sum_{u \in Z_2} |[u, Z_2 \setminus \{t_i\}]| \geq \sum_{r=1}^{k} (k-r)|Z_2^r|. \tag{4.9}$$

Therefore, the number of edges in $\delta(\pi)$ is such that

$$\begin{aligned}
|\delta_H(\pi)| &\geq |[Z_3, Z_2]| + |[Z_2, Z_1]| + |[s, Z_1]| + |E(Z_2)| + |E(Z_1)| \\
&\geq k|Z_3| + |[Z_2, Z_1]| + |[s, Z_1]| + \left\lceil \frac{\sum_{u \in Z_2} |[u, Z_2 \setminus \{u\}]| + \sum_{u \in Z_1} |[u, Z_1 \setminus \{u\}]|}{2} \right\rceil, \\
&\geq \left\lceil \frac{2k|Z_3| + 2|[Z_2, Z_1]| + 2|[s, Z_1]| + \sum_{u \in Z_2} |[u, Z_2 \setminus \{u\}]| + \sum_{u \in Z_1} |[u, Z_1 \setminus \{u\}]|}{2} \right\rceil.
\end{aligned}$$

The above inequality, together with (4.6), (4.7), (4.8) and (4.9), yield

$$\begin{aligned}
|\delta_H(\pi)| &\geq \left\lceil \frac{2k|Z_3| + 2\sum_{r=1}^{k} r|Z_2^r| + \sum_{r=1}^{k} (k-r)|Z_2^r| + \sum_{r=1}^{k} r|Z_1^r| + \sum_{r=1}^{k} |Z_1^r|}{2} \right\rceil \\
&\geq \left\lceil \frac{2k|Z_3| + k\sum_{r=1}^{k} |Z_2^r| + \sum_{r=1}^{k} r|Z_2^r| + \sum_{r=1}^{k} r|Z_1^r| + \sum_{r=1}^{k} |Z_1^r|}{2} \right\rceil.
\end{aligned} \tag{4.10}$$

Now remark that

$$\sum_{r=1}^{k} r|Z_1^r| + \sum_{r=1}^{k} |Z_1^r| \geq 2\sum_{r=1}^{k} |Z_1^r|$$
$$\sum_{r=2}^{k} r|Z_2^r| \geq 2\sum_{r=2}^{k} |Z_2^r|.$$

These remarks together with inequality (4.10) imply that

$$|\delta_H(\pi)| \geq \left\lceil \frac{2k|Z_3| + k|Z_2^1| + 2\sum_{r=1}^{k}|Z_1^r| + (k+2)\sum_{r=2}^{k}|Z_2^r|}{2} \right\rceil$$

$$\geq \left\lceil \frac{(k-2)|Z_2^1| + 2k|Z_3| + 2\left(|Z_2^1| + \sum_{r=1}^{k}|Z_1^r|\right) + (k+2)\sum_{r=2}^{k}|Z_2^r|}{2} \right\rceil.$$

Since $p = |Z_3| + |Z_2^1| + \sum_{r=2}^{k}|Z_2^r| + \sum_{r=1}^{k}|Z_1^r|$, we can replace $|Z_2^1| + \sum_{r=1}^{k}|Z_1^r|$ in the above expression by $p - |Z_3| - \sum_{r=2}^{k}|Z_2^r|$. We thus obtain

$$|\delta_H(\pi)| \geq \left\lceil \frac{(k-2)|Z_2^1| + 2k|Z_3| + 2\left(p - |Z_3| - \sum_{r=2}^{k}|Z_2^r|\right) + (k+2)\sum_{r=2}^{k}|Z_2^r|}{2} \right\rceil$$

$$\geq \left\lceil \frac{2p + k\sum_{r=2}^{k}|Z_2^r| + 2(k-1)|Z_3|}{2} \right\rceil. \tag{4.11}$$

Now, as $k \geq 2$, we have that $2(k-1) \geq k$. Thus, inequality (4.11) implies that

$$|\delta_H(\pi)| \geq \left\lceil \frac{2p + k\sum_{r=2}^{k}|Z_2^r| + k|Z_3|}{2} \right\rceil. \tag{4.12}$$

Finally, since by assumption $\sum_{r=2}^{k}|Z_2^r| + |Z_3| > \frac{(k-1)p}{k+1}$, inequality (4.12) implies that

$$|\delta_H(\pi)| \geq \left\lceil \frac{2p + k\frac{(k-1)p}{k+1}}{2} \right\rceil = \left\lceil \frac{(k + \frac{2}{k+1})p}{2} \right\rceil,$$

which ends the proof of the theorem.

4.5 Proof of Theorem 4.2 for disjoint demands and $L = 2$

We start the proof by the following observation. Since $L = 2$, if an edge t_it_j is used simultaneously by a 2-s_it_i-path and a 2-s_jt_j-path, then necessarily, these paths are of the form (s_i, t_j, t_i) and (s_j, t_i, t_j). In other words, if edge t_it_j is used simultaneously in 2-paths for both t_i and t_j, then the edges s_it_j and s_jt_i are taken in the solution. Also, remark that edges of $[s_i, t_i]$ are never used in a 2-path for another demand (s_j, t_j), for every $i, j \in \{1, ..., p\}$, with $j \neq i$.

For every $i \in \{1, ..., p\}$, we let $C_i = [s_i, t_i]$, $E_i = [t_i, Z \setminus \{t_i\}]$, $B_i = [t_i, V_0 \setminus \{s_i\}]$ and $A_i = [s_i, Z_1 \setminus \{t_i\}]$. Also, we let $e_i = |E_i|$, $c_i = |C_i|$, $b_i = |B_i|$ and $a_i = |A_i|$, for every $i \in \{1, ..., p\}$. Figure 11 below gives an illustration of the above observations and notations.

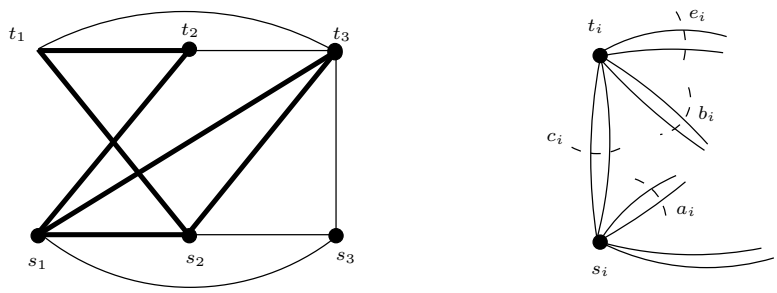

Figure 11: Illustrations of notations A_i, B_i, C_i, E_i, for $L = 2$ and $p = 3$. Thiner edges are edges t_it_j which are not used simultaneouly in 2-paths for t_i and t_j.

Now, for every $i \in \{1, ..., p\}$, we let $\overline{E}_i$ be the set of edges t_it_j of E_i which are used simultaneously in a 2-path for both t_i and t_j. For sake of readability, we assume that $\overline{E}_i = E_i$, that is for every destination node t_i, every edge t_it_j in the solution is used in 2-paths for both t_i and t_j. As a consequence, $e_i = a_i = b_i$, for every $i \in \{1, ..., p\}$. Notice that the above assumption is not too restrictive. Indeed, the proof will follow the same lines for the general case, that is when $|E_i| > |\overline{E}_i|$, for some $i \in \{1, ..., p\}$. The only difference will be the addition of extra-terms corresponding to the fact that $|E_i| > |\overline{E}_i|$ which does not change the main arguments of our proof. Thus, in the remaining, we assume that $\overline{E}_i = E_i$ and $e_i = a_i = b_i$, for all $i \in \{1, ..., p\}$.

Remember that, from the cut inequality induced by each node set $\{t_i\}$, we have

$$|\delta(t_i)| = e_i + b_i + c_i \geq k, \text{for every } i \in \{1, ..., p\}. \tag{4.13}$$

From this, it is easy to see that

$$|\delta_H(\pi)| \geq \left\lceil \frac{kp + |\delta_H(V_0)|}{2} \right\rceil \geq \left\lceil \frac{kp + \sum_{i=1}^{p}(b_i + c_i)}{2} \right\rceil. \tag{4.14}$$

Clearly, to prove the theorem, it suffices to show that

$$\sum_{i=1}^{p}(b_i + c_i) \geq \left\lceil \frac{k}{2} \right\rceil p.$$

For this purpose, let

$$P_1 = \{i \in \{1, ..., p\} \text{ such that } e_i < \left\lceil \frac{k}{2} \right\rceil\}$$

and

$$P_2 = \{i \in \{1, ..., p\} \text{ such that } e_i \geq \left\lceil \frac{k}{2} \right\rceil\}.$$

The definition of P_1 implies that $e_i = \lceil \frac{k}{2} \rceil - \alpha_i$, for some $\alpha_i \geq 1$.

Also, as by our above assumption, $e_i = b_i$, for every $i \in \{1, ..., p\}$, and especially for $i \in P_1$, inequality (4.13) for $i \in P_1$ implies that

$$c_i \geq k - 2e_i = k - 2(\left\lceil \frac{k}{2} \right\rceil - \alpha_i) = \left\lfloor \frac{k}{2} \right\rfloor - \left\lceil \frac{k}{2} \right\rceil + 2\alpha_i, \text{for every } i \in P_1. \tag{4.15}$$

Thus,

$$\begin{aligned} \sum_{i=1}^{p}(b_i + c_i) &= \sum_{i=1}^{k}(e_i + c_i) = \sum_{i \in P_1}(e_i + c_i) + \sum_{i \in P_2}(e_i + c_i), \\ &\geq \sum_{i \in P_1}(e_i + c_i) + \sum_{i \in P_2} e_i. \end{aligned} \tag{4.16}$$

As mentioned before, $e_i \geq \lceil \frac{k}{2} \rceil$, for every $i \in P_2$, and $e_i = \lceil \frac{k}{2} \rceil - \alpha_i$, for every $i \in P_1$. Hence, inequality (4.16) implies that

$$\sum_{i=1}^{p}(b_i + c_i) \geq \left\lceil \frac{k}{2} \right\rceil |P_1| + \sum_{i \in P_1}(c_i - \alpha_i) + \left\lceil \frac{k}{2} \right\rceil |P_2|.$$

This latter inequality, together with (4.15) and the fact that $p = |P_1| + |P_2|$, imply that

$$\sum_{i=1}^{p}(b_i + c_i) \geq \left\lceil \frac{k}{2} \right\rceil p + \sum_{i \in P_1}(\left\lfloor \frac{k}{2} \right\rfloor - \left\lceil \frac{k}{2} \right\rceil + \alpha_i). \tag{4.17}$$

Finally, since for every $k \geq 2$, $\lfloor \frac{k}{2} \rfloor - \lceil \frac{k}{2} \rceil \in \{-1, 0\}$, and since $\alpha_i \geq 1$, for every $i \in P_1$,

$$\left\lfloor \frac{k}{2} \right\rfloor - \left\lceil \frac{k}{2} \right\rceil + \alpha_i \geq 0, \text{for every } i \in P_1.$$

Therefore, inequality (4.17) implies that

$$\sum_{i=1}^{p}(b_i + c_i) \geq \left\lceil \frac{k}{2} \right\rceil p,$$

which completes the proof of the theorem.

4.6 Proof of Theorem 4.2 for disjoint demands and $L = 3$

Remember that we assume in the proof that, the demand set is composed of pairs (s_i, t_i), $i = 1, ..., p$, the partition π is such that $V_i = \{t_i\}$, for every $i \in \{1, ..., p\}$, and that $V_0 = \{s_1, ..., s_p\}$. Consider a solution H which is feasible for the kHNDP with $L = 3$ and $k \geq 2$, and denote by H the subgraph of G induced by this solution.

As before, we consider the graph G_π induced by π and let Z_1 (resp. Z_2) (resp. Z_3) be the set of nodes at distance 1 (resp. 2) (resp. 3) from s. Also let Z_1^r be the set of nodes $t_i \in Z_1$ such that $|[t_i, V_0]| + |[t_i, Z_1 \setminus \{t_i\}]| = r \leq k - 1$, and Z_1^k the set of nodes of Z_1 such that $|[t_i, V_0]| + |[t_i, Z_1 \setminus \{t_i\}]| \geq k$. In a similar way, let Z_2^r be the set of nodes $t_i \in Z_2$ such that $|[t_i, Z_1]| = r \leq k - 1$ and Z_2^k be the set of nodes $t_i \in Z_2$ such that $|[t_i, Z_1]| \geq k$. Finally, we denote by S_1^r (resp. S_2^r) (resp. S_3) the set of source nodes associated with the nodes of Z_1^r (resp. Z_2^r) (resp. Z_3).
Figure 12 gives an illustration of these node sets for $k = 3$.

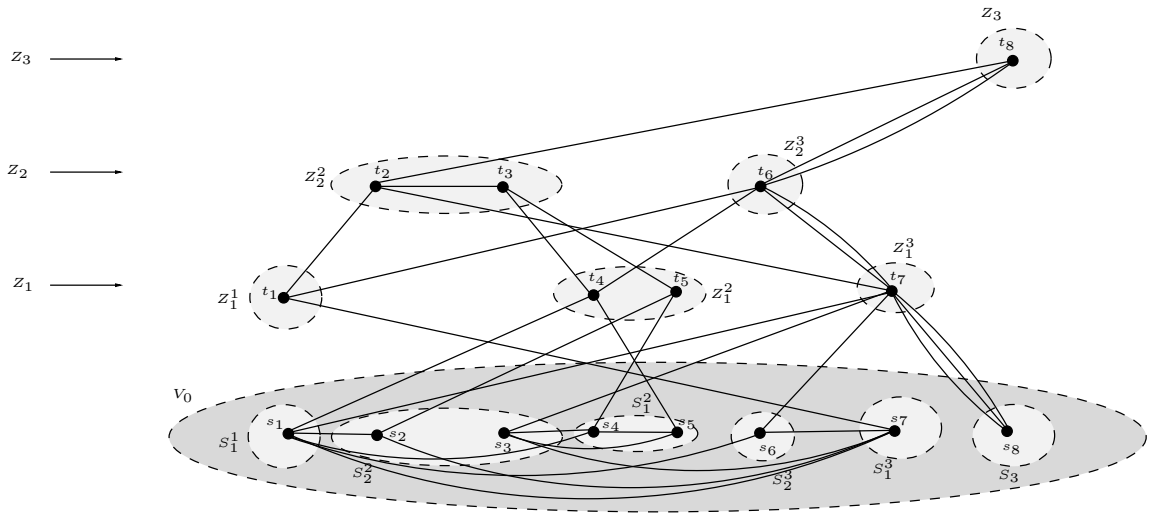

Figure 12: Support graph of a Hop-Constrained partition inequality for disjoint demands, $L = 3$ and $k = 3$. Here $p = 7$, $V_i = \{t_i\}$, $i = 1, .., 7$, $V_0 = \{s_1, ..., s_7\}$. $Z_1^1 = \{t_1\}$, $Z_1^2 = \{t_4, t_5\}$, $Z_1^3 = \{t_7\}$, $Z_2^2 = \{t_2, t_3\}$, $Z_2^3 = \{t_6\}$, $Z_3 = \{t_8\}$, $S_1^1 = \{s_1\}$, $S_2^2 = \{s_2, s_3\}$, $S_1^2 = \{s_4, s_5\}$, $S_2^3 = \{s_6\}$, $S_1^3 = \{s_7\}$, $S_3 = \{s_8\}$.

First remark that if $\sum_{r=1}^{k} |Z_1^r| \geq \frac{kp}{k+1}$, then, clearly,

$$\begin{aligned} |\delta_H(\pi)| &\geq \left\lceil \frac{kp + |[Z_1, V_0]|}{2} \right\rceil, \\ &\geq \left\lceil \frac{kp + \frac{kp}{k+1}}{2} \right\rceil \end{aligned} \tag{4.18}$$

and the result of the theorem holds. Hence, for the remain of the proof, we assume that $\sum_{r=1}^{k} |Z_1^r| < \frac{kp}{k+1}$. As a consequence, $|Z_3| + \sum_{r=1}^{k} |Z_2^r| > \frac{p}{k+1}$.

Now observe that

1. for every $t_i \in Z_1^r$, $r = 1, ..., k$, at least $k - r$ 3-$s_i t_i$-paths use edges from $[t_i, Z_2]$. Each of these paths also use edges from $[s_i, Z_1]$. Thus, $|[Z_1^{'r}, Z_1]| \geq (k - r)|Z_1^r|$.

2. for every $t_i \in Z_2^r$, $r = 1, ..., k$, at least $k - r$ 3-$s_i t_i$-paths use edges from $[t_i, Z_2 \setminus \{t_i\}]$, and each of these paths also use edges from $[s_i, Z_1]$. Thus, $|[Z_2^{'r}, Z_1]| \geq (k - r)|Z_2^r|$.

3. for every $t_i \in Z_3$, any 3-$s_i t_i$-path does not use edges passing through another node of Z_3 and uses an edge of $[s_i, Z_1]$. Thus, $|[Z_3, Z_2]| \geq k|Z_3|$ and $|[Z_3^{'}, Z_1]| \geq k|Z_3|$.

These observations imply that

$$|[Z_1, V_0]| \geq \sum_{r=1}^{k} (k - r)(|Z_1^r| + |Z_2^r|). \tag{4.19}$$

Also, by the definition of node sets Z_2^r, we have

$$|[Z_2, Z_1]| \geq \sum_{r=1}^{k} r|Z_2^r|, \tag{4.20}$$

$$|[Z_1, V_0]| + \sum_{u \in Z_1} |[u, Z_1 \setminus \{u\}]| \geq \sum_{r=1}^{k} r|Z_1^r|, \tag{4.21}$$

$$\sum_{u \in Z_2} |[u, Z_2 \setminus \{u\}]| \geq \sum_{r=1}^{k} (k - r)|Z_2^r|. \tag{4.22}$$

Thus, the number of edges involved in $\delta_H(\pi)$ is

$$\begin{aligned}
|\delta_H(\pi)| &\geq |[Z_3, Z_2]| + |[Z_2, Z_1]| + |[Z_1, V_0]| + |E(Z_2)| + |E(Z_1)|, \\
&\geq k|Z_3| + |[Z_2, Z_1]| + |[Z_1, V_0]| + \left\lceil \frac{\sum_{u \in Z_2} |[u, Z_2 \setminus \{u\}]| + \sum_{u \in Z_1} |[u, Z_1 \setminus \{u\}]|}{2} \right\rceil \\
&\geq \left\lceil \frac{2k|Z_3| + 2|[Z_2, Z_1]| + 2|[Z_1, V_0]| + \sum_{u \in Z_2} |[u, Z_2 \setminus \{u\}]| + \sum_{u \in Z_1} |[u, Z_1 \setminus \{u\}]|}{2} \right\rceil.
\end{aligned}$$

From the above inequality, together with (4.20), (4.21) and (4.22), we obtain

$$\begin{aligned}
|\delta_H(\pi)| &\geq \left\lceil \frac{2k|Z_3| + 2\sum_{r=1}^{k} r|Z_2^r| + |[Z_1, V_0]| + \sum_{r=1}^{k} (k - r)|Z_2^r| + \sum_{r=1}^{k} r|Z_1^r|}{2} \right\rceil \\
&\geq \left\lceil \frac{2k|Z_3| + k\sum_{r=1}^{k} |Z_2^r| + \sum_{r=1}^{k} r|Z_2^r| + \sum_{r=1}^{k} r|Z_1^r| + |[Z_1, V_0]|}{2} \right\rceil.
\end{aligned} \tag{4.23}$$

By combining inequalities (4.23) and (4.19), we get

$$
\begin{aligned}
|\delta_H(\pi)| &\geq \left\lceil \frac{2k|Z_3| + k\sum_{r=1}^{k}|Z_2^r| + \sum_{r=1}^{k} r|Z_2^r| + \sum_{r=1}^{k} r|Z_1^r| + \sum_{r=1}^{k}(k-r)(|Z_1^r| + |Z_2^r|)}{2} \right\rceil \\
&\geq \left\lceil \frac{2k|Z_3| + k\sum_{r=1}^{k}|Z_2^r| + k\sum_{r=1}^{k}(|Z_1^r| + |Z_2^r|)}{2} \right\rceil \\
&\geq \left\lceil \frac{k\sum_{r=1}^{k}|Z_2^r| + k\left(\sum_{r=1}^{k}(|Z_1^r| + |Z_2^r|) + |Z_3|\right) + k|Z_3|}{2} \right\rceil \\
&\geq \left\lceil \frac{kp + k\left(\sum_{r=1}^{k}|Z_2^r| + |Z_3|\right)}{2} \right\rceil . \qquad (4.24)
\end{aligned}
$$

Finally, since, by assumption $|Z_3| + \sum_{r=1}^{k}|Z_2^r| > \frac{p}{k+1}$, inequality (4.24) implies that

$$
|\delta(\pi)| \geq \left\lceil \frac{kp + \frac{kp}{k+1}}{2} \right\rceil ,
$$

which ends the proof of the theorem.

5 Steiner-SP-partition inequalities for kHNDP$_{Nat}(G, D)$

In [11], Chopra study the kECSP and introduces a class of inequalities called SP-partition inequalities (see also [26]). These inequalities are defined as follows. Let $\pi = (V_1, ..., V_p)$, $p \geq 2$, be a partition of G such that the graph G_π, induced by π, is series-parallel. Then, the inequality

$$
x(\delta(V_1, ..., V_p)) \geq \left\lceil \frac{k}{2} \right\rceil p - 1 \qquad (5.1)
$$

is called a SP-partition inequality. Chopra [11] showed that SP-partition inequalities (5.1) are valid for the kECSP polytope. In what follows, we introduce a similar class of inequalities and show that they are valid for the kHNDP polytopes.

Let $\pi = (V_1, ..., V_p)$, $p \geq 3$, be a partition of V such that the graph $G_\pi = (V_\pi, E_\pi)$ is series-parallel. Suppose that $V_\pi = \{v_1, ..., v_p\}$ where v_i is the node of G_π corresponding to the set V_i,

$i = 1, ..., p$. The partition π is said to be a *Steiner-SP-partition* if and only if π is a Steiner-partition and either

1. $p = 3$ or
2. $p \geq 4$ and there exists a node $v_{i_0} \in V_\pi$ incident to exactly two nodes v_{i_0-1} and v_{i_0+1} such that the partitions π_1 and π_2 obtained from π by contracting respectively the sets V_{i_0}, V_{i_0-1}, V_{i_0} and V_{i_0+1} are themselves Steiner-SP-partitions.

The procedure to check if a partition is a Steiner-SP-partition is recursive. It stops when the partition obtained after the different contractions is either a Steiner-partition and of size three or it is not a Steiner-partition.

In the following theorem, we give necessary and sufficient condition for a Steiner-partition to be a Steiner-SP-partition. Remind that the demand graph is denoted by $G_D = (R_D, E_D)$, where R_D is the set of terminal nodes of G. The edge set E_D is obtained by adding an edge between two nodes of R_D if and only if $\{u, v\} \in D$.

Theorem 5.1 *Let $\pi = (V_1, ..., V_p)$, $p \geq 3$, be a partition of V such that G_π is series-parallel. The partition π is a Steiner-SP-partition of G if and only if the subgraph of G_D induced by π is connected.*

Proof. First observe that, as π is a SP-partition of G, one can obtain from π a two-size partition by applying repeatidly the following operation. Let $\pi^j = (V_1^j, ..., V_{p_j}^j)$ be a SP-partition of G. Suppose that $V_{i_0}^j$, for some i_0, is incident to exactly two elements $V_{i_0-1}^j$ and $V_{i_0+1}^j$. Then, the operation consists in contracting the sets $V_{i_0-1}^j$ and $V_{i_0}^j$ and consider the partition $\pi^{j+1} = (V_1^{j+1}, ..., V_{p_{j+1}}^{j+1})$ where

$$\begin{aligned} V_i^{j+1} &= V_i^j \text{ for } i = 1, ..., i_0 - 2, \\ V_{i_0-1}^{j+1} &= V_{i_0-1}^j \cup V_{i_0}^j, \\ V_i^{j+1} &= V_{i+1}^j \text{ for } i = i_0, ..., p_j - 1. \end{aligned}$$

Note that the new partition π^{j+1} induces a SP-partition of G and that we have $p - 2$ iterations to obtain a two-size partition from π.
Now, we have that π is not a Steiner-SP-partition if and only if there exists an integer $q \leq p - 2$ such that the partition $\pi^q = (V_1^q, ..., V_{p_q}^q)$, obtained by application of the above operation, is not a Steiner-partition, that is the node set $V_{i_0}^q$ of π^q obtained by the contraction procedure to the partition π^{q-1} is such that $\delta_{G_D}(V_{i_0}^q) = \emptyset$. Thus, if $V_{i_1}, ..., V_{i_r}$, $r \geq 2$, are the node sets of π that have been reduced to $V_{i_0}^q$ during the different steps of the contraction procedure, then we have that $\delta_{G_D}(\bigcup_{i=1}^{r} V_{i_r}) = \emptyset$. Therefore, the subgraph of G_D induced by π is not connected, which ends the proof. □

As a consequence of Theorem 5.1, if the demand graph is connected (this is the case when, for instance, all the demands are rooted in the same node), then every Steiner-partition of V, inducing a series-parallel subgraph of G, is a Steiner-SP-partition of V.
With a Steiner-SP-partition $(V_1, ..., V_p)$, $p \geq 3$, we associate the following inequality

$$x(\delta(V_1, ..., V_p)) \geq \left\lceil \frac{k}{2} \right\rceil p - 1. \tag{5.2}$$

Inequalities of type (5.2) will be called *Steiner-SP-partition inequalities*. We have the following.

Theorem 5.2 *Inequality (5.2) is valid for* $k\text{HNDP}_{Ag}(G,D)$, $kHNDP^{Sep}_{Cu}(G,D)$, $kHNDP^{Sep}_{Flo}(G,D)$ *and* $kHNDP^{Sep}_{Pa}(G,D)$.

Proof. Let $\pi = (V_1, ..., V_p)$, $p \geq 3$ be a Steiner-SP-partition. The proof is by induction on p. If $p = 3$, then, as π is a Steiner-partition, the inequality

$$x(\delta(V_1, V_2, V_3)) \geq \left\lceil \frac{3k}{2} \right\rceil = 3 \left\lceil \frac{k}{2} \right\rceil - 1$$

is valid.
Now suppose that every inequality (5.2), induced by a Steiner-SP-partition of p elements, $p \geq 3$, is valid for the kHNDP polytopes, and consider a Steiner-SP-partition $\pi = (V_1, ..., V_p, V_{p+1})$. As G_π is series-parallel, there exists a node set V_{i_0} of π which is incident to exactly two elements of π, say V_{i_0-1} and V_{i_0+1}. We let $F_1 = [V_{i_0}, V_{i_0-1}]$ and $F_2 = [V_{i_0}, V_{i_0+1}]$. Since π is a Steiner-SP-partition and hence is a Steiner-partition, V_{i_0} induces a valid st-cut inequality, for some $\{s,t\} \in D$. Hence we have that

$$x(F_1) + x(F_2) \geq k.$$

W.l.o.g., we will suppose that

$$x(F_1) \geq \left\lceil \frac{k}{2} \right\rceil. \tag{5.3}$$

Consider the partition $\pi' = (V_1, ..., V_{i_0-2}, V_{i_0-1} \cup V_{i_0}, V_{i_0+1}, ..., V_{p+1})$. As π is a Steiner-SP-partition containing more than three elements, π' is also a Steiner-SP-partition which contains p elements. Thus, by the induction hypothesis, the Steiner-SP-partition inequality induced by π', that is

$$x(\delta(V_1, ..., V_{i_0-2}, V_{i_0-1} \cup V_{i_0}, V_{i_0+1}, ..., V_{p+1})) \geq \left\lceil \frac{k}{2} \right\rceil p - 1 \tag{5.4}$$

is valid. By summing the inequalities (5.3) and (5.4), we get

$$x(\delta(V_1, ..., V_p, V_{p+1})) \geq \left\lceil \frac{k}{2} \right\rceil (p+1) - 1,$$

which ends the proof of the theorem. □

Inequality (5.2) expresses the fact that, in a solution of the kHNDP, the multicut induced by a Steiner-SP-partition contains at least $\lceil \frac{k}{2} \rceil p - 1$ edges, since this solution contains k edge-disjoint paths between every pair of nodes $\{s,t\} \in D$.

Chopra [11] also described a lifting procedure for inequalities (5.1) for the kECSP. This procedure can be easily extended, for the kHNDP, to inequalities of type (5.2). It is described as follows. Let $G = (V, E)$ be a graph and $k \geq 3$ an odd integer. Let $G' = (V, E \cup E')$ be a graph obtained from G by adding an edge set E'. Let $\pi = (V_1, ..., V_p)$ be a Steiner-SP-partition of G. Then the following inequality is valid for $k\text{HNDP}_{Ag}(G,D)$, $k\text{HNDP}^{Sep}_{Cu}(G,D)$, $k\text{HNDP}^{Sep}_{Flo}(G,D)$ and $k\text{HNDP}^{Sep}_{Pa}(G,D)$

$$x(\delta_G(V_1, ..., V_p)) + \sum_{e \in E' \cap \delta_{G'}(V_1, ..., V_p)} a(e)x(e) \geq \left\lceil \frac{k}{2} \right\rceil p - 1, \tag{5.5}$$

where $a(e)$ is the length (in terms of edges) of a shortest path in G_π between the endnodes of e, for all $e \in E' \cap \delta_{G'}(V_1, ..., V_p)$. We will call inequalities of type (5.5) *lifted Steiner-SP-partition inequalities.*

Steiner-SP-partition inequalities (5.2) are shown to be valid for the kHNDP polytopes for any $L \geq 2$. However, we can show that when $L = 2$, they are dominated by partition inequalities (4.1) and (4.3).

Theorem 5.3 *When $L = 2$, the Steiner-SP-partition inequalities (5.2) are dominated by the partition inequalities (4.1) and (4.3).*

Proof. To prove the theorem, it suffices to show, for a given Steiner-SP-partition, that the right-hand-side of the Steiner-SP-partition inequality is lower than that of the partition inequality induced by this partition. In this proof, we do this only for $L = 2$ and rooted demands, the proof being the same for disjoint demands. Thus, consider the case of rooted demands and a partition π of V into $p \geq 2$ node sets. Recall that the partition inequality (4.1) is of the form

$$x(\delta(\pi)) \geq \left\lceil \frac{(k+1)p}{2} \right\rceil, \tag{5.6}$$

and Steiner-SP-partition inequality (5.2) is of form

$$x(\delta(\pi)) \geq \left\lceil \frac{k}{2} \right\rceil p - 1.. \tag{5.7}$$

We assume, w.l.o.g., that k is odd. We have that

$$\left\lceil \frac{(k+1)p}{2} \right\rceil - (\left\lceil \frac{k}{2} \right\rceil p - 1) \geq \frac{(k+1)p}{2} - \frac{k+1}{2}p - 1 = 1 > 0,$$

which shows that inequality (5.6) dominates (5.7). □

6 Futher valid inequalities for kHNDP$_{Nat}(G, D)$

In this section, we present two other classes of inequalities which are valid for the polytope associated with the natural formulation of the kHNDP. These inequalities are generalizations of those introduced by Huygens et al. [47] for the 2HNDP.

6.1 Double cut inequalities

In the following we introduce a class of inequalities that are valid for the kHNDP polytopes for $L \geq 2$ and $k \geq 2$. They are given by the following theorem.

Theorem 6.1 *Let $\{s, t\}$ be a demand, $i_0 \in \{0, ..., L\}$ and $\Pi = \{V_0, ..., V_{i_0-1}, V_{i_0}^1, V_{i_0}^2, V_{i_0+1}, ..., V_{L+1}\}$ a family of node sets of V such that $\pi = (V_0, ..., V_{i_0-1}, V_{i_0}^1, V_{i_0}^2 \cup V_{i_0+1}, V_{i_0+2}, ..., V_{L+1})$ induces a partition of V. Suppose that*

1. *$V_{i_0}^1 \cup V_{i_0}^2$ induces an $s_{j_1}t_{j_1}$-cut of G with $\{s_{j_1}, t_{j_1}\} \in D$ and $s_{j_1} \in V_{i_0}^1$ or $t_{j_1} \in V_{i_0}^1$ (note that s_{j_1} and t_{j_1} cannot be simultaneously in $V_{i_0}^1$ and are not in $V_{i_0}^2$. Also note that $V_{i_0}^2$ may be empty);*

2. V_{i_0+1} induces an $s_{j_2}t_{j_2}$-cut of G with $\{s_{j_2}, t_{j_2}\} \in D$ (note that j_1 and j_2 may be equal);

3. π induces an L-st-path-cut of G with $s \in V_0$ (resp. $t \in V_0$) and $t \in V_{L+1}$ (resp. $s \in V_{L+1}$).

Let $\overline{E} = [V_{i_0-1}, V_{i_0}^1] \cup [V_{i_0+2}, V_{i_0}^2 \cup V_{i_0+1}] \cup \left(\bigcup_{k,l \notin \{i_0, i_0+1\}, |k-l|>1} [V_k, V_l] \right)$ *and* $F \subseteq \overline{E}$ *such that* $|F|$ *and* k *have different parities.*

Let also $\hat{E} = (\bigcup_{i=0}^{i_0-2} [V_i, V_{i+1}]) \cup (\bigcup_{i=i_0+2}^{L} [V_i, V_{i+1}]) \cup F$. *Then, the inequality*

$$x(\delta(\pi) \setminus \hat{E}) \geq \left\lceil \frac{3k - |F|}{2} \right\rceil, \tag{6.1}$$

is valid for kHNDP$_{Ag}(G, D)$, $kHNDP^{Sep}_{Cu}(G, D)$, $kHNDP^{Sep}_{Flo}(G, D)$ *and* $kHNDP^{Sep}_{Pa}(G, D)$ *(recall that* $\delta(\pi)$ *is the set of edges of the* E *having their endnodes in different elements of* π*).*

Proof. Let T be the L-st-path-cut of G induced by the partition π. As T is an L-st-path-cut, and $V_{i_0}^1 \cup V_{i_0}^2$ and V_{i_0+1} induce st-cut with $\{s, t\} \in \{\{s_{j_1}, t_{j_1}\}, \{s_{j_2}, t_{j_2}\}\}$, the inequalities below are valid for the kHNDP polytopes

$$\begin{aligned}
& x(T) \geq k, \\
& x(\delta(V_{i_0}^1 \cup V_{i_0}^2)) \geq k, \\
& x(\delta(V_{i_0+1})) \geq k, \\
& -x(e) \geq -1 && \text{for all } e \in F, \\
& x(e) \geq 0 && \text{for all } e \in \overline{E} \setminus F.
\end{aligned}$$

By summing these inequalities, dividing by 2 and rounding up the right hand side, we obtain inequality (6.1). □

Inequalities of type (6.1) are called *double cut inequalities*. They generalize those introduced by Huygens and Mahjoub [46] for the kHNDP when $k = 2$. We discuss in the following special cases for these inequalities. This concerns the case where $L \in \{2, 3\}$ and $i_0 = 0$.
The set of edges having a positive coefficient in inequality (6.1) plus the edges of F is called a *double cut*. Figure 13 gives an example for $L = 3$ and $i_0 = 0$.
Let $L = 2$, $\{s, t\} \in D$ and $\Pi = \{V_0^1, V_0^2, V_1, V_2, V_3\}$ be a family of node sets of V such that $\pi = (V_0^1, V_0^2 \cup V_1, V_2, V_3)$ induces a 2-st-path-cut, and V_1 induces a valid s_1t_1-cut in G, for some $\{s_1, t_1\} \in D$. If $F \subseteq [V_0^2 \cup V_1, V_2]$ is chosen such that $|F|$ and k have different parities, then the double cut inequality induced by Π and F in this case can be written as

$$\begin{aligned}
x([V_0^1, V_1 \cup V_2 \cup V_3]) + x([V_0^2, V_1 \cup V_3]) + x([V_1, V_3]) \\
+ x([V_0^2 \cup V_1, V_2] \setminus F) \geq \left\lceil \frac{3k - |F|}{2} \right\rceil.
\end{aligned} \tag{6.2}$$

Now let $L = 3$, $\{s, t\} \in D$ and $\Pi = \{V_0^1, V_0^2, V_1, V_2, V_3, V_4\}$ be a family of node sets of V such that $\pi = (V_0^1, V_0^2 \cup V_1, V_2, V_3, V_4)$ induces a 3-st-path-cut, and V_1 induces a valid s_1t_1-cut in G. If $F \subseteq [V_0^2 \cup V_1 \cup V_4, V_2]$ is chosen such that $|F|$ and k have different parities, then the double cut inequality induced by Π and F can be written as

$$\begin{aligned}
x([V_0^1, V_1 \cup V_2 \cup V_3 \cup V_4]) + x([V_0^2, V_1 \cup V_3 \cup V_4]) + x([V_1, V_3 \cup V_4]) \\
+ x([V_0^2 \cup V_1 \cup V_4, V_2] \setminus F) \geq \left\lceil \frac{3k - |F|}{2} \right\rceil.
\end{aligned} \tag{6.3}$$

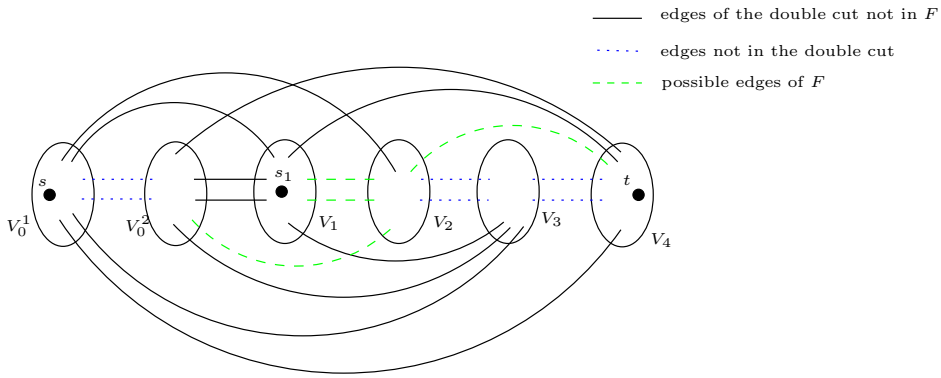

Figure 13: A double cut with $L = 3$ and $i_0 = 0$

As it will turn out, inequalities (6.2) and (6.3) are very effective in the Branch-and-Cut algorithms we developed for the problem.

6.2 Triple path-cut inequalities

Here is a further class of valid inequalities. They also generalize inequalities given by Huygens and Mahjoub [46]. We distinguish the cases where $L = 2$ and $L = 3$. We have the following theorem.

Theorem 6.2 *i) Let $L = 2$ and $\{V_0, V_1, V_2, V_3^1, V_3^2, V_4^1, V_4^2\}$ be a family of node sets of V such that $(V_0, V_1, V_2, V_3^1 \cup V_3^2, V_4^1 \cup V_4^2)$ induces a partition of V and there exist two demands $\{s_1, t_1\}$ and $\{s_2, t_2\}$ with $s_1, s_2 \in V_0$, $t_1 \in V_3^2$ and $t_2 \in V_4^2$. The sets V_3^1 and V_4^1 may be empty and s_1 and s_2 may be the same. Let also $V_3 = V_3^1 \cup V_3^2$, $V_4 = V_4^1 \cup V_4^2$ and $F \subseteq [V_3^2, V_1 \cup V_4^1] \cup [V_3^1, V_4^2]$ such that $|F|$ and k have different parities. Then, the inequality*

$$2x([V_0, V_2]) + x([V_0, V_3 \cup V_4]) + x([V_4^2, V_1 \cup V_3^2]) + x(([V_3^2, V_1 \cup V_4^1] \cup [V_3^1, V_4^2]) \setminus F) \geq \left\lceil \frac{3k - |F|}{2} \right\rceil \tag{6.4}$$

is valid for kHNDP$_{Ag}(G, D)$, $kHNDP^{Sep}_{Cu}(G, D)$, $kHNDP^{Sep}_{Flo}(G, D)$ and $kHNDP^{Sep}_{Pa}(G, D)$.
ii) Let $L = 3$ and $(V_0, ..., V_3, V_4^1, V_4^2, V_5^1, V_5^2)$ be a family of node sets of V such that $(V_0, ..., V_3, V_4^1 \cup V_4^2, V_5^1 \cup V_5^2)$ induces a partition of V and there exist two demands $\{s_1, t_1\}$ and $\{s_2, t_2\}$ with $s_1, s_2 \in V_0$, $t_1 \in V_4^2$ and $t_2 \in V_5^2$. The sets V_4^1 and V_5^1 may be empty and s_1 and s_2 may be the same. Let also $V_4 = V_4^1 \cup V_4^2$, $V_5 = V_5^1 \cup V_5^2$ and $F \subseteq [V_2, V_4^2] \cup [V_3, V_4 \cup V_5]$ such that $|F|$ and k have different parities. Then, the inequality

$$2x([V_0, V_2]) + 2x([V_0, V_3]) + 2x([V_1, V_3]) + x([V_0 \cup V_1, V_4 \cup V_5]) + x([V_4, V_5]) + x([V_2, V_5^2]) + x(([V_2, V_4^2] \cup [V_3, V_4 \cup V_5]) \setminus F) \geq \left\lceil \frac{3k - |F|}{2} \right\rceil \tag{6.5}$$

is valid for kHNDP$_{Ag}(G, D)$, $kHNDP^{Sep}_{Cu}(G, D)$, $kHNDP^{Sep}_{Flo}(G, D)$ and $kHNDP^{Sep}_{Pa}(G, D)$.

Proof.
i) Let T_1 be the 2-s_1t_1-path-cut induced by the partition $(V_0, V_1 \cup V_4, V_2 \cup V_3^1, V_3^2)$ and T_2 and

T_3 the 2-s_2t_2-path-cuts induced by the partitions $(V_0, V_1 \cup V_3, V_2 \cup V_4^1, V_4^2)$ and $(V_0, V_1, V_2 \cup V_3 \cup V_4^1, V_4^2)$, respectively. The following inequalities are valid for the kHNDP polytopes

$$\begin{aligned}
&x(T_1) \geq k,\\
&x(T_2) \geq k,\\
&x(T_3) \geq k,\\
&-x(e) \geq -1, \text{ for all } e \in F,\\
&x(e) \geq 0, \text{ for all } e \in ([V_3^2, V_1 \cup V_4^1] \cup [V_3^1, V_4^2]) \setminus F.
\end{aligned}$$

By summing these inequalities, dividing by 2 and rounding up the right hand side, we get inequality (6.4).
ii) Let T_1 be the 3-s_1t_1-path-cut induced by the partition $(V_0, V_1 \cup V_5, V_2, V_3 \cup V_4^1, V_4^2)$, and T_2 and T_3 be the 3-s_2t_2-path-cuts induced by the partitions $(V_0, V_1 \cup V_4, V_2, V_3 \cup V_5^1, V_5^2)$ and $(V_0, V_1, V_2, V_3 \cup V_4 \cup V_5^1, V_5^2)$, respectively. The following inequalities are valid for the kHNDP polytopes

$$\begin{aligned}
&x(T_1) \geq k,\\
&x(T_2) \geq k,\\
&x(T_3) \geq k,\\
&-x(e) \geq -1, && \text{for all } e \in F,\\
&x(e) \geq 0, && \text{for all } e \in ([V_2, V_4^2] \cup [V_3, V_4 \cup V_5]) \setminus F.
\end{aligned}$$

By adding these inequalities, dividing by 2 and rounding up the right hand side, we get inequality (6.5). □

Inequalities of type (6.4) and (6.5) will be called *triple path-cut inequalities*. The set of edges having a positive coefficient in inequality (6.4) ((6.5)) plus the edges of F will be called a *triple path-cut* (see Figure 14 for an example with $L = 2$).

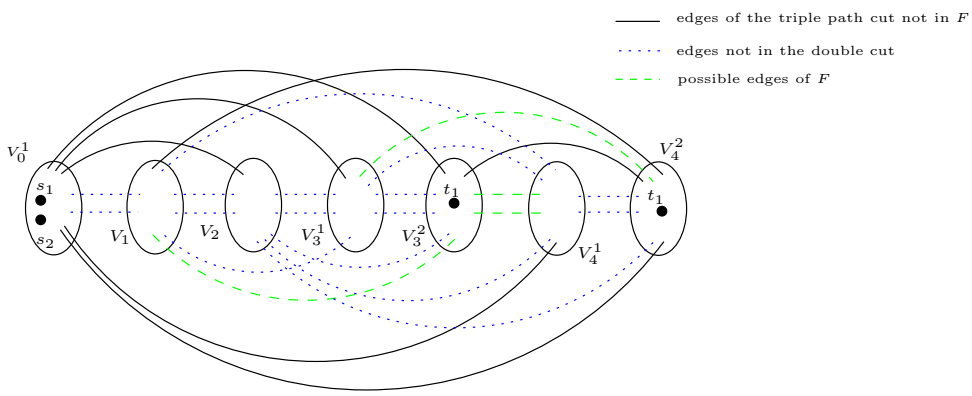

Figure 14: A triple path-cut with $L = 2$

7 Valid inequalities for kHNDP$_{Ag}(G,D)$ and kHNDP$^{Sep}_{Cu}(G,D)$

In this section, we introduce a class of inequalities that are valid for kHNDP$_{Ag}(G,D)$ and kHNDP$^{Sep}_{Cu}(G,D)$. These inequalities are inspired from those introduced by Dahl [16] for the polytope of the Survivable Directed Network Design Problem (kDNDP). The kDNDP consists, given a directed graph $\widetilde{H}$, a set of demands D and an integer $k \geq 2$, in finding a minimum weight subgraph of $\widetilde{H}$ which contains k arc-disjoint st-dipaths for every demand $\{s,t\} \in D$. We will first describe these inequalities for kHNDP$_{Ag}(G,D)$ and then extend them to kHNDP$^{Sep}_{Cu}(G,D)$.

7.1 Aggregated cut inequalities for kHNDP$_{Ag}(G,D)$

Let $\{\widetilde{W}_1,...,\widetilde{W}_p\}$, $p \geq 2$, be a family of node sets of $\widetilde{V}$ such that each set $\widetilde{W}_i$ induces an st-dicut of $\widetilde{G}$, for some $\{s,t\} \in D$, and $\widetilde{F}^0_i \subseteq \delta^+_{\widetilde{G}}(\widetilde{W}_i)$, for $i = 1,...,p$. Let $\widetilde{F} = \bigcup_{i=1}^{p}[\delta^+_{\widetilde{G}}(\widetilde{W}_i) \setminus \widetilde{F}^0_i]$ and, for an arc $a \in \widetilde{A}$, let $r(a)$ be the number of sets $\delta^+_{\widetilde{G}}(\widetilde{W}_i) \setminus \widetilde{F}^0_i$ which contain the arc a. Note that if $a \in \widetilde{A}$ does not belong to any set $\delta^+_{\widetilde{G}}(\widetilde{W}_i) \setminus \widetilde{F}^0_i$, then $r(a) = 0$. For an edge $e \in E$ and an arc subset $\widetilde{U} \subseteq \widetilde{A}$, we let

$$r'(e,\widetilde{U}) = \sum_{a \in \widetilde{A}(e) \cap \widetilde{U}} r(a).$$

The inequalities below are valid for kHNDP$_{Ag}$

$$\begin{aligned} & y(\delta^+_{\widetilde{G}}(\widetilde{W}_i)) \geq k \text{ for } i = 1,...,p, \\ & -y(a) \geq -1 \quad \text{for all } a \in \widetilde{F}^0_i,\ i = 1,...,p. \end{aligned}$$

By summing these inequalities, we obtain

$$\sum_{a \in \widetilde{F}} r(a)y(a) \geq kp - \sum_{i=1}^{p} |\widetilde{F}^0_i|.$$

If $\widetilde{F}_1$ (resp. $\widetilde{F}_2$) denotes the set of arcs $a \in \widetilde{F}$ such that $r(a)$ is odd (resp. even), then the previous inequality can be written as

$$\sum_{a \in \widetilde{F}_1} r(a)y(a) + \sum_{a \in \widetilde{F}_2} r(a)y(a) \geq kp - \sum_{i=1}^{p} |\widetilde{F}^0_i|. \tag{7.1}$$

Let $\widetilde{F}^2_1 \subseteq \widetilde{F}_1$ such that, for every edge $e \in E$ corresponding to an arc of $\widetilde{F}_1$, $r'(e,\widetilde{F}^2_1)$ is even. Let E_2 be the set of edges corresponding to the arcs of $\widetilde{F}^2_1$. By summing inequality (7.1) together with the inequalities

$$r(a)x(e) \geq r(a)y(a), \text{ for all } a \in \widetilde{F}^2_1 \text{ and } e \text{ corresponding to } a,$$

we get

$$\sum_{e \in E_2} r'(e,\widetilde{F}^2_1)x(e) + \sum_{a \in \widetilde{F}_1 \setminus \widetilde{F}^2_1} r(a)y(a) + \sum_{a \in \widetilde{F}_2} r(a)y(a) \geq kp - \sum_{i=1}^{p} |\widetilde{F}^0_i|. \tag{7.2}$$

By dividing by 2 and rounding up the right hand side of inequality (7.2), we obtain the following inequality

$$\sum_{e\in E_2}\frac{r'(e,\widetilde{F}_1^2)}{2}x(e)+\sum_{a\in\widetilde{F}_1\setminus\widetilde{F}_1^2}\frac{r(a)+1}{2}y(a)+\sum_{a\in\widetilde{F}_2}\frac{r(a)}{2}y(a)\geq\left\lceil\frac{kp-\sum_{i=1}^{p}|\widetilde{F}_i^0|}{2}\right\rceil. \tag{7.3}$$

Inequalities of type (7.3) will be called *aggregated cut inequalities*. We give the following result which directly comes from the above description.

Theorem 7.1 *The aggregated cut inequalities (7.3) are valid for* kHNDP$_{Ag}(G,D)$ *when* $L\in\{2,3\}$.

Inequalities (7.3) come from families of st-dicuts of $\widetilde{G}$ which may have different forms of configurations for the node sets $\widetilde{W}_1,...,\widetilde{W}_p$, $p\geq 2$, and the arc sets $\widetilde{F}_i^0\subseteq\delta^+_{\widetilde{G}}(\widetilde{W}_i)$, $i=1,...,p$. In the following, we discuss a special case of these inequalities.
Let $\{\widetilde{W}_1,...,\widetilde{W}_p\}$, $p\geq 2$, be a family of node sets of $\widetilde{V}$ such that each set $\widetilde{W}_i$, $i=1,...,p$, induces an st-dicut, for some $\{s,t\}\in D$, and let $\widetilde{F}_i^0\subseteq\delta^+_{\widetilde{G}}(\widetilde{W}_i)$, $i=1,...,p$, be arc sets such that $0\leq r(a)\leq 2$ for all $a\in\widetilde{A}$. Let $\widetilde{F}_2$ (resp. $\widetilde{F}_1$) be the set of arcs such that $r(a)=2$ (resp. $r(a)=1$). Let $\widetilde{F}_1^2$ be the set of arcs $a\in\widetilde{F}_1$ for which there is another arc $a'\in\widetilde{F}_1$ which corresponds to the same edge of E, and let E_2 be the set of the corresponding edges. The inequality of type (7.3) associated with this configuration can be written as

$$\sum_{a\in\widetilde{F}_2}y(a)+\sum_{e\in E_2}x(e)+\sum_{a\in\widetilde{F}_1\setminus\widetilde{F}_1^2}y(a)\geq\left\lceil\frac{kp-\sum_{i=1}^{p}|\widetilde{F}_i^0|}{2}\right\rceil. \tag{7.4}$$

As it will turn out, inequalities (7.4) may define facets under certain conditions and will be useful for solving the kHNDP using a Branch-and-Cut algorithm (Section **??**).

7.2 Aggregated cut inequalities for kHNDP$^{Sep}_{Cu}(G,D)$

The aggregated cut inequalities can be defined for the polytope kHNDP$^{Sep}_{Cu}(G,D)$ in a similar way. Let $\widetilde{G}_{st}=(\widetilde{V}_{st},\widetilde{A}_{st})$, $\{s,t\}\in D$, be the directed graphs associated with G and $\{s,t\}\in D$ in Formulation (2.23). Let $\{\{s_1,t_1\},...,\{s_q,t_q\}\}$ be a subset of demands. Consider a family of node sets $\{\widetilde{W}_1^{s_1t_1},...,\widetilde{W}_{p_1}^{s_1t_1},...,\widetilde{W}_1^{s_qt_q},...,\widetilde{W}_{p_q}^{s_qt_q}\}$, with $p_i\geq 1$, for all $i\in\{1,...,q\}$ and $p=\sum_{i=1}^{q}p_i\geq 2$, where $\widetilde{W}_j^{s_it_i}$, $j=1,...,p_i$, induces an s_it_i-dicut in $\widetilde{G}_{s_it_i}$. Let $\widetilde{F}_j^{s_it_i,0}\subseteq\delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}_j^{s_it_i})$. Let $\widetilde{F}^{s_it_i}=\bigcup_{j=1}^{p_i}[\delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}_j^{s_it_i})\setminus\widetilde{F}_j^{s_it_i,0}]$, for every $i\in\{1,...,q\}$, and for a given arc $a\in\widetilde{A}_{s_it_i}$, $i=1,...,q$, we let $r_{s_it_i}(a)$ be the number of sets $\delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}_j^{s_it_i})\setminus\widetilde{F}_j^{s_it_i,0}$ containing arc a. If a does not belong to any of these sets, then $r_{s_it_i}(a)=0$. Given an edge $e\in E$ and an arc subset $\widetilde{U}_i\subseteq\widetilde{A}_{s_it_i}$, we let $r'(e,\widetilde{U}_i)=\sum_{a\in\widetilde{A}_{s_it_i}(e)\cap\widetilde{U}_i}r_{s_it_i}(a)$.

The inequalities below are valid for $k\text{HNDP}^{Sep}_{Cu}(G,D)$,

$$\begin{aligned} & y_{s_it_i}(\delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}^{s_it_i}_j)) \geq k \text{ for } j=1,...,p_i,\ i=1,...,q, \\ & -y_{s_it_i}(a) \geq -1 \qquad \text{for } a \in \widetilde{F}^{s_it_i}_j,\ j=1,...,p_i,\ i=1,...,q. \end{aligned}$$

By summing these inequalities, we get

$$\sum_{i=1}^{q}\left(\sum_{a\in\widetilde{F}^{s_it_i}} r_{s_it_i}(a)y_{s_it_i}(a)\right) \geq kp - \sum_{i=1}^{q}\sum_{j=1}^{p_i}|\widetilde{F}^{s_it_i,0}_j|.$$

Let $\widetilde{F}^{s_it_i,1}$ (resp. $\widetilde{F}^{s_it_i,2}$) be the set of arcs $a \in \widetilde{F}^{s_it_i}$ having $r_{s_it_i}(a)$ odd (resp. even). The inequality above can then be written as

$$\sum_{i=1}^{q}\left(\sum_{a\in\widetilde{F}^{s_it_i,1}} r_{s_it_i}(a)y_{s_it_i}(a) + \sum_{a\in\widetilde{F}^{s_it_i,2}} r_{s_it_i}(a)y_{s_it_i}(a)\right) \geq kp - \sum_{i=1}^{q}\sum_{j=1}^{p_i}|\widetilde{F}^{s_it_i,0}_j|. \tag{7.5}$$

Now we let $\widetilde{F}^{s_it_i,1}_2 \subseteq \widetilde{F}^{s_it_i,1}$, $i=1,...,q$, be the arc sets such that, for every edge $e \in E$ associated with an arc of $\widetilde{F}^{s_it_i,1}_2$, $\sum_{i=1}^{q} r'(e,\widetilde{F}^{s_it_i,1}_2)$ is even. If E_2 denotes the set of edges corresponding to the arcs of $\widetilde{F}^{s_it_i,1}_2$, $i=1,...,q$, then by adding inequality (7.5) and the inequalities

$$r_{s_it_i}(a)x(e) \geq r_{s_it_i}(a)y_{s_it_i}(a) \text{ for all } a \in \widetilde{F}^{s_it_i,1}_2 \text{ where } e \text{ corresponds to } a,$$

we get

$$\begin{aligned} \sum_{i=1}^{q}\left(\sum_{a\in\widetilde{F}^{s_it_i,1}\setminus\widetilde{F}^{s_it_i,1}_2} r_{s_it_i}(a)y_{s_it_i}(a) + \sum_{a\in\widetilde{F}^{s_it_i,2}} r_{s_it_i}(a)y_{s_it_i}(a)\right) + \\ \sum_{e\in E_2}(\sum_{i=1}^{q} r'(e,\widetilde{F}^{s_it_i,1}_2))x(e) \geq kp - \sum_{i=1}^{q}\sum_{j=1}^{p_i}|\widetilde{F}^{s_it_i,0}_j|. \end{aligned} \tag{7.6}$$

Finally, by dividing inequality (7.6) by 2 and rounding up the right hand side of the resulting inequality, we obtain

$$\begin{aligned} \sum_{i=1}^{q}\left(\sum_{a\in\widetilde{F}^{s_it_i,1}\setminus\widetilde{F}^{s_it_i,1}_2} \frac{r_{s_it_i}(a)+1}{2}y_{s_it_i}(a) + \sum_{a\in\widetilde{F}^{s_it_i,2}} \frac{r_{s_it_i}(a)}{2}y_{s_it_i}(a)\right) + \\ \sum_{e\in E_2}\frac{\sum_{i=1}^{q} r'(e,\widetilde{F}^{s_it_i,1}_2)}{2}x(e) \geq \left\lceil \frac{kp - \sum_{i=1}^{q}\sum_{j=1}^{p_i}|\widetilde{F}^{s_it_i}_j|}{2} \right\rceil. \end{aligned} \tag{7.7}$$

We then have the following result.

Theorem 7.2 *Inequality (7.7) is valid for $kHNDP^{Sep}_{Cu}(G,D)$.*

Inequalities (7.7) will be also called *aggregated cut inequalities.*
We are going to describe a special case for inequalities (7.7), which will be utile in the Branch-and-Cut algorithm based on the Cut formulation (see Section **??**).
Let $\{\widetilde{W}_1^{s_1t_1}, ..., \widetilde{W}_{p_1}^{s_1t_1}, ..., \widetilde{W}_1^{s_qt_q}, ..., \widetilde{W}_{p_q}^{s_qt_q}\}$, with $p_i \geq 1$, for $i = 1, ..., q$, and $p = \sum_{i=1}^{q} p_i \geq 2$, be a family of node sets such that $\widetilde{W}_j^{s_it_i}$ induces s_it_i-dicut of $\widetilde{G}_{s_it_i}$, $i = 1, ..., q$. Let $\widetilde{F}_j^{s_it_i,0} \subseteq \delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}_j^{s_it_i})$ be arc sets and
$\widetilde{F}^{s_it_i} = \bigcup_{i=1}^{p}[\delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}_j^{s_it_i}) \setminus \widetilde{F}_j^{s_it_i,0}]$. Suppose that $0 \leq r_{s_it_i}(a) \leq 2$ for all $a \in \widetilde{A}_{s_it_i}$, $i = 1, ..., q$. Let $\widetilde{F}^{s_it_i,2}$ be the set of arcs of $\widetilde{F}^{s_it_i}$ having $r_{s_it_i}(a) = 2$ and $\widetilde{F}^{s_it_i,1}$ the set of arcs of $\widetilde{F}^{s_it_i}$ having $r_{s_it_i}(a) = 1$. Let $\widetilde{F}_2^{s_it_i,1}$ be the subset of arcs $a \in \widetilde{F}^{s_it_i,1}$ such that there exists another arc $a' \in \widetilde{F}^{s_it_i,1}$ which corresponds to the same edge of E, and let E_2 be the set of the corresponding edges.
Then the inequality (7.7), induced by this configuration, can be written as

$$\sum_{i=1}^{q}\left(\sum_{a\in\widetilde{F}^{s_it_i,2}} y_{s_it_i}(a) + \sum_{a\in\widetilde{F}^{s_it_i,1}\setminus\widetilde{F}_2^{s_it_i,1}} y_{s_it_i}(a)\right) + \sum_{e\in E_2} x(e) \geq \left\lceil \frac{kp - \sum_{i=1}^{q}\sum_{j=1}^{p_i}|\widetilde{F}_j^{s_it_i}|}{2} \right\rceil. \tag{7.8}$$

7.3 Lifting procedure for aggregated cut inequalities

In what follows we define a lifting procedure for the aggregated cut inequalities for both Aggregated and Cut formulations, (7.4) and (7.8). This will permit to extend these inequalities to a more general class of valid inequalities.
Consider first the polytope $k\text{HNDP}_{Ag}(G, D)$. The lifting procedure is given in the following theorem.

Theorem 7.3 *Let $G = (V, E)$ be an undirected graph, $D \subseteq V \times V$ and $\widetilde{G} = (\widetilde{V}, \widetilde{A})$ be the directed graph associated with G in the Aggregated formulation. Let*

$$\sum_{e\in E}\alpha(e)x(e) + \sum_{a\in\widetilde{A}}\beta(a)y(a) \geq \gamma$$

be an inequality of type (7.4) induced by a family of node sets $\Pi = \{\widetilde{W}_1, ..., \widetilde{W}_p\}$ and arc sets $\widetilde{F}_i^0 \subseteq \delta_i^0$, $p \geq 2$, which is valid for $k\text{HNDP}_{Ag}(G, D)$. Let $G' = (V, E \cup E')$ be a graph obtained by adding to G an edge set E' and let $\widetilde{G}' = (\widetilde{V}, \widetilde{A} \cup \widetilde{A}')$ be the directed graph associated with G' in the Aggregated formulation ($\widetilde{A}'$ is the set of arcs corresponding to the edges of E'). Then, the inequality

$$\sum_{e\in E}\alpha(e)x(e) + \sum_{a\in\widetilde{A}}\beta(a)y(a) + \sum_{a\in\widetilde{A}'}\left\lceil\frac{q(a)}{2}\right\rceil y(a) \geq \gamma, \tag{7.9}$$

is valid for $kHNDP_{Ag}(G', D)$, where $q(a)$ is the number of dicuts $\delta^+_{\widetilde{G}'}(\widetilde{W}_i)$ containing the arc a, for all $a \in \widetilde{A}'$.

Proof. W.l.o.g., we will suppose that $E' = \{e_0\}$. The proof is similar in the case where more than one edge are added to G. Also, for more clarity, we will consider that only one arc, say a_0, is associated with e_0 in $\widetilde{G}'$, that is we will consider that $\widetilde{A}' = \{a_0\}$.
We are going to show that for every solution $(\overline{x}, \overline{y}) \in k\text{HNDP}_{Ag}(G, D)$,

$$\sum_{e \in E} \alpha(e)\overline{x}(e) + \sum_{a \in \widetilde{A}} \beta(a)\overline{y}(a) + \left\lceil \frac{q(a_0)}{2} \right\rceil \overline{y}(a_0) \geq \left\lceil \frac{kp - \sum_{i=1}^{p} |\widetilde{F}_i^0|}{2} \right\rceil.$$

First, let $\Delta(x, y) = \alpha x + \beta y$, that is

$$\Delta(x, y) = \sum_{a \in \widetilde{F}_2} y(a) + \sum_{e \in E_2} x(e) + \sum_{a \in \widetilde{F}_1 \setminus \widetilde{F}_1^2} y(a),$$

where $\widetilde{F}_2$, $\widetilde{F}_1$, $\widetilde{F}_1^2$ and E_2 are the arc and edge sets involved in $\alpha x + \beta y \geq \gamma$. The lifted inequality can hence be written as

$$\Delta(x, y) + \left\lceil \frac{q(a_0)}{2} \right\rceil \overline{y}(a_0) \geq \left\lceil \frac{kp - \sum_{i=1}^{p} |\widetilde{F}_i^0|}{2} \right\rceil. \tag{7.10}$$

If $\overline{y}(a_0) = 0$, then obviously the restriction of $(\overline{x}, \overline{y})$ to E and $\widetilde{A}$ is in $k\text{HNDP}_{Ag}(G, D)$. Thus, $\Delta(\overline{x}, \overline{y}) \geq \left\lceil \frac{kp - \sum_{i=1}^{p} |\widetilde{F}_i^0|}{2} \right\rceil$, and hence (7.10) is satisfied.
Now suppose that $\overline{y}(a_0) = 1$. We have that

$$\begin{aligned}
\sum_{i=1}^{p} \overline{y}(\delta_{\widetilde{G}}^{+}(\widetilde{W}_i) \setminus \widetilde{F}_i^0) &= \sum_{i=1}^{p} \overline{y}(\delta_{\widetilde{G}}^{+}(\widetilde{W}_i)) - \overline{y}(\widetilde{F}_i^0) \\
&= 2 \sum_{a \in \widetilde{F}_2} \overline{y}(a) + \sum_{a \in \widetilde{F}_1^2} \overline{y}(a) + \sum_{a \in \widetilde{F}_1 \setminus \widetilde{F}_1^2} \overline{y}(a) \\
&\leq 2 \sum_{a \in \widetilde{F}_2} \overline{y}(a) + 2 \sum_{e \in E_2} \overline{x}(e) + \sum_{a \in \widetilde{F}_1 \setminus \widetilde{F}_1^2} \overline{y}(a) \\
&= 2\Delta(\overline{x}, \overline{y}) - \sum_{a \in \widetilde{F}_1 \setminus \widetilde{F}_1^2} \overline{y}(a).
\end{aligned}$$

Thus we get

$$\begin{aligned}
\Delta(\overline{x}, \overline{y}) &\geq \frac{1}{2} \left[\sum_{i=1}^{p} \overline{y}(\delta_{\widetilde{G}}^{+}(\widetilde{W}_i)) - \sum_{i=1}^{p} \overline{y}(\widetilde{F}_i^0) + \sum_{a \in \widetilde{F}_1 \setminus \widetilde{F}_1^2} \overline{y}(a) \right], \\
&\geq \frac{1}{2} \left[\sum_{i=1}^{p} \overline{y}(\delta_{\widetilde{G}}^{+}(\widetilde{W}_i)) - \sum_{i=1}^{p} \overline{y}(\widetilde{F}_i^0) \right].
\end{aligned}$$

Hence,

$$\Delta(\overline{x},\overline{y}) \geq \left\lceil \frac{\sum_{i=1}^{p} \overline{y}(\delta^{+}_{\widetilde{G}}(\widetilde{W}_i)) - \sum_{i=1}^{p} |\widetilde{F}^0_i|}{2} \right\rceil. \tag{7.11}$$

If $\widetilde{W}_i$, $i = 1, ..., q(a_0)$, are the node sets of Π such that the dicut $\delta^{+}_{\widetilde{G}'}(\widetilde{W}_i)$ contains a_0, then we have that

$$\begin{array}{rcll} \overline{y}(\delta^{+}_{\widetilde{G}}(\widetilde{W}_i)) & = & \overline{y}(\delta^{+}_{\widetilde{G}'}(\widetilde{W}_i)) - \overline{y}(a_0), & i = 1, ..., q(a_0), \\ \overline{y}(\delta^{+}_{\widetilde{G}}(\widetilde{W}_i)) & = & \overline{y}(\delta^{+}_{\widetilde{G}'}(\widetilde{W}_i)), & i = q(a_0)+1, ..., p. \end{array}$$

As $(\overline{x},\overline{y})$ induces a solution of kHNDP$_{Ag}$ on G', we have that $\overline{y}(\delta^{+}_{\widetilde{G}'}(\widetilde{W}_i)) \geq k$, $i = 1, ..., p$. Moreover, since $\overline{y}(a_0) = 1$, we have that

$$\overline{y}(\delta^{+}_{\widetilde{G}}(\widetilde{W}_i)) \geq k - 1, \ i = 1, ..., q(a_0). \tag{7.12}$$

Thus, from (7.11) and (7.12), we obtain

$$\begin{aligned} \Delta(\overline{x},\overline{y}) &\geq \left\lceil \frac{k(p - q(a_0)) + (k-1)q(a_0) - \sum_{i=1}^{p} |\widetilde{F}^0_i|}{2} \right\rceil, \\ &\geq \left\lceil \frac{kp - \sum_{i=1}^{p} |\widetilde{F}^0_i| - q(a_0)}{2} \right\rceil, \\ &\geq \left\lceil \frac{kp - \sum_{i=1}^{p} |\widetilde{F}^0_i|}{2} \right\rceil - \left\lceil \frac{q(a_0)}{2} \right\rceil. \end{aligned}$$

Therefore, since $\overline{y}(a_0) = 1$, we get

$$\Delta(\overline{x},\overline{y}) + \left\lceil \frac{q(a_0)}{2} \right\rceil \overline{y}(a_0) \geq \left\lceil \frac{kp - \sum_{i=1}^{p} |\widetilde{F}^0_i|}{2} \right\rceil,$$

which ends the proof of the theorem. □ □

Now we give a lifting procedure for aggregated cut inequalities (7.8) when the Cut formulation is considered. This procedure is similar to that introduced for inequalities (7.4) for the Aggregated formulation. It is given in the theorem below.

Theorem 7.4 *Let $G = (V, E)$ be an undirected graph, $D \subseteq V \times V$ and $\widetilde{G}_{st}$ be the directed graph associated with G and a demand $\{s, t\} \in D$ in the cut formulation, for all $\{s, t\} \in D$. Let*

$$\sum_{e \in E} \alpha(e)x(e) + \sum_{i=1}^{q} \sum_{a \in \widetilde{A}_{s_i t_i}} \beta_{s_i t_i}(a) y_{s_i t_i}(a) \geq \gamma,$$

be an inequality of type (7.8) induced by a demand set $\{\{s_1, t_1\}, ..., \{s_q, t_q\}\}$, a family of node sets $\{\widetilde{W}_1^{s_1 t_1}, ..., \widetilde{W}_{p_1}^{s_1 t_1}, ..., \widetilde{W}_1^{s_q t_q}, ..., \widetilde{W}_{p_q}^{s_q t_q}\}$, with $p_i \geq 1$, for all $i \in \{1, ..., q\}$ and $p = \sum_{i=1}^{q} p_i \geq 2$, and arc sets $\widetilde{F}_j^{s_i t_i, 0} \subseteq \delta_{\widetilde{G}_{s_i t_i}}(\widetilde{W}_j^{s_i t_i})$, $j = 1, ..., p_i$, $i = 1, ..., q$. Let $G' = (V, E \cup E')$ and $\widetilde{G}'_{st} = (\widetilde{V}_{st}, \widetilde{A}_{st} \cup \widetilde{A}'_{st})$ be the directed graph associated with G' in the Cut formulation, for all $\{s, t\} \in D$ ($\widetilde{A}'_{st}$ is the set of arcs corresponding to the edges of E').
Then the inequality

$$\sum_{e \in E} \alpha(e)x(e) + \sum_{i=1}^{q} \sum_{a \in \widetilde{A}_{s_i t_i}} \beta_{s_i t_i}(a) y_{s_i t_i}(a) + \sum_{i=1}^{q} \sum_{a \in \widetilde{A}'_{s_i t_i}} \left\lceil \frac{q_{s_i t_i}(a)}{2} \right\rceil y_{s_i t_i}(a) \geq \gamma \tag{7.13}$$

is valid for $kHNDP_{Cu}^{Sep}(G', D)$, where $q_{s_i t_i}(a)$ is the number of dicuts $\delta^+_{\widetilde{G}'_{s_i t_i}}(\widetilde{W}_j^{s_i t_i})$, in $\widetilde{G}'_{s_i t_i}$, containing the arc a, for every $a \in \widetilde{A}'_{s_i t_i}$, $i = 1, ..., p$.

Proof. Similar to that of Theorem 7.3. □

8 Facets

Throughout this section, we consider a complete graph $G = (V, E)$ and suppose that $|V| \geq k+2$.

The first result concerns necessary conditions for the aggregated cut inequalities (7.4) to define facets for kHNDP$_{Ag}(G, D)$. To this end, we first give the following lemma.

Lemma 8.1 *Consider an inequality of type (7.4) induced by a family of node sets $\Pi = \{\widetilde{W}_1, ..., \widetilde{W}_p\}$, $p \geq 2$, and arc subsets $\widetilde{F}_i^0 \subseteq \delta^+_{\widetilde{G}}(\widetilde{W}_i)$, $i = 1, ..., p$. Let $\widetilde{F}_2$, $\widetilde{F}_1$, $\widetilde{F}_1^2$ and E_2 be the arc and edge sets involved in this inequality. Then (7.4) can be written as*

$$\sum_{i=1}^{p} y(\delta^+(\widetilde{W}_i)) + 2 \sum_{e \in E_2} x(e) - \sum_{a \in \widetilde{F}_1^2} y(a) + \sum_{i=1}^{p} (|\widetilde{F}_i^0| - y(\widetilde{F}_i^0)) + \sum_{a \in \widetilde{F}_1 \setminus \widetilde{F}_1^2} y(a) \geq kp + 1. \tag{8.1}$$

Moreover, (7.4) is tight for a solution $(x_0, y_0) \in k$HNDP$_{Ag}(G, D)$ if and only if one of the following conditions holds

i)

$$2 \sum_{e \in E_2} x_0(e) - \sum_{a \in \widetilde{F}_1^2} y_0(a) + \sum_{i=1}^{p} (|\widetilde{F}_i^0| - y_0(\widetilde{F}_i^0)) + \sum_{a \in \widetilde{F}_1 \setminus \widetilde{F}_1^2} y_0(a) = 1 \tag{8.2}$$

and $y_0(\delta^+(\widetilde{W}_i)) = k$, for $i = 1, ..., p$;

ii)

$$2\sum_{e\in E_2} x_0(e) - \sum_{a\in \widetilde{F}_1^2} y_0(a) + \sum_{i=1}^{p}(|\widetilde{F}_i^0| - y_0(\widetilde{F}_i^0)) + \sum_{a\in \widetilde{F}_1\setminus \widetilde{F}_1^2} y_0(a) = 0 \tag{8.3}$$

and there exists $i_0 \in \{1,...,p\}$ *such that* $y_0(\delta^+(\widetilde{W}_i)) = k$, *for* $i \in \{1,...,p\}\setminus\{i_0\}$ *and* $y_0(\delta^+(\widetilde{W}_{i_0})) = k+1$.

Proof. First we show that $\alpha x + \beta y \geq \gamma$ is equivalent to (8.1). As kp and $\sum_{i=1}^{p}|\widetilde{F}_i^0|$ have different parities, $\alpha x + \beta y \geq \gamma$ is equivalent to

$$2\sum_{e\in E_2} x(e) + 2\sum_{a\in \widetilde{F}_2} y(a) + 2\sum_{a\in \widetilde{F}_1\setminus \widetilde{F}_1^2} y(a) \geq kp - \sum_{i=1}^{p}|\widetilde{F}_i^0| + 1. \tag{8.4}$$

From the st-dicuts induced by the sets $\widetilde{W}_i$, we have that

$$\begin{aligned}\sum_{i=1}^{p} y(\delta^+(\widetilde{W}_i)\setminus \widetilde{F}_i^0) &= 2\sum_{a\in \widetilde{F}_2} y(a) + \sum_{a\in \widetilde{F}_1^2} y(a) + \sum_{a\in \widetilde{F}_1\setminus \widetilde{F}_1^2} y(a),\\ &= 2\sum_{a\in \widetilde{F}_2} y(a) + 2\sum_{e\in E_2} x(e) - 2\sum_{e\in E_2} x(e) + \sum_{a\in \widetilde{F}_1^2} y(a) + \sum_{a\in \widetilde{F}_1\setminus \widetilde{F}_1^2} y(a).\end{aligned}$$

together with (8.4), we get

$$\sum_{i=1}^{p} y(\delta^+(\widetilde{W}_i)\setminus \widetilde{F}_i^0) + 2\sum_{e\in E_2} x(e) - \sum_{a\in \widetilde{F}_1^2} y(a) + \sum_{a\in \widetilde{F}_1\setminus \widetilde{F}_1^2} y(a) \geq kp - \sum_{i=1}^{p}|\widetilde{F}_i^0| + 1. \tag{8.5}$$

By combining (8.5) and $y(\delta^+(\widetilde{W}_i)\setminus \widetilde{F}_i^0) = y(\delta^+(\widetilde{W}_i)) - y(\widetilde{F}_i^0)$, $i = 1,...,p$, we get (8.1).
Now consider a solution $(x_0, y_0) \in k\text{HNDP}_{Ag}(G, D)$ satisfying (7.4) with equality. By the previous result, we have that

$$\sum_{i=1}^{p} y_0(\delta^+(\widetilde{W}_i)) + \sum_{i=1}^{p}(|\widetilde{F}_i^0| - y_0(\widetilde{F}_i^0)) + 2\sum_{e\in E_2} x_0(e) - \sum_{a\in \widetilde{F}_1^2} y_0(a) + \sum_{a\in \widetilde{F}_1\setminus \widetilde{F}_1^2} y_0(a) = kp + 1. \tag{8.6}$$

As (x_0, y_0) induces a solution of the kHNDP, we have that $y_0(\delta^+(\widetilde{W}_i)) \geq k$, $i = 1,...,p$. Therefore, $\sum_{i=1}^{p} y_0(\delta^+(\widetilde{W}_i)) \geq kp$, and hence,

$$\sum_{i=1}^{p}(|\widetilde{F}_i^0| - y_0(\widetilde{F}_i^0)) + 2\sum_{e\in E_2} x_0(e) - \sum_{a\in \widetilde{F}_1^2} y_0(a) + \sum_{a\in \widetilde{F}_1\setminus \widetilde{F}_1^2} y_0(a) \leq 1. \tag{8.7}$$

If (8.7) is satisfied with equality, then, clearly $y_0(\delta^+(\widetilde{W}_i)) = k$, $i = 1,...,p$. If not, as $y_0(\delta^+(\widetilde{W}_i)) \geq k$, $i = 1,...,p$, this yields $y_0(\delta^+(\widetilde{W}_{i_0})) = k+1$ for some $i_0 \in \{1,...,p\}$ and $y_0(\delta^+(\widetilde{W}_i)) = k$, for $i \in \{1,...,p\}\setminus\{i_0\}$.

Conversely, if (8.7) is tight for (x_0, y_0) and $y_0(\delta^+(\widetilde{W}_i)) = k$ for all $i \in \{1, ..., p\}$, then clearly, (8.1) is tight for (x_0, y_0) and hence $\alpha x + \beta y \geq \gamma$ is tight for (x_0, y_0). If (8.7) is not tight for (x_0, y_0), that is

$$\sum_{i=1}^{p}(|\widetilde{F}_i^0| - y_0(\widetilde{F}_i^0)) + 2\sum_{e\in E_2} x_0(e) - \sum_{a\in\widetilde{F}_1^2} y_0(a) + \sum_{a\in\widetilde{F}_1\setminus\widetilde{F}_1^2} y_0(a) = 0,$$

$y_0(\delta^+(\widetilde{W}_{i_0})) = k + 1$ for some $i_0 \in \{1, ..., p\}$ and $y_0(\delta^+(\widetilde{W}_{i_0})) = k$ for $i \in \{1, ..., p\} \setminus \{i_0\}$, then clearly, (8.1) is also tight for (x_0, y_0). Thus, $\alpha x + \beta y \geq \gamma$ is tight for (x_0, y_0). □ □

Corollary 8.1 *Consider an inequality of type (7.4) induced by a family of node sets $\{\widetilde{W}_1, ..., \widetilde{W}_p\}$, $p \geq 2$, and arc subsets $\widetilde{F}_i^0 \subseteq \delta^+_{\widetilde{G}}(\widetilde{W}_i)$, $i = 1, ..., p$. Let $\widetilde{F}_2$, $\widetilde{F}_1$, $\widetilde{F}_1^2$ and E_2 be the arc and edge sets involved in this inequality. If (7.4) is tight for a solution (x_0, y_0) of kHNDP$_{Ag}(G, D)$ then,*

$$2\sum_{e\in E_2} x_0(e) - \sum_{a\in\widetilde{F}_1^2} y_0(a) + \sum_{i=1}^{p}(|\widetilde{F}_i^0| - y_0(\widetilde{F}_i^0)) + \sum_{a\in\widetilde{F}_1\setminus\widetilde{F}_1^2} y_0(a) \leq 1. \tag{8.8}$$

Theorem 8.1 *Let $\Pi = \{\widetilde{W}_1, ..., \widetilde{W}_p\}$, $p \geq 2$, be a family of node sets of $\widetilde{V}$ such that each set $\widetilde{W}_i$, $i = 1, ..., p$, induces an s_it_i-dicut of $\widetilde{G}$, for some $\{s_i, t_i\} \in D$, and $\widetilde{F}_i^0 \subseteq \delta^+_{\widetilde{G}}(\widetilde{W}_i)$. Suppose that every arc of $\widetilde{A}$ belongs to at most two sets $\delta^+_{\widetilde{G}}(\widetilde{W}_i) \setminus \widetilde{F}_i^0$. Then, the aggregated cut inequality (7.4) induced by Π and $\widetilde{F}_i^0$, $i = 1, ..., p$, defines a facet of kHNDP$_{Ag}(G, D)$ different from the trivial and s_it_i-dicut inequalities, only if for all $i \in \{1, ..., p\}$, the following conditions hold*

1. *$|\widetilde{W}_i \cap S_D| = |(\widetilde{V} \setminus \widetilde{W}_i) \cap T_D| = 1$;*
2. *$|\widetilde{W}_i \cap S_D| \geq 2$ and for all $s \in (\widetilde{W}_i \setminus \{s_i\}) \cap S_D$, $[s, \widetilde{V} \setminus \widetilde{W}_i] = \emptyset$;*
3. *$|(\widetilde{V} \setminus \widetilde{W}_i) \cap T_D| \geq 2$ and for all $t \in [(\widetilde{V} \setminus \widetilde{W}_i) \setminus \{t_i\}] \cap T_D$, $[\widetilde{W}_i, t] = \emptyset$.*

Proof. Let us denote by $\alpha x + \beta y \geq \gamma$ the inequality (7.4) induced by Π and $\widetilde{F}_i^0$, $i = 1, ..., p$, and suppose that it defines a facet of kHNDP$_{Ag}(G, D)$. We will show that $|\widetilde{W}_i \cap S_D| = 1$, for $i = 1, ..., p$. The proof follows the same lines for $|(\widetilde{V} \setminus \widetilde{W}_i) \cap T_D| = 1$. Also the proof for 2) and 3) is similar.
Suppose, on the contrary, that there exists $i_0 \in \{1, ..., p\}$ such that $\widetilde{W}_{i_0}$ induces an st-dicut of $\widetilde{G}$ and that $(\widetilde{W}_{i_0} \setminus \{s\}) \cap S_D \neq \emptyset$. Let s' be a node of $(\widetilde{W}_{i_0} \setminus \{s\}) \cap S_D$, and suppose that $[s', \widetilde{V} \setminus \widetilde{W}_{i_0}] \neq \emptyset$ (see Figure 15).

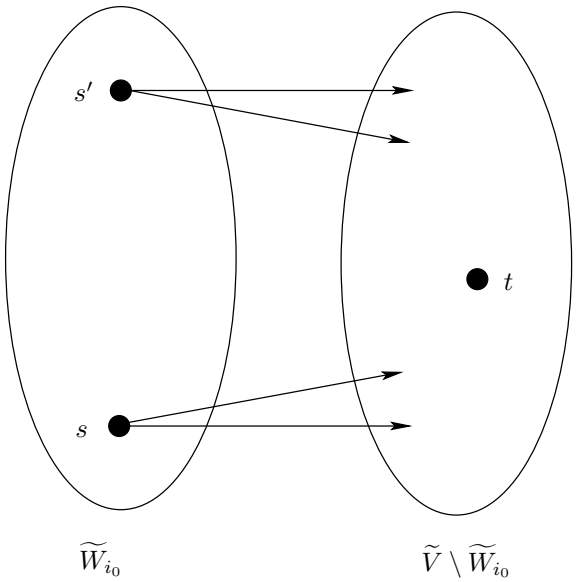

Figure 15: A set $\widetilde{W}_{i_0}$ containing two nodes of S

Let $\widetilde{H}_0 = \widetilde{F}_2 \cap [s', \widetilde{V} \setminus \widetilde{W}_{i_0}]$ and $\widetilde{H}_1 = (\widetilde{F}_1 \setminus \widetilde{F}_1^2) \cap [s', \widetilde{V} \setminus \widetilde{W}_{i_0}]$. Also let $\widetilde{H}_2 = \widetilde{F}_1^2 \cap [s', \widetilde{V} \setminus \widetilde{W}_{i_0}]$, $\widetilde{H}_3$ be the set of arcs of $\widetilde{F}_1^2$ corresponding to the same edges as the arcs of $\widetilde{H}_2$. Let E_0 be the edge set corresponding to the arcs of $\widetilde{H}_2$ and $\widetilde{H}_3$. Consider now the aggregated cut inequality induced by $\{\widetilde{W}_1', ..., \widetilde{W}_p'\}$ and $\widetilde{F}_i^{0'}$, $i = 1, ..., p$, where $\widetilde{W}_i' = \widetilde{W}_i$, $\widetilde{F}_i^{0'} = \widetilde{F}_i^0$, for $i \in \{1, ..., p\} \setminus \{i_0\}$, and $\widetilde{W}_{i_0}' = \widetilde{W}_{i_0} \setminus \{s'\}$, $\widetilde{F}_i^{0'} = \widetilde{F}_i^0 \setminus [s', \widetilde{V} \setminus \widetilde{W}_{i_0}]$. Let $\widetilde{F}_2'$, $\widetilde{F}_1'$, $\widetilde{F}_1^{2'}$ and E_2' be the set of arcs and edges involved in this inequality. By the above observation, as the arcs of $\widetilde{H}_3$ correspond to those of $\widetilde{H}_2$, we have that $\widetilde{H}_3 \cap [s', \widetilde{V} \setminus \widetilde{W}_{i_0}] = \emptyset$. Also, by the same observation, no arc of $\widetilde{H}_0$ may correspond to an arc of $\widetilde{H}_2$ and $\widetilde{H}_3$. Thus, we have that

$$\begin{aligned}
&\widetilde{F}_2' = \widetilde{F}_2 \setminus \widetilde{H}_0, \\
&\widetilde{F}_1^{2'} = \widetilde{F}_1^2 \setminus (\widetilde{H}_2 \cup \widetilde{H}_3), \\
&\widetilde{F}_1' \setminus \widetilde{F}_1^{2'} = [(\widetilde{F}_1 \setminus \widetilde{F}_1^2) \setminus \widetilde{H}_1] \cup \widetilde{H}_0 \cup \widetilde{H}_3. \\
&E_2' = E_2 \setminus E_0.
\end{aligned}$$

Therefore, the inequality (7.4) induced by $\{\widetilde{W}_1', ..., \widetilde{W}_p'\}$ and $\widetilde{F}_i^{0'}$, $i = 1, ..., p$, can be written as

$$\sum_{a \in \widetilde{F}_2 \setminus \widetilde{H}_0} y(a) + \sum_{e \in E_2 \setminus E_0} x(e) + \sum_{a \in (\widetilde{F}_1 \setminus \widetilde{F}_1^2) \setminus \widetilde{H}_1} + \sum_{a \in \widetilde{H}_0} y(a) + \sum_{a \in \widetilde{H}_3} y(a) \geq \left\lceil \frac{kp - \sum_{i=1}^{p} |\widetilde{F}_i^{0'}|}{2} \right\rceil. \quad (8.9)$$

By summing up inequality (8.9) and the inequalities

$$x(e) \geq y(a), \text{ for all } a \in \widetilde{H}_3, \text{ where } e \text{ is the edge of } E_0 \text{ corresponding to } a. \quad (8.10)$$

$$y(a) \geq 0, \text{ for all } a \in \widetilde{H}_1, \quad (8.11)$$

we get

$$\sum_{a\in\widetilde{F}_2} y(a) + \sum_{e\in E_2} x(e) + \sum_{a\in\widetilde{F}_1\setminus\widetilde{F}_1^2} y(a) \geq \left\lceil \frac{kp - \sum_{i=1}^{p} |\widetilde{F}_i^{0\prime}|}{2} \right\rceil. \tag{8.12}$$

Clearly, if $\widetilde{F}_{i_0} \cap [s', \widetilde{V} \setminus \widetilde{W}_{i_0}] = \emptyset$, then $\widetilde{F}'_{i_0} = \widetilde{F}_{i_0}$ and inequality (8.12) is the same as $\alpha x + \beta y \geq \gamma$. Thus $\alpha x + \beta y \geq \gamma$ is redundant with respect to (8.9)-(8.11), and hence cannot define a facet of kHNDP$_{Ag}(G, D)$. If $\widetilde{F}_{i_0} \cap [s', \widetilde{V} \setminus \widetilde{W}_{i_0}] \neq \emptyset$, then the right hand side of inequality (8.12) is greater than that of $\alpha x + \beta y \geq \gamma$. Thus, $\alpha x + \beta y \geq \gamma$ is dominated by (8.9)-(8.11), and hence cannot define a facet of kHNDP$_{Ag}(G, D)$. □ □

The next theorems give necessary conditions for the double cut and triple path-cut inequalities to define facets of the kHNDP polytopes. Before each theorem, we give two technical lemmas which will be useful to prove the theorem. The first lemma, Lemma 8.2, is given without proof since it is obtained from Theorem 3.4 of [9].

Lemma 8.2 [9] *Let $\{s, t\} \in D$, T be a 3-st-path-cut ineduced by a partition $(V_0, ..., V_4)$. If an edge set $F \subseteq E$ induces a solution of the kHNDP and is such that $|F \cap T| = k$, then $|F \cap ([s, V_1] \cup [V_3, t] \cup [s, t])| \geq k$.*

Lemma 8.3 *Let $\alpha x \geq \gamma$ be a double cut inequality induced by a family of node sets $\Pi = (V_0^1, V_0^2, V_1, ..., V_{L+1})$ of V, $F \subseteq E$ and $\{s, t\} \in D$ with $s \in V_0^1$ and $t \in V_{L+1}$ (here $i_0 = 0$). Then, $\alpha x \geq \gamma$ can be written as*

$$x(T) + x(\delta(V_0^1 \cup V_0^2)) + x(\delta(V_1)) + x(\overline{E} \setminus F) + |F| - x(F) \geq 3k + 1, \tag{8.13}$$

where T is the L-st-path-cut induced by the partition $(V_0^1, V_0^2 \cup V_1, V_2, ..., V_{L+1})$.
Moreover, $\alpha x \geq \gamma$ is tight for a solution x_0 of $kHNDP_{Ag}$, $kHNDP_{Cut}$, $kHNDP_{Flo}^{Sep}$, $kHNDP_{Pa}^{Sep}$, where $x_0 \in \mathbb{R}^E$, if and only if one of the following conditions holds.

i) $x_0(\overline{E} \setminus F) + |F| - x_0(F) = 1$ and $x_0(T) = x_0(\delta(V_0^1 \cup V_0^2)) = x_0(V_1) = k$;

ii) $x_0(\overline{E} \setminus F) + |F| - x_0(F) = 0$ and

a) $x_0(T) = k + 1$, $x_0(\delta(V_0^1 \cup V_0^2)) = k$ and $x_0(V_1) = k$;
b) $x_0(T) = k$, $x_0(\delta(V_0^1 \cup V_0^2)) = k + 1$ and $x_0(V_1) = k$;
c) $x_0(T) = k$, $x_0(\delta(V_0^1 \cup V_0^2)) = k$ and $x_0(V_1) = k + 1$;

Proof. W.l.o.g., we will consider the polytope kHNDP$_{Ag}(G, D)$. The proof is similar for kHNDP$_{Cut}(G, D)$, kHNDP$_{Flo}^{Sep}(G, D)$ and kHNDP$_{Pa}^{Sep}(G, D)$.
Let H denote the double cut induced by Π. The inequality $\alpha x \geq \gamma$ is equivalent to

$$x(H \setminus \overline{E}) + x(\overline{E} \setminus F) \geq \frac{3k - |F| + 1}{2}.$$

This implies that

$$2x(H \setminus \overline{E}) + 2x(\overline{E}) - 2x(F) \geq 3k - |F| + 1. \tag{8.14}$$

From the L-st-path-cut T and cuts $\delta(V_0^1 \cup V_0^2)$ and $\delta(V_1)$, we have that

$$x(T) + x(\delta(V_0^1 \cup V_0^2)) + x(\delta(V_1)) = 2x(H \setminus \overline{E}) + x(\overline{E}). \tag{8.15}$$

By combining (8.14) and (8.15), we get

$$x(T) + x(\delta(V_0^1 \cup V_0^2)) + x(\delta(V_1)) + x(\overline{E}) - 2x(F) \geq 3k - |F| + 1,$$

and hence

$$x(T) + x(\delta(V_0^1 \cup V_0^2)) + x(\delta(V_1)) + x(\overline{E} \setminus F) + |F| - x(F) \geq 3k + 1.$$

Therefore, $\alpha x \geq \gamma$ is equivalent to (8.13).
Now suppose that $\alpha x \geq \gamma$ is tight for (x_0, y_0). From the development above, we have that inequality (8.13) is also tight for (x_0, y_0), that is

$$x_0(T) + x_0(\delta(V_0^1 \cup V_0^2)) + x_0(\delta(V_1)) + x_0(\overline{E} \setminus F) + |F| - x_0(F) = 3k + 1.$$

Since $x_0(T) \geq k$, $x_0(\delta(V_0^1 \cup V_0^2)) \geq k$ and $x_0(\delta(V_1)) \geq k$, it is clear that $x_0(\overline{E} \setminus F) + |F| - x_0(F) \leq 1$. Hence, if $x_0(\overline{E} \setminus F) + |F| - x_0(F) = 1$, we have that $x_0(T) = x_0(\delta(V_0^1 \cup V_0^2)) = x_0(\delta(V_1)) = k$. If $x_0(\overline{E} \setminus F) + |F| - x_0(F) = 0$, then, clearly, either $x_0(T)$, $x_0(\delta(V_0^1 \cup V_0^2))$ or $x_0(\delta(V_1))$ is equal to $k+1$ and the others are equal to k.
Consider now a solution $(x_0, y_0) \in k\text{HNDP}_{Ag}(G, D)$ such that $x_0(\overline{E} \setminus F) + |F| - x_0(F) = 1$ and $x_0(T) = x_0(\delta(V_0^1 \cup V_0^2)) = x_0(\delta(V_1)) = k$. Then, clearly, inequality (8.13) is satisfied with equality, and hence, $\alpha x \geq \gamma$ is tight for (x_0, y_0). Similarly, if $x_0(\overline{E} \setminus F) + |F| - x_0(F) = 0$ and either $x_0(T)$, $x_0(\delta(V_0^1 \cup V_0^2))$ or $x_0(\delta(V_1))$ is equal to $k+1$ with the others equal to k, then (8.13) is satisfied with equality by x_0 and hence, $\alpha x \geq \gamma$ is tight for (x_0, y_0), which ends the proof of the lemma. □ □

Theorem 8.2 *Suppose that $L \geq 2$ and $k \geq 2$, and let $\{s, t\} \in D$.*
Let $\Pi = \{V_0^1, V_0^2, V_1, ..., V_{L+1}\}$ be a family of node sets of V and $F \subseteq E$ which induce a double cut of G with respect to $\{s, t\}$, $s \in V_0^1$ and $t \in V_{L+1}$ (here $i_0 = 0$). Then, the double cut inequality induced by Π and F defines a facet of $k\text{HNDP}_{Ag}(G, D)$, $kHNDP_{Cu}^{Sep}(G, D)$, $kHNDP_{Flo}^{Sep}(G, D)$, $kHNDP_{Pa}^{Sep}(G, D)$ different from the trivial inequalities and inequalities (2.1)-(2.4) only if the following conditions hold

i) $|V_0^1| = |V_{L+1}| = 1$;

ii) if $L = 3$, then $|[V_0^1, V_0^2 \cup V_1] \cup [V_3, V_4] \cup [V_0^1, V_4]| \geq k$.

Proof. The proof will be done for $k\text{HNDP}_{Ag}(G, D)$ as it is similar for $k\text{HNDP}_{Cu}^{Sep}(G, D)$, $k\text{HNDP}_{Flo}^{Sep}(G, D)$ and $k\text{HNDP}_{Pa}^{Sep}(G, D)$. We will denote by $\alpha x \geq \gamma$ the double cut inequality induced by Π and F, and assume that it defines a facet of $k\text{HNDP}_{Ag}(G, D)$ different from those induced by the trivial, cut and L-st-path-cut inequalities. Let $\mathcal{F} = \{(x, y) \in k\text{HNDP}_{Ag}(G, D)$ such that $\alpha x = \gamma\}$ and let T denote the L-st-path-cut induced by the partition $(V_0^1, V_0^2 \cup V_1, V_2, ..., V_{L+1})$.
i) Let us denote by H the double cut induced by Π and F. Suppose first that $|V_0^1| \geq 2$. By considering the family of node sets $\Pi' = \{\{s\}, V_0^2 \cup V_0^1 \setminus \{s\}, V_1, ..., V_{L+1}\}$, the double cut H'

induced by Π' and F is such that $H = H' \cup [V_0^1 \setminus \{s\}, V_1]$. Thus, the double cut inequality induced by H is redundant with respect to

$$x(H' \setminus F) \geq \left\lceil \frac{3k - |F|}{2} \right\rceil$$
$$x(e) \geq, \text{ for all } e \in [V_0^1 \setminus \{s\}, V_1], \tag{8.16}$$

and hence, cannot define a facet.
ii) We will show that $\mathcal{F} \neq \emptyset$ only if ii) holds. As $\alpha x \geq \gamma$ defines a facet different from $x(\delta(V_0^1 \cup V_0^2)) \geq k$, there exists a solution $(\overline{x}, \overline{y}) \in \mathcal{F}$ such that $\overline{x}(\delta(V_0^1 \cup V_0^2)) \geq k+1$. Thus, by Lemma 8.3, $\overline{x}(T) = k$. Therefore, the graph induced by $\overline{x}$ contains exactly k edge-disjoint L-st-paths. Moreover, each L-st-path intersects T only once. Thus, by Lemma 8.2, we have that $|[V_0^1, V_{L+1}]| + |[V_0^1, V_0^2 \cup V_1]| + |[V_L, V_{L+1}]| \geq k$. □ □

Lemma 8.4 *Let $\alpha x \geq \gamma$ be a triple path-cut inequality induced by a family of node set $\Pi = \{V_0, ..., V_L, V_{L+1}^1, V_{L+1}^2, V_{L+2}^1, V_{L+2}^2\}$ and $F \subseteq E$. Then $\alpha x \geq \gamma$ can be written as*

$$x(T_1) + x(T_2) + x(T_3) + x(\overline{E} \setminus F) + |F| - x(F) \geq 3k + 1 \tag{8.17}$$

where T_1, T_2 and T_3 are the triple path-cuts induced by the partitions $(V_0, V_1 \cup V_4, V_2 \cup V_3^1, V_3^2)$, $(V_0, V_1 \cup V_3, V_2 \cup V_4^1, V_4^2)$ and $(V_0, V_1, V_2 \cup V_3 \cup V_4^1, V_4^2)$, respectively, and $\overline{E} = [V_3^2, V_1 \cup V_4^1] \cup [V_3^1, V_4^2]$ (resp. $\overline{E} = [V_2, V_4^2] \cup [V_3, V_4 \cup V_5]$) if $L = 2$ (resp. $L = 3$).
Moreover, $\alpha x \geq \gamma$ is tight for a solution x_0 of the kHNDP, where $x_0 \in \mathbb{R}^E$, if and only if one of the following inequalities holds

i) $x_0(\overline{E} \setminus F) + |F| - x_0(F) = 1$ and $x_0(T_1) = x_0(T_2) = x_0(T_3) = k$;

ii) $x_0(\overline{E} \setminus F) + |F| - x_0(F) = 0$ and, for some $i_0 \in \{1, 2, 3\}$, $x_0(T_{i_0}) = k + 1$ and $x_0(T_i) = k$ for $i \in \{1, 2, 3\} \setminus \{i_0\}$.

Proof. Similar to that of Lemma 8.3. □

Theorem 8.3 *Let $L \in \{2, 3\}$ and consider $\Pi = \{V_0, ..., V_L, V_{L+1}^1, V_{L+1}^2, V_{L+2}^1, V_{L+2}^2\}$ be a family of node sets of V and $F \subseteq E$ which induce a triple path-cut of G with respect to demands $\{s_1, t_1\}$ and $\{s_2, t_2\}$. Then, the triple path-cut inequality induced by Π and F defines a facet of $k\text{HNDP}_{Ag}(G, D)$, $kHNDP_{Cu}^{Sep}(G, D)$, $kHNDP_{Flo}^{Sep}(G, D)$, $kHNDP_{Pa}^{Sep}(G, D)$ only if the following conditions hold*

i) $V_0 \setminus \{s_1, s_2\} = \emptyset$;

ii) $|V_{L+1}^2| = 1$;

iii) $|V_{L+2}^2| = 1$;

iv) if $L = 3$, then

a) $|[\{s_1, s_2\}, V_1 \cup V_5^1 \cup \{t_2\}]| + |[V_3 \cup V_4^1, t_1]| + |[\{s_1, s_2\}, t_1]| \geq k$;
b) $|[\{s_1, s_2\}, V_1 \cup V_4^1 \cup \{t_1\}]| + |[V_3 \cup V_5^1, t_2]| + |[\{s_1, s_2\}, t_2]| \geq k$;
c) $|[\{s_1, s_2\}, V_1]| + |[V_3 \cup V_4^1 \cup \{t_1\} \cup V_5^1, t_2]| + |[\{s_1, s_2\}, t_2]| \geq k$.

Proof. For the proof of Conditions i)-iii), we will consider, w.l.o.g., that $L = 3$. We will denote by $\alpha x \geq \gamma$ the triple-cut inequality induced by Π and F and let $\mathcal{F} = \{(x, y) \in k\text{HNDP}(G, D)$ such that $\alpha x = \gamma\}$.

i) Suppose that $V_0 \setminus \{s_1, s_2\} \neq \emptyset$ and denote by H the triple path-cut induced by Π and F. Consider the family of node sets $\Pi' = \{\{s_1, s_2\}, V_0 \setminus \{s_1, s_2\} \cup V_1, V_2, V_3, V_4^1, V_4^2, V_5^1, V_5^2\}$ and $F' = F$. If H' denotes the triple path-cut induced by Π' and F', we have that $H' = H \setminus [V_0 \setminus \{s_1, s_2\}, V_2]$. Thus, as $V_0 \setminus \{s_1, s_2\} \neq \emptyset$, inequality (6.5) induced by Π and F is redundant with respect to the inequalities

$$\begin{aligned}
&2x([\{s_1, s_2\}, V_2]) + 2x([\{s_1, s_2\}, V_3]) + 2x([V_1 \cup (V_0 \setminus \{s_1, s_2\}), V_3]) + \\
&\qquad x([\{s_1, s_2\} \cup V_1 \cup (V_0 \setminus \{s_1, s_2\}), V_4 \cup V_5]) + x([V_4, V_5]) + x([V_2, V_5^2]) + \\
&\qquad\qquad x(([V_2, V_4^2] \cup [V_3, V_4 \cup V_5]) \setminus F) \geq \left\lceil \frac{3k - |F|}{2} \right\rceil, \\
&x(e) \geq 0, \text{ for all } e \in [V_0 \setminus \{s_1, s_2\}, V_2].
\end{aligned}$$

Therefore, the triple path-cut inequality induced by Π and F cannot define a facet of the kHNDP polytopes.
ii) Now we show that $|V_4^2| = 1$. Suppose, on the contrary, that $|V_4^2| \geq 2$. Let $\Pi' = \{V_0, ..., V_3, V_4^1 \cup V_4^2 \setminus \{t_1\}, \{t_1\}, V_5^1, V_5^2\}$. First suppose that $F \cap [V_2, V_4^2 \setminus \{t_1\}] = \emptyset$ and let H' be the triple path-cut induced by Π' and F. As $F \cap [V_2, V_4^2 \setminus \{t_1\}] = \emptyset$, we have that $H' = H \setminus [V_2, V_4^2 \setminus \{t_1\}]$. If $\alpha' x \geq \gamma'$ denotes the triple path-cut inequality induced by Π' and F, then it is not hard to see that $\alpha'(e) = \alpha(e)$, for all $e \in H' \setminus F$, and that $\gamma' = \gamma$. Thus, $\alpha x \geq \gamma$ is redundant with respect to the following inequalities

$$\begin{aligned}
&\alpha' x \geq \gamma, \\
&x(e) \geq 0, \text{ for all } e \in [V_2, V_4^2 \setminus \{t_1\}],
\end{aligned}$$

and hence, cannot define a facet of the kHNDP poytopes.
If $F \cap [V_2, V_4^2 \setminus \{t_1\}] \neq \emptyset$, then we consider $F' = F \setminus (F \cap [V_2, V_4^2 \setminus \{t_1\}])$ and let $\alpha' x \geq \gamma'$ be the triple path-cut inequality induced by Π' and F'. Also let H' denote this triple path-cut. As before, we have that $H' = H \setminus [V_2, V_4^2 \setminus \{t_1\}]$ and, for all $e \in H' \setminus F'$, $\alpha'(e) = \alpha(e)$. Moreover, $\gamma = \left\lceil \frac{3k-|F|}{2} \right\rceil$ and $\gamma' = \left\lceil \frac{3k-|F|+|F \cap [V_2, V_4^2 \setminus \{t_1\}]|}{2} \right\rceil$. As $|F \cap [V_2, V_4^2 \setminus \{t_1\}]| \geq 1$, we have that $\gamma' \geq \gamma$. This implies that $\alpha x \geq \gamma$ is dominated by the inequalities

$$\begin{aligned}
&\alpha' x \geq \gamma', \\
&x(e) \geq 0, \text{ for all } e \in [V_2, V_4^2 \setminus \{t_1\}] \setminus F.
\end{aligned}$$

Thus, it cannot define a facet of the kHNDP poytopes.
iii) Suppose that $|V_5^2| \geq 2$. Consider $\Pi' = \{V_0, ..., V_3, V_4^1, V_4^2, V_5^1 \cup V_5^2 \setminus \{t_2\}, \{t_2\}\}$ and let H and H' denote the triple path-cuts induced by Π and F, and by Π' and F respectively. If $F \cap [V_3, V_5^2 \setminus \{t_2\}] = \emptyset$, then, clearly, $H' = H \setminus [V_2, V_5^2 \setminus \{t_2\}]$. If $F \cap [V_3, V_5^2 \setminus \{t_2\}] \neq \emptyset$, then it is also not hard to see, as before, that $H' = H \setminus [V_2, V_5^2 \setminus \{t_2\}]$.
This implies that the triple path-cut inequality induced by H is redundant with respect to that induced by H' and the inequalities $x(e) \geq 0$, for all $e \in [V_2, V_5^2 \setminus \{t_2\}]$. Thus, it cannot define a facet.
iv) To show that conditions iv) are necessary for $\alpha x \geq \gamma$ to define a facet, we show that the sets $\mathcal{F}_i = \{x \in \mathbb{R}^E$ such that x induces a solution of the kHNDP and $x(T_i) = k\}$, $i = 1, 2, 3$, are

non empty only if conditions iv) are satisfied. As $\mathcal{F}$ is different from the hyperspace induced by the inequality $x(e) \geq 0$ for some $e \in F$, there exists a solution $(\overline{x}, \overline{y}) \in \mathcal{F}$ such that $\overline{x}(e) = 0$. Thus, $|F| - \overline{x}(F) \geq 1$. By Lemma 8.4, this implies that $\overline{x}(\overline{E} \setminus F) + |F| - \overline{x}(F) = 1$ and hence, $\overline{x}(T_i) = k$, for $i = 1, 2, 3$. Therefore, from Lemma 8.2, we obtain that

$$
\begin{aligned}
&|[\{s_1, s_2\}, V_1 \cup V_5^1 \cup \{t_2\}]| + |[V_3 \cup V_4^1, t_1]| + |[\{s_1, s_2\}, t_1]| \geq k, \\
&|[\{s_1, s_2\}, V_1 \cup V_4^1 \cup \{t_1\}]| + |[V_3 \cup V_5^1, t_2]| + |[\{s_1, s_2\}, t_2]| \geq k, \\
&|[\{s_1, s_2\}, V_1]| + |[V_3 \cup V_4^1 \cup \{t_1\} \cup V_5^1, t_2]| + |[\{s_1, s_2\}, t_2]| \geq k,
\end{aligned}
$$

which ends the proof of the theorem. □ □

In the following section, we discuss the separation problem associated with the inequalities present before and present separation algorithm for each of them.

9 Separation Procedures

9.1 Separation of st-cut and L-st-path-cut inequalities

The separation problem of st-cut and L-st-path-cut inequalities, when $L = 2, 3$, can be reduced to computing a minimum weight st-dicut in the graph $\widetilde{G}_{st}$, for every $(s, t) \in D$, or in the aggregated graph $\widetilde{G}$. The reduction relies on the correspondance between st-cut and L-st-path-cuts, on one hand, and particular st-dicuts in $\widetilde{G}_{st}$ or in $\widetilde{G}$ on the other hand (see [24] for more details). This correspondance implies that, for a solution $\overline{x} \in \mathbb{R}^E$, there exists a violated st-cut or L-st-path-cut inequality if and only if, the weight of a minimum st-dicut in $\widetilde{G}_{st}$ or in $\widetilde{G}$ is $< k$.

The weights given to the arcs of $\widetilde{G}_{st}$ or $\widetilde{G}$ are as follows

- for an arc a of $\widetilde{G}_{st}$ or $\widetilde{G}$ corresponding to an edge e in G, we give the weight $\overline{x}(e)$ to arc a.
- for an arc a of $\widetilde{G}_{st}$ or $\widetilde{G}$ which does not correspond to any edge of G, we give a weight sufficiently large.

Therefore, the separation problem of st-cut and L-st-path-cut inequalities can be solved in polynomial time.

9.2 Separation of Hop-Constrained partition inequalities

In this section, we discuss our separation procedure for the Hop-Constrained partition inequalities (4.1), (4.2), (4.3) and (4.4). The idea of our separation procedure is to reduce the separation problem of these inequalities to computing a minimum weight cut in a suitable graph. Before going further, it is worth noting that we believe that the separation problem of Inequalities (4.1), (4.2), (4.3) and (4.4) is NP-hard. In fact, with the transformations we will present below, one can see that, under some conditions, the separation problem of the Hop-Constrained partition inequalities can be written as that of rounded capacity inequalities involved in the Capacitated Vehicle Routing Poblem. On the other hand, Diarrassouba [23] showed that the separation problem associated with these latter inequalities is strongly NP-hard.This explains why we believe that the separation problem of the Hop-Constrained partition inequalities is NP-hard in the strong sense, and why we devise a separation heuristic.

We discuss first our separation procedure for the Hop-Constrained partition inequalities in the case of rooted demands (both for $L = 2$ and $L = 3$). For our purpose, we consider the Hop-Constrained partition inequalities (4.1) and (4.2) under a generic form. Let $\pi = (V_0, V_1, ..., V_p)$, $p \geq 2$, with $s \in V_0$ and $V_i \cap T_D \neq \emptyset$, for all $i \in \{1, ..., p\}$. The Hop-Constrained partition inequality induced by π can be written as

$$x(\delta(\pi)) \geq \left\lceil \frac{\gamma p}{C} \right\rceil, \tag{9.1}$$

where

$$\begin{cases} \gamma = k + 1 \text{ and } C = 2 \text{ if } L = 2, \\ \gamma = k(k+1) + 2 \text{ and } C = 2(k+1) \text{ if } L = 3. \end{cases}$$

Also, we assume that there is no steiner node in the graph, that is the node set of the graph is $V = \{s\} \cup T_D$, and that $|V_i| = 1$, for all $i \in \{1, ..., p\}$ (recall that the demand set is rooted). We denote by t_i the only node of V_i, for every $i \in \{1, ..., p\}$. Now, for a solution $\overline{x} \in \mathbb{R}^E$, the inequality (9.1) can be written as

$$\sum_{i=1}^{p} \overline{x}(\delta(V_i)) + \overline{x}(\delta(V_0)) = \sum_{i=1}^{p} \overline{x}(\delta(t_i)) + \overline{x}(\delta(V_0)) \geq 2 \left\lceil \frac{\gamma p}{C} \right\rceil. \tag{9.2}$$

Now, if we let $W = \{t_1, ..., t_p\}$, since, the graph G is undirected, it appears that $\overline{x}(V_0) = \overline{x}(\delta(W))$. Thus, the inequality (9.2) reduces to

$$\sum_{i=1}^{p} \overline{x}(\delta(V_i)) + \overline{x}(\delta(V_0)) = \sum_{u \in W} \overline{x}(\delta(u)) + \overline{x}(\delta(W)) \geq 2 \left\lceil \frac{\gamma |W|}{C} \right\rceil, \tag{9.3}$$

Also, observe that $\left\lceil \frac{\gamma|W|}{C} \right\rceil = \frac{\gamma|W|}{C} + \frac{C - r_W}{C}$, where

$$r_W = \begin{cases} \gamma|W| \pmod C, & \text{if } \gamma|W| \pmod C \neq 0, \\ C, & \text{if } \gamma|W| \pmod C = 0. \end{cases}$$

Hence, Inequality (9.3) reduces to

$$\sum_{u \in W} \overline{x}(\delta(u)) + \overline{x}(\delta(W)) \geq 2\frac{\gamma|W|}{C} + 2\frac{C - r_W}{C}. \tag{9.4}$$

Finally, by replacing $|W|$ by $|V| - |\overline{W}|$ in (9.4), we obtain

$$\sum_{u \in W} \overline{x}(\delta(u)) + \overline{x}(\delta(W)) + \frac{2\gamma}{C}|\overline{W}| \geq 2\frac{\gamma|V|}{C} + 2\frac{C - r_W}{C}. \tag{9.5}$$

Notice that the node set W is such that $s \notin W$.

Now we give the main piece of our separation algorithm. Consider a node $t \in T_D$ and assume that $t \in W$. We construct the graph $\overline{G}_t = (V, \overline{E}_t)$ with

$$\overline{E}_t = E \cup \{su, \text{ for all } u \in V \setminus \{s\}\} \cup \{ut, \text{ for all } u \in V \setminus \{t\}\}.$$

We also give the weight $\overline{z}(e)$ to each edge $e \in \overline{E}_t$, where

- $\overline{z}(e) = \overline{x}(e)$, for every $e \in E$,

- $\overline{z}(su) = \frac{2\gamma}{C}$, for every $u \in V \setminus \{s\}$,
- $\overline{z}(tu) = \overline{x}(\delta_G(u))$, for every $u \in V \setminus \{t\}$.

It is not hard to see that for a node set $W \subseteq V$ with $t \in W$ and $s \notin W$,

$$\overline{z}(\delta_{\overline{G}_t}(W)) = \sum_{u \in W} \overline{x}(\delta_G(u)) + \overline{x}(\delta_G(W)) + \frac{2\gamma}{C}|\overline{W}|.$$

Clearly, $\overline{z}(\delta_{\overline{G}_t}(W))$ corresponds to the left-hand-side of Inequality (9.5). Thus, if the graph G does not contain steiner nodes and the demands are rooted, then there exists a violated Hop-Constrained partition inequality, if and only if there exists a node $t \in T_D$ and a node set $W \subseteq V$ with $t \in W$ and $s \notin W$ such that $\overline{z}(\delta_{\overline{G}_t}(W)) < 2\frac{\gamma|W|}{C} + 2\frac{C-r_W}{C}$.

We are now ready to describe our separation procedure. First, we contract all the steiner nodes of G, if there is any, with a terminal node, in such a way that the reduced graph does not contain any steiner node. Let $\hat{G} = (\hat{V}, \hat{E})$ be the reduced graph thus obtained. For convenience, we will denote by $\hat{V} = \{s, t_1, ..., t_d\}$. For every, $t \in \hat{V} \setminus \{s\}$, we construct the graph $\overline{G}_t$, as defined above, and define the weight vector $\overline{z}$ as described above. Then, we compute a minimum weight cut $\delta_{\overline{G}_t}(W)$, with $t \in W$ and $s \notin W$, and w.r.t. weight $\overline{z}$. As mentionned above, if $\overline{z}(\delta_{\overline{G}_t}(W)) < 2\frac{\gamma|W|}{C} + 2\frac{C-r_W}{C}$, then there is a Hop-Constrained partition inequality violated by $\overline{x}$. If $t_1, ..., t_p$ are the nodes of $\hat{G}$ composing W, we build the partition $\pi = (V_0, V_1, ..., V_p)$, such that

- V_i, for all $i \in \{1, ..., p\}$, is composed of the destination $t_i \in W$ together with all the nodes of G which have been contracted with t_i in $\hat{G}$,
- $V_0 = V \setminus \bigcup_{i=1}^{p} V_i$.

The algorithm stops as soon as it finds a violated Hop-Constrained partition inequality, or when every node $t \in T_D$ has been selected without finding any violated inequality.

Now, we address the case of disjoint demands. As for the case of rooted demands, we start by writing the Hop-Constrained partition inequality under a generic form which is

$$x(\delta(\pi)) \geq \left\lceil \frac{\gamma p}{C} \right\rceil, \tag{9.6}$$

where

$$\begin{cases} \gamma = k + \left\lceil \frac{k}{2} \right\rceil, \text{ and } C = 2 \text{ if } L = 2, \\ \gamma = k(k+2) \text{ and } C = 2(k+1) \text{ if } L = 3. \end{cases}$$

Then, we contract all the steiner nodes with a terminal node so that the reduced graph does not contain any steiner node. After this, we contract together all the source nodes $s_1, ..., s_d$, into a super node s, and let $\hat{G} = (\hat{V}, \hat{E})$ denote the reduced rooted graph thus obtained. Our separation procedure for the Hop-Constrained partition inequalities in the case of disjoint demands then follows the same lines as for the rooted demands. For every node $t \in \hat{V} \setminus \{s\}$, we build the graph $\overline{G}_t$ with the weight vector $\overline{z}$, and compute a minimim weight cut $\delta_{\overline{G}_t}(W)$ such that $t \in W$ and $s \notin W$, w.r.t. weight $\overline{z}$. If $\overline{z}(\delta_{\overline{G}_t}(W)) < 2\frac{\gamma|W|}{C} + 2\frac{C-r_W}{C}$, then there exists a Hop-Constrained partition inequality violated by $\overline{x}$, and this partition is obtained in the same way as for the rooted demands. The algorithm stops as soon as a violated inequality is found

or all the nodes $t \in \hat{V} \setminus \{s\}$ have been explored without finding any violated inequality.

Clearly, all the steps of our procedure can be implemented to run in polynomial time and the whole algorithm can be bounded by $|D| \times$ (Runtime of a maximum Flow). By using the efficient algorithm of Golberg and Tarjan [35] for computing maximum flows in the graphs $\overline{G}_t$, our separation algorithm can be implmented to run in $O(|D||V|^3)$.

As it will turn out in the next sections, our separation procedure for the Hop-Constrained partition inequalities is effective for finding violated inequalities, and this for every $k \geq 2$, $L = 2$ and $L = 3$, and for both rooted and disjoint demands.

9.3 Separation of Steiner-SP-partition inequalities

Now we turn our attention to the separation of the Steiner-SP-partition inequalities. We devise the following heuristic to separate inequalities (5.2). The main idea is to determine a Steiner-partition $\pi = (V_1, ..., V_p)$, $p \geq 3$, of V which induces an outerplanar subgraph of $G(\overline{x})$ and such that the subgraph of G_D (the demand graph) induced by π is connected. By Theorem 5.1, such a partition is a Steiner-SP-partition. Also, the partitions we are looking for are such that $|[V_i, V_{i+1}]| \geq \lceil \frac{k}{2} \rceil$, $i = 1, ..., p$, (modulo p) and for every consecutive sets V_i and V_{i+1}, the edge set $[V_i, V_{i+1}]$ contains at least one edge with fractional value.
The heuristic works as follows. We first contract every pair of nodes t and u, where t is a terminal node, u is a steiner node and $\overline{x}(\delta_{G(\overline{x})}(u) \setminus \{ut\}) \leq \overline{x}(ut)$. The node resulting from that contraction is considered as a terminal. Let $G(\overline{x})' = (V', E')$ be the reduced graph thus obtained.
We look in $G(\overline{x})'$ for a path $\Gamma = \{v'_1v'_2, v'_2v'_3, ..., v'_{p-2}v'_{p-1}\}$, $p \geq 3$, such that $v'_1, ..., v'_{p-1}$ are terminal nodes, $|[v'_i, v'_{i+1}]| \geq \lceil \frac{k}{2} \rceil$ and $[v'_i, v'_{i+1}]$ contains one edge or more with fractional value, for $i = 1, ..., p-2$. The partition $\pi = (V_1, ..., V_p)$, $p \geq 3$, is constructed such that V_i is the node set of G corresponding to v'_i, $i = 1, ..., p-1$, and $V_p = V \setminus (\bigcup_{i=1}^{p-1} V_i)$.
Afterwards, we check by a simple heuristic if the graph $G_\pi(\overline{x})'$ is outerplanar and if the subgraph of G_D induced by π is connected. If this is the case, then, we check if the Steiner-SP-partition inequality induced by π is violated. If not, then we compute from π new partitions $\pi_i = (V_i, V_{i+1}, V \setminus (V_i \cup V_{i+1}))$, $i = 1, ..., p-2$. Clearly, these new partitions are Steiner-partitions and, since they are of size 3, they induce Steiner-SP-partitions. We then check if the Steiner-SP-partition inequality induced by π_i is violated, for $i = 1, ..., p-2$.
If none of these inequalities is violated by $\overline{x}$, we apply again the procedure by looking for another path. In order to avoid the detection of the same path, we label the nodes we met during the search of the previous ones so that they won't be considered again in the search of the new path. This process is iterated until either we find a violated Steiner-SP-partition inequality or all the nodes of V' are labeled. The heuristic can be implemented to run in $O(|E'||V'| + |D|)$ time.

9.4 Separation of double cut inequalities

The separation of double cut inequalities is performed by looking for inequalities of type (6.2) for $L = 2$ and of type (6.3) for $L = 3$ that are violated by the current solution.

The idea of the procedure is to find a partition $\pi = (V_0, ..., V_L, V_{L+1})$, $L \in \{2, 3\}$, of G and an edge set $F \subseteq E$, with $|V_0| = |V_1| = 1$ and $[V_0, V_1] \neq \emptyset$, which induces a double cut, with

$i_0 = 0$, and whose weight is minimum with respect to $\overline{x}$. The procedure works as follows. For all $\{s,t\} \in D$, we compute the st-cut $\delta_G(s)$. If $\overline{x}(\delta_G(s)) = k$, then for every terminal $s' \in R_D$ such that $\overline{x}([s,s']) > 0$ and $\overline{x}(\delta_G(s')) = k$, we compute an L-st-path-cut T of G induced by a partition $\pi = (V_0, ..., V_L, V_{L+1})$ with $V_0 = \{s\}$ and $V_1 = \{s'\}$. For this, we use the correspondance between L-st-path-cuts in G and st-dicuts in the aggregated graph $\widetilde{G}$, given by Lemma 2.1. Since the desired partition π must be such that $V_0 = \{s\}$ and $V_1 = \{s'\}$, we must have $T \cap [s,s'] = \emptyset$ and $\delta_G(s) \setminus [s,s'] \subseteq T$. Thus, any st-dicut of $\widetilde{G}$ corresponding to T must contain arcs corresponding to the edges of $\delta_G(s) \setminus [s,s']$ and no arcs corresponding to the edges of $[s,s']$. Also remark that this st-dicut does not contain any arc of the form (u',u''), $u \in V$ or of the form (t',t), $t \in T_D$. Therefore, to compute an st-dicut of $\widetilde{G}$ corresponding to the desired L-st-path-cut, we start by giving the arcs corresponding to the edges of $[s,s']$ an infinit capacity and removing all the arcs corresponding to the edges of $\delta_G(s) \setminus [s,s']$. Then, we give to every arc of the form (u',u''), $u \in V$ and (t',t), $t \in T_D$, an infinit capacity. Afterwards, we compute a maximum flow between s and t with respect to these capacities. Let $\delta^+_{\widetilde{G}}(\widetilde{W})$ denote the st-dicut thus obtained.
To check that this dicut corresponds to an L-st-path-cut of G, we apply the following procedure. We first remove from G all the edges corresponding to the arcs of $\delta^+_{\widetilde{G}}(\widetilde{W})$. Then, we compute the shortest paths between s and every node of $V \setminus \{s\}$ with respect to length 1 on the remaining edges. Let $l(u)$ denotes the length of a shortest path between s and u, $u \in V \setminus \{s\}$. If $l(t)$ is finite, then $\delta^+_{\widetilde{G}}(\widetilde{W})$ corresponds to an L-st-path-cut of G. In this case, we construct the partition π such that $V_0 = \{s\}$, $V_1 = \{s'\}$, $V_i = \{u \in V \setminus \{s,s',t\} \mid l(u) = i\}$, $i = 2, ..., L$, and $V_{L+1} = V \setminus (\bigcup_{i=0}^{L} V_i)$.
Let $\hat{E}$ be the edge set $[V_1, V_2]$ (resp. $[V_1 \cup V_4, V_2]$) if $L = 2$ (resp. $L = 3$) having a positive value with respect to $\overline{x}$. We choose the edges of F among those of $\hat{E}$ having the highest value and such that $|F|$ and k have different parities. If $|\hat{E}| \geq k-1$, then F consists of the $k-1$ edges having the highest value. If $|\hat{E}| < k-1$ and $|\hat{E}|$ has a parity different from that of k, then we let $F = \hat{E}$. If $|\hat{E}| < k-1$ and $|\hat{E}|$ has the same parity as k, then we let $F = \hat{E} \setminus \{e_0\}$ where e_0 is the edge of $\hat{E}$ having the smallest value.
Finally, we check if the inequality (6.2) (resp. (6.3)) for $L = 2$ (resp. $L = 3$) induced by π and F is violated or not.
We repeat this procedure for every demand $\{s,t\} \in D$, and the violated inequalities found are added to the constraint pool. To compute the maximum flow in $\widetilde{G}$ we use the algorithm of Goldberd and Tarjan [35] which runs in $O(|\widetilde{A}||\widetilde{V}| \log \frac{|\widetilde{V}|^2}{|\widetilde{A}|})$ time. If G is complete and $L = 3$, we have that $|\widetilde{V}| = 2|V| + |S_D| + |T_D|$ and $|\widetilde{A}| = (|V|-1)(|V| + |S_D| + |T_D|)$. Thus, the maximum flow algorithm runs in $O(|V|^3 \log \frac{(2|V|+|S_D|+|T_D|)^2}{(|V|-1)(|V|+|S_D|+|T_D|)})$. To compute the shortest paths in G between s and the other nodes of V, we use the algorithm of Dijkstra [28] which is implemented to run is $O(|V||E| \log(|V|))$ time. As the computation of a cut in the graph G requires at most $|E|$ iterations, our separation procedure runs in $O(|V|^3 \log |V| \frac{(2|V|+|S_D|+|T_D|)^2}{(|V|-1)(|V|+|S_D|+|T_D|)}))$ time, and hence is polynomial. If $L = 2$, the algorithm is also polynomial.
For the case of the separated flow formulations, the procedure is the same except that the computation of the L-st-path-cut, induced by the partition π, is performed using the directed graph $\widetilde{G}_{st}$ associated with the demand $\{s,t\}$. We remove from $\widetilde{G}_{st}$ all the arcs corresponding to the edges of $\delta_G(s) \setminus [s,s']$, and those corresponding to the edges of $[s,s']$ are given an infinit capacity. In the same way, we give an infinit capacity to every arc of the form (u,u'), with $u \in \widetilde{V}_{st}$. Then, we compute a maximum flow between s and t in $\widetilde{G}_{st}$. Also, for this formulations, the algorithm remains polynomial.

9.5 Separation of triple path-cut inequalities

To separate triple path-cut inequalities, we devise a heuristic. This heuristic is based on Theorem 8.3. The procedure is given for $L = 3$. It is similar for $L = 2$. The main idea is to compute, given two demands $\{s, t_1\}$ and $\{s, t_2\}$, a family $\Pi = \{V_0, V_1, V_2, V_3, V_4^1, V_4^2, V_5^1, V_5^2\}$ of node sets from a 3-st_1-path-cut T induced by a partition of the form $(V_0, V_1 \cup V_4^1 \cup V_4^2, V_2, V_3 \cup V_5^1, V_5^2)$. In fact, from this latter partition, one can obtain a whole triple path-cut by fixing the sets V_4^1, V_4^2, V_5^1 and V_5^2. In our procedure, we will look for those triple path-cuts such that $V_4^1 = \emptyset$, $V_4^2 = \{t_2\}$, $V_5^1 = \emptyset$ and $V_5^2 = \{t_1\}$.

The procedure works as follows. For each source $s \in S_D$, we apply the following steps. Let $\{s, t_1\}$ and $\{s, t_2\}$ be two demands associated with s. We first look for a partition $\pi = (V_0', V_1', V_2', V_3', V_4')$ which induces an L-st_1-path-cut of G, denoted by T, and such that $V_0' = \{s\}$ and $t_2 \in V_1'$. For this, we use the correspondance between the L-st_1-path-cuts in G and st_1-dicuts in $\widetilde{G}$. Since $t_2 \in V_1'$ and $V_0' = \{s\}$, we have that $T \cap [s, t_2] = \emptyset$ and any arc of $\widetilde{G}$, corresponding to the edges of $[s, t_2]$, does not appear in an st_1-dicut of $\widetilde{G}$ corresponding to T. Thus, computing T reduces to compute a minimum weight st_1-dicut in $\widetilde{G}$. To do this, we compute a maximum flow in $\widetilde{G}$ between s and t_1 with respect to the following capacities:

- for every arc of $\widetilde{A}([s, t_2])$ or of the form (u', u'') or (t', t), with $u \in N$ and $t \in T_D$, we give an infinit capacity;
- for every arc of $\widetilde{A}(e)$, with $e \in E \setminus [s, t_2]$, we give the capacity $\overline{x}(e)$.

Let $\delta^+_{\widetilde{G}}(\widetilde{W})$ denote the directed cut thus obtained. We check if it corresponds to an L-st_1-path-cut by performing the following steps. First, we remove from G all the edges corresponding to the arcs of $\delta^+_{\widetilde{G}}(\widetilde{W})$ and compute all the shortest paths between s and the other nodes of G with respect to the length 1 on the remaining edges. Let $l(u)$ denote the length of the shortest path between s and u, for all $u \in V \setminus \{s\}$. If $l(t_1)$ is finite, then $\delta^+_{\widetilde{G}}(\widetilde{W})$ corresponds to an L-st_1-path-cut, denoted by T. In this case, we construct the partition π such that $V_0' = \{s\}$, $V_i' = \{u \in V \mid l(u) = i\}$, for $i \in \{1, 2, 3\}$, and for all the nodes $u \in V \setminus \{t_1\}$ such that $l(u) \geq 4$ or $l(u) = +\infty$, we assign them alternatively to V_1' and V_3'. Finally, $V_4' = V \setminus (\bigcup_{i=0}^{3} V_i')$. Note that $t_1 \in V_4'$ as $l(t_1) > 3$ and $t_2 \in V_1'$. Now the family of node sets Π is such that $V_0 = V_0' = \{s\}$, $V_1 = V_1' \setminus \{t_2\}$, $V_2 = V_2', V_3 = V_3'$, $V_4^1 = \emptyset$, $V_4^2 = \{t_2\}$, $V_5^1 = \emptyset$ and $V_5^2 = \{t_1\}$.

Let $\hat{E}$ be the set of edges of $[V_3^2, V_1 \cup V_4^1] \cup [V_3^1, V_4^2]$ having a positive value with respect to $\overline{x}$. We choose the edges of F among those of $\hat{E}$ having the highest value and such that $|F|$ and k have different parities. If $|\hat{E}| \geq k - 1$, then F consists of the $k - 1$ edges having the highest value. If $|\hat{E}| < k - 1$ and $|\hat{E}|$ has a parity different from that of k, then we let $F = \hat{E}$. If $|\hat{E}| < k - 1$ and $|\hat{E}|$ has the same parity as k, then we let $F = \hat{E} \setminus \{e_0\}$ where e_0 is the edge of $\hat{E}$ having the smallest value.

Finally, we check if the triple path-cut inquality induced by Π and F is violated or not.

Our algorithm runs in polynomial time, as it consists, for every pair $\{\{s, t_1\},\{s, t_2\}\}$ of demands, in computing a maximum flow and shortest paths between s and the other nodes of G. In our implementation, we use the algorithm of Goldberg and Tarjan [35] for the maximum flow and the algorithm of Dijkstra [28] for the shortest paths which run in $O(|V|^3 \log \frac{(2|V|+|S_D|+|T_D|)^2}{(|V|-1)(|V|+|S_D|+|T_D|)})$ and $O(|V|^3 \log |V|)$ time, respectively. Thus, the procedure runs in $O(|D|^2(|V|^3 \log |V| \frac{(2|V|+|S_D|+|T_D|)^2}{(|V|-1)(|V|+|S_D|+|T_D|)}))$ time, and thus, is polynomial.

For the case of the separated formulations, the procedure is the same except that the computation of the L-st-path-cut inducing the partition π is performed using the directed graph $\widetilde{G}_{st_1}$

associated with the demand $\{s, t_1\}$. All the arcs corresponding to the edges of $[s, t_2]$ are given an infinit capacity. In the same way, we give an infinit capacity to every arc of the form (u, u'), with $u \in \widetilde{V}_{st}$ and every arc corresponding to an edge $e \in E \setminus [s, t_2]$ is given the capacity $\overline{x}(e)$. Then, we compute a maximum flow between s and t_1 in $\widetilde{G}_{st_1}$.

9.5.1 Separation of aggregated cut inequalities

To separate the aggregated cut inequalities, we consider the inequalities of type (7.4) and (7.8) and devise a heuristic to separate them. In particular, we consider the inequalities of type (7.4) and (7.8) described in the two following lemmas. The separation procedure relies on a special graph (introduced later) defined with respect to $\widetilde{G}$ ($\widetilde{G}_{st}$, $\{s, t\} \in D$) and a fractional solution. Recall that these inequalities are valid for the polytopes kHNDP$_{Ag}(G, D)$ and kHNDP$_{Cu}(G, D)$.

Lemma 9.1 *Consider an inequality $\alpha x + \beta y \geq \gamma$ of type (7.4) induced by a node set family $\Pi = \{\widetilde{W}_1, ..., \widetilde{W}_p\}$, $p \geq 2$, and arc subsets $\widetilde{F}_i^0 \subseteq \delta^+_{\widetilde{G}}(\widetilde{W}_i)$ such that $|\widetilde{F}_i^0| = k - 1$. Let $\widetilde{F} = \bigcup_{i=1}^{p} (\delta^+_{\widetilde{G}}(\widetilde{W}_i) \setminus \widetilde{F}_i^0)$, $\widetilde{F}_2$ be the set of arcs of $\widetilde{A}$ which appear twice in $\widetilde{F}$ and $\widetilde{F}_1$ those which appear once in $\widetilde{F}$. Suppose that for all arc $a \in \widetilde{F}_1$ there is another arc $a' \in \widetilde{F}_1$ which corresponds to the same edge of G as a. Let E_2 be the set of edges of G corresponding to the arcs of $\widetilde{F}_1$. If $(\overline{x}, \overline{y}) \in \mathbb{R}^E \times \mathbb{R}^{\widetilde{A}}$ is a fractional solution of $kHNDP_{Ag}(G, D)$ such that $\overline{y}(\delta^+_{\widetilde{G}}(\widetilde{W}_i)) = k$ and $\overline{y}(a) = 1$, for all $a \in \widetilde{F}_i^0$, $i = 1, ..., p$, then $\alpha x + \beta y \geq \delta$ is violated by $(\overline{x}, \overline{y})$ if and only if*

$$2 \sum_{e \in E_2} \overline{x}(e) - \sum_{a \in \widetilde{F}_1} \overline{y}(a) < 1. \tag{9.7}$$

Proof. First observe that inequality $\alpha x + \beta y \geq \delta$ is violated by $(\overline{x}, \overline{y})$ if and only if

$$\sum_{a \in \widetilde{F}_2} \overline{y}(a) + \sum_{e \in E_2} \overline{x}(e) < \frac{p+1}{2}. \tag{9.8}$$

Since $\overline{y}(\delta^+_{\widetilde{G}}(\widetilde{W}_i)) = k$, $|\widetilde{F}_i^0| = k - 1$ and $\overline{y}(a) = 1$ for all $a \in \widetilde{F}_i^0$, we have that

$$\overline{y}(\delta^+_{\widetilde{G}}(\widetilde{W}_i) \setminus \widetilde{F}_i^0) = 1 \text{ for } i = 1, ..., p.$$

Thus, $\sum_{i=1}^{p} \overline{y}(\delta^+_{\widetilde{G}}(\widetilde{W}_i) \setminus \widetilde{F}_i^0) = 2 \sum_{a \in \widetilde{F}_2} \overline{y}(a) + \sum_{a \in \widetilde{F}_1} \overline{y}(a) = p$ and hence,

$$\sum_{a \in \widetilde{F}_2} \overline{y}(a) = \frac{p}{2} - \frac{1}{2} \sum_{a \in \widetilde{F}_1} \overline{y}(a). \tag{9.9}$$

From (9.8) and (9.9), we get

$$p - \sum_{a \in \widetilde{F}_1} \overline{y}(a) + 2 \sum_{e \in E_2} \overline{x}(e) < p + 1.$$

and the result follows. □

Lemma 9.2 *Consider an inequality* $\alpha x + \sum_{\{s,t\}\in D} y_{st}\beta_{st} \geq \gamma$ *of type (7.8) induced by a family of node sets* $\Pi = \{\widetilde{W}_1^{s_1t_1}, ..., \widetilde{W}_{p_1}^{s_1t_1}, ..., \widetilde{W}_1^{s_qt_q}, ..., \widetilde{W}_{p_q}^{s_qt_q}\}$, *with* $p_i \geq 1$, *for* $i = 1, ..., q$, *and* $p = \sum_{i=1}^{q} p_i \geq 2$, *and arc subsets* $\widetilde{F}_j^{s_it_i,0} \subseteq \delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}_j^{s_it_i})$ *such that* $|\widetilde{F}_j^{s_it_i,0}| = k-1$, $j = 1, ..., p_i$, $i = 1, ..., q$. *Let* $\widetilde{F}^{s_it_i} = \bigcup_{j=1}^{p_i}[\delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}_j^{s_it_i}) \setminus \widetilde{F}_j^{s_it_i,0}]$, $i = 1, ..., q$. *Also let* $\widetilde{F}^{s_it_i,2}$ *be the set of arcs of* $\widetilde{A}_{s_it_i}$ *which appear twice in* $\widetilde{F}^{s_it_i}$ *and* $\widetilde{F}^{s_it_i,1}$ *those which appear once in* $\widetilde{F}^{s_it_i}$. *Suppose that for all arc* $a \in \widetilde{F}^{s_it_i,1}$, *there exists a unique arc* $a' \in \widetilde{F}^{s_{i'}t_{i'},1}$ *for some* $i' \in \{1, ..., q\}$ *which corresponds to the same edge of* G *as* a. *Let* E_2 *be the set of edges of* G *corresponding to these arcs.*
If $(\overline{x}, \overline{y}_{s_1t_1}, ..., \overline{y}_{s_dt_d})$ *is a fractional solution of* $kHNDP_{Cu}(G, D)$ *such that* $\overline{y}_{s_it_i}(\delta^+_{\widetilde{G}_{s_it_i}}(\widetilde{W}_{s_it_i})) = k$ *and* $\overline{y}_{s_it_i}(a) = 1$, *for all* $a \in \widetilde{F}^{s_it_i,0}$, $i = 1, ..., q$, *then* $\alpha x + \sum_{\{s,t\}\in D} y_{st}\beta_{st} \geq \gamma$ *is violated by* $(\overline{x}, \overline{y}_{s_1t_1}, ..., \overline{y}_{s_dt_d})$ *if and only if*

$$2\sum_{e\in E_2} \overline{x}(e) - \sum_{i=1}^{q} \sum_{a\in \widetilde{F}^{s_it_i}} \overline{y}_{s_it_i}(a) < 1. \tag{9.10}$$

Proof. Similar to the proof of Lemma 9.1. □

In the following, we are going to discuss the separation of the aggregated cut inequalities (7.4) for kHNDP$_{Ag}$. After that, we will describe the separation procedure for the aggregated cut inequalities (7.8) related to kHNDP$_{Cu}$.
To this end, we introduce an undirected graph, denoted by $H(\overline{x}, \overline{y})$, obtained from $\widetilde{G}$ and defined with respect to $(\overline{x}, \overline{y})$. As we will see, the main property of this graph is that there is a matching between some particular cycles of $H(\overline{x}, \overline{y})$ and inequalities of type (7.4), described as in Lemma 9.1. The graph $H(\overline{x}, \overline{y})$ is obtained as follows.
For each arc of $\widetilde{A}$ having a fractional value with respect to $\overline{y}$, we add a node in $H(\overline{x}, \overline{y})$. For convenience, we will denote by a the node of $H(\overline{x}, \overline{y})$ corresponding to an arc a of $\widetilde{G}$. We add an edge in $H(\overline{x}, \overline{y})$ between two nodes a_1 and a_2 if one of the conditions below is satisfied.

1. There exists an st-dicut of $\widetilde{G}(\overline{y})$, say $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$, for some $\{s, t\} \in D$, which contains a_1 and a_2, and such that $\overline{y}(\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})) = k$, $|\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \cap \widetilde{A}_1(\overline{y})| = k-1$ and $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \cap \widetilde{A}_f(\overline{y}) = \{a_1, a_2\}$.

2. The arcs a_1 and a_2 correspond to the same edge of G.

The edges added by Condition 1 will be said of *type 1* and those added by Condition 2 will be said of *type 2*. As we will see later, both edges of type 1 and 2 can be find in polynomial time. Figures 16 and 17 give respectively the support graph $\widetilde{G}(\overline{y})$ of a fractional solution $(\overline{x}, \overline{y})$ of kHNDP$_{Ag}(G, D)$ and the graph $H(\overline{x}, \overline{y})$ associated with that solution.

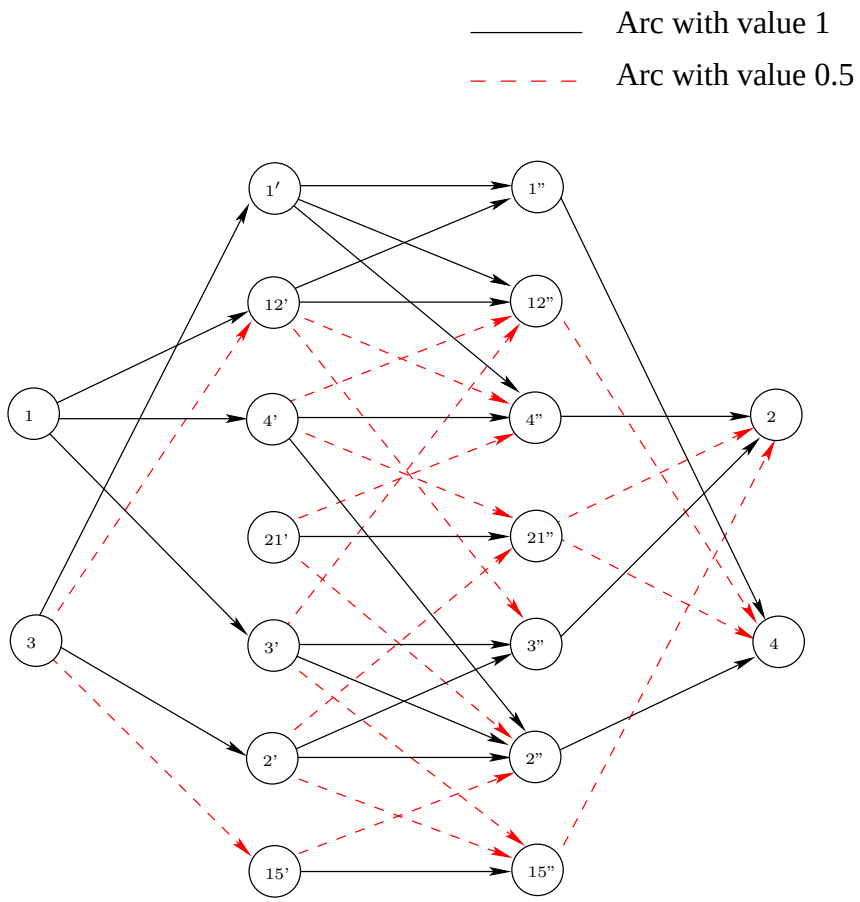

Figure 16: The support graph $\widetilde{G}(\overline{y})$ of a fractional solution $(\overline{x}, \overline{y})$ for $L = 3$ and $k = 3$

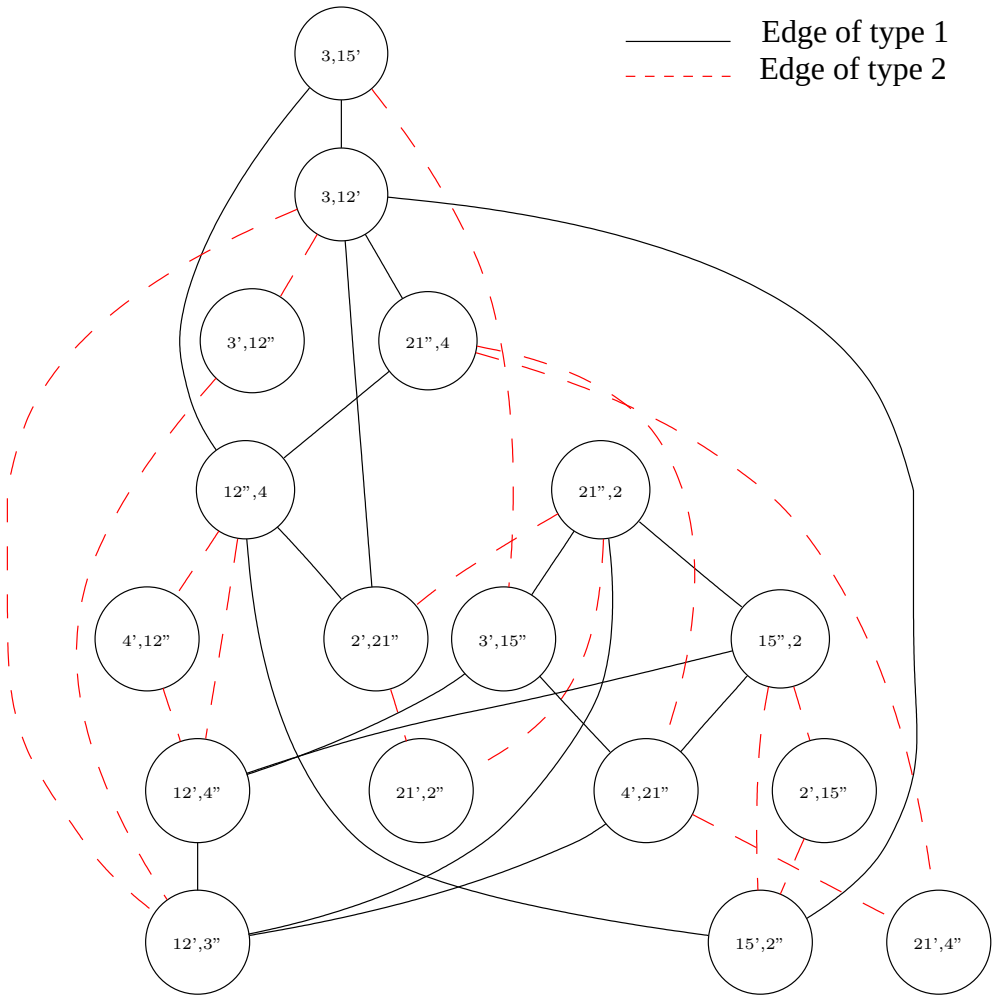

Figure 17: Graph $H(\overline{x}, \overline{y})$ obtained from $\widetilde{G}(\overline{y})$

Note that in the case where there is an edge of type 1 in $H(\overline{x},\overline{y})$ between two nodes a_1 and a_2, we have that $\overline{y}(a_1)+\overline{y}(a_2)=1$. Also, if there is an edge of type 2 between two nodes a_1 and a_2, then $\overline{x}(e)>0$ where e is the edge of G corresponding to a_1 and a_2. Also it is not hard to see that, if in $H(\overline{x},\overline{y})$ there are two edges of type 2 of the form a_1a_2 and a_2a_3, then there is also an edge of type 2 between a_1 and a_3 (a_1, a_2 and a_3 form a triangle).
Now we give the main property of $H(\overline{x},\overline{y})$.

Lemma 9.3 *Let $C=\{a_1a_2,a_2a_3,...,a_{|C|}a_1\}$ be a cycle of $H(\overline{x},\overline{y})$ and $\{a_{i_1}a_{j_1},...,a_{i_p}a_{j_p}\}$ the set of edges of C of type 1. Also, let V_1 be the set of nodes of C incident to two consecutive edges of type 1. Suppose that $p\geq 2$ and that C does not contain two consecutive edges of type 2. Then, C yields an inequality of type (7.4) defined by $\Pi=\{\widetilde{W}_1,...,\widetilde{W}_p\}$ and $\widetilde{F}^0_r=\delta_{\widetilde{G}(\overline{y})}(\widetilde{W}_r)\setminus\{a_{i_r},a_{j_r}\}$, $r=1,...,p$, where $\widetilde{W}_r$ is the node set of $\widetilde{G}$ associated with the edge $a_{i_r}a_{j_r}$ in $H(\overline{x},\overline{y})$.*

Proof. First observe that the arcs of $\widetilde{A}(\overline{y})$ which appear twice in $\widetilde{F}=\bigcup_{i=1}^{p}[\delta_{\widetilde{G}(\overline{y})}(\widetilde{W}_r)\setminus\widetilde{F}^0_i]$ are those of $\widetilde{G}(\overline{y})$ corresponding to the nodes of V_1, while the arcs which appear once in $\widetilde{F}$ are those of $\widetilde{A}(\overline{y})$ corresponding to the nodes of $\{a_1,...,a_{|C|}\}\setminus V_1$. Thus we let $\widetilde{F}_2$ and $\widetilde{F}_1$ be these two sets of arcs, respectively. Since every node $a\in\{a_1,...,a_{|C|}\}\setminus V_1$ is incident to one edge of C of type 2, say aa', the arcs a and a' are in $\widetilde{F}_1$ and correspond to the same edge of G. Thus, the aggregated cut inequality associated with this configuration can be written as

$$\sum_{a\in\widetilde{F}_2}y(a)+\sum_{e\in E_2}x(e)\geq\left\lceil\frac{p}{2}\right\rceil,$$

where E_2 is the edge set of G corresponding to the arcs of $\widetilde{F}_1$. □

To illustrate that lemma, on Figure 17, the cycle

$$C=\{(3,15')(3,12'),(3,12')(21'',4),(21'',4)(4',21''),(4',21'')(3',15''),(3',15'')(3,15')\}$$

contains three edges of type 1, $(3,15')(3,12')$, $(3,12')(21'',4)$ and $(4',21'')(3',15'')$, and two edges of type 2, $(21'',4)(4',21'')$ and $(3',15'')(3,15')$, that are not incident. One can see on Figure 16 that the node sets $\widetilde{W}_1=\{3\}$, $\widetilde{W}_2=\{3,2',15',21'',3'',2'',15'',2\}$ and $\widetilde{W}_3=\{1,12',3',4',1'',12'',4'',3'',2'',4\}$ induce two $3-4$-dicuts and one $1-2$-dicut of $\widetilde{G}(\overline{y})$, and that these dicuts contain respectively the pairs of arcs $\{(3,15'),(3,12')\}$, $\{(3,12'),(21'',4)\}$ and $\{(4',21''),(3',15'')\}$. Moreover, they are such that $\overline{y}(\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}_i))=k$ and $|\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}_i)\cap\widetilde{A}_1(\overline{y})|=k-1$, $i\in\{1,2,3\}$. Finally, it obviously follows that $\Pi=\{\widetilde{W}_1,\widetilde{W}_2,\widetilde{W}_3\}$ and $\widetilde{F}^0_1=\{(3,1'),(3,2')\}$, $\widetilde{F}^0_2=\{(3,1'),(2'',4)\}$ and $\widetilde{F}^0_3=\{(4'',2),(3'',2)\}$ induce an aggregated cut inequality of type (7.4). Furthermore, this inequality is violated by $(\overline{x},\overline{y})$.
Before describing the construction procedure for $H(\overline{x},\overline{y})$, we give the following lemma.

Lemma 9.4 *Let $(\overline{x},\overline{y})$ be a fractional solution of $kHNDP_{Ag}(G,D)$, and let a_1 and a_2 be two arcs of $\widetilde{G}$ with fractional values and $\{s,t\}\in D$. If there exists a minimum weight st-dicut of $\widetilde{G}(\overline{y})$, say $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$, such that $\{a_1,a_2\}\subseteq\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ and $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})\setminus\{a_1,a_2\}\subseteq\widetilde{A}_1(\overline{y})$, then $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ can be considered in such a way that every arc $a\in\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})\setminus\{a_1,a_2\}$ is either in $\delta^+_{\widetilde{G}(\overline{y})}(s)$ or in $\delta^-_{\widetilde{G}(\overline{y})}(t)\setminus[t',t]_{\widetilde{G}(\overline{y})}$.*

Proof. Let $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ be a minimum weight st-dicut of $\widetilde{G}(\overline{y})$ containing a_1 and a_2 and such that $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \setminus \{a_1, a_2\} \subseteq \widetilde{A}_1(\overline{y})$. Suppose also that there is an arc $a \in \delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \setminus \{a_1, a_2\}$ which is not in $\delta^+_{\widetilde{G}(\overline{y})}(s) \cup [\delta^-_{\widetilde{G}(\overline{y})}(t) \setminus \{(t', t)\}]$. Hence, a is either of the form (u', v''), with $u' \in N'$, $v'' \in N''$ and u and v may be the same, or of the form (t', t). If $a = (u', v'')$, then $u' \in \widetilde{W}$ and the node set $\widetilde{W}' = \widetilde{W} \setminus \{u'\}$ induces an st-dicut. Since $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ is a minimum weight st-dicut, $[s, u']_{\widetilde{G}(\overline{y})} \neq \emptyset$ and therefore, $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}') = (\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \setminus \{(u', v'')\}) \cup \{(s, u')\}$. Since $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ is of minimum weight with respect to $\overline{y}$, we have that $\overline{y}(s, u') \geq \overline{y}(u', v'')$. As $\overline{y}(u', v'') = 1$, we also have that $\overline{y}(s, u') = 1$ and that $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}')$ is a minimum weight st-dicut. If $a = (t', t)$, then since $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ is of minimum weight in $\widetilde{G}(\overline{y})$, there is an arc of the form (s, t'). Thus, $\widetilde{W}' = \widetilde{W} \setminus \{t'\}$ induces an st-dicut of $\widetilde{G}(\overline{y})$. Moreover, as the weight of $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ is minimum with respect to $\overline{y}$, we have that $\overline{y}(s, t') \geq \overline{y}(t', t) = 1$. Hence, $\overline{y}(s, t') = 1$ and $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}')$ is also of minimum weight. By repeating this operation until $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ does not contain any arc of the form (u', v'') or (t', t), we obtain a minimum weight st-dicut of $\widetilde{G}(\overline{y})$ which contains a_1 and a_2, such that $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \setminus \{a_1, a_2\} \subseteq \widetilde{A}_1(\overline{y})$ and such that every arc of $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \setminus \{a_1, a_2\}$ is either in $\delta^+_{\widetilde{G}(\overline{y})}(s)$ or in $\delta^-_{\widetilde{G}(\overline{y})}(t) \setminus [t', t]_{\widetilde{G}(\overline{y})}$, which ends the proof of the Lemma. □

A consequence of Lemma 9.4 is that an st-dicut $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ of $\widetilde{G}(\overline{y})$ containing two arcs a_1 and a_2 with fractional values, such that $\overline{y}(\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})) = k$ and $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \cap \widetilde{A}_f(\overline{y}) = \{a_1, a_2\}$ can be obtained by computing st-dicuts of $\widetilde{G}(\overline{y})$ containing a_1 and a_2 and such that $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \setminus \{a_1, a_2\} \subseteq \delta^+_{\widetilde{G}(\overline{y})}(s) \cup (\delta^-_{\widetilde{G}(\overline{y})}(t) \setminus \{(t', t)\})$.

The construction of the graph $H(\overline{x}, \overline{y})$ is performed by computing first the edges of type 2. For every pair of arcs $(a, a') \in \widetilde{A}(\overline{y}) \times \widetilde{A}(\overline{y})$, corresponding to the same edge of E and having a fractional value, we add an edge of type 2 between the corresponding nodes in $H(\overline{x}, \overline{y})$. To compute the edges of type 1, we use a procedure based on Lemma 9.4. The idea is to compute a maximum flow in $\widetilde{G}(\overline{y})$ with respect to appropriate capacities separating s and t. Given two arcs a_1 and a_2 such that $\overline{y}(a_1) + \overline{y}(a_2) = 1$ and a pair $\{s, t\} \in D$, we first give 0 as capacity to a_1 and a_2. Then, we give an infinit capacity to every other arc of $\widetilde{G}(\overline{y})$ having a fractional value. This ensures that a_1 and a_2 are the only arcs of fractional values present in the st-dicut we will obtain. We give an infinit capacity to every arc of $\delta^+_{\widetilde{G}(\overline{y})}(s)$ and $\delta^-_{\widetilde{G}(\overline{y})}(t)$ indicent to a_1 and a_2 and having value 1. We also give an infinit capacity to every arc of $[t', t]_{\widetilde{G}(\overline{y})}$. For all other arc a, we give $\overline{y}(a)$ as capacity (note that for these arcs, $\overline{y}(a) = 1$). Then, we compute a maximum flow between s and t with respect to these capacities. Let $\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})$ denote the st-dicut thus obtained. By Lemma 9.4, we have that $\delta^+_{\widetilde{G}(\overline{y})} \setminus \{a_1, a_2\} \subseteq \widetilde{A}_1(\overline{y})$. We then check if $\overline{y}(\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W})) = k$ and $|\delta^+_{\widetilde{G}(\overline{y})}(\widetilde{W}) \setminus \{a_1, a_2\}| = k - 1$. If this is the case, then we add an edge of type 1 between the nodes of $H(\overline{x}, \overline{y})$ corresponding to a_1 and a_2. We repeat this procedure for all pairs of arcs (a_1, a_2) having fractional value and such that $\overline{y}(a_1) + \overline{y}(a_2) = 1$, and for all demand $\{s, t\} \in D$.

Now we describe the separation procedure of the aggregated cut inequalities. The procedure is based on Lemma 9.1. Thus we generate inequalities of type (7.4) which satisfy the conditions of that lemma. First, we compute $H(\overline{x}, \overline{y})$ as described above. Then we compute one or more

cycles of $H(\overline{x},\overline{y})$ which contain an odd number of edges of type 1 and which do not contain two consecutive edges of type 2. By Lemma 9.3, every cycle satisfying these conditions yields an aggregated cut inequality of type (7.4). We then check if for each inequality thus obtained, $(\overline{x},\overline{y})$ satisfies inequality (9.7). If this is the case, then by Lemma 9.1, this inequality is violated by $(\overline{x},\overline{y})$ and added to the set of violated inequalities. If no cycle is found or if for every inequality of type (7.4) obtained, $(\overline{x},\overline{y})$ does not satisfy inequality (9.7), then the procedure ends with failure.

To detect cycles of $H(\overline{x},\overline{y})$ satisfying the conditions of Lemma 9.3, we use a procedure in which we compute shortest paths in an auxiliary graph obtained from $H(\overline{x},\overline{y})$. Let H_b be the undirected graph obtained as follows. The node set of H_b is composed of two copies, denoted by V_b' and V_b'', of the node set of $H(\overline{x},\overline{y})$. The copies of a node a of $H(\overline{x},\overline{y})$ are denoted by a' and a'' with $a' \in V_b'$ and $a'' \in V_b''$. For every edge a_1a_2 of $H(\overline{x},\overline{y})$ of type 1, we add in H_b two edges of the form $a_1'a_2''$ and $a_2'a_1''$ and give them 1 as length. For every edge a_1a_2 of $H(\overline{x},\overline{y})$ of type 2, we add in H_b two edges of the form $a_1'a_2'$ and $a_1''a_2''$ and give them a length M sufficiently large. Figure 18 shows an example of graph H_b obtained from a subgraph of $H(\overline{x},\overline{y})$ given in Figure 17. It is not hard to see that a path between two nodes a' and a'' of H_b corresponds to a cycle of $H(\overline{x},\overline{y})$ containing node a and an odd number of edges of type 1, and does not contain two consecutive edges of type 2, and vice versa.

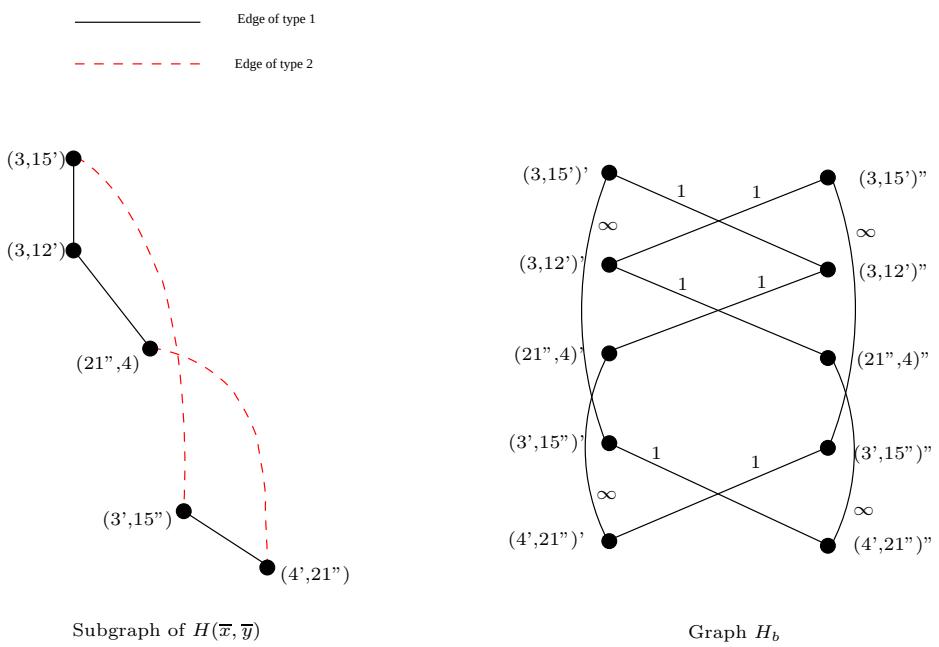

Figure 18: Graph H_b obtained from a subgraph of $H(\overline{x},\overline{y})$

For our separation procedure, we compute the shortest paths between each pair of nodes (a', a'') of H_b, for every node a of $H(\overline{x},\overline{y})$.

Now we turn to the aggregated cut inequalities for the Cut formulation. The separation procedure for these inequalities is similar to that described above for kHNDP$_{Ag}$. Given a fractional solution $(\overline{x},\overline{y}_{s_1t_1},...,\overline{y}_{s_dt_d})$ of kHNDP$_{Cu}(G,D)$, we construct the graph $H(\overline{x},\overline{y}_{s_1t_1},...,\overline{y}_{s_dt_d})$ in a similar way as $H(\overline{x},\overline{y})$, that is for all $\{s,t\} \in D$, and for every arc $a \in \widetilde{A}^f_{st}(\overline{y}_{st})$ we associate a node in $H(\overline{x},\overline{y}_{s_1t_1},...,\overline{y}_{s_dt_d})$. We add an edge, called of type 1, between two nodes a_1

and a_2 if they belong to the same graph $\widetilde{G}_{st}$, $\overline{y}_{st}(a_1) + \overline{y}_{st}(a_2) = 1$ and there exists an st-dicut $\delta^+_{\widetilde{G}_{st}(\overline{y}_{st})}(\widetilde{W})$ containing a_1 and a_2 and such that $\delta^+_{\widetilde{G}_{st}(\overline{y}_{st})}(\widetilde{W}) \cap \widetilde{A}^f_{st}(\overline{y}_{st}) = \{a_1, a_2\}$ and $|(\delta^+_{\widetilde{G}_{st}(\overline{y}_{st})}(\widetilde{W}) \setminus \{a_1, a_2\}) \cap \widetilde{A}^1_{st}(\overline{y}_{st})| = k - 1$. We also add an edge, called of type 2, between two nodes $a_1 \in \widetilde{A}^f_{s_i t_i}(\overline{y}_{s_i t_i})$ and $a_2 \in \widetilde{A}^f_{s_{i'} t_{i'}}(\overline{y}_{s_{i'} t_{i'}})$ if the arcs a_1 and a_2 correspond to the same edge of G.
The st-dicut $\delta^+_{\widetilde{G}_{st}(\overline{y}_{st})}(\widetilde{W})$ used to set edges of type 1 can be computed with the procedure used for kHNDP$_{Ag}$. As before, every cycle of $H(\overline{x}, \overline{y}_{s_1 t_1}, ..., \overline{y}_{s_d t_d})$ which contains an odd number of edges of type 1 and which does not contain two consecutive edges of type 2 yields an inequality of type (7.8). These cycles are computed by looking for shortest paths in a graph H_b obtained in a similar way as for kHNDP$_{Ag}$. Finally, for each cycle thus obtained, we check if $(\overline{x}, \overline{y}_{s_1 t_1}, ..., \overline{y}_{s_d t_d})$ satisfies or not inequality (9.10) with respect to the sets E_2 and $\widetilde{F}^{s_i t_i, 1}$ obtained from that cycle. If this is the case, then by Lemma 9.2, the corresponding inequality of type (7.8) is violated by $(\overline{x}, \overline{y}_{s_1 t_1}, ..., \overline{y}_{s_d t_d})$ and hence added to the set of violated inequalities.
The procedure is the same for all the formulations and is described below.

10 Concluding remarks

In this paper, we have studied the kHNDP and particularly focused on the polytope associated with the problem. We have introduced several classes of inequalities that are valid for the polytope associated with the natural formulation, as well as inequalities valid for an extended formulation of the kHNDP. We have also investigated conditions for the various inequalities to define facets. Then, we have presented separation algorithms for each class of inequalities.
The next step of this study is to devise a Branch-and-Cut algorithm using the inequalities presented in this report, and conduct a computational study for assessing the efficiency of these inequaliyties. The litterature shows that some computational effort must be done for efficiently solving the kHNDP, espcially for large scale instances.

Finally, it appears that the polytope associated with the natural formulation of the kHNDP remains not well known when $L \geq 4$. To the best of our knowledge, except Huygens and Mahjoub [46] who explicitly described the inequalities which are necessary in the natural formulation of the kHNDP for $L = 4$ and $k = 2$, no further class of valid inequalities is known for that formulation when $L \geq 4$ and $k \geq 2$. Thus, identifiying new classes of facet-defining inequalities for any $L \geq 4$ could represent an interesting challenge from a theoretical point of view but also for practical purposes, as these inequalities may help in efficiently solving the kHNDP for any $L \geq 4$.

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