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► **To cite this version:**

Elena Di Bernardino, Thomas Laloë, Cambyse Pakzad. Estimation of extreme L1 -multivariate expectiles with functional covariates. 2023. hal-04051376

HAL Id: hal-04051376

<https://hal.science/hal-04051376>

Preprint submitted on 29 Mar 2023

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Estimation of extreme L^1 -multivariate expectiles with functional covariates

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Abstract

The present article is devoted to the semi-parametric estimation of multivariate expectiles for extreme levels. The considered multivariate risk measures also include the possible conditioning with respect to a functional covariate, belonging to an infinite-dimensional space. By using the first order optimality condition, we interpret these expectiles as solutions of a multidimensional nonlinear optimum problem. Then the inference is based on a minimization algorithm of gradient descent type, coupled with consistent kernel estimations of our key statistical quantities such as conditional quantiles, conditional tail index and conditional tail dependence functions. The method is valid for equivalently heavy-tailed marginals and under a multivariate regular variation condition on the underlying unknown random vector with arbitrary dependence structure. Our main result establishes the consistency in probability of the optimum approximated solution vectors with a speed rate. This allows us to estimate the global computational cost of the whole procedure according to the data sample size.

Keywords: Dependence, Extreme value theory, Multivariate Expectiles, Multivariate Regular Variation, Optimization, Multivariate Risk measures
2020 MSC: 60G70, 62H12, 90C53, 91G70

1. Introduction

Risk measurement theory is an active branch of research with numerous applications in the fields of finance, insurance, economics and for the environment (hydrology, geology, . . .).

To study the extreme risk of a random phenomena, e.g. of a big loss on a financial position, the most popular way is to estimate quantiles, also known as Value-at-Risk (VaR), at high level. But it has been argued that it lacks of sub-additivity property [1], i.e. the quantile of the sum of two portfolios can exceed the sum of the quantiles of the two portfolios; which is contradiction with the principle of diversification. Consequently, the quantile is an incoherent risk measure in view of [2]. Another drawback is that it relies on the frequency of tail events and not on their real magnitudes which is precisely what one would like to know. On the other hand, the second most famous risk measure, the expected shortfall, is not elicitable in the sense of [23], meaning that it is not defined as the minimization of the expectation of some score function. This is though a desirable property since it allows backtesting procedure: one periodically compares the expected risk measure with the actual value of the variable of interest in order to evaluate the accuracy of the forecasting methodology, see [5]. The univariate expectiles are then introduced by [30] and turn out to be the only law invariant risk measures which are elicitable by construction and coherent for a threshold level range, see [6, 38]. Concerning the univariate expectile, we also refer to [4, 10, 11]. Conversely to quantiles, they depend on both the realisations and probabilities of the underlying random variable. In economic terms, they may be interpreted as ratios of expected gain/loss which found to be recognized in portfolio management, see [4], and represent the quantity of money to inject in a position to reach a prescribed ratio gain/loss.

A fundamental question for potential practice is the statistical estimation in the extreme regime of the risk measures at our disposal. In this context, the extreme regime is modelled by a risk level tending towards zero or one, and an assumption of heavy-tails on the underlying distributions, typically of Pareto-type, which best captures rare phenomena. The proper mathematical framework is the regular variation notion. Besides, it often happens in practical

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applications that the observations are recorded along with auxiliary information represented by a random covariate. Then, one would preferably take advantage of the extra information by focusing on the conditional extremes. This line of research was carried by [20, 21] when the covariate is a random vector. A valuable improvement is [22] for univariate extreme expectiles when the covariate belongs to an infinite-dimensional or functional space.

Furthermore, in several situations, one needs to simultaneously manage risks over different positions, requiring a multivariate version of risk measures which would take the underlying dependence structure into account. Recently, several possible multivariate extensions of expectiles emerged in the literature such as geometric expectiles in [24] and L^p -expectiles in [28]. In the present work, we focus on the specific case of the multivariate L^1 -expectiles of [28] in the extreme regime, which we abbreviate with the notation MEEs for *Multivariate Extreme Expectiles*. Herein, in [28] the authors construct Σ -expectiles as another extension possibility based on correlation matrices Σ that reduces to the L^1 -version if $\Sigma \equiv \mathbf{1}$. To estimate multivariate expectiles, they exploit in [28] the elicibility by focusing on the first order optimality condition, namely MEEs are points in the d -dimensional Euclidean space for which the score gradient vanishes. The latter only involving tail expectations through positive and negative parts, the authors achieve the estimation by means of Robbins-Monro's stochastic approximation method for moderate levels of risk. Conversely, in the extreme regime, the same authors use in [29] classic regular variation tools to express MEEs ratios as solutions of a system of coupled nonlinear equations. In addition, an estimation procedure is given for L^1 -MEEs with equivalent regularly varying marginal tails, provided the tail dependence is either comonotonic or asymptotically independent. Then, the approach in [29] works well for specific dependence structures.

In this paper, we address the estimation of the L^1 -MEEs when a conditional covariate lying in a possibly infinite-dimensional space is available. We assume the equivalent regularly varying marginal tails hypothesis and that the underlying dependence structure and the marginal distributions are unknown. Again, the first order optimality condition yields a system of equations for which functional MEEs ratios are solutions. Equivalently, functional MEEs may be seen as roots of a certain loss function which can be turned into an optimization problem. Crucial quantities are involved regarding the tail behaviour such as the conditional tail index and the conditional tail dependence function. Inspired by [3], plugging their empirical counterparts in the optimization problem results in the approximated loss function. Finally, we propose to apply a BFGS-gradient method. In our main result (see Theorem 5.1), we prove the consistency in probability with rate of the approximated loss function and of the associated optimum solution in this conditional functional setting. Contrarily to [3], in the present work, a special attention is devoted to the rates of convergence in the approximation of the underlying optimization problem. It allows to link the loss function approximation quality with the steps of the used gradient descent algorithm. As a result, we explicitly provide the speed rate at which the approximated optimum converges to its theoretical value, which essentially quantifies the estimation quality according to the sample size.

The paper is organized as follows. In Section 2, we present some necessary notation and the setting of our model. We introduce in Section 3 the formal definition of functional multivariate L^1 -expectiles and how they can be theoretically related to a non-linear optimum problem. We subsequently develop in Section 4 the statistical tools of the present paper. After introducing the different estimators, we construct the associated approximated optimum problem. In Section 5 we present the required hypotheses and we state our main result (see Theorem 5.1) about the convergence with rate of the approximated optimum problem to the theoretical one. We outline the sketch of the proof of our main result and we give a crucial intermediate convergence result on the loss function and its gradient is also given (see Proposition 5.1). We devote the last part of Section 5 to a discussion about the required hypotheses. Section 6 is devoted to the proof of the main result. Auxiliary proofs and supplementary lemmas are postponed to Section 7. Further material about the second order regular variation condition is provided in Appendix A.

2. Notation and preliminaries

In this work, we consider a Polish space $(E, \|\cdot\|_E)$ endowed with its Borel σ -algebra and a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $2 \leq d < +\infty$ and a random pair $(X, Y) \in \mathbb{R}^d \times E$, with $X = (X_j)_{1 \leq j \leq d}$, defined on Ω such that $X \in (L^1(\Omega))^d$. On account of [26, Theorem 3.2], such topological features on E ensures the existence of regular conditional probabilities $\mathbb{P}_y(\cdot) := \mathbb{P}(\cdot | Y = y)$ on \mathcal{A} for \mathbb{P}_Y -almost all $y \in E$, where $\mathbb{P}_Y = \mathbb{P} \circ Y^{-1}$ is the pushforward measure. As a result, we

may denote the conditional cumulative distribution function (cdf) of X given $Y = y$ by $F_{X,y}(\mathbf{x}) := \mathbb{P}_y(\{X \leq \mathbf{x}\}) = \mathbb{P}(X \leq \mathbf{x} | Y = y)$, for $\mathbf{x} \in \mathbb{R}^d$ and $y \in E$, where the inequality is to be understood component-wise on \mathbb{R}^d .

We denote the conditional marginal distributions, which we suppose to be continuous, by

$$x_j \mapsto F_{j,y}(x_j) := F_{X,y}(+\infty, \dots, x_j, +\infty, \dots), \quad 1 \leq j \leq d,$$

and for $1 \leq j < k \leq d$, the conditional bivariate marginal distributions by

$$(x_j, x_k) \mapsto F_{j,k,y}(x_j, x_k) := F_{X,y}(+\infty, \dots, x_j, +\infty, \dots, x_k, +\infty, \dots). \quad (1)$$

We also introduce the conditional marginal quantile function of X_j given $Y = y$ and its conditional marginal quantile tail function by, respectively,

$$\begin{aligned} q_{j,y}(\alpha) &:= \inf \{x > 0, F_{j,y}(x) \geq \alpha\}, \quad \alpha \in (0, 1), \\ U_{j,y}(x) &:= q_{j,y}(1 - x^{-1}) = \inf \{t \in \mathbb{R}, F_{j,y}(t) \geq 1 - x^{-1}\}, \quad x > 0. \end{aligned} \quad (2)$$

Under the hypothesis of continuity of conditional marginal distributions and by Sklar's Theorem (1959) (see [35]), there exists a unique copula C_y such that, for $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$,

$$C_y(\mathbf{u}) = F_{X,y}(F_{1,y}^{-1}(u_1), \dots, F_{d,y}^{-1}(u_d)) = \mathbb{P}\left(\bigcap_{j=1}^d \{F_{j,y}(X_j) \leq u_j\} | Y = y\right),$$

for which we assume the regularity condition from [19, 34]; namely, for each $j = 1, \dots, d$,

$$\mathbf{u} \mapsto \frac{\partial C_y}{\partial u_j}(\mathbf{u}) \text{ exists and is continuous on the set } \{\mathbf{u} \in [0, 1]^d : 0 < u_j < 1\}. \quad (3)$$

For $y \in E$ and $\mathbf{x} \in \mathbb{R}_+^d$, we define the conditional stable tail dependence function

$$L_y(\mathbf{x}) := \lim_{t \downarrow 0} t^{-1} \mathbb{P}\left(\bigcup_{j=1}^d \{1 - F_{j,y}(X_j) \leq tx_j\} | Y = y\right) = \lim_{t \downarrow 0} t^{-1} (1 - C_y(\mathbf{1} - t\mathbf{x})). \quad (4)$$

For any $y \in E$ and $\mathbf{x} \in [0, +\infty]^d \setminus \{(+\infty, \dots, +\infty)\}$, the associated conditional upper tail dependence function is given by

$$\lambda_y(\mathbf{x}) = \lim_{t \downarrow 0} t^{-1} \mathbb{P}\left(\bigcap_{j=1}^d \{1 - F_{j,y}(X_j) \leq tx_j\} | Y = y\right) = \|\mathbf{x}\|_1 - L_y(\mathbf{x}). \quad (5)$$

Let $1 \leq j < k \leq d$, $\mathbf{u} = (u_1, \dots, u_d) \in [0, 1]^d$ and $\mathbf{u}^{jk} = (\mathbf{1}_{\ell \notin \{j,k\}} + u_\ell \mathbf{1}_{\ell \in \{j,k\}})_{1 \leq \ell \leq d}$. By using (5) and for $\mathbf{x} \in \mathbb{R}_+^2$, we introduce the bivariate restrictions of the conditional copula and tail dependence functions as following

$$\begin{aligned} C_{j,k,y}(u_j, u_k) &:= C_y(\mathbf{u}^{jk}) = \mathbb{P}(F_{j,y}(X_j) \leq u_j, F_{k,y}(X_k) \leq u_k | Y = y), \\ L_{j,k,y}(\mathbf{x}) &:= \lim_{t \downarrow 0} t^{-1} (1 - C_{j,k,y}(\mathbf{1} - t\mathbf{x})), \quad \lambda_{j,k,y}(\mathbf{x}) = \|\mathbf{x}\|_1 - L_{j,k,y}(\mathbf{x}) = \lim_{t \downarrow 0} t^{-1} \bar{C}_{j,k,y}(t\mathbf{x}), \end{aligned}$$

where $\bar{C}_{j,k,y}$ is the survival copula defined by $\bar{C}_{j,k,y}(u, v) = u + v - 1 + C_{j,k,y}(1 - u, 1 - v)$ on $[0, 1]^2$. In the subsequent paper, we suppose that each bivariate tail dependence function is continuous. Furthermore, we quantify the convergence rate in (4), akin to [13, 33], by assuming that for any $1 \leq j \neq k \leq d$, there exists $\mu_{j,k,y} > 0$ such that, as $t \downarrow 0$, and for any $T > 0$,

$$\sup_{\mathbf{x} \in [0, T]^d} \left| t^{-1} (1 - C_{j,k,y}(\mathbf{1} - t\mathbf{x})) - L_{j,k,y}(\mathbf{x}) \right| = O(t^{\mu_{j,k,y}}).$$

In particular, defining $\mu_y := \max_{1 \leq j \neq k \leq d} \{\mu_{j,k,y}\}$, we may write

$$\max_{1 \leq j \neq k \leq d} \sup_{\mathbf{x} \in [0, T]^d} \left| t^{-1} \left(1 - C_{j,k,y}(\mathbf{1} - t\mathbf{x}) \right) - L_{j,k,y}(\mathbf{x}) \right| = O(t^{\mu_y}). \quad (6)$$

We now state hypothesis on the considered random vector \mathbf{X} given $Y = y$: it has equivalent regularly varying marginal tails. More precisely, as in [3, 29], we assume that each conditional marginal function behaves the same way in the extreme regime.

Definition 2.1 (Considered conditional marginal tails model). *For any $1 \leq j \leq d$ and $y \in E$, there exists $\gamma_y > 0$, $\rho_{j,y} \leq 0$, $c_{j,y} > 0$, such that*

$$(\mathcal{H}1) \quad \bar{F}_{j,y}(\cdot) = \mathbb{P}(X_j > \cdot | Y = y) \in 2\text{RV}_{-\frac{1}{\gamma_y}, \frac{\rho_{j,y}}{\gamma_y}}(+\infty),$$

$$(\mathcal{H}2) \quad \lim_{x \rightarrow \infty} \frac{\bar{F}_{j,y}(x)}{\bar{F}_{1,y}(x)} = c_{j,y} < +\infty.$$

Assumptions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ are classical in the extreme value literature. For a discussion about Assumptions $(\mathcal{H}1)$ and $(\mathcal{H}2)$ the reader is referred to Appendix A.

3. Functional MEEs and optimum system

The multivariate risk measure studied in the present paper is introduced in the following definition.

Definition 3.1 (Multivariate L^1 -expectile with functional covariate). *Let $y \in E$. We define the multivariate L^1 -expectile with functional covariate Y at risk level $\alpha \in (0, 1)$ as the random vector in \mathbb{R}^d , such that*

$$\begin{aligned} \mathbf{e}_\alpha(\mathbf{X}, y) &:= \arg \min_{\mathbf{x} \in \mathbb{R}^d} \mathbb{E} \left[\alpha \|(X - \mathbf{x})_+\|_1^2 + (1 - \alpha) \|(X - \mathbf{x})_-\|_1^2 | Y = y \right] \\ &= \arg \min_{(x_1, \dots, x_d) \in \mathbb{R}^d} \mathbb{E} \left[\alpha \left(\sum_{j=1}^d (X_j - x_j)_+ \right)^2 + (1 - \alpha) \left(\sum_{j=1}^d (X_j - x_j)_- \right)^2 | Y = y \right]. \end{aligned} \quad (7)$$

Definition 3.1 can be seen as a functional conditional extension of the multivariate L^1 -expectiles recently introduced in [3] (for further details the interested reader is referred to Equation (2) and Conclusion section in [3]). Observe that the equality in (7) is assured by the strict convexity of $s_\alpha(t, x) = \alpha \|(t - x)_+\|_1^2 + (1 - \alpha) \|(t - x)_-\|_1^2$ in the variable x which transfers to $\mathbb{E}(s_\alpha(X, x))$ as well by Jensen inequality. Obviously, the conditional L^1 -expectile in (7) is a random vector in \mathbb{R}^d and we write $\mathbf{e}_\alpha(\mathbf{X}, y) = (e_\alpha^j(\mathbf{X}, y))_{1 \leq j \leq d}$. Furthermore, concerning the endpoint, one has $x_F = \sup\{x, F(x) < 1\} = +\infty$, so that $\mathbf{e}_\alpha^j(\mathbf{X}, y) \rightarrow \infty$, for $\alpha \rightarrow 1$ and $1 \leq j \leq d$.

Definition in (7) can be reformulated in terms of first order optimality condition, giving rise to a system in \mathbb{R}^d . As suggested in [3], we quickly convert the work of [29] to the functional setting to provide an expression of the optimum system. This involves other quantities such as the conditional tail index in the heavy-tailed case and the conditional bivariate tail dependence function. One may without difficulty transpose Propositions 3.1, 3.3 and 5.1 in [29] to our functional setup. In particular, suppose that the random vector \mathbf{X} given $Y = y$ is regularly varying (MRV) (see Appendix A) of index $1/\gamma_y$ with conditional marginal tails satisfying Conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$. Then, as $\alpha \rightarrow 1$, any limiting vector $(\eta, \beta_2, \dots, \beta_d)$ of $\left(\frac{1-\alpha}{\bar{F}_{1,y}(e_\alpha^1(\mathbf{X}, y))}, \frac{e_\alpha^2(\mathbf{X}, y)}{e_\alpha^1(\mathbf{X}, y)}, \dots, \frac{e_\alpha^d(\mathbf{X}, y)}{e_\alpha^1(\mathbf{X}, y)} \right)$, for $y \in E$, satisfies the system with $k = 1, \dots, d$,

$$\frac{1}{1/\gamma_y - 1} - \eta \frac{\beta_k^{1/\gamma_y}}{c_{k,y}} = - \sum_{j \neq k} \left(\int_{\frac{\beta_j}{\beta_k}}^{+\infty} \lambda_{j,k,y} \left(\frac{c_{j,y}}{c_{k,y}} t^{-1/\gamma_y}, 1 \right) dt - \eta \frac{\beta_k^{1/\gamma_y - 1}}{c_{k,y}} \beta_j \right). \quad (8)$$

In the case where $\Theta := (\eta, \beta_2, \dots, \beta_d)$ is unique, by using the quantile in (2), as $\alpha \rightarrow 1$, we have

$$\mathbf{e}_\alpha(\mathbf{X}, y) \sim q_{1,y}(\alpha) \eta^{\gamma_y} (1, \beta_2, \dots, \beta_d)^t.$$

For a comprehensive review of the different definitions available for the MRV property, we refer to [32]. This assumption on $X|Y$ mainly ensures the existence of the conditional bivariate upper tail dependence functions which we suppose continuous, see Section 2.3 in [29]. Notice the system (8) only displays pairwise interactions through the conditional bivariate tail dependence function. In view of (8), we introduce the associated optimum problem.

Definition 3.2 (Loss function for the optimum problem of $e_\alpha(X, y)$). *Let $y \in E$. We consider $\Theta := (\eta, \beta_2, \dots, \beta_d) > 0$ as (8). Consider also*

$$\xi_y = (\gamma_y, c_{2,y}, \dots, c_{d,y}, \lambda_y) \quad (9)$$

with λ_y as in (5), γ_y as in Condition (H1) and $c_{j,y}$ as in Condition (H2). Define the vector $\varphi_y(\Theta, \xi_y) = (\varphi_{k,y}(\Theta, \xi_y))_{1 \leq k \leq d}$ where

$$\varphi_{k,y}(\Theta, \xi_y) := \frac{\gamma_y}{\gamma_y - 1} - \eta \frac{\beta_k^{1/\gamma_y}}{c_{k,y}} \left(1 + \sum_{j \neq k} \frac{\beta_j}{\beta_k} \right) + \sum_{j \neq k} \int_{\frac{\beta_j}{\beta_k}}^{+\infty} \lambda_{j,k,y} \left(\frac{c_{j,y}}{c_{k,y}} t^{-1/\gamma_y}, 1 \right) dt. \quad (10)$$

With these notations, one can formulate the system in (8) as $\varphi_y(\Theta, \xi_y) = \mathbf{0}$ (the equality being understood in \mathbb{R}^d). We introduce the loss function via the squared Euclidean norm

$$\mathcal{L}_{\xi_y}(\Theta) := \frac{1}{2} \left\| \varphi_y(\Theta, \xi_y) \right\|_2^2 = \frac{1}{2} \sum_{k=1}^d \varphi_{k,y}(\Theta, \xi_y)^2, \quad (11)$$

and an optimal vector Θ_y^* obtained by minimizing the loss function \mathcal{L}_{ξ_y} in (11),

$$\Theta_y^* \in \arg \min_{\Theta} \mathcal{L}_{\xi_y}(\Theta). \quad (12)$$

To minimize the optimum problem (12), we turn to the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm which belongs to the family of quasi-Newton optimization methods. This choice is motivated by practicality while, from a rigorous point of view, its global convergence is denied due to the loss function lacking of convexity property. A particular interest comes from the fact that no second derivatives computations are needed. In fact, we will rather consider the L-BFGS-B method, a refinement of the classic BFGS that incorporates bound constraints.

Since the underlying distribution is not known in practice, direct application of any minimization algorithm on the optimization problem (11)-(12) is not feasible. The next section is devoted to overcome this drawback.

4. Approximated optimum problem

Let (X_i, Y_i) for $1 \leq i \leq n$, with $X_i = (X_{ij})_{1 \leq j \leq d}$, be n independent copies of $(X, Y) \in \mathbb{R}^d \times E$. In the following the convergence in probability and in distribution, as n goes to infinity, are respectively denoted by $\xrightarrow{\mathbb{P}}$ and \xrightarrow{d} . Relying on this data sample, we consider an approximated version of the loss function for the optimum problem of $e_\alpha(X, y)$ previously introduced in Definition 3.2.

Definition 4.1 (Approximated loss function of $e_\alpha(X, y)$). *By using (11) the approximated loss function can be written as $\mathcal{L}_{\hat{\xi}_{n,y}}(\Theta) = \frac{1}{2} \left\| \varphi_y(\Theta, \hat{\xi}_{n,y}) \right\|_2^2$, where $\hat{\xi}_{n,y}$ is an estimator vector of ξ_y in (9). Furthermore, we define the associated optimum problem as*

$$\hat{\Theta}_y^* \in \arg \min_{\Theta} \mathcal{L}_{\hat{\xi}_{n,y}}(\Theta). \quad (13)$$

We now aim to consistently estimate the loss function $\mathcal{L}_{\xi_y}(\Theta)$. To this purpose, we firstly build a consistent estimator vector $\hat{\xi}_{n,y} = (\hat{\gamma}_{n,y}, \hat{c}_{2,n,y}, \dots, \hat{c}_{d,n,y}, \hat{\lambda}_y)$ of ξ_y . According to (8), we need to estimate $\lambda_{j,k,y}$ (see Section 4.1 below) and $\gamma_y, \mathbf{c}_y = (c_{2,y}, \dots, c_{d,y})$ (see Section 4.2). The interested reader is referred to the framework in [19, 22] (see also [17]).

4.1. Empirical counterpart for $\lambda_{j,k,y}$

Let $1 \leq j < k \leq d$. We consider the empirical counterpart of the bivariate conditional cdf $F_{j,k,y}$ in (1) (see, e.g., [8, 15, 22])

$$\hat{F}_{j,k,n,y}(x_j, x_k) := \sum_{i=1}^n w_{i,n} \mathbf{1}_{\{X_{ij} \leq x_j, X_{ik} \leq x_k\}}, \quad \text{for } x_j, x_k \in \mathbb{R}, \quad (14)$$

where the Nadaraya-Watson weights are given by

$$w_{i,n} := w_{i,n}(Y, y, h_n) = \frac{K\left(\frac{\|Y_i - y\|_E}{h_n}\right)}{\sum_{s=1}^n K\left(\frac{\|Y_s - y\|_E}{h_n}\right)}, \quad \text{with } \sum_{i=1}^n w_{i,n} = 1, \quad (15)$$

with $K : \mathbb{R} \rightarrow \mathbb{R}^+$ a kernel function, i.e. positive and measurable, having support in $[0, 1]$. We define the empirical estimator of the conditional copula and its bivariate restriction

$$\hat{C}_{n,y}(\mathbf{u}) := \sum_{i=1}^n w_{i,n} \mathbf{1}_{\{\bigcap_{j=1}^d \{\hat{F}_{j,n,y}(X_{ij}) \leq u_j\}\}}, \quad \hat{C}_{j,k,n,y}(u_j, u_k) := \hat{C}_{n,y}(\mathbf{u}^{j,k}) = \sum_{i=1}^n w_{i,n} \mathbf{1}_{\{\hat{F}_{j,n,y}(X_{ij}) \leq u_j, \hat{F}_{k,n,y}(X_{ik}) \leq u_k\}}, \quad (16)$$

where $\hat{F}_{j,n,y}, \hat{F}_{k,n,y}$ are the corresponding marginal distribution functions of $\hat{F}_{j,k,n,y}$ in (14). Furthermore, for $\mathbf{x} \in \mathbb{R}_+^2$ and k_n a deterministic intermediate sequence, i.e. $1 \ll k_n \ll n$, we consider the empirical estimators of the bivariate conditional stable tail dependence function and of the associated upper tail dependence function:

$$\hat{L}_{j,k,n,y}(\mathbf{x}) := \frac{n}{k_n} \left(1 - \hat{C}_{j,k,n,y}(\mathbf{1} - \frac{k_n}{n} \mathbf{x})\right), \quad \hat{\lambda}_{j,k,n,y}(\mathbf{x}) = \|\mathbf{x}\|_1 - \hat{L}_{j,k,n,y}(\mathbf{x}).$$

4.2. Empirical counterparts for γ_y and \mathbf{c}_y

Let $y \in E$. Assume $h_n \ll 1$ be a non-random positive sequence such that the small ball probability function is positive, i.e.,

$$\psi_y(h_n) := \mathbb{P}(\|Y - y\|_E \leq h_n) > 0. \quad (17)$$

The latter quantity has a major impact in the functional estimation framework as one can see for instance in our hypotheses in Section 5.1 below. Note that it heavily relies on the intrinsic metric on E .

Our plug-in estimation relies on functional estimators of the marginal quantile and tail index previously introduced by [17, 22]. Let us then define

$$\hat{q}_{j,y,n}(\alpha) := \inf \{x > 0, \hat{F}_{j,y,n}(x) \geq \alpha\}, \quad \alpha \in [0, 1],$$

the natural empirical counterpart of the conditional quantile function in (2), where $\hat{F}_{j,y,n}$ is the marginal distribution function associated to (14).

We pick an integer $1 \leq J < +\infty$ and a subdivision $0 < \tau_J < \tau_{J-1} < \dots < \tau_i < \dots < \tau_2 < \tau_1 \leq 1$, e.g. $\tau_i = 1/i$ and $J = 9$ are motivated by the discussion after Corollary 2 in [9]. The tail index being the same over all margins, we restrict our attention to $j = 1$ and consider the so-called functional Hill estimator, first introduced by [17],

$$\hat{\gamma}_{n,y} := \frac{\sum_{i=1}^J \ln \hat{q}_{1,n,y}(1 - \tau_i \bar{\alpha}_n) - \ln \hat{q}_{1,n,y}(\alpha_n)}{-\sum_{i=1}^J \ln(\tau_i)}. \quad (18)$$

Concerning the tail ratio $c_{j,y}$ in Condition $(\mathcal{H}2)$, we simply extend in a functional framework the estimator proposed in [3, 29]

$$\hat{c}_{j,n,y} := \left(\frac{\hat{q}_{j,n,y}(\alpha_n)}{\hat{q}_{1,n,y}(\alpha_n)} \right)^{1/\hat{\gamma}_{n,y}} \quad \text{for } j = 2, \dots, d. \quad (19)$$

By using the inference procedure presented in Sections 4.1 and 4.2, we now focus on the convergence of the approximated optimum problem in Equation (13).

5. Convergence of the approximated optimum problem for functional MEEs

In Section 5.1 below, we introduce conditions required in our main convergence result, *i.e.*, Theorem 5.1. These conditions fall into porous categories. Because of the bias, the scaling sequence α_n and the smoothing parameter h_n are linked through different regimes involving in particular the small ball probability $\psi_y(h_n)$.

5.1. Required assumptions

We organize the presentation of our assumptions by separating them in three classes (I, II and III). The first class below is related to the bandwidth h_n for the small ball probability in (17), the risk level α_n and the marginal distribution function of X_1 .

(I.1) The bandwidth $h := h_n \ll 1$ satisfies the intermediate regime $n\bar{\alpha}_n\psi_y(h_n) \gg 1$.

(I.2) For any $1 \leq j \leq d$, there exists $\delta_j > 0$ such that for any $y \in E$,

$$\sqrt{n\bar{\alpha}_n\psi_y(h_n)\ln(\bar{\alpha}_n)} \sup_{\substack{z \geq (1-\delta_j)q_{j,y}(\alpha_n) \\ y' \in B(y, h_n)}} \frac{1}{\ln z} \left| \ln \frac{\bar{F}_{1,y'}(z)}{\bar{F}_{1,y}(z)} \right| \xrightarrow{n \rightarrow +\infty} 0.$$

(I.3) For any $y \in E$, it holds that $\sqrt{n\bar{\alpha}_n\psi_y(h_n)A_{1,y}(\bar{\alpha}_n^{-1})} \xrightarrow{n \rightarrow +\infty} p_1 \in \mathbb{R}$, where $A_{1,y}$ is the auxiliary function of X_1 from Condition (H1) (see also Equation (A.1)).

(I.4) For any $s \in [0, 1]$ and $0 < h_n \ll 1$ deterministic, the following pointwise convergence holds

$$\tau_{y,h_n}(s) := \frac{\psi_y(h_n s)}{\psi_y(h_n)} = \mathbb{P}(\|Y - y\|_E < sh_n \mid \|Y - y\|_E < h_n) \xrightarrow{h_n \rightarrow 0_+} \tau_{y,0}(s).$$

A second class of hypothesis involves the conditional bivariate distribution functions and the conditional bivariate upper tail dependence functions.

(II.1) For any $1 \leq j < k \leq d$ and $y \in E$, there exists $\eta_{j,k,y} > 0$ such that, for n large enough, as $t \rightarrow 0$,

$$\lambda_{j,k,y}(t, 1) = O(t^{\eta_{j,k,y}}), \quad \hat{\lambda}_{j,k,n,y}(t, 1) = O_{\mathbb{P}}(t^{\eta_{j,k,y}}).$$

Furthermore, it holds that $\gamma_y < \eta_y := \min_{j \neq k} \eta_{j,k,y}$.

(II.2) For any $1 \leq j < k \leq d$ and $y \in E$, it holds that $\sup_{y' \in E, \|y'\| \leq 1} \sup_{\mathbf{x} \in \mathbb{R}^2} |F_{j,k,y+\delta y'}(\mathbf{x}) - F_{j,k,y}(\mathbf{x})| \xrightarrow{\delta \rightarrow 0_+} 0$.

(II.3) For any $1 \leq j < k \leq d$, analogously as in [16] we consider

$$\phi_{j,k,\mathbf{x},y}^F(s) := \mathbb{E}_Y [F_{j,k,Y}(\mathbf{x}) - F_{j,k,y}(\mathbf{x}) \mid \|Y - y\|_E = s], \quad \text{for } \mathbf{x} = (x_j, x_k),$$

quantifying the expected difference $F_{j,k,Y}(\mathbf{x}) - F_{j,k,y}(\mathbf{x})$ when Y is forced to be at a distance s from the point $y \in E$. Assume that there exists $\beta > 0$ such that the function $s \mapsto \phi_{j,k,\mathbf{x},y}^F(s)$ with $\mathbf{x} = (x_j, x_k)$ satisfies, uniformly in \mathbf{x} ,

$$\phi_{j,k,\mathbf{x},y}^F(s) = s^\beta \nu_{j,k}(\mathbf{x}) + o(s^\beta), \quad s \rightarrow 0_+,$$

for some continuous function $\mathbf{u} \mapsto \nu_{j,k}(F_{j,y}^{-1}(u_j), F_{k,y}^{-1}(u_k))$ on $[0, 1]^2$. Furthermore, the bandwidth $h := h_n \ll 1$ satisfies $n\psi_y(h_n)h_n^{2\beta} = O(1)$.

A third class hypothesis focuses on the Kernel K in (15).

(III.1) K has support $[0, 1]$, is bounded in $(0, 1)$, has a continuous derivative on $[0, 1)$ with $K'(s) \leq 0$ and

$$M_1 := K(1) - \int_0^1 K'(s)\tau_{y,0}(s)ds > 0,$$

where $\tau_{y,0}$ is the pointwise limit as $h \rightarrow 0_+$ of $\tau_{y,h}$ in ((I.4)).

A discussion on the previous required assumptions is postponed to Section 5.3. We are now ready to state and prove our main result.

5.2. Main result

We begin by enunciating the multiple hypotheses needed for our main result. It is then followed by the different speed rates associated to the estimation part of the optimum problem and to the actual minimization algorithm (BFGS family). These scaling sequences are expressed in terms of copula estimation, small ball probability and regular variation.

Let a covariate value $y \in E$. Regarding the inference, we require that:

- the random vector $X = (X_1, \dots, X_d)$ given $Y = y$ is multivariate regularly varying with index $1/\gamma_y$ (see Appendix A) and the conditional marginal tails satisfy Conditions $(\mathcal{H}1)$ and $(\mathcal{H}2)$.
- The condition in (3) and the first order copula condition in (6).
- Assumptions of classes I, II and III in Section 5.1.

Next, denote by $\hat{\Theta}_y^{(m)}$ the m^{th} iteration step of the minimization algorithm applied to the approximated optimum problem (13). We make the following assumptions on the optimization procedure:

- for any $y \in E$, there exists a hyperrectangle $\mathcal{K} := \prod_{r=1}^d [\varepsilon_r, M_r] \subset (0, +\infty)^d$ such that for any $m \geq 1$ and $n \geq 1$ large enough, $\hat{\Theta}_y^{(m)} \in \mathcal{K}$.
- The minimization algorithm solves for the global minimum of optimum problem (12) and has complexity $O(\delta_{(m)})$, i.e., the computational cost of m steps of the algorithm is proportional to $\delta_{(m)}$.

Concerning the speed rates, for any $1 \leq j \leq d$, any deterministic $\alpha_n = 1 + o(1)$, $1 \ll k_n \ll n$ and $m_n \gg 1$, we define the following deterministic sequences, with $\mu_y > 0$ as in (6),

$$\tilde{\delta}_{-1,n,y} := \left(\frac{n}{k_n}\right)^{\mu_y}, \quad \tilde{\delta}_{0,n,y} := \sqrt{n\bar{\alpha}_n\psi_y(h)}, \quad \tilde{\delta}_{j,n,y} := A_{j,y}^{-1}(\bar{\alpha}_n^{-1}). \quad (20)$$

Next, we introduce the speed rate $\delta_{n,y}$ as a scaling sequence such that

$$1 \ll \delta_{n,y} \ll \min_{-1 \leq j \leq d} \tilde{\delta}_{j,n,y}. \quad (21)$$

We may now state our main result under the previous assumptions.

Theorem 5.1 (Approximated optimum convergence with rate). *Let $y \in E$. Let Θ_y^* the optimum of (12) and $\hat{\Theta}_y^{(m_n)}$ the m_n^{th} iteration step in the minimization algorithm applied to (13) where $m_n \gg 1$ is deterministic. Let also two scaling sequences $\delta_{n,y}$ and $\delta_{(m_n)}$. Then, under the previous hypotheses,*

$$\min(\delta_{n,y}, \delta_{(m_n)}) \|\hat{\Theta}_y^{(m_n)} - \Theta_y^*\|_1 \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

The proof of Theorem 5.1 is postponed below.

In order to estimate functional MEEs, we draw our attention on (8). A direct consequence is that, if unique, the limit $\Theta = (\eta, \beta_2, \dots, \beta_d)$ of $(\frac{1-\alpha}{F_{1,y}(e_\alpha^1(X,y))}, \frac{e_\alpha^2(X,y)}{e_\alpha^1(X,y)}, \dots, \frac{e_\alpha^d(X,y)}{e_\alpha^1(X,y)})$, as $\alpha \rightarrow 1$, satisfies the optimum problem (12) and

$$e_\alpha(X, y) \sim F_{1,y}^{-1}(\alpha)\eta^{\gamma_y} (1, \beta_2, \dots, \beta_d)^t, \quad \alpha \rightarrow 1.$$

Notice that the optimum convergence in [3] is given *via* a non-interchangeable two limits result. The first concerns the convergence of the statistical approximation of the loss function, the second the convergence of the descent gradient algorithm. The interested reader is referred to Algorithm 2 and Corollary 3 in [3]. Conversely, our result unifies both limits $n, m \rightarrow +\infty$ into a single limit, i.e.,

$$\hat{\Theta}_y^{(m_n)} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} \Theta_y^*. \quad (22)$$

The speed rate $\delta_{n,y}$ in Theorem 5.1, is associated to the statistical approximation of the loss function. The second rate $\delta_{(m)}$ is inherited of the used minimization algorithm. Considering the double limit $m := m_n \gg 1$, the global speed rate of (22) naturally involves a compromise between the two scaling sequences. For sake of clarity, in Remark 5.2 below, we present some more practical expressions on the speed rate $\delta_{n,y}$.

Remark 5.2 (About the rate $\delta_{n,y}$). Under Assumption (I.3), the condition on the scaling sequence $\delta_{n,y}$ in (21) becomes

$$1 \ll \delta_{n,y} \ll \min_{\substack{-1 \leq j \leq d \\ j \neq -1}} \tilde{\delta}_{j,n,y}. \quad (23)$$

Notice that explicit expressions for the auxiliary functions $A_{k,y}$ are available (e.g., [27]) for particular distributions. For instance, one can consider the Lomax (or Pareto II) conditional marginal distributions, which we generically denote by $X_j|Y = y \sim \text{Lomax}(1/\gamma_y, s_j)$ where $\gamma_y, s_j > 0$ for $y \in E$ and $1 \leq j \leq d$. For this model, the survival conditional distribution function is $\bar{F}_{j,y}(x) = (1 + x/s_j)^{-1/\gamma_y} \in 2\text{RV}_{-1/\gamma_y, -1}$, or equivalently, the conditional tail quantile function is $U_{j,y}(t) = s_j(t^{\gamma_y} - 1) \in 2\text{RV}_{\gamma_y, -\gamma_y}$ with auxiliary function $A_y(t) = \gamma_y t^{-\gamma_y}$. In this case, the speed rate in (23) has a simpler expression

$$1 \ll \delta_{n,y} \ll \min \left\{ \delta_{-1,n,y}, \delta_{0,n,y}, \gamma_y (1 - \alpha_n)^{-\gamma_y} \right\}.$$

Further, if one considers $k_n = n\psi_y(h_n)$ which satisfies $1 \ll k_n \ll n$, one may write $\min(\delta_{0,n,y}, \delta_{-1,n,y}) = (n\psi_y(h_n))^{\min(\frac{1}{2}, \mu_y)}$ and the relation

$$1 \ll \delta_{n,y} \ll \min \left(\gamma_y (1 - \alpha_n)^{-\gamma_y}, (n\psi_y(h_n))^{\min(\frac{1}{2}, \mu_y)} \right).$$

Another possibility is to suppress the auxiliary functions dependence by considering $\delta_{0,n,y} = \delta_{-1,n,y}(1 + o(1))$ as $n \rightarrow +\infty$, or equivalently $k_n \sim n(n\psi_y(h_n))^{-1/(2\mu_y)}$. This particular choice of k_n gives under the condition $1 \ll k_n \ll n$ the new regime $\psi_y(h_n)^{1/(2\mu_y)} \ll n^{1-1/(2\mu_y)}$.

Remark 5.3 (About the minimization algorithm and rate $\delta_{(m)}$). At the moment, we are essentially facing two possibilities for quasi-Newton methods (which have less complexity than the classic Newton ones). The first one is the classic BFGS method as described by Algorithm 6.1 in [31] or Algorithm 1 in [3]. For such algorithm, each iteration costs $O(d^2)$ arithmetic operations. Hence, the whole computational cost is $O(d^2 m)$ where m is the number of iterations, and we set $\delta_{(m)} = d^2 m$. On the other hand, one may preferably use the L-BFGS-B version, which is designed for bound constrained optimization with limited memory storage. The motivations for this particular algorithm are three-fold. First, it seems reasonable to consider MEEs only in a certain range. Second, no informations about the Hessian matrix or about the structure of the loss function are required. Besides, when $m := m_n \gg 1$ and n is the number of observations, the upcoming proof of Theorem 5.1 requires uniform convergence results. The computational cost of one iteration in the L-BFGS-B algorithm is linear in d , namely $O(p^2 d)$ with $3 \leq p \leq 20$ controlling the amount of storage (see [37]). Thus, we associate the speed rate $\delta_{(m)} = p^2 d m$ to this case.

To prove Theorem 5.1, we proceed in a two-step demonstration.

- i) Firstly, we build the approximated minimization problem in (13) by considering an estimator vector $\hat{\xi}_{n,y} = (\hat{\gamma}_{n,y}, \hat{c}_{2,n,y}, \dots, \hat{c}_{d,n,y}, \hat{\lambda}_y)$. Once the consistency of $\hat{\xi}_{n,y}$ is assured, we prove uniform convergence with associated speed rate of the approximated loss function in Definition 4.1 and its gradient function to their theoretical counterparts.
- ii) Then we link the minimization algorithm with the estimation procedure by inducing a dependency of the iteration steps on the data sample size n . Namely, we allow m to depend on n so that $m = m_n \gg 1$. This translates the fact that we are simultaneously consistently approximating the optimum problem (12) and solving the resulting approximated optimum problem (13) using the BFGS minimization algorithm.

Proof of Theorem 5.1. The proof of Theorem 5.1 heavily relies on the convergence of the loss function and its gradient. Moreover, the validity of step ii) can be proved if the gradient convergence is uniform. These results are gathered in the following intermediate proposition. The proof of Proposition 5.1 is postponed to Section 6.

Proposition 5.1 (Approximated loss function convergence with rate). *Assume that assumptions of Theorem 5.1 hold true. Then,*

$$\delta_{n,y} \left| \mathcal{L}_{\xi_y}(\Theta) - \mathcal{L}_{\hat{\xi}_{n,y}}(\Theta) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0, \quad (24)$$

$$\delta_{n,y} \left\| \nabla \mathcal{L}_{\xi_y}(\Theta) - \nabla \mathcal{L}_{\hat{\xi}_{n,y}}(\Theta) \right\|_1 \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0. \quad (25)$$

Furthermore if we restrict $\Theta \in \prod_{j=1}^d [\varepsilon_j, M_j]$ with $0 < \varepsilon_j < M_j < +\infty$ for any $1 \leq j \leq d$, we get

$$\delta_{n,y} \sup_{\Theta \in \prod_{j=1}^d [\varepsilon_j, M_j]} \left| \mathcal{L}_{\xi_y}(\Theta) - \mathcal{L}_{\hat{\xi}_{n,y}}(\Theta) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0, \quad (26)$$

$$\delta_{n,y} \sup_{\Theta \in \prod_{j=1}^d [\varepsilon_j, M_j]} \left\| \nabla \mathcal{L}_{\xi_y}(\Theta) - \nabla \mathcal{L}_{\hat{\xi}_{n,y}}(\Theta) \right\|_1 \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0. \quad (27)$$

Moreover, in Theorem 5.1, we assume that the minimization algorithm solves for the global minimum of (12). Once (26) and (27) are established (see also Equation (32)), one may decompose with the triangular inequality

$$\left\| \hat{\Theta}_y^{(m_n)} - \Theta_y^* \right\|_1 \leq \left\| \hat{\Theta}_y^{(m_n)} - \Theta_y^{(m_n)} \right\|_1 + \left\| \Theta_y^{(m_n)} - \Theta_y^* \right\|_1$$

The first quantity translates the cost of estimating the m_n^{th} step of the minimization algorithm, while the second quantity is purely inherent to the nature of the used minimization algorithm. Concerning the minimization algorithm, we may consider any suitable gradient descent algorithm with complexity $\delta_{(m)}$, so the whole cost is

$$\left\| \Theta_y^{(m_n)} - \Theta_y^* \right\|_1 = O(\delta_{(m_n)}^{-1}).$$

By using the classical steps in the BFGS quasi-Newton optimization algorithm (see, e.g., Algorithm 1 in [3]), once the uniform convergence (27) holds, it immediately follows that, independently of m_n ,

$$\left\| \hat{\Theta}_y^{(m_n)} - \Theta_y^{(m_n)} \right\|_1 = O_{\mathbb{P}}(\delta_{n,y}^{-1}).$$

Hence the desired result. \square

5.3. Hypothesis discussion

Quantiles are the building blocks of the estimation procedure as they play a crucial role in the estimation of the tail index and ratios. If Conditions $(\mathcal{H}1)$, (III.1), (I.1), (I.2) are satisfied, one can prove the following asymptotic normality of the extreme conditional quantile estimator (see for instance Proposition 1 in [22] and Theorem 2 in [17]),

$$\tilde{\delta}_{0,n,y} \left(\frac{\hat{q}_{k,n,y}(\alpha_n)}{q_{k,y}(\alpha_n)} - 1 \right) \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N}(0, 1), \quad 1 \leq k \leq d, \quad (28)$$

where $\tilde{\delta}_{0,n,y}$ is as in (20). If in addition (I.3) is fulfilled, [22, Theorem 7] extends the Gaussian fluctuations for the functional tail index (see also Theorem 4 in [17]),

$$\tilde{\delta}_{0,n,y} (\hat{\gamma}_{n,y} - \gamma_y) \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N}\left(\frac{p_1}{\ln(J!)} \sum_{j=2}^J \frac{j^{\rho_{1,y}} - 1}{\rho_{1,y}}, \frac{J(J-1)(2J-1)}{6 \log(J!)} \gamma^2(y) \right), \quad (29)$$

with $\tilde{\delta}_{0,n,y}$ as in (20), γ_y as in Condition $(\mathcal{H}1)$ with associated estimator $\hat{\gamma}_{n,y}$ as in (18), $\rho_{1,y}$ the second order tail index from Condition $(\mathcal{H}1)$, $p_1 \in \mathbb{R}$ as in Condition (I.3) and J a positive integer (e.g., $J = 9$).

Concerning the fluctuations of the empirical copula in this functional setting, one may invoke [19, Theorem 3]. When each marginal $F_{j,y}$ is continuous and Assumptions (3), (I.1), (I.4), (II.2), (II.3), (III.1) are satisfied, it yields

$$\sqrt{n\psi_y(h_n)} \sup_{\mathbf{u} \in [0,1]^2} |\hat{C}_{j,k,n,y}(\mathbf{u}) - C_{j,k,y}(\mathbf{u})| = O_{\mathbb{P}}(1), \quad (30)$$

where $\psi_y(h_n)$ is the small probability in (17), $C_{j,k,y}$ the conditional bivariate copula (of the margins j, k) with associated estimator $\hat{C}_{j,k,n,y}$ defined in (16).

We devote the rest of this section to illustrate the role of the required hypothesis in our Theorem 5.1, in particular in order to guarantee the crucial convergences in (28)-(29)-(30).

- In view of [17, Lemma 4], the regime in (I.1) is a necessary and sufficient condition for the almost sure presence of at least one sample point in the region $B(y, h_n) \times (q_{j,y}(\alpha_n), +\infty)$ for any $1 \leq j \leq d$, that is, $q_{j,y}(\alpha_n)$ is within the sample. Therefore, it makes sure that, as the number of data sample points increases, the number of observations larger than the conditional quantile increases.
- The hypothesis (III.1) gathers conditions on the kernel for the simultaneous application of convergences in (28)-(29)-(30). It is hence a combination of Type I kernel, according to [14, Definition 4.1], required for (28)-(29), and of continuous kernel versions.
- Concerning univariate quantiles and tail index, we require Conditions (I.2) and (I.3) for applying (28)-(29) and proving Proposition 6.1 and Lemma 6.2, 7.2, 7.3. The first Condition (I.2) allows to control the oscillations of the survival distribution function when the covariate is varying. For a deeper discussion, we refer to the end of Section 2 in [22]. On the other hand, as well-known in the extreme literature, the Hill type estimators require a second order regular variation condition (see also Condition (H2)) in order to determine their rate convergence and fluctuations. The estimator in (18) of the tail index is built upon the first marginal empirical quantile without any second order related object involved. This lacking in the tail approximation generates a bias that we should assume *via* (I.3) to be negligible in the scale $\tilde{\delta}_{n,y,0}$. Inherited from Theorem 7 in [22], Condition (I.3) links the asymptotic of the first marginal auxiliary function $A_{1,y}$ with the small ball probability regime.
- Condition (II.1) is an integrability requirement, used in Proposition 6.1, for manipulating conditional stable tail dependence functions and their estimators as they appear through integrals in the loss function (11).
- Assumptions (I.4), (II.2), (II.3) are technical requirements for (30) related to the functional aspect and appearing in Proposition 6.1 and Lemma 7.3. For instance, the authors of [19] used Condition (II.2) for tightness and equicontinuity in the considered empirical processes. In particular, the object $\tau_{y,h}$ in Assumption (I.4) was first introduced in [16] with the same condition on the limiting function $\tau_{y,0}$. The connection with assumption on M_1 in Condition (III.1) comes down from the fact that kernel expectations are key objects (see [17, 22]) in functional kernel methods and may be written as $\mathbb{E}_Y[K(h_n^{-1}\|Y - y\|_E)] = K(1)\psi_y(h_n) - \int_0^1 K'(s)\psi_y(sh_n)ds$ (see [16, Lemma 2]). The identification of $\tau_{y,0}$ is driven by the asymptotic behaviour of $s \mapsto \psi_y(s) := \mathbb{P}(\|Y - y\|_E \leq s)$. As stated in [14], the class of fractal (infinite-dimensional) processes is the one for which the small ball probabilities behave as $\psi_y(s) \sim C_y s^a$, for $s \rightarrow 0$ and $C_y, a > 0$. For such processes, we have $\tau_{y,0}(s) = s^a$. This includes the multivariate Euclidean case $E = \mathbb{R}^p$, with $a = p$. On the other hand, for the class of processes such that $\psi_y(s) \sim C_y s^a \exp(-C/s^b)$, for $s \rightarrow 0$ and some $C_y, C, a, b > 0$, one has $\tau_{y,0}(s) = \delta_1(s)$ with δ_1 being the Dirac delta function at point 1 (see [16]). As noted in [19], for those processes, $\int_0^1 K'(s)\tau_{y,0}(s) ds$ is zero so that one usually assumes $K(1) > 0$ in this case. In the general case, the condition $K'(\cdot) \leq 0$ is asked. For more details about $\tau_{y,h}$ and β in Condition (II.3), the interested reader is referred to Section 2.3, especially Proposition 1 and Section 4, in [16], and Section 3.2 in [19].

6. Proof of Proposition 5.1

Soliciting Δ -method (see, e.g., Theorem 3.8 in [36]), the proof of (24) and (25) is reduced to the establishment of the building block

$$\delta_{n,y} \max_{1 \leq k \leq d} |\varphi_{k,y}(\Theta, \xi_y) - \varphi_{k,n,y}(\Theta, \hat{\xi}_{n,y})| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0. \quad (31)$$

The proof of (26) and (27) relies upon, for $0 < \varepsilon_r < M_r < +\infty$, $1 \leq r \leq d$,

$$\delta_{n,y} \max_{1 \leq k \leq d} \sup_{\Theta \in \prod_{r=1}^d [\varepsilon_r, M_r]} |\varphi_{k,y}(\Theta, \xi_y) - \varphi_{k,n,y}(\Theta, \hat{\xi}_{n,y})| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0. \quad (32)$$

In view of (10), the proof of (31) and (32) comes down by proving the following convergence

$$\delta_{n,y} \max \left(\left| \Delta_{n,y}^{(1)} \right|, \left| \Delta_{k,n,y}^{(2)}(\Theta) \right|, \left| \Delta_{j,k,n,y}^{(3)}(\Theta) \right| \right) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0, \quad 1 \leq j \neq k \leq d, \quad y \in E, \quad (33)$$

and its uniform counterpart

$$\delta_{n,y} \sup_{\Theta \in \prod_{r=1}^d [\varepsilon_r, M_r]} \max \left(|\Delta_{n,y}^{(1)}|, |\Delta_{k,n,y}^{(2)}(\Theta)|, |\Delta_{j,k,n,y}^{(3)}(\Theta)| \right) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0, \quad (34)$$

where $\Theta = (\eta, \beta_2, \dots, \beta_d) \in (0, +\infty)^d$, $\beta_1 = 1$ and

$$\begin{aligned} \Delta_{n,y}^{(1)} &:= \frac{\gamma_y}{\gamma_y - 1} - \frac{\hat{\gamma}_{n,y}}{\hat{\gamma}_{n,y} - 1}, \quad \Delta_{k,n,y}^{(2)}(\Theta) = \eta \left(1 + \sum_{j \neq k} \frac{\beta_j}{\beta_k} \right) \left(\frac{\beta_k^{1/\gamma_y}}{c_{k,y}} - \frac{\beta_k^{1/\hat{\gamma}_{n,y}}}{\hat{c}_{k,n,y}} \right), \\ \Delta_{j,k,n,y}^{(3)}(\Theta) &= \sum_{j \neq k} \left(\int_{\frac{\beta_j}{\beta_k}}^{+\infty} \hat{\lambda}_{j,k,n,y} \left(\frac{\hat{c}_{j,n,y}}{\hat{c}_{k,n,y}} t^{-1/\hat{\gamma}_{n,y}}, 1 \right) - \lambda_{j,k,y} \left(\frac{c_{j,y}}{c_{k,y}} t^{-1/\gamma_y}, 1 \right) dt \right). \end{aligned} \quad (35)$$

A straightforward application of the Δ -method and (29) implies that for any deterministic $1 \ll \delta_{0,n,y} \ll \tilde{\delta}_{0,n,y}$ and independently on $\Theta \in (0, +\infty)^d$,

$$\delta_{0,n,y} |\Delta_{n,y}^{(1)}| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

This readily provides (33) and (34) for $\Delta_{n,y}^{(1)}$. The quantities $\Delta_{k,n,y}^{(2)}$ and $\Delta_{j,k,n,y}^{(3)}$ are respectively tackled in the upcoming Lemma 6.2 and Proposition 6.1. One can obtain the identities for the gradient loss function by adapting the setting in Appendix A.3. in [3] to our functional covariate case. By using these gradient expressions we can show the gradient convergence in (26)-(27). Indeed, the random quantities $\hat{\gamma}_y$ and $\hat{c}_{k,y}$ appear through algebraic expressions in the gradient loss function in the same form as in (10).

The Karamata's results, Theorem B.1.4 - B.1.5, Equation (B.1.14), Remark B.1.7 in [12], see also Theorem 1.3.1 in [7], allow us to write

$$\ell_{U_{j,y}}(x) := \ell_{j,y}(x) = \kappa_j(y) b_{j,y}(x) \exp \left(\int_{x_0}^x \frac{b_{j,y}(t) - 1}{t} dt \right), \quad (36)$$

for some arbitrary $x_0 > 0$, some constant $\kappa_j(y) \in (0, +\infty)$ and

$$b_{j,y}(x) = \frac{x \ell_{j,y}(x)}{\int_{x_0}^x \ell_{j,y}(t) dt}. \quad (37)$$

Studying the Karamata representation (36) of the slowly varying part $\ell_{j,y}$ through the point of view of Π -Class (see Definition 7.1), we may show in the following Lemma 6.1 an asymptotic expansion on $\ell_{j,y}$.

Notice that Lemma 6.1 in the case $\rho_{j,y} < 0$ is a straightforward generalization to the conditional setting of Lemma 1 in [25]. The proof in the specific case where $\rho_{j,y} = 0$ is postponed to Section 7.1. For examples of distributions satisfying the second order regular variation condition with $\rho_{j,y} = 0$ we interested reader is referred to Remark A.4.

Lemma 6.1. Assume that for any $y \in E$, for any $1 \leq j \leq d$, $U_{j,y} \in 2RV_{-\gamma_y, \rho_{j,y}}(+\infty)$, $\gamma_y > 0$, $\rho_{j,y} \leq 0$, with auxiliary function $A_{j,y} \in RV_{\rho_{j,y}}(+\infty)$ or equivalently Condition $(\mathcal{H}1)$. Then, for some constant $\kappa_j(y) \in (0, +\infty)$ independent of x ,

$$\ell_{j,y}(x) = \kappa_j(y) \left(1 + O\left(A_{j,y}(x)\right) \right), \quad x \rightarrow +\infty.$$

Once the slowly varying part of the tail quantile function is determined, we may derive, under the second order regular variation property (see Assumption $(\mathcal{H}2)$), the speed rate in the consistency of the tail ratios.

Lemma 6.2. Let $\Theta = (\eta, \beta_2, \dots, \beta_d) > 0$, $y \in E$, $\alpha_n = 1 + o(1)$, $h_n \ll 1$ deterministic such that the intermediate regime $n\bar{\alpha}_n \psi_y(h_n) \gg 1$ holds true. For each marginal $2 \leq k \leq d$, let $\delta_{k,n,y}$ be a scaling sequence such that

$$1 \ll \delta_{k,n,y} \ll \min_{s \in \{0,1,k\}} \tilde{\delta}_{s,n,y}, \quad \text{where } \tilde{\delta}_{s,n,y} \text{ is defined in Equation (20)}. \quad (38)$$

Then, as $n \rightarrow +\infty$, under Assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$, (III.1), (I.1), (I.2), (I.3),

$$\delta_{k,n,y} |c_{k,y} - \hat{c}_{k,y}| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0. \quad (39)$$

Furthermore, it holds that, for $1 \leq k \leq d$,

$$\delta_{k,n,y} \left| \Delta_{k,n,y}^{(2)}(\Theta) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0, \quad \delta_{k,n,y} \sup_{\Theta \in \prod_{r=1}^d [\varepsilon_r, M_r]} \left| \Delta_{k,n,y}^{(2)}(\Theta) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0,$$

for $0 < \varepsilon_r < M_r < +\infty$, $1 \leq r \leq d$.

As defined in (38), the auxiliary marginal scaling sequences $\delta_{k,n,y}$ may be combined to recover the global rate $\delta_{n,y}$ defined in (21) by considering the minimum over $\tilde{\delta}_{-1,n,y}$ and over every $\delta_{k,n,y}$ for $k \geq 2$. The proof of Lemma 6.2 is postponed to Section 7.1. Next, we consider the integral of stable tail dependence functions and prove the following convergences.

Proposition 6.1. Let $\delta_{n,y}$ as in (21). Let $y \in E$, $1 \leq j \neq k \leq d$, $\Theta = (\eta, \beta_2, \dots, \beta_d) \in (0, +\infty)^d$ and $\beta_1 = 1$. If Assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$, (3), (6) and Assumptions of classes I, II, III in Section 5.1 are satisfied, then

$$\delta_{n,y} \left| \Delta_{j,k,n,y}^{(3)}(\Theta) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0,$$

and for any $0 < \varepsilon_r < M_r < +\infty$, $1 \leq r \leq d$,

$$\delta_{n,y} \sup_{\Theta \in \prod_{r=1}^d [\varepsilon_r, M_r]} \left| \Delta_{j,k,n,y}^{(3)}(\Theta) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

The proof of Proposition 6.1 can be found in Section 7.1.

7. Auxiliary proofs and results

We gather in this section the proofs of the results needed for Proposition 5.1 as well as further auxiliary results required for the intermediate steps of the proof.

7.1. Proofs of lemmas of Section 6

We introduce a particular proxy version to the second order regular variation property to be connected to slowly varying functions when the second order tail index is zero. In this context, we constructively specify the auxiliary function and shall make use of an uniform Drees-type inequality. This is of importance in order to address in depth the Karamata representation applied to the slowly varying part of the conditional marginal quantile tail function.

Definition 7.1 (Class II). [Definition B.2.4 in [12]] A measurable function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ belongs to the class $\Pi(a)$ if there exists an auxiliary function $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for $x > 0$,

$$\lim_{t \rightarrow +\infty} \frac{f(tx) - f(t)}{a(t)} = \ln(x). \quad (40)$$

Remark 7.1. The auxiliary function is slowly varying, namely $a \in \text{RV}_0(+\infty)$, by Theorem B.2.7 in [12]. By Theorem B.2.12 in [12], $f \in \Pi(a)$ is equivalent to $f \in \Pi(\tilde{a})$ with $\tilde{a}(t) = f(t) - \frac{1}{t} \int_0^t f(s) ds$. Hence, if $f \in \Pi(a)$ for some auxiliary function a , we can always choose instead the alternative auxiliary function \tilde{a} . More can be said, if in addition $x^\gamma f(x) \in 2\text{RV}_{\gamma,0}$, $\gamma > 0$. In this case, the auxiliary function A in Definition A.2 can be chosen as $A(t) = \tilde{A}(t) := \frac{\tilde{a}(t)}{\tilde{f}(t)}$. According to Proposition B.2.17 in [12], if $f \in \Pi(a)$, then for any $\varepsilon, \delta > 0$, there exists $x_0 = x_0(\varepsilon, \delta) \geq 0$, such that for any t and x with $tx \geq x_0$, the following uniform bound holds

$$\left| \frac{\ell(tx) - \ell(x)}{\tilde{a}(x)} - \ln(t) \right| \leq \varepsilon \left(\max(t^\delta, t^{-\delta}) \right), \quad (41)$$

with the relation $\tilde{A}\ell = \tilde{a}$.

Proof of Lemma 6.1. Let $1 \leq j \leq d$ and $y \in E$. The case $\rho_{j,y} < 0$ is a straightforward generalization to the conditional setting of Lemma 1 in [25]. We may now assume the second order tail index of $U_{j,y}$ to be zero, i.e., $\rho_{j,y} = 0$. According to the definition of the second order regular variation, there exists auxiliary functions $A_{j,y} := A_{U_{j,y}} \in \text{RV}_0(+\infty)$ for $1 \leq j \leq d$ with ultimately constant sign and $\lim_{t \rightarrow +\infty} A_{j,y}(t) = 0$ such that

$$\lim_{t \rightarrow +\infty} \frac{1}{A_{j,y}(t)\ell_{j,y}(t)} \left(\ell_{j,y}(tx) - \ell_{j,y}(t) \right) = \ln(x), \quad x > 0.$$

Since $\ell_{j,y}$ is non-negative, we immediately deduce that

$$\ell_{j,y} \in \Pi(a_{j,y}), \quad a_{j,y}(t) := \text{sign} \{A_{j,y}(t)\} A_{j,y}(t)\ell_{j,y}(t) \in \text{RV}_0(+\infty), \quad a_{j,y} \geq 0.$$

By Remark 7.1, the definition of the second order regular variation and the relation $U_{j,y}(x) = x^{\gamma_y} \ell_{j,y}(x)$, we can modify the auxiliary functions so that

$$\ell_{j,y}(\cdot) \in \Pi(\tilde{a}_{j,y}), \quad \tilde{a}_{j,y}(t) := \ell_{j,y}(t) - \frac{1}{t} \int_0^t \ell_{j,y}(s) ds,$$

and $U_{j,y}(\cdot) \in 2\text{RV}_{-1/\gamma_y, 0}(+\infty)$, with auxiliary function

$$A_{j,y}(t) = \frac{\tilde{a}_{j,y}(t)}{\ell_{j,y}(t)} = 1 - \int_0^1 \frac{\ell_{j,y}(tu)}{\ell_{j,y}(t)} du \in \text{RV}_0(+\infty).$$

Fix $\varepsilon > 0$ and $\delta \in (0, 1)$. Let $x_0(\varepsilon, \delta) > 0$ such that (41) holds. From Potter's Theorem B.1.9 in [12] with $\delta_1 = 1$ and $\delta_2 = 1/2$, let $x_0(\delta_1, \delta_2) > 0$ such that Potter's bounds hold as well. We finally set $x_0 := \max(x_0(\varepsilon, \delta), x_0(\delta_1, \delta_2))$. In the rest of the proof, we consider this particular choice of x_0 in (36) and investigate (37).

With the substitution $u = \frac{t}{x}$ in (37), one can express $b_{j,y}^{-1}(x) = \int_{\frac{x_0}{x}}^1 \frac{\ell_{j,y}(ux)}{\ell_{j,y}(x)} du$. Then, with $A_{j,y} \in \text{RV}_0(+\infty)$,

$$\frac{1}{A_{j,y}(x)} \left(b_{j,y}^{-1}(x) - 1 + \frac{x_0}{x} \right) = \int_0^1 \frac{1}{A_{j,y}(x)} \left[\frac{\ell_{j,y}(ux)}{\ell_{j,y}(x)} - 1 \right] \mathbf{1}_{[\frac{x_0}{x}, 1]}(u) du.$$

For $u \in (0, 1)$ and x such that $ux \geq x_0$, the bound (41) yields

$$\left| \frac{1}{A_{j,y}(x)} \left[\frac{\ell_{j,y}(ux)}{\ell_{j,y}(x)} - 1 \right] \mathbf{1}_{[\frac{x_0}{x}, 1]}(u) \right| \leq -\ln(u) + \varepsilon u^{-\delta} := f_{\varepsilon, \delta}(u).$$

The function $u \mapsto f_{\varepsilon, \delta}(u)$ is integrable on $(0, 1)$ as $-1 < -\delta < 0$. Altogether with the pointwise convergence (40), the dominated convergence theorem gives

$$\int_0^1 \frac{1}{A_{j,y}(x)} \left[\frac{\ell_{j,y}(ux)}{\ell_{j,y}(x)} - 1 \right] \mathbf{1}_{[\frac{x_0}{x}, 1]}(u) du \xrightarrow{x \rightarrow +\infty} \kappa_0 := \int_0^1 \ln(u) du = -1 < +\infty.$$

The previous convergence holds with limiting constant $-\kappa_0$ instead of κ_0 by accordingly adjusting the sign of the auxiliary function $A_{j,y}$. Opting for this convention, we subsequently set $\kappa_0 = 1$. One may then write

$$b_{j,y}^{-1}(x) = 1 + A_{j,y}(x) - \frac{x_0}{x} + o\left(A_{j,y}(x)\right).$$

Since $x \mapsto \frac{1}{x} \in \text{RV}_{-1}(+\infty)$ and $A_{j,y} \in \text{RV}_0(+\infty)$, it follows that $b_{j,y}(x) = 1 + A_{j,y}(x) + o\left(A_{j,y}(x)\right)$, $x \rightarrow +\infty$. Going back to (36),

$$\ell_{j,y}(x) = \kappa_j(y) \left(1 + A_{j,y}(x) + o\left(A_{j,y}(x)\right) \right) \exp \left(\int_{x_0}^x \frac{A_{j,y}(t) + o\left(A_{j,y}(t)\right)}{t} dt \right).$$

Repeating the same argument of dominated convergence but in the first order regular variation context for the auxiliary function $A_{j,y} \in \text{RV}_{0(+\infty)}$ with Potter's Theorem B.1.9 in [12],

$$\int_{x_0}^x \frac{A_{j,y}(t)}{t} dt = A_{j,y}(x) + o(A_{j,y}(x)), \quad x \rightarrow +\infty.$$

Therefore, a Taylor expansion as $x \rightarrow +\infty$ yields the asymptotic representation

$$\ell_{j,y}(x) = \kappa_j(y) \left(1 + O(A_{j,y}(x))\right), \quad A_{j,y} \in \text{RV}_{0(+\infty)}, \quad \kappa_j(y) \in (0, +\infty).$$

□

Proof of Lemma 6.2. Let $y \in E$ and $2 \leq k \leq d$. First, observe that straight consequences of (28) as well as Δ -method applied to (29) are

$$\frac{\hat{q}_{k,n,y}(\alpha_n)}{q_{k,y}(\alpha_n)} = 1 + O_{\mathbb{P}}(\tilde{\delta}_{0,n,y}^{-1}), \quad \hat{\gamma}_{n,y}^{-1} - \gamma_y^{-1} = O_{\mathbb{P}}(\tilde{\delta}_{0,n,y}^{-1}). \quad (42)$$

One may write

$$\ln \left(\frac{\hat{q}_{k,n,y}(\alpha_n)}{\hat{q}_{1,n,y}(\alpha_n)} \right) = \ln \left(\frac{\hat{q}_{k,n,y}(\alpha_n)}{q_{k,y}(\alpha_n)} \right) + \ln \left(\frac{q_{k,y}(\alpha_n)}{q_{1,y}(\alpha_n)} \right) + \ln \left(\frac{q_{1,y}(\alpha_n)}{\hat{q}_{1,n,y}(\alpha_n)} \right).$$

In view of (42) and after Taylor expanding, this becomes

$$\ln \left(\frac{\hat{q}_{k,n,y}(\alpha_n)}{\hat{q}_{1,n,y}(\alpha_n)} \right) = \ln \left(\frac{q_{k,y}(\alpha_n)}{q_{1,y}(\alpha_n)} \right) + O_{\mathbb{P}}(\tilde{\delta}_{0,n,y}^{-1}). \quad (43)$$

We investigate the logarithm term in the RHS of (43) and precise its asymptotic behaviour. Regarding the quantiles, it is clear that, for $\alpha_n \ll 1$ and each $1 \leq j \leq d$, we have $q_{j,y}(\alpha_n) \gg 1$ and $q_{j,y} \in \text{RV}_{\gamma_y}(1)$. As a functional version of Lemma 3.2 in [29] which may be proved along the same steps, we have that, if $t = O(s)$, for any $1 \leq i, j \leq d$,

$$\frac{\bar{F}_{i,y}(s)}{\bar{F}_{j,y}(t)} \sim \frac{c_{i,y}}{c_{j,y}} \left(\frac{s}{t} \right)^{-1/\gamma_y}, \quad t \rightarrow +\infty.$$

It follows that

$$1 = \frac{\bar{F}_{k,y}(q_{k,y}(\alpha_n))}{\bar{F}_{1,y}(q_{1,y}(\alpha_n))} = c_{k,y} \left(\frac{q_{1,y}(\alpha_n)}{q_{k,y}(\alpha_n)} \right)^{1/\gamma_y} + o(1), \quad n \rightarrow +\infty.$$

Hence,

$$c_{k,y} = \left(\frac{q_{k,y}(\alpha_n)}{q_{1,y}(\alpha_n)} \right)^{1/\gamma_y} + o(1), \quad \text{i.e.} \quad \frac{q_{k,y}(\alpha_n)}{q_{1,y}(\alpha_n)} = c_{k,y}^{\gamma_y} + o(1).$$

On the other hand, one may write the ratio in terms of slowly varying functions

$$\frac{q_{k,y}(\alpha_n)}{q_{1,y}(\alpha_n)} = \frac{U_{k,y}(\bar{\alpha}_n^{-1})}{U_{1,y}(\bar{\alpha}_n^{-1})} = \frac{\ell_{k,y}(\bar{\alpha}_n^{-1})}{\ell_{1,y}(\bar{\alpha}_n^{-1})}.$$

By Lemma 6.1, for $1 \leq j \neq k \leq d$ and as $x \rightarrow +\infty$, the following asymptotic holds after a Taylor expansion

$$\frac{\ell_{j,y}(x)}{\ell_{k,y}(x)} = \frac{\kappa_j(y)}{\kappa_k(y)} \left[1 + O \left(\max(A_{j,y}(x), A_{k,y}(x)) \right) \right].$$

We deduce the tail ratio representation $c_{k,y} = (\kappa_k(y)/\kappa_1(y))^{1/\gamma_y}$ and that for $1 \leq k \leq d$,

$$0 \leq \limsup_{n \rightarrow +\infty} \min(\tilde{\delta}_{1,n,y}, \tilde{\delta}_{k,n,y}) \left(\frac{q_{k,y}(\alpha_n)}{q_{1,y}(\alpha_n)} - c_{k,y}^{\gamma_y} \right) < +\infty. \quad (44)$$

We can state (44) in an alternative way $c_{k,y}^{\gamma_y} \frac{q_{1,y}(\alpha_n)}{q_{k,y}(\alpha_n)} = 1 + O\left(\max\left(\tilde{\delta}_{1,n,y}^{-1}, \tilde{\delta}_{k,n,y}^{-1}\right)\right)$, $n \rightarrow +\infty$. Combining (29), (42), (43) and (44) with the definition of $\hat{c}_{k,y}$ in (19), it yields

$$\ln(\hat{c}_{j,n,y}) = \frac{1}{\hat{\gamma}_{n,y}} \ln\left(\frac{\hat{q}_{k,n,y}(\alpha_n)}{\hat{q}_{1,n,y}(\alpha_n)}\right) = \frac{1}{\hat{\gamma}_{n,y}} \ln\left(\frac{q_{k,y}(\alpha_n)}{q_{1,y}(\alpha_n)}\right) + O_{\mathbb{P}}\left(\tilde{\delta}_{0,n,y}^{-1}\right) = \frac{\gamma_y}{\hat{\gamma}_{n,y}} \ln c_{k,y} + o_{\mathbb{P}}\left(\max\left(\tilde{\delta}_{1,n,y}^{-1}, \tilde{\delta}_{k,n,y}^{-1}\right)\right) + O_{\mathbb{P}}\left(\tilde{\delta}_{0,n,y}^{-1}\right).$$

Limit in (39) is proved after an application of the Δ -method. We now consider the quantity

$$\frac{\beta_k^{1/\hat{\gamma}_{n,y}}}{\hat{c}_{k,n,y}} \left(\frac{\beta_k^{1/\gamma_y}}{c_{k,y}}\right)^{-1} = \exp\left[-\hat{\gamma}_{n,y}^{-1} \ln\left(\frac{\hat{q}_{k,n,y}(\alpha_n)}{\hat{q}_{1,n,y}(\alpha_n)}\right) + (\hat{\gamma}_{n,y}^{-1} - \gamma_y^{-1}) \ln(\beta_k) + \ln(c_{k,y})\right]. \quad (45)$$

Combining again (29), (42), (43) and (44), and Taylor expanding in (45), it yields

$$\frac{\beta_k^{1/\hat{\gamma}_{n,y}}}{\hat{c}_{k,n,y}} - \frac{\beta_k^{1/\gamma_y}}{c_{k,y}} = \frac{c_{k,y}^{\gamma_y}}{\gamma_y} O_{\mathbb{P}}\left(\max\left(\tilde{\delta}_{1,n,y}^{-1}, \tilde{\delta}_{k,n,y}^{-1}\right)\right) + O_{\mathbb{P}}\left(\tilde{\delta}_{0,n,y}^{-1}\right).$$

The previous asymptotic equality is valid for any $1 \leq k \leq d$. Moreover, the asymptotic holds uniformly over $\beta_k \in [\varepsilon_k, M_k]$ with $0 < \varepsilon_k < M_k < +\infty$. Indeed, in view of the term $\ln(\beta_k)$ in (45), we discriminate two cases: when $0 < \beta_k \neq 1$ and when $\beta_k = 1$.

In the former case, it is clear that $|\ln \beta_k|$ is positive and bounded by a constant depending on ε_k, M_k .

In the second case, the speed rate $\tilde{\delta}_{0,n,y}$ induced by the difference $\hat{\gamma}_{n,y}^{-1} - \gamma_y^{-1}$ apparently vanishes but is reintroduced by the term $\ln\left(\frac{\hat{q}_{1,n,y}(\alpha_n)}{\hat{q}_{k,n,y}(\alpha_n)}\right)$. Now, the term $\eta(1 + \sum_{j \neq k} \beta_j / \beta_k)$ is positive and bounded when $\Theta = (\eta, \beta_2, \dots, \beta_d) \in \prod_{k=1}^d [\varepsilon_k, M_k]$ and $\beta_1 = 1$. Hence, the second part of Lemma 6.2 is proved. \square

Proof of Proposition 6.1. In the following, for the sake of readability, we drop the finite sum in (35). We start by decomposing $\Delta_{j,k,n,y}^{(3)}$ defined in (35) as $\Delta_{j,k,n,y}^{(3)} = \Delta_{j,k,n,y}^{(3,1)} + \Delta_{j,k,n,y}^{(3,2)}$ with

$$\begin{aligned} \Delta_{j,k,n,y}^{(3,1)} &= \int_{\frac{\beta_j}{\beta_k}}^{+\infty} \lambda_{j,k,y} \left(\frac{\hat{c}_{j,n,y}}{\hat{c}_{k,n,y}} t^{-1/\hat{\gamma}_{n,y}}, 1 \right) dt - \int_{\frac{\beta_j}{\beta_k}}^{+\infty} \lambda_{j,k,y} \left(\frac{c_{j,y}}{c_{k,y}} t^{-1/\gamma_y}, 1 \right) dt, \\ \Delta_{j,k,n,y}^{(3,2)} &= \int_{\frac{\beta_j}{\beta_k}}^{+\infty} \hat{\lambda}_{j,k,n,y} \left(\frac{\hat{c}_{j,n,y}}{\hat{c}_{k,n,y}} t^{-1/\hat{\gamma}_{n,y}}, 1 \right) dt - \int_{\frac{\beta_j}{\beta_k}}^{+\infty} \lambda_{j,k,y} \left(\frac{\hat{c}_{j,n,y}}{\hat{c}_{k,n,y}} t^{-1/\hat{\gamma}_{n,y}}, 1 \right) dt, \end{aligned}$$

where $\hat{\gamma}_{n,y}$ and $\hat{c}_{j,n,y}$, $1 \leq j \leq d$, are defined in (18) and (19). Note that $\lambda_{j,k,y}(0, 1) = 1 - L_{j,k,y}(0, 1) = 0$.

In view of Assumption (II.1), the mapping $t \mapsto t^{-(\gamma_y+1)} \lambda_{j,k,y}(t, 1)$ is integrable in the proximity of 0 so we may consider the change of variable

$$\frac{c_{j,y}}{c_{k,y}} t^{-1/\gamma_y} = u \iff t = \left(\frac{c_{k,y}}{c_{j,y}} u\right)^{-\gamma_y}, \quad dt = -\gamma_y \left(\frac{c_{k,y}}{c_{j,y}}\right)^{-\gamma_y} u^{-(\gamma_y+1)} du.$$

It yields the alternative form

$$\Delta_{j,k,n,y}^{(3,1)} = -v_y \int_0^{M_y(\Theta)} \left\{ \lambda_{j,k,y}(\hat{s}_{j,k,n,y}(t), 1) - \lambda_{j,k,y}(t, 1) \right\} dt, \quad \Delta_{j,k,n,y}^{(3,2)} = -\hat{v}_{n,y} \int_0^{\hat{M}_{n,y}(\Theta)} \left\{ \hat{\lambda}_{j,k,n,y}(t, 1) - \lambda_{j,k,y}(t, 1) \right\} dt,$$

with

$$\hat{s}_{j,k,n,y}(t) := \frac{\hat{c}_{j,n,y}}{\hat{c}_{k,n,y}} \left(\frac{c_{k,y}}{c_{j,y}}\right)^{\gamma_y/\hat{\gamma}_{n,y}} \cdot t^{\gamma_y/\hat{\gamma}_{n,y}} := \hat{\sigma}_{n,y} \cdot t^{\gamma_y/\hat{\gamma}_{n,y}}, \quad (46)$$

$$M_y(\Theta) := \frac{c_{j,y}}{c_{k,y}} \left(\frac{\beta_j}{\beta_k}\right)^{-1/\gamma_y}, \quad v_y := \gamma_y \left(\frac{c_{k,y}}{c_{j,y}}\right)^{-\gamma_y}, \quad \hat{M}_{n,y}(\Theta) := \frac{\hat{c}_{j,n,y}}{\hat{c}_{k,n,y}} \left(\frac{\beta_j}{\beta_k}\right)^{-1/\hat{\gamma}_{n,y}}, \quad \hat{v}_{n,y} := \hat{\gamma}_{n,y} \left(\frac{\hat{c}_{k,n,y}}{\hat{c}_{j,n,y}}\right)^{-\hat{\gamma}_{n,y}}.$$

In order to prove Proposition 6.1, it is enough to show that, for $s \in \{1, 2\}$,

$$\delta_{n,y} \left| \Delta_{j,k,n,y}^{(3,s)}(\Theta) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0, \quad (47)$$

and the uniform counterparts, for $0 < \varepsilon_r < M_r < +\infty$, $1 \leq r \leq d$, $\delta_{n,y} \sup_{\Theta \in \prod_{r=1}^d [\varepsilon_r, M_r]} \left| \Delta_{j,k,n,y}^{(3,s)}(\Theta) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0$.

From (29) and (39), for any $y \in E$, $1 \leq j \leq d$ and $t \geq 0$, we have $\hat{\gamma}_{n,y} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} \gamma_y$, $\hat{c}_{j,n,y} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} c_{j,y}$.

In particular, the continuous mapping theorem readily implies that $\hat{s}_{j,k,n,y}(t) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} t$ and also

$$\hat{M}_{n,y}(\Theta) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} M_y(\Theta), \quad \hat{v}_{n,y}(t) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} v_y. \quad (48)$$

It is classic knowledge that the stable tail dependence function $\lambda_{j,k,y}$ is 1-Lipschitz, hence, the case $\Delta_{n,y}^{(1)}$ in (47) follows from Lemma 7.2. On the other hand, the case $\Delta_{k,n,y}^{(2)}$ in (47) is a mere consequence of the uniform boundedness over $\beta := (\beta_1, \beta_2, \dots, \beta_d) \in \prod_{j=1}^d [\varepsilon_j, M_j]$ with $0 < \varepsilon_j < M_j < +\infty$, for any j , of the quantities in (48) and the uniform convergence in probability of Lemma 7.3. \square

7.2. Auxiliary lemmas

In this section we introduce and prove three auxiliary results. Recall the scaling sequence $\delta_{k,n,y}$ for each margins $1 \leq k \leq d$ defined in (38). Firstly we may have a closer look to the previous convergence of $\hat{s}_{j,k,n,y}$ in (46). It follows from (29) and (39) that

$$\delta_{k,n,y} (\hat{\sigma}_{n,y} - 1) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

An application of the multivariate Δ -method yields the following pointwise convergence in probability

$$\delta_{k,n,y} (\hat{s}_{j,k,n,y}(t) - t) \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

Leaning on the monotonicity of the mapping $t \mapsto \hat{s}_{j,k,n,y}(t)$, we improve the pointwise convergence in probability $\hat{s}_{j,k,n,y}(t) - t = o_{\mathbb{P}}(1)$ to an uniform version by a probabilistic Dini-type argument.

Lemma 7.1. *For any $0 < T < +\infty$ and $1 \leq j \neq k \leq d$, it holds that $\sup_{t \in [0, T]} |\hat{s}_{j,k,n,y}(t) - t| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0$.*

Proof of Lemma 7.1. Let $T > 0$ and $\varepsilon > 0$. Let also an integer $N := N(\varepsilon) \geq 1$ such that $T < \varepsilon N$. We consider a subdivision with step less than ε of the compact set $[0, T] = \cup_{\ell=0}^{m-1} [\frac{\ell}{N}T, \frac{\ell+1}{N}T]$. In particular, for any $t \in [0, T]$, there exists $0 \leq \ell_1 < \ell_2 \leq N$ such that $\frac{\ell_1}{N}T \leq t \leq \frac{\ell_2}{N}T$. Fix $\omega \in \Omega$ for the moment and assume that for any $0 \leq \ell \leq N$, $|\hat{s}_{j,k,n,y}(\frac{\ell}{N}T)(\omega) - \frac{\ell}{N}T| < \varepsilon$. Since the mapping $x \mapsto \hat{s}_{j,k,n,y}(x)(\omega)$ is non-decreasing, the following inequalities hold

$$\hat{s}_{j,k,n,y}(t)(\omega) - t \leq \hat{s}_{j,k,n,y}\left(\frac{\ell_2}{N}T\right)(\omega) - \frac{\ell_1}{N}T \leq s_{j,k,n,y}\left(\frac{\ell_2}{N}T\right)(\omega) - \frac{\ell_2}{N}T + \varepsilon < 2\varepsilon,$$

$$\hat{s}_{j,k,n,y}(t)(\omega) - t \geq \hat{s}_{j,k,n,y}\left(\frac{\ell_1}{N}T\right)(\omega) - \frac{\ell_2}{N}T \geq \hat{s}_{j,k,n,y}\left(\frac{\ell_1}{N}T\right)(\omega) - \frac{\ell_1}{N}T - \varepsilon > -2\varepsilon.$$

Thus, $\sup_{t \in [0, T]} |\hat{s}_{j,k,n,y}(t)(\omega) - t| < 2\varepsilon$. As a consequence,

$$\mathbb{P} \left(\left| \hat{s}_{j,k,n,y}\left(\frac{\ell}{N}T\right) - \frac{\ell}{N}T \right| < \varepsilon, \forall \ell = 0, \dots, N \right) \leq \mathbb{P} \left(\sup_{t \in [0, T]} |\hat{s}_{j,k,n,y}(t) - t| < 2\varepsilon \right).$$

Taking the complementary, using union bound and the previous pointwise convergence in probability of $\hat{s}_{j,k,n,y}$, as $N, T, \varepsilon < +\infty$ are independent of n ,

$$\mathbb{P} \left(\sup_{t \in [0, T]} |\hat{s}_{j,k,n,y}(t) - t| > \varepsilon \right) \leq \sum_{\ell=0}^N \mathbb{P} \left(\left| \hat{s}_{j,k,n,y}\left(\frac{\ell}{N}T\right) - \frac{\ell}{N}T \right| > \frac{\varepsilon}{2} \right) \xrightarrow[n \rightarrow +\infty]{} 0.$$

\square

In the following lemma we provide the speed rate of convergence in Lemma 7.1.

Lemma 7.2. Let $\delta_{k,n,y}$ be the scaling sequence for each margins $1 \leq k \leq d$ defined in (38). Let $y \in E$ and $0 < T < +\infty$. Under Assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$ and (I.1), (I.2), (I.3), (III.1), for $j \neq k$,

$$\delta_{k,n,y} \sup_{t \in [0, T]} \left| \hat{s}_{j,k,n,y}(t) - t \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

Proof of Lemma 7.2. Recall $\hat{\sigma}_{n,y} := \frac{\hat{c}_{j,n,y}}{\hat{c}_{k,n,y}} \left(\frac{c_{k,y}}{c_{j,y}} \right)^{\gamma_y / \hat{\gamma}_{n,y}}$. Define $g_{j,k,n,y}(t) := \hat{s}_{j,k,n,y}(t) - t$ for $t \geq 0$ with $\hat{s}_{j,k,n,y}$ as in (46).

Consider $t_{n,y} := \left(\frac{\gamma_y}{\hat{\gamma}_{n,y}} \hat{\sigma}_{n,y} \right)^{\frac{\hat{\gamma}_{n,y}}{\hat{\gamma}_{n,y} - \gamma_y}}$ the solution of

$$g'_{j,k,n,y}(t) = \hat{\sigma}_{n,y} \frac{\gamma_y}{\hat{\gamma}_{n,y}} t^{\gamma_y / \hat{\gamma}_{n,y} - 1} - 1 = 0, \quad t \geq 0.$$

This corresponds to the maximum value of $g_{j,k,n,y}$ when $\gamma_y < \hat{\gamma}_{n,y}$ or its minimum value when $\gamma_y > \hat{\gamma}_{n,y}$. This is a consequence of the 2nd order optimality condition and

$$g''_{j,k,n,y}(t) = \hat{\sigma}_{n,y} \frac{\gamma_y}{\hat{\gamma}_{n,y}} (\gamma_y / \hat{\gamma}_{n,y} - 1) t^{\gamma_y / \hat{\gamma}_{n,y} - 2}, \quad t \geq 0.$$

Using (29) and (39), we easily see that

$$t_{n,y} = \left(\frac{\gamma_y}{\hat{\gamma}_{n,y}} \hat{\sigma}_{n,y} \right)^{\frac{\hat{\gamma}_{n,y}}{\hat{\gamma}_{n,y} - \gamma_y}} = \exp \left(\frac{\hat{\gamma}_{n,y}}{\hat{\gamma}_{n,y} - \gamma_y} \ln \left(\frac{\gamma_y}{\hat{\gamma}_{n,y}} \hat{\sigma}_{n,y} \right) \right) = \exp \left(\frac{\hat{\gamma}_{n,y}}{\hat{\gamma}_{n,y} - \gamma_y} o_{\mathbb{P}} \left(\delta_{k,n,y}^{-1} \right) \right) = 1 + o_{\mathbb{P}}(1).$$

For fixed $\omega \in \Omega$, $0 < T < +\infty$ and n large enough, we may write

$$\sup_{t \in [0, T]} |g_{j,k,n,y}(t)|(\omega) \leq \max \left(|g_{j,k,n,y}(t_{n,y})|(\omega), |g_{j,k,n,y}(T)|(\omega) \right).$$

By Taylor expansion and multivariate Δ -method with (29) and (39), one readily shows that for $0 < T < +\infty$ fixed, $g_{j,k,n,y}(T) = o_{\mathbb{P}}(1/\delta_{k,n,y})$. Now, we compute

$$g_{j,k,n,y}(t_{n,y}) = t_{n,y} \left(\frac{\hat{\gamma}_{n,y}}{\gamma_y} - 1 \right).$$

Then, using (29) and (39) once more yields $g_{j,k,n,y}(t_{n,y}) = o_{\mathbb{P}}(1/\delta_{k,n,y})$. \square

Lemma 7.3. Let $M \in (0, +\infty)$. Let $1 \leq j \neq k \leq d$, $y \in E$ and $1 \ll k_n \ll n$ be an intermediate sequence. Under Assumptions (3), (6), (I.4), (II.2), (II.3), (III.1), we define

$$1 \ll \delta_{0,n,y} \ll \tilde{\delta}_{0,n,y} = \sqrt{n \bar{\alpha}_n \psi_y(h)} \quad \text{and} \quad 1 \ll \delta_{-1,n,y} \ll \tilde{\delta}_{-1,n,y} = \left(\frac{n}{k_n} \right)^{\mu_y}$$

as in Equation (20) with $\mu_y > 0$ as in Condition (6). Then, it holds that

$$\min(\delta_{0,n,y}, \delta_{-1,n,y}) \sup_{\mathbf{x} \in [0, M]^2} |\lambda_{j,k,y}(\mathbf{x}) - \hat{\lambda}_{j,k,n,y}(\mathbf{x})| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

Proof of Lemma 7.3. By using Equation (5), we write

$$\sup_{\mathbf{x} \in [0, M]^2} |\lambda_{j,k,y}(\mathbf{x}) - \hat{\lambda}_{j,k,n,y}(\mathbf{x})| = \sup_{\mathbf{x} \in [0, M]^2} |L_{j,k,y}(\mathbf{x}) - \hat{L}_{j,k,n,y}(\mathbf{x})|.$$

For n large enough, one can get $\frac{k_n}{n} M \leq 1$. Then, the triangular inequality allows us to bound the latter quantity by

$$\sup_{\mathbf{x} \in [0, M]^2} |L_{j,k,y}(\mathbf{x}) - \tilde{L}_{j,k,n,y}(\mathbf{x})| + \sup_{\mathbf{x} \in [0, M]^2} |\hat{L}_{j,k,n,y}(\mathbf{x}) - \tilde{L}_{j,k,n,y}(\mathbf{x})|,$$

where $\tilde{L}_{j,k,n,y}(\mathbf{x}) := \frac{n}{k_n} \left(1 - C_{j,k,y}(\mathbf{1} - \frac{k_n}{n} \mathbf{x})\right)$. Concerning the second term, since $\bar{\alpha}_n = o(1)$, it readily follows from (30) that for any deterministic $1 \ll \delta_{0,n,y} \ll \tilde{\delta}_{0,n,y} \ll \sqrt{n\psi_y(h_n)}$,

$$\delta_{0,n,y} \sup_{\mathbf{x} \in [0,M]^2} \left| \hat{L}_{j,k,n,y}(\mathbf{x}) - \tilde{L}_{j,k,n,y}(\mathbf{x}) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0.$$

Lastly, using the rate condition in (6), we have $\delta_{-1,n,y} \sup_{\mathbf{x} \in [0,M]^2} \left| L_{j,k,y}(\mathbf{x}) - \tilde{L}_{j,k,n,y}(\mathbf{x}) \right| \xrightarrow[n \rightarrow +\infty]{\mathbb{P}} 0$. \square

Conclusion

In this paper, we present a semi-parametric method for estimating functional multivariate L^1 -expectiles at extreme risk levels (i.e., functional L^1 -MEEs). Going beyond [3], we establish, in this functional setting, the consistency with rate of the approximated loss function by using empirical kernel-based estimators for the tail index, the tail ratio and the upper tail dependence function. Then, we couple the loss function estimation with a gradient descent algorithm (e.g., BFGS-family). As a result, we give the consistency with rate of the approximated optimum problem for solving functional L^1 -MEEs.

An immediate line of work is the implementation of the MEEs estimation method and testing its finite-sample performance. Adopting the BFGS family method as gradient descent algorithm, one may proceed to the simulation study for Lomax marginals and survival Clayton as dependence structure. Such model is closely related to [3, 17, 22] but taking into account that Pareto I is not second order regularly varying.

Furthermore, one may raise the question of how to calibrate the parameters (especially from the kernel smoothing). A cross-validation procedure such as [22] is in consideration for the choice of the bandwidth associated to the small ball probability hypothesis of the considered functional space and the risk level of the MEEs.

Another possible line of research would be to consider the L^p -norm in Equation (7), i.e., to consider the L^p -expectiles defined in [28] in this functional setting. The question of their estimation may also be approached by the same semi-parametric methodology although it would display more intricate algebra. Along the same lines, one may consider a possible extension of [29] to functional extreme Σ -expectiles for $\Sigma \neq \mathbf{1}$.

While the specific case of one dominant marginal is studied in [29], an interesting problem is to remove the multivariate regular variation hypothesis on the random vector \mathbf{X} with equivalent marginals tails.

Finally, a more ambitious future work would be to connect our inference method with the online optimization theory. Indeed, we propose in our work to simultaneously estimate and optimize the loss function associated to the considered functional L^1 -MEEs, which may essentially be seen as updating with new data-points a multidimensional optimization problem.

Acknowledgement

This work has been supported by the project ANR McLaren (ANR-20-CE23-0011). This work has been partially supported by the French government, through the 3IA Côte d'Azur Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR-19-P3IA-0002.

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A. Regular variation framework

Definition A.1 (Multivariate Regular Variation). A random vector $\mathbf{X} \in \mathbb{R}^d$ is regularly varying with index $\alpha \geq 0$ if there exists a random vector Θ on the unit sphere \mathbb{S}^{d-1} such that for any $x \in (0, +\infty)$, the following vague convergence holds:

$$\frac{\mathbb{P}\left(\|\mathbf{X}\| > xz, \frac{\mathbf{X}}{\|\mathbf{X}\|} \in \cdot\right)}{\mathbb{P}(\|\mathbf{X}\| > z)} \xrightarrow{z \rightarrow +\infty} x^{-\alpha} \mathbb{P}(\Theta \in \cdot).$$

Definition A.2 (Second order regular variation [12]). Let $\gamma > 0$ and $\rho \leq 0$. We say that $U \in 2RV_{-\gamma, \rho}(+\infty)$ if there exists an auxiliary function A with ultimately constant sign and $\lim_{t \rightarrow +\infty} A(t) = 0$ such that

$$\lim_{t \rightarrow +\infty} \frac{1}{A(t)} \left(\frac{U(tx)}{U(t)} - x^\gamma \right) = x^\gamma H_\rho(x) := x^\gamma \int_1^x u^{\rho-1} du, \quad x > 0. \quad (\text{A.1})$$

Note that for $x > 0$,

$$H_\rho(x) = \ln(x) \mathbf{1}_{\{\rho=0\}} + \frac{x^\rho - 1}{\rho} \mathbf{1}_{\{\rho < 0\}}.$$

By Theorem 2.3.9 in [12], this is equivalent to

$$\lim_{t \rightarrow +\infty} \frac{1}{A(1/\bar{F}(t))} \left(\frac{\bar{F}(tx)}{\bar{F}(t)} - x^{-\frac{1}{\gamma}} \right) = H_{\rho, \gamma}(x) := x^{-\frac{1}{\gamma}} \frac{x^{\frac{\rho}{\gamma}} - 1}{\gamma \rho}, \quad x > 0. \quad (\text{A.2})$$

Namely,

$$U \in 2RV_{\gamma, \rho}(+\infty) \iff \bar{F} \in 2RV_{-\frac{1}{\gamma}, \frac{\rho}{\gamma}}(+\infty).$$

Besides, on the auxiliary functions level, Theorem 2.3.3 in [12] provides

$$A \in RV_\rho(+\infty), \quad A \circ \frac{1}{\bar{F}} \in RV_{\frac{\rho}{\gamma}}(+\infty).$$

Remark A.3. Condition $(\mathcal{H}1)$ entails a second order regular variation behaviour. Note that $c_{1,y} \equiv 1$ for any $y \in E$. For any $1 \leq j \leq d$, since $\bar{F}_{j,y} \in RV_{-1/\gamma_y}(+\infty)$, there exists $\ell_{\bar{F}_{j,y}} \in RV_0(+\infty)$, such that

$$\bar{F}_{j,y}(x) = x^{-1/\gamma_y} \ell_{\bar{F}_{j,y}}(x), \quad x > 0. \quad (\text{A.3})$$

Also, we have $U_{j,y} \in RV_{\gamma_y}(+\infty)$ and thus, there also exists a slowly varying function $\ell_{U_{j,y}} := \ell_{j,y}$ such that $U_{j,y}(x) = x^{\gamma_y} \ell_{j,y}(x)$, for $x > 0$. Condition $(\mathcal{H}2)$ is inherited from [3, 29] and states that each marginal behaves the same way in the extreme regime. Moreover, since $\bar{F}_{j,y} \in 2RV_{-1/\gamma_y, \rho_{j,y}/\gamma_y}(+\infty)$, it follows that $U_{j,y} \in 2RV_{\gamma_y, \rho_{j,y}}(+\infty)$. According to Definition A.2, there exists auxiliary functions $A_{j,y} := A_{U_{j,y}} \in RV_{\rho_{j,y}}(+\infty)$ satisfying Equation (A.1).

Remark A.4. Examples of distributions satisfying the second order regular variation condition are:

- Log-gamma distribution defined as the exponential of the sum of two independent standard exponential random variables: $\bar{F}(x) = \frac{1}{x} (1 + \log(x)) \in 2RV_{-1,0}(+\infty)$ with $A(1/\bar{F}(t)) = 1/\log(t)$ (see [18]).
- For $x > e$, $\bar{F}(x) = \frac{\log x}{x} e \in 2RV_{-1,0}(+\infty)$, with $A(1/\bar{F}(t)) = 1/\log(t)$.
- Hall-Weiss class: $\bar{F}(x) = \frac{1}{2} x^{-\alpha} (1 + x^\rho) \in 2RV_{-\alpha, \rho}(+\infty)$ for $\alpha > 0$ and $\rho < 0$ with $A(1/\bar{F}(t)) = \rho t^\rho$ (see [18]).
- Cauchy distribution: $\bar{F}(x) = \frac{1}{\pi} \tan^{-1}(1/x) + \frac{1}{2} (1 - \text{sign}(x)) \in 2RV_{-1,-2}(+\infty)$ (see [17]).
- Fréchet distribution: $\bar{F}(x) = 1 - e^{-x^{-\frac{1}{\gamma}}} \in 2RV_{-\gamma, -\gamma}(+\infty)$, $\gamma > 0$ (see [17]).
- Burr distribution: $\bar{F}(x) = (1 + x^\tau)^{-\lambda} \in 2RV_{-\frac{1}{\tau}, -\tau}(+\infty)$, $\tau, \lambda > 0$ (see [17]).
- Student t_ν , $\nu > 1$, distribution with density $f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$. Then, $\bar{F} \in 2RV_{-\nu, -2}(+\infty)$.

The interested reader can find more 2RV type distributions in Section 5 of [27].