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► **To cite this version:**

Marta Bílková, Sabine Frittella, Daniil Kozhemiachenko, Ondrej Majer, Krishna Manoorkar. Describing and Quantifying Contradiction between Pieces of Evidence via Belnap Dunn Logic and Dempster-Shafer Theory. 2023. hal-04051113

HAL Id: hal-04051113

<https://hal.science/hal-04051113>

Preprint submitted on 29 Mar 2023

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Describing and Quantifying Contradiction between Pieces of Evidence via Belnap Dunn Logic and Dempster-Shafer Theory

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Abstract

Belnap Dunn logic is a four-valued logic introduced to model reasoning with incomplete or contradictory information. In this article, we show how Dempster-Shafer theory can be used over Belnap Dunn logic in order to formalise reasoning with incomplete and/or contradictory pieces of evidence. First, we discuss how to encode different kinds of evidence, and how to interpret the resulting belief and plausibility functions. Then, we discuss the behavior of Dempster's rule in this framework and present a variation of the rule. Finally, we show how to construct credal sets of classical probability measures based on this kind of evidence.

Keywords: Dempster-Shafer theory, Belnap Dunn logic, contradictory evidence.

1. Introduction

Combination of evidence and conflict in Dempster-Shafer theory. Combination of conflicting or contradictory sources has been a major topic of study in Dempster-Shafer (DS) theory [8, 11, 10, 3]. In Dempster's original combination rule (DS-rule) [8], it was assumed that the sources are completely reliable, and hence any conflict between them is considered impossible. Zadeh [11] gives an example to show that DS-rule can lead to counter intuitive results when it is used to aggregate pieces of evidence that are not fully reliable and with significant degree of conflict between them. Several modifications of DS-rule have been proposed and studied in the literature to aggregate evidences both from not fully reliable sources and from sources strongly contradicting each other. Shafer [8] describes *discounting* or *tradeoff* method to deal with conflict. In this method, when the sources have a conflict between them, the analyst discounts sources based on their reliability

before using DS-rule. Yager [10] proposes a combination rule which is similar to DS-rule but the mass attached to conflicting evidence is assigned to the whole frame of discernment. That is, having conflicting evidence is considered equivalent to having no information. Dubois and Prade [3] propose instead that if two sources attach mass to sets A and B , with $A \cap B = \emptyset$, then in the combination the mass $m(A) \cdot m(B)$ is attached to the set $A \cup B$. Intuitively this corresponds to the idea that if sources are contradictory, then the analyst concludes that at least one of them is correct. In this work, we use an expansion of Belnap Dunn logic (BD) to represent and combine conflicting evidence. BD allows us to represent and reason about incomplete and contradictory information.

Belnap Dunn logic. BD was introduced to reason about information rather than about truth [1]. In classical logic a statement p is either *true* or *false*, meaning that p is *true* (resp. *false*) iff the statement p is true (resp. false) in the world. In BD, a statement p is either “supported by the available information”, or “contradicted by the available information”, or “neither supported nor contradicted by the available information”, or “both supported and contradicted by the available information”. These four truth values are respectively denoted **T** (*true*), **F** (*false*), **N** (*neither*), **B** (*both*) and are interpreted over the 4-valued De Morgan algebra (Figure 1).

The four elements ordered from bottom to top define the so-called *truth lattice*. The truth tables of the conjunction \wedge and disjunction \vee of this lattice are given in Definition 1. By going from **F** to **T**, one goes from a situation where the information available fully supports the falsity of the statement, to a situation where the information available fully supports the truth of the statement. The four elements ordered from left to right define the so-called *information*

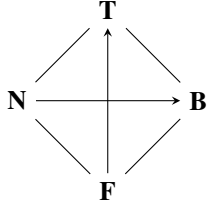


Figure 1: Belnap–Dunn square

lattice. By going from **N** to **B**, one goes from a situation where there is no information about the statement, to a situation where there is contradictory information about the statement. The truth tables of the disjunction \sqcup and the conjunction \sqcap of the information lattice are given in Definition 10.

A logic such that $p \vee \neg p$ (resp. $p \wedge \neg p$) is not an axiom is called *paracomplete* (resp. *paraconsistent*). BD is a weakening of classical propositional logic that is both paracomplete and paraconsistent.

Paraconsistent probabilities. In the classical case, $p(\phi)$ (resp. $p(\neg\phi)$) encodes the probability that ϕ is true (resp. false). Dunn [4] introduces paraconsistent probabilities that describe the information available about ϕ via four numbers (b, d, u, c) . They encode the degree of belief b , disbelief d , uncertainty u (ignorance), and conflict c (contradiction) about ϕ . Klein et al. [6] presents a probabilistic extension of BD with a sound and complete axiomatization.

Our project. In Bílková et al. [2], we introduce belief functions over BD models and present logics to reason both with probabilities and belief functions over BD. This work is a first step towards understanding (imprecise) probabilities within a paracomplete and paraconsistent framework. In this article, we show how situations, where highly contradictory information is available, can be formalised using BD and DS-theory.

Structure of the paper. First, in Section 2, we provide preliminaries on the semantics of BD, paraconsistent probabilities, and in Section 3, we discuss how to encode evidence via mass functions over BD-models and how to interpret the resulting belief and plausibility functions. Then, in Section 4, we discuss a variation of DS-rule over BD-models and its interpretation in BD. Finally, in Section 5 and Section 6, we introduce different notions of support of a statement that induce different belief functions over BD formulas, and we show that some of them allow to deduce credal sets of classical probabilities based on mass functions over BD-models.

2. An expansion of Belnap–Dunn logic and non-standard probabilities

Semantics of Belnap–Dunn logic. In this paragraph, we present an expansion of BD with an additional connective Δ from Sano and Omori [7]. Throughout the paper, Prop will denote a *finite* set of propositional variables and $\text{Lit} = \text{Prop} \cup \{\neg p \mid p \in \text{Prop}\}$ the associated set of literals.

Definition 1 (Semantics of $\text{BD}\Delta$) Let $\mathcal{V} = \{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}$ and define the following grammar in the Backus–Naur form.

$\mathcal{L}_{\text{BD}\Delta} \ni \phi := \perp \mid \top \mid p \in \text{Prop} \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \Delta\phi$.
Let \mathcal{L}_{BD} denote the Δ -free fragment of $\mathcal{L}_{\text{BD}\Delta}$.

A **4-valuation** is a map $v_4 : \text{Prop} \rightarrow \mathcal{V}$ that is extended to complex formulas using the following definitions.

\neg	\wedge	\vee	Δ
\top	\top	\top	\top
\mathbf{B}	\mathbf{B}	\mathbf{B}	\mathbf{B}
\mathbf{N}	\mathbf{N}	\mathbf{N}	\mathbf{N}
\mathbf{F}	\mathbf{F}	\mathbf{F}	\mathbf{F}

We say that ϕ BD-entails χ ($\phi \models_{\text{BD}} \chi$) iff it holds that if $v_4(\phi) \in \{\mathbf{T}, \mathbf{B}\}$, then $v_4(\chi) \in \{\mathbf{T}, \mathbf{B}\}$ for every 4-valuation. ϕ $\text{BD}\Delta$ -entails χ ($\phi \models_{\text{BD}\Delta} \chi$) iff it holds that if $v_4(\phi) \in \{\mathbf{T}, \mathbf{B}\}$, then $v_4(\chi) \in \{\mathbf{T}, \mathbf{B}\}$ and if $v_4(\phi) \in \{\mathbf{F}, \mathbf{B}\}$, then $v_4(\chi) \in \{\mathbf{F}, \mathbf{B}\}$ for every 4-valuation.

We will use the fact that BD and classical logic (CL) share the same language and that $\models_{\text{BD}} \subseteq \models_{\text{CL}}$ (see e.g. [9]).

Observe that \neg , \wedge , and \vee preserve **B** and **N**, thus there is no $\{\Delta, \perp, \top\}$ -free formula that is always true or always non-false. For instance, saying that the formula $p \vee \neg p$ is always true would be interpreted as “it is always the case that we have either the information that p is true or the information that $\neg p$ is true”. On the other hand, for every v_4 , $v_4(\Delta p \vee \neg\Delta p) = \mathbf{T}$ and $v_4(\Delta p \wedge \neg\Delta p) = \mathbf{F}$. Indeed, the formula $\Delta p \vee \neg\Delta p$ is interpreted as “either we have information stating that p is true or we do not have information stating that p is true”.

Definition 2 (Lindenbaum algebras) Let \cong be the congruence relation on \mathcal{L}_{BD} defined as $\phi \cong \phi'$ iff $\phi \models_{\text{BD}} \phi'$ and $\phi' \models_{\text{BD}} \phi$. The BD Lindenbaum algebra over \mathcal{L}_{BD} is the De Morgan algebra $\langle \{[\phi]_{\phi \in \mathcal{L}_{\text{BD}}}\}, \wedge, \vee, \neg \rangle$, where $[\phi]$ is the equivalence class of the formula ϕ , $\neg[\phi] = [\neg\phi]$ and $[\phi] \odot [\psi] = [\phi \odot \psi]$ for $\odot \in \{\wedge, \vee\}$. The CL Lindenbaum algebra over \mathcal{L}_{BD} is the Boolean algebra defined similarly using \models_{CL} .

Definition 3 (BD-model) A BD-model is a tuple $\mathfrak{M} = \langle W, v_4 \rangle$ with W a finite nonempty set and $v_4 : \text{Prop} \times W \rightarrow \mathcal{V}$. v_4 is extended to complex formulas using tables from Definition 1. For every nonempty $V \subseteq \mathcal{V}$, the V -extension of a formula ϕ on \mathfrak{M} is $|\phi|^V = \{w \in W : v_4(\phi, w) \in V\}$.

For instance, $v_4(p, w) = \mathbf{B}$ means that “the state w has information supporting both the truth and the falsity of p ” and $|\phi|^{\mathbf{T}, \mathbf{F}}$ is the set of states that either provide only information supporting the truth of ϕ or only information supporting the falsity of ϕ . Note also that if all variables of ϕ are in scope of Δ , then $|\phi|^{\mathbf{B}, \mathbf{N}} = \emptyset$.

Definition 4 (Canonical model) The canonical BD-model over Prop is a tuple $\mathfrak{M}_c = \langle W_c, v_4 \rangle$, where $W_c = \mathcal{P}(\text{Lit})$ and the valuation $v_4 : \text{Prop} \times W_c \rightarrow \mathcal{V}$ is

$$\begin{aligned} v_4(p, w) &= \mathbf{N} \text{ if } p, \neg p \notin w, \\ v_4(p, w) &= \mathbf{T} \text{ if } p \in w \text{ and } \neg p \notin w, \\ v_4(p, w) &= \mathbf{F} \text{ if } p \notin w \text{ and } \neg p \in w, \\ v_4(p, w) &= \mathbf{B} \text{ if } p, \neg p \in w. \end{aligned}$$

defined as follows:

Intuitively, W_c is the set of all possible valuations v_4 over Prop . Similarly, $\mathcal{P}(\text{Prop})$ can be seen as the set of all classical valuations: i.e. p is true at $S \in \mathcal{P}(\text{Prop})$ iff $p \in S$. A state $w \in W_c$ is said *incomplete* (resp. *classical*) if $v_4(p, w) \in \{\mathbf{T}, \mathbf{N}, \mathbf{F}\}$ (resp. $v_4(p, w) \in \{\mathbf{T}, \mathbf{F}\}$) for every $p \in \text{Prop}$.

Definition 5 We consider formulas in \mathcal{L}_{BD} . A conjunctive clause is *irredundant* if it contains each literal at most once. A formula φ is in *irredundant disjunctive normal form (iDNF)* if it is a disjunction of irredundant conjunctive clauses, and moreover, if $\varphi = \bigvee_{i \in I} \varphi_i$ implies $\varphi_i \not\vdash_{\text{BD}} \varphi_j$ for every $i \neq j$.

Every formula in \mathcal{L}_{BD} is BD-equivalent to a *unique* (up to permutation of clauses and literals) formula in iDNF.

Definition 6 We consider formulas in $\mathcal{L}_{\text{BD}\Delta}$. A \mathcal{V} -atom is one of the following formulas.

$$\begin{aligned} \mathbf{T}p &:= \Delta p \wedge \neg \Delta \neg p & \mathbf{B}p &:= \Delta p \wedge \Delta \neg p \\ \mathbf{N}p &:= \neg \Delta p \wedge \neg \Delta \neg p & \mathbf{F}p &:= \neg \Delta p \wedge \Delta \neg p \end{aligned}$$

A \mathcal{V} -literal is a formula of the form $\mathbf{X}p := \bigvee_{\mathbf{X} \in \mathcal{V}} \mathbf{X}p$ with $\mathcal{V} \subseteq \mathcal{V}$. A witnessed clause is a conjunction of \mathcal{V} -literals. A clause is *reduced* iff no variable occurs twice. A witnessed clause is *complete* iff it is reduced, every variable occurs once, and it contains only \mathcal{V} -atoms. A witnessed disjunctive normal form (wDNF) is a disjunction of witnessed clauses. A wDNF $\bigvee_{i \in I} c_i$ is *maximal* iff it is composed of complete clauses and $|c_i|^{\mathbf{T}} = |c_j|^{\mathbf{T}}$ implies $i = j$.

Note that a \mathcal{V} -witness $\mathbf{X}p$ means ‘ p has value \mathbf{X} ’. A \mathcal{V} -literal over p designates the set of values p can have. A wDNF specifies the possible values of variables, e.g., $\phi = (\{\mathbf{T}, \mathbf{N}\}p \wedge \{\mathbf{B}, \mathbf{N}\}q \wedge \{\mathbf{T}, \mathbf{F}\}r) \vee (\{\mathbf{N}\}p \wedge \{\mathbf{N}\}q \wedge \{\mathbf{N}\}r)$ means that ‘either p is non-false, q is non-classical, and r is classical, or $p, q,$ and r have value \mathbf{N} .’ For sake of readability, we write $\mathbf{TN}p$ instead of $\{\mathbf{T}, \mathbf{N}\}p$. Using this convention for ϕ we get $\phi = (\mathbf{TN}p \wedge \mathbf{BN}q \wedge \mathbf{TF}r) \vee (\mathbf{N}p \wedge \mathbf{N}q \wedge \mathbf{N}r)$.

For any formula $\phi \in \mathcal{L}_{\text{BD}\Delta}$ and for any nonempty set of truth values $\mathcal{V} \subseteq \mathcal{V}$, one can write a formula $\mathbf{V}\phi \in \mathcal{L}_{\text{BD}\Delta}$

such that $|\mathbf{V}\phi|^{\mathbf{T}} = |\phi|^{\mathbf{V}}$. To reduce the use of superscripts, whenever there is no ambiguity, we will write $|\phi|$ instead of $|\phi|^{\mathbf{T}}$ and $|\mathbf{V}\phi|$ instead of $|\phi|^{\mathbf{V}}$.

Notice that the \mathbf{T} -extension on the canonical model of a complete clause is a singleton, that is, any state w of the canonical model can be uniquely (up to permutation of the \mathcal{V} -atoms) represented by a complete witnessed clause c_w and every set X of states can be uniquely (up to permutation of the clauses) represented by the corresponding maximal wDNF formula $\phi_X = \bigvee_{w \in X} c_w$.

Paraconsistent probabilities Several equivalent representations of paraconsistent probabilities have been introduced in the literature. In this article, we focus our attention on probabilities over BD-models by Klein et al. [6].

Definition 7 (Probability assignments) A (paraconsistent) probability assignment is a function $p : \mathcal{L}_{\text{BD}} \rightarrow [0, 1]$ satisfying, for all $\phi, \psi \in \mathcal{L}_{\text{BD}}$,

1. $p(\perp) = 0$ and $p(\top) = 1$,
2. monotonicity. if $\phi \models_{\text{BD}} \psi$, then $p(\phi) \leq p(\psi)$,
3. import-export rule. $p(\phi \vee \psi) + p(\phi \wedge \psi) = p(\phi) + p(\psi)$.

A classical probability assignment is a function $p : \mathcal{L}_{\text{BD}} \rightarrow [0, 1]$ satisfying, for all $\phi, \psi \in \mathcal{L}_{\text{CL}}$,

1. normalization. $p(\top) = 1$,
2. monotonicity. if $\phi \models_{\text{CL}} \psi$, then $p(\phi) \leq p(\psi)$,
3. finite additivity. $p(\phi \vee \psi) = p(\phi) + p(\psi)$, for $(\phi \wedge \psi) \models_{\text{CL}} \perp$.

Monotonicity ensures that equivalent formulas have the same probability. For classical probabilities, if we define probability measures over the CL Lindenbaum algebra, monotonicity follows from additivity. In addition, one can easily show that any classical probability assignment is also a probability assignment.

Definition 8 (Probabilistic BD-models) A probabilistic BD-model is a tuple $\mathfrak{M} = \langle W, v_4, \mu \rangle$ such that $\langle W, v_4 \rangle$ is a BD-model and $\mu : \mathcal{P}(W) \rightarrow [0, 1]$ is a probability measure on $\mathcal{P}(W)$.

Then, the *induced probability assignment* is defined as follows: for any formula $\phi \in \mathcal{L}_{\text{BD}}$,

$$p(\phi) = \mu(|\mathbf{TB}\phi|).$$

Klein et al. [6] present the results over the two-valuation semantics of BD. Here, we choose to work with an equivalent semantics using a 4-valued valuation. We straightforwardly rephrase their results on that semantics.

While working with BD, some subsets of states of the canonical model cannot be represented by a formula, however using the axioms of probabilities, one can compute the measure of each subset of states. We use $\text{BD}\Delta$ as a tool to designate each subset of states via a formula. This will be useful to describe the focal elements of mass functions and

to present aggregation rules in a reader friendly manner. Probabilities over the logic $\text{BD}\Delta$ are not known in the literature and the algebraic semantic of this logic is not yet fully understood. Therefore, the study of (imprecise) probabilities over $\text{BD}\Delta$ is an open problem we do not discuss in this article. We now define the notion of belief and plausibility over BD formulas.

Definition 9 A belief and a plausibility assignment over BD formulas are functions $\text{Bel}, \text{Pl} : \mathcal{L}_{\text{BD}} \rightarrow [0, 1]$ s.t.

- (1) they are monotone w.r.t. BD entailment,
- (2) they satisfy respectively: for every $n \in \mathbb{N}$, for all $\phi_1 \dots \phi_n \in \mathcal{L}_{\text{BD}}$,

$$\text{Bel}(\phi_1 \vee \dots \vee \phi_n) \geq \sum_{\emptyset \neq I \subseteq [1, n]} (-1)^{|I|+1} \text{Bel} \left(\bigwedge_{i \in I} \phi_i \right), \quad (1)$$

and

$$\text{Pl}(\phi_1 \wedge \dots \wedge \phi_n) \leq \sum_{\emptyset \neq I \subseteq [1, n]} (-1)^{|I|+1} \text{Pl} \left(\bigvee_{i \in I} \phi_i \right). \quad (2)$$

A belief (resp. plausibility) assignment is said to be normal if $\text{Bel}(\perp) = 0$ and $\text{Bel}(\top) = 1$ (resp. $\text{Pl}(\perp) = 0$ and $\text{Pl}(\top) = 1$).

3. Dempster-Shafer theory and BD.

In this section, we discuss how to encode information via a mass function over the canonical model. Then we discuss the behaviour of DS-rule on the canonical model via examples. $m : \mathcal{P}(S) \rightarrow [0, 1]$ is a mass function over S if $\sum_{s \in S} m(s) = 1$. m is normal if $m(\emptyset) = 0$. Let m be a mass function over S . We note bel_m and pl_m the maps from $\mathcal{P}(S)$ to $[0, 1]$ defined as follows:

$$\text{bel}_m(X) = \sum_{Y \subseteq X} m(Y), \quad \text{and} \quad \text{pl}_m(X) = 1 - \text{bel}_m(X^c)$$

where X^c is the set complement of X . bel_m (resp. pl_m) is a belief (resp. plausibility) function.

In this section, we consider the canonical model $\mathfrak{M}_c = \langle W_c, v_4 \rangle$ over the finite set of propositional variables Prop .

Encoding of evidence on the canonical model. Within $\text{BD}\Delta$, the statement “there is information supporting p ” is encoded by the formula p and equivalently by the \vee -literal $\mathbf{T}p \vee \mathbf{B}p$ noted $\mathbf{TB}p$. Indeed, the formula p is satisfied by every state w such that $v_4(w)(p) \in \{\mathbf{T}, \mathbf{B}\}$ which is equivalent to $v_4(w)(\mathbf{TB}p) = \mathbf{T}$. Therefore, one encodes the statement “the information 100% supports p ” via the mass function $m_p : \mathcal{P}(W_c) \rightarrow [0, 1]$ such that $m(|\mathbf{TB}p|) = 1$. The mass function $m_{\mathbf{T}p}$ such that $m_{\mathbf{T}p}(|\mathbf{T}p|) = 1$ encodes classical evidence supporting p , i.e. “the information 100% supports p and there is no information available supporting $\neg p$ ”. We say that “the information supports exactly p .”

Encoding of contradiction. In this framework, we can encode a source stating “I have information supporting p with certainty 0.3, $\neg p$ with certainty 0.3 and the remaining of my information supports exactly q , and is contradictory about p ” via the mass function m such that $m(|\mathbf{TB}p|) = m(|\mathbf{TB}\neg p|) = 0.3$ and $m(|\mathbf{T}q \wedge \mathbf{B}p|) = 0.4$.

Encoding absence of evidence. One can differentiate between the statement “I have no information about p ” and transmitting no information by either assigning the mass to the set $|\mathbf{N}p|$ or to the set $W_c = |\top|$.

Encoding incomplete sources. A source is said to be incomplete if, for every $p \in \text{Prop}$, it says “either there is information supporting exactly p or there is information supporting exactly $\neg p$ or there is no information about p ”. For instance, a coherent database, that contains no contradictory information but has no information about some entries.

Encoding classical sources. A source is said classical, if it affirms that, for every $p \in \text{Prop}$, “the information supports exactly p or supports exactly $\neg p$.”

How to encode lack of information? When the source announces “ p with certainty 0.8”, where do we assign the remaining 0.2? The formulas $\psi_{\text{CL}} = \bigwedge_{p \in \text{Prop}} \mathbf{TF}p$ and $\psi_{\text{PCL}} = \bigwedge_{p \in \text{Prop}} \mathbf{TNF}p$ correspond to the set of states assigning values in the set $\{\mathbf{T}, \mathbf{F}\}$ or $\{\mathbf{T}, \mathbf{N}, \mathbf{F}\}$ to every variable, that is the so-called classical and incomplete states. The formula $\psi_{W_c} = \top$ corresponds to the set of all states. The choice between these alternatives depend on the nature of the source (e.g. is the source contradictory?) and the reasoning framework of the analyst (e.g. does the analyst accept contradictory evidence?).

Interpreting belief and plausibility over BD-models.

As the mass function is defined on the powerset algebra of the canonical model, bel_m and pl_m are respectively belief and plausibility functions and every combination rule using only the fact that the underlying algebra is a powerset algebra can still be used.

The BD-negation \neg does not correspond to the set theoretic complement, however the map $\text{pl}_m^- : \mathcal{P}(W_c) \rightarrow [0, 1]$ such that $\text{pl}_m^-(X_\phi) = 1 - \text{bel}_m(X_{\neg\phi})$ with $X_\phi = |\mathbf{TB}\phi|$ and $X_{\neg\phi} = |\mathbf{TB}\neg\phi|$ is still a plausibility function. $\text{pl}_m(|\mathbf{TB}\phi|)$ and $\text{pl}_m^-(|\mathbf{TB}\phi|)$ represent respectively the amount of evidence coherent with the truth of ϕ and not supporting the falsity of ϕ . But one can have models such that $\text{pl}_m(|\mathbf{TB}\phi|) \neq \text{pl}_m^-(|\mathbf{TB}\phi|)$ and $\text{bel}_m(|\mathbf{TB}\phi|) > \text{pl}_m^-(|\mathbf{TB}\phi|)$. See Bílková et al. [2] for more details.

Belief and plausibility as lower and upper paraconsistent probabilities. Recall that a probabilistic BD-model induces a probability assignment on BD-formulas s.t. $p(\phi) = \mu(|\phi|^{\mathbf{TB}})$. That is, the measure of the set of states

that support the truth of ϕ , but can be contradictory about ϕ . m induces the following lower (resp. upper) bound $\text{Bel}_{\text{TB}}(\phi)$ (resp. $\text{Pl}_{\text{TB}}(\phi)$) on $p(\phi)$:

$$\text{Bel}_{\text{TB}}(\phi) := \text{bel}_m(|\phi|^{\text{TB}}), \quad (3)$$

$$\text{Pl}_{\text{TB}}(\phi) := \text{pl}_m(|\phi|^{\text{TB}}) = 1 - \text{bel}_m((|\phi|^{\text{TB}})^c). \quad (4)$$

Since $|\phi \vee \psi|^{\text{TB}} = |\phi|^{\text{TB}} \cup |\psi|^{\text{TB}}$, $|\phi \wedge \psi|^{\text{TB}} = |\phi|^{\text{TB}} \cap |\psi|^{\text{TB}}$, and bel_m and pl_m are belief and plausibility function, Bel_{TB} and Pl_{TB} are respectively belief and plausibility assignments over BD formulas. Moreover, $\text{Bel}_{\text{TB}}(\top) = 1$ and $\text{Bel}_{\text{TB}}(\perp) = 0$. Therefore, Bel_{TB} and Pl_{TB} are normal.

Let \mathcal{F}_m be the set of probability measures on $\mathcal{P}(W)$ such that bel_m and pl_m are the lower and upper probabilities given by \mathcal{F}_m . Every probability measure $\mu : \mathcal{P}(W) \rightarrow [0, 1]$ induces a probability assignment p_μ on BD formulas such that, for any formula ϕ , $p_\mu(\phi) := \mu(|\phi|^{\text{TB}})$ as described in Section 2. Then Bel_{TB} and Pl_{TB} are the lower and upper probabilities given by the set $\mathcal{F}_{\text{TB}} = \{p_\mu \mid \mu \in \mathcal{F}_m\}$.

Behaviour of DS-rule. Since we are considering mass functions over powerset algebras, one may still apply existing combination rules. However, the results for non-classical sources may be unintuitive, as to be expected. Let m_1, m_2 be two mass functions over S . Their aggregation $m_{1\oplus 2}$ via DS-rule is defined as follows: $m_{1\oplus 2}(\emptyset) = 0$, otherwise

$$m_{1\oplus 2}(X) = \frac{\sum\{m_1(X_1) \cdot m_2(X_2) \mid X_1 \cap X_2 = X\}}{1 - \sum\{m_1(X_1) \cdot m_2(X_2) \mid X_1 \cap X_2 \neq \emptyset\}}.$$

Example 1 (Two disagreeing doctors) It is assumed that a patient can have one and only one of the diseases $S = \{a, b, c\}$ (the events a, b and c are incompatible).

Doctor 1 thinks that the patient has disease a with certainty 0.9 and that it is very unlikely the patient has disease b , therefore assigning certainty 0.1 to that option. Doctor 2 thinks that the patient has disease c with certainty 0.9 and that it is very unlikely the patient has disease b , therefore assigning certainty 0.1 to that option.

An analyst can interpret the claim of Doctor 1 as ‘‘I have information that $a \wedge \neg b \wedge \neg c$ is true and that $\neg(a \wedge \neg b \wedge \neg c)$ is false with certainty 0.9’’, which corresponds to assigning mass 0.9 to the set $|\text{Ta} \wedge \text{Fb} \wedge \text{Fc}|^{\text{T}}$. Following the same kind of reasoning for Doctor 2, the analyst gets the following mass functions:

$$\begin{aligned} m_1(|\text{Ta} \wedge \text{Fb} \wedge \text{Fc}|) &= 0.9, & m_1(|\text{Fa} \wedge \text{Tb} \wedge \text{Fc}|) &= 0.1, \\ m_2(|\text{Fa} \wedge \text{Fb} \wedge \text{Tc}|) &= 0.9, & m_2(|\text{Fa} \wedge \text{Tb} \wedge \text{Fc}|) &= 0.1. \end{aligned}$$

Since $\text{Ta} \wedge \text{Fa}$ is a contradictory statement in $\text{BD}\Delta$, one will get exactly the same conclusion as in the classical case by applying DS-rule, that is: $m_{1\oplus 2}(|\text{Fa} \wedge \text{Tb} \wedge \text{Fc}|) = 1$.

This conclusion may be perfectly justified if Doctor 1 (resp. 2) is an expert on disease c (resp. a), therefore

when they say it cannot be disease c (resp. a), they are necessarily correct. However, one may want to be able to consider doctors who might be wrong (without having to evaluate how reliable they are) or to describe situations in which the doctors refuse to give an opinion about the diseases they are not knowledgeable about.

Example 2 (Reasoning with BD) We still consider our two doctors, but we do not assume that they are fully reliable experts. Therefore, when they announce p we assign the corresponding mass to $|\text{TB}p|$ rather than $|\text{Tp}|$. In this framework, this means that two contradictory pieces of information $|\text{TB}p|$ and $|\text{FB}p|$ are not inconsistent. When doctor 1 (resp. 2) announces the evidence supports disease x , they still imply that they reject the other diseases. We get the following mass functions

$$m_1(|\text{TB}a \wedge \text{FB}b \wedge \text{FB}c|) = 0.9$$

$$m_1(|\text{FB}a \wedge \text{TB}b \wedge \text{FB}c|) = 0.1$$

$$m_2(|\text{FB}a \wedge \text{FB}b \wedge \text{TB}c|) = 0.9$$

$$m_2(|\text{FB}a \wedge \text{TB}b \wedge \text{FB}c|) = 0.1$$

and the following aggregated mass function

$$m_{1\oplus 2}(X) = \begin{cases} 0.81 & \text{if } X = |\text{Ba} \wedge \text{FB}b \wedge \text{Bc}| \\ 0.09 & \text{if } X = |\text{Ba} \wedge \text{Bb} \wedge \text{FB}c| \\ & \text{or } X = |\text{FB}a \wedge \text{Bb} \wedge \text{Bc}| \\ 0.01 & \text{if } X = |\text{FB}a \wedge \text{TB}b \wedge \text{FB}c| \\ 0 & \text{otherwise.} \end{cases}$$

$\text{bel}_{1\oplus 2}(|\text{TB}\phi|)$ and $\text{pl}_{1\oplus 2}(|\text{TB}\phi|) = 1 - \text{bel}_{1\oplus 2}(|\text{NF}\phi|)$ provide the following lower and upper probabilities:

ϕ	a	$\neg a$	b	$\neg b$	$a \wedge \neg a$	$b \wedge \neg b$
$\text{bel}_{1\oplus 2}$	0.9	1	0.19	0.99	0.9	0.18
$\text{pl}_{1\oplus 2}$	1	1	1	1	1	1

Notice that $v_4(p \wedge \neg p, w) = \mathbf{B}$ if $v_4(p, w) = \mathbf{B}$ and $v_4(p \wedge \neg p, w) = \mathbf{N}$ otherwise. The values for a (resp. $\neg a$, $a \wedge \neg a$) are the same as those for c (resp. $\neg c$, $c \wedge \neg c$).

We have that $\text{bel}_{1\oplus 2}(|\text{TB}a|) = \text{bel}_{1\oplus 2}(|\text{TB}c|) = 0.9$ is 4.7 times larger than $\text{bel}_{1\oplus 2}(|\text{TB}b|) = 0.19$ where $\text{bel}_{1\oplus 2}(|\text{TB}a|)$ encodes how much evidence supports the truth of a . In addition, $\text{bel}_{1\oplus 2}(|\text{Ba}|) = \text{bel}_{1\oplus 2}(|\text{Bc}|) = 0.9$ is 5 times larger than $\text{bel}_{1\oplus 2}(|\text{Bb}|) = 0.18$ which means that the evidence is highly contradictory regarding a and c and 5 times less contradictory regarding b . This is confirmed by the fact that $\text{bel}_{1\oplus 2}(|\text{FB}a|) = \text{bel}_{1\oplus 2}(|\text{FB}c|) = 1$ - that is all the evidence supports the fact that it is not disease a or c - and $\text{bel}_{1\oplus 2}(|\text{TB}a|) = \text{bel}_{1\oplus 2}(|\text{Ba}|) = 0.9$ - that is, all the evidence supporting the truth of a is contradictory.

The mass function highlights that (1) doctors agree the patient most likely does not have disease b , (2) evidence strongly supports the hypothesis that they have disease a or c and (3) the evidence is highly contradictory which may suggest that further investigation is necessary.

Example 3 (Reasoning with incomplete information)

Now, Doctor 1 considers only diseases a and b and states

that it is disease a with certainty 0.9 and disease b with certainty 0.1. They do not provide any information about disease c . Doctor 2 states that it is disease b with certainty 0.1, disease c with certainty 0.9 and that they could not find any conclusive information about disease a and cannot estimate the induced uncertainty. If one assumes that the sources reason classically, one can encode them as follows:

$$\begin{aligned} m_1(|\mathbf{T}a \wedge \mathbf{F}b|) &= 0.9 & m_2(|\mathbf{N}a \wedge \mathbf{F}b \wedge \mathbf{T}c|) &= 0.9 \\ m_1(|\mathbf{F}a \wedge \mathbf{T}b|) &= 0.1 & m_2(|\mathbf{N}a \wedge \mathbf{T}b \wedge \mathbf{F}c|) &= 0.1. \end{aligned}$$

Here, it is impossible to aggregate using DS-rule because $\mathbf{T}a \wedge \mathbf{N}a = \mathbf{F}a \wedge \mathbf{N}a = \perp$. That is, DS-rule behaves as if the information about a was contradictory. However, if one source says ‘I have information supporting a ’ and one source says ‘I have no information about a ’, one would expect to conclude ‘the sources have information supporting a ’. This problem will be resolved by the variation of DS-rule proposed in Section 4.

Example 1 shows that one can still encode classical reasoning and computation in the proposed BD framework, by assigning non-zero mass only to sets of classical states. Example 2 shows that if one assigns non-zero mass only to sets representable as the \mathbf{TB} -extensions of BD formulas, then DS-rule allows to describe the nature (contradictory or not) of the available information. Example 3 shows that DS-rule may not be suitable for combination of more general types of sources. In the next section we propose a variation of DS-rule more suitable for our framework.

4. Aggregation rule based on the information lattice

In the previous section, we discussed the behaviour of DS-rule within the framework of BD. If one works only with set of states that are the \mathbf{TB} extension of BD-formulas, then DS-rule provides a natural way to aggregate information: if two sources agree about the truth of ϕ , the corresponding mass is assigned to $|\mathbf{TB}\phi|$, otherwise it is assigned to $|\mathbf{B}\phi|$. However, if one wants to work with mass functions whose focal elements are arbitrary sets of states, behaviour of DS-rule is less intuitive (see Example 3).

As shown in Example 1, classical pieces of information can be encoded in BD framework by assigning mass to the set $|\mathbf{T}\phi|$. In this case, DS-rule will behave as in the classical framework. However, here we get more options to assign the mass $m_1(X) \cdot m_2(Y)$ when $X \cap Y = \emptyset$ (i.e. mass of contradictory evidence). As discussed previously (page 1), the algebra of truth values of BD (Figure 1) contains two lattices: the *truth lattice* and the *information lattice*. In this section, we use the information lattice to propose a variation of DS-rule.

First, we present the behavior of the logical connectives of the information lattice. Then, we define a fusion operation \odot

on maximal wDNF’s using \sqcup and use this operation to define an aggregation rule on mass functions over the canonical model. Finally, we discuss the mathematical properties and the behavior of this aggregation rule.

Definition 10 We denote \sqcup (resp. \sqcap) the disjunction (resp. conjunction) of the information lattice of Figure 1. It is called information join (resp. information meet). These operators on \mathcal{V} are defined as follows:

\sqcup	\mathbf{T}	\mathbf{B}	\mathbf{N}	\mathbf{F}	\sqcap	\mathbf{T}	\mathbf{B}	\mathbf{N}	\mathbf{F}
\mathbf{T}	\mathbf{T}	\mathbf{B}	\mathbf{T}	\mathbf{B}	\mathbf{T}	\mathbf{T}	\mathbf{T}	\mathbf{N}	\mathbf{N}
\mathbf{B}	\mathbf{B}	\mathbf{B}	\mathbf{B}	\mathbf{B}	\mathbf{B}	\mathbf{T}	\mathbf{B}	\mathbf{N}	\mathbf{F}
\mathbf{N}	\mathbf{T}	\mathbf{B}	\mathbf{N}	\mathbf{F}	\mathbf{N}	\mathbf{N}	\mathbf{N}	\mathbf{N}	\mathbf{N}
\mathbf{F}	\mathbf{B}	\mathbf{B}	\mathbf{F}	\mathbf{F}	\mathbf{F}	\mathbf{N}	\mathbf{F}	\mathbf{N}	\mathbf{F}

We propose the following fusion operator on formulas. If two formulas convey non-contradictory information, then the fusion returns the common ground. For instance, $\mathbf{TB}p \odot \mathbf{TF}p = \mathbf{T}p$. Indeed, the first formula states ‘the evidence supports the truth of p or it is contradictory about p ’, and the second states ‘the evidence supports the truth of p or the falsity of p ’. Both agree on the fact that ‘the evidence supports the truth of p ’. This behavior is exactly the same as DS-rule on non-contradictory pieces of evidence.

If two formulas convey contradictory information about p , then the fusion returns the information join on the information about p . For instance, $\mathbf{T}p \odot \mathbf{F}p = \mathbf{B}p$ and $(\mathbf{T}p \wedge \mathbf{T}q) \odot (\mathbf{F}p \wedge \mathbf{T}q) = \mathbf{B}p \wedge \mathbf{T}q$. One can also consider more complex formulas such as $\mathbf{TB}p \odot \mathbf{FN}p = \mathbf{TB}p$. The first formula states ‘the information supports the truth of p or it is contradictory about p ’ and the second formula states ‘the information supports the falsity of p or is inconclusive about p ’. The fusion conveys the fact that ‘the aggregated information either supports the truth of p or is contradictory about p ’.

In the following, we define the fusion on maximal wDNF. This is enough because every set of states is uniquely represented by a maximal wDNF and vice versa.

Definition 11 (Fusion) Let \odot be a binary operation defined on $\text{BD}\Delta$ formulas in maximal wDNF as follows. Let $\phi = \bigvee_{i \in I} c_i$ and $\psi = \bigvee_{j \in J} d_j$ be $\text{BD}\Delta$ formulas in maximal wDNF. If $\phi \wedge \psi \not\equiv_{\text{BD}} \perp$, then $\phi \odot \psi = \phi \wedge \psi$, else

$$\phi \odot \psi = \bigvee_{i \in I, j \in J} c_i \odot d_j \quad \text{with} \quad c_i \odot d_j = \bigwedge_{p \in \text{Prop}} (X_i \sqcup X_j)p.$$

c_i and d_j are complete clauses, therefore each variable appears exactly once in a \mathcal{V} -atom.

Proposition 12 \odot is idempotent and commutative. The neutral element is the mass function assigning 1 to W_c .

Notice that the empty set is represented by the empty disjunction and that the fusion between a formula and the empty disjunction is the empty disjunction. In addition,

since \wedge is idempotent, \odot is idempotent. Since, \wedge and \sqcup are commutative, \odot is commutative. However, it is not associative. For instance, consider the formulas: $\phi = (\mathbf{T} \wedge \mathbf{N}q) \vee (\mathbf{N}p \wedge \mathbf{F}q)$, $\chi = (\mathbf{N}p \wedge \mathbf{T}q) \vee (\mathbf{N}p \wedge \mathbf{N}q)$ and $\psi = (\mathbf{T}p \wedge \mathbf{N}q) \vee (\mathbf{N}p \wedge \mathbf{T}q) \vee (\mathbf{T}p \wedge \mathbf{F}q)$.

Now, we can define the aggregation rule \odot over mass functions on canonical models.

Definition 13 Let $m_1, m_2 : \mathcal{P}(W_c) \rightarrow [0, 1]$ be two mass functions on the canonical model. Let ϕ_X denote the maximal wDNF such that $|\phi_X| = X$ for any $X \subseteq W_c$. Let $X \odot Y$ denote the set $|\phi_X \odot \phi_Y|$. The aggregation $m_{1 \odot 2} : \mathcal{P}(W_c) \rightarrow [0, 1]$ is defined as follows.

$$m_{1 \odot 2}(Z) = \sum_{X, Y \in \mathcal{P}(W_c) \text{ s.t. } X \odot Y = Z} m_1(X) \cdot m_2(Y). \quad (5)$$

Example 1. Two disagreeing doctors. One gets the following aggregated mass function

$$m_{1 \odot 2}(X) = \begin{cases} 0.81 & \text{if } X = |\mathbf{B}a \wedge \mathbf{F}b \wedge \mathbf{B}c| \\ 0.09 & \text{if } X = |\mathbf{B}a \wedge \mathbf{B}b \wedge \mathbf{F}c| \\ & \text{or } X = |\mathbf{F}a \wedge \mathbf{B}b \wedge \mathbf{B}c| \\ 0.01 & \text{if } X = |\mathbf{F}a \wedge \mathbf{T}b \wedge \mathbf{F}c| \\ 0 & \text{otherwise.} \end{cases}$$

The focal elements of $m_{1 \odot 2}$ are singletons, therefore it induces a probability assignment p such that

ϕ	a	$\neg a$	b	$\neg b$	$a \wedge \neg a$	$b \wedge \neg b$
p	0.9	1	0.19	0.99	0.9	0.18

and $p(\phi) = \text{bel}_{1 \odot 2}(|\mathbf{T}\mathbf{B}\phi|) = \text{pl}_{1 \odot 2}(|\mathbf{T}\mathbf{B}\phi|)$. Notice that the conclusions are very similar to Example 2. The mass function encodes the fact that the information is highly contradictory about a and c with $p(|\mathbf{B}a|) = p(|\mathbf{B}c|) = 0.9$ and that the evidence supporting b is rather weak and mostly non-contradictory: $p(|\mathbf{T}b|) = 0.01$, $p(|\mathbf{F}b|) = 0.81$, $p(|\mathbf{B}b|) = 0.18$. Thus, unlike in the case of DS-rule where all the contradictory evidence is discarded, our combination rule describes the extent of contradictions between doctors about different diseases.

Example 2. Two disagreeing doctors. Here, the aggregation will be identical to DS-rule, since the mass functions do not contradict each other in the sense of BD.

Example 3. Reasoning with incomplete information. One gets the following aggregated mass function.

$$m_{1 \odot 2}(X) = \begin{cases} 0.81 & \text{if } X = |\mathbf{T}a \wedge \mathbf{F}b \wedge \mathbf{T}c| \\ 0.09 & \text{if } X = |\mathbf{T}a \wedge \mathbf{B}b \wedge \mathbf{F}c| \\ & \text{or } X = |\mathbf{F}a \wedge \mathbf{B}b \wedge \mathbf{T}c| \\ 0.01 & \text{if } X = |\mathbf{F}a \wedge \mathbf{T}b \wedge \mathbf{F}c| \\ 0 & \text{otherwise.} \end{cases}$$

The focal elements of $m_{1 \odot 2}$ are singletons, therefore it induces a probability assignment p such that

ϕ	a	$\neg a$	b	$\neg b$	$a \wedge \neg a$	$b \wedge \neg b$
p	0.9	0.1	0.19	0.99	0	0.18

the values for a (resp. $\neg a$, $a \wedge \neg a$) are the same as those for c (resp. $\neg c$, $c \wedge \neg c$). Here, the mass function encodes the fact that the available information strongly supports the truth of a and c . This result comes from the fact that \sqcup tells us the nature of the available information: incomplete, contradictory, supporting the statement.

Thus, the proposed combination rule \odot has the following advantages over DS-rule: (1) It behaves more in accordance with intuition than DS-rule when dealing with pieces of evidence which cannot be encoded using BD-formulas as shown in Example 3. (2) It provides a more nuanced framework to pinpoint topics leading to conflict between classical sources and quantify the conflict on different topics rather than overall conflict as shown in Example 1.

5. What would be a ‘good classical’ piece of evidence?

In this section, we discuss what can be considered as a *good classical* piece of evidence for a statement ϕ in the paraconsistent framework, and therefore which notions - based on the evidence- of “belief” and “plausibility” would be the most pertinent to estimate a lower and upper bound on an unknown classical probability assignment p on the formulas.

Let $\mathfrak{M}_c = \langle W_c, v_4 \rangle$ be the canonical model over Prop. Let $m : \mathcal{P}(W_c) \rightarrow [0, 1]$ be a mass function and bel_m (resp. pl_m) its associated belief (resp. plausibility) function (cf. Section 3). Consider a formula $\phi \in \mathcal{L}_{\text{BD}}$.

T-support. A state $w \in W_c$ provides *T-support* to ϕ , if $w \in |\phi|^{\mathbf{T}}$. That is, w supports the truth of ϕ , and does not support falsity of ϕ . Notice that if $v_4(p, w) = \mathbf{N}$ and $v_4(q, w) = \mathbf{B}$, then $v_4(p \vee q, w) = \mathbf{T}$. Thus, in general $|\phi \vee \psi|^{\mathbf{T}} \neq |\phi|^{\mathbf{T}} \cup |\psi|^{\mathbf{T}}$. We define the following lower (resp. upper) bound $\text{Bel}_{\mathbf{T}}(\phi)$ (resp. $\text{Pl}_{\mathbf{T}}(\phi)$) on $p(\phi)$:

$$\text{Bel}_{\mathbf{T}}(\phi) := \text{bel}_m(|\phi|^{\mathbf{T}}) \text{ and } \text{Pl}_{\mathbf{T}}(\phi) := 1 - \text{bel}_m(|\phi|^{\mathbf{F}}).$$

Since $\phi \models_{\text{BD}} \psi$ implies that $|\phi|^{\mathbf{T}} \subseteq |\psi|^{\mathbf{T}}$, $\text{Bel}_{\mathbf{T}}$ and $\text{Pl}_{\mathbf{T}}$ are monotone w.r.t. \models_{BD} . In addition, we have $|\phi|^{\mathbf{T}} \cup |\psi|^{\mathbf{T}} \subseteq |\phi \vee \psi|^{\mathbf{T}}$ and $|\phi|^{\mathbf{T}} \cap |\psi|^{\mathbf{T}} = |\phi \wedge \psi|^{\mathbf{T}}$. Thus, for every $n \in \mathbb{N}$, for every $\phi_1, \dots, \phi_n \in \mathcal{L}_{\text{BD}}$,

$$\begin{aligned} \text{Bel}_{\mathbf{T}}\left(\bigvee_{1 \leq i \leq n} \phi_i\right) &= \text{bel}_m\left(|\bigvee_{1 \leq i \leq n} \phi_i|^{\mathbf{T}}\right) \geq \text{bel}_m\left(\bigcup_{1 \leq i \leq n} |\phi_i|^{\mathbf{T}}\right), \\ \text{Bel}_{\mathbf{T}}\left(\bigwedge_{1 \leq i \leq n} \phi_i\right) &= \text{bel}_m\left(|\bigwedge_{1 \leq i \leq n} \phi_i|^{\mathbf{T}}\right) = \text{bel}_m\left(\bigcap_{i \in I} |\phi_i|^{\mathbf{T}}\right). \end{aligned}$$

Moreover, $\text{Bel}_{\mathbf{T}}(\perp) = 0$ and $\text{Bel}_{\mathbf{T}}(\top) = 1$. Hence, $\text{Bel}_{\mathbf{T}}$ is a belief assignment over BD. In addition, we have $1 - \text{bel}_m(|\phi|^{\mathbf{F}}) = 1 - \text{bel}_m(|\neg\phi|^{\mathbf{T}}) = 1 - \text{Bel}_{\mathbf{T}}(\neg\phi)$. As,

\neg is a De Morgan negation, $\text{Pl}_{\mathbf{T}}$ is a normal plausibility function [5, Lemma 2.14]. Thus, $\text{Bel}_{\mathbf{T}}$ and $\text{Pl}_{\mathbf{T}}$ are respectively normal belief and plausibility assignments. Note that since $\text{Bel}_{\mathbf{T}} \leq \text{Bel}_{\mathbf{TB}} \leq \text{Pl}_{\mathbf{TB}} \leq \text{Pl}_{\mathbf{T}}$, every probability assignment $\mathbf{p} \in \mathcal{F}_{\mathbf{TB}}$ satisfies $\text{Bel}_{\mathbf{T}} \leq \mathbf{p} \leq \text{Pl}_{\mathbf{T}}$. Thus, $\text{Bel}_{\mathbf{T}}$ and $\text{Pl}_{\mathbf{T}}$ define a non-empty credal set of probabilities.

This notion of support in a formula ϕ does not require for a piece of evidence to be consistent about every subformula of ϕ , it simply requires for the evidence to support ϕ without supporting its negation.

Support based on classical ‘proofs’. Let $Q_w := \{l \in \text{Lit} \mid v_4(l, w) = \mathbf{T}\}$. A state $w \in W$ supports ϕ if it provides a classical ‘proof’ of ϕ , i.e. if $(\bigwedge_{l \in Q_w} l) \models_{\text{CL}} \phi$. For any formula ϕ , let $|\phi|^{\text{CP}}$ be the set of states which provide a classical ‘proof’ of ϕ . We obtain the following lower bound $\text{Bel}_{\text{CP}}(\phi)$ on $\mathbf{p}(\phi)$:

$$\text{Bel}_{\text{CP}}(\phi) = \text{bel}_{\mathbf{m}}(|\phi|^{\text{CP}}).$$

The associated plausibility function is defined by $\text{Pl}_{\text{CP}}(\phi) = 1 - \text{Bel}_{\text{CP}}(\neg\phi)^{\text{CP}}$. By the properties of classical entailment we have $|\phi|^{\text{CP}} \cup |\psi|^{\text{CP}} \subseteq |\phi \vee \psi|^{\text{CP}}$ and $|\phi \wedge \psi|^{\text{CP}} = |\phi|^{\text{CP}} \cap |\psi|^{\text{CP}}$. This implies that Bel_{CP} satisfies equation (1). As $\text{Pl}_{\text{CP}}(\phi) = 1 - \text{bel}_{\mathbf{m}}(\neg\phi)^{\text{CP}}$, pl_{CP} satisfies equation (2). Moreover, since \perp and \top are falsum and tautology, we have $\text{Bel}_{\text{CP}}(\perp) = 0$ and $\text{Bel}_{\text{CP}}(\top) = 1$. Thus, Bel_{CP} and Pl_{CP} are belief and plausibility assignments.

This notion of support is natural when the analyst only considers evidence as support for ϕ if the part of the evidence which is classical entails ϕ .

Remark 14 Note that for any propositional variable p , we have $|p|^{\text{CP}} = |p|^{\text{T}}$. Thus, we have $\text{Bel}_{\text{CP}}(p) = \text{Bel}_{\mathbf{T}}(p)$. Moreover, for any formula ϕ , if ϕ has a classical proof, then it must be exactly true. Therefore, $\text{Bel}_{\text{CP}} \leq \text{Bel}_{\mathbf{T}}$.

Support from incomplete (resp. classical) states. Here, for the support of ϕ , we consider only incomplete (resp. classical) states. Let $\mathbf{IC} \subseteq W_c$ (resp. $\mathbf{C} \subseteq W_c$), be the set of incomplete (resp. classical) states. These notions of support induce respectively the following lower bounds on $\mathbf{p}(\phi)$,

$$\text{Bel}_{\mathbf{IC}}(\phi) := \text{bel}_{\mathbf{m}}(|\phi|^{\text{T}} \cap \mathbf{IC}),$$

and

$$\text{Bel}_{\mathbf{C}}(\phi) := \text{bel}_{\mathbf{m}}(|\phi|^{\text{T}} \cap \mathbf{C}).$$

The associated upper bounds are given by $\text{Pl}_{\mathbf{IC}}(\phi) = 1 - \text{bel}_{\mathbf{m}}(|\phi|^{\text{F}} \cap \mathbf{IC})$ and $\text{Pl}_{\mathbf{C}}(\phi) = 1 - \text{bel}_{\mathbf{m}}(|\phi|^{\text{F}} \cap \mathbf{C})$. As $\text{Bel}_{\mathbf{T}}$ and $\text{Pl}_{\mathbf{T}}$ satisfy equations (1) and (2) respectively, $\text{Bel}_{\mathbf{IC}}$ (resp. $\text{Bel}_{\mathbf{C}}$) and $\text{Pl}_{\mathbf{IC}}$ (resp. $\text{Pl}_{\mathbf{C}}$) must also satisfy these equations. Therefore, $\text{Bel}_{\mathbf{IC}}$ (resp. $\text{Bel}_{\mathbf{C}}$) and $\text{Pl}_{\mathbf{IC}}$ (resp. $\text{Pl}_{\mathbf{C}}$) are belief and plausibility assignments.

However, since we only consider incomplete (resp. classical) states when we calculate $\text{Bel}_{\mathbf{IC}}$ (resp. $\text{Bel}_{\mathbf{C}}$),

$\text{Bel}_{\mathbf{IC}}(\top)$ (resp. $\text{Bel}_{\mathbf{C}}(\top)$) need not be 1. Thus, in general, $\text{Bel}_{\mathbf{IC}}$ and $\text{Bel}_{\mathbf{C}}$ are non-normal belief functions.

This notion of support is natural when the analyst wants to restrict themselves to the incomplete or classical states. That is, they ignore completely any evidence which is contradictory in evaluating belief and plausibility.

Comparison of the different supports.

Proposition 15 For any mass function \mathbf{m} and any $\phi \in \mathcal{L}_{\text{BD}}$, we have the following chain of inclusions.

$$\begin{aligned} [\text{Bel}_{\text{BD}}(\phi), \text{Pl}_{\text{BD}}(\phi)] &\subseteq [\text{Bel}_{\mathbf{T}}(\phi), \text{Pl}_{\mathbf{T}}(\phi)] \\ &\subseteq [\text{Bel}_{\text{CP}}(\phi), \text{Pl}_{\text{CP}}(\phi)] \subseteq [\text{Bel}_{\mathbf{IC}}(\phi), \text{Pl}_{\mathbf{IC}}(\phi)] \\ &\subseteq [\text{Bel}_{\mathbf{C}}(\phi), \text{Pl}_{\mathbf{C}}(\phi)]. \end{aligned}$$

For any formula ϕ , the interval $[\text{Bel}(\phi), \text{Pl}(\phi)]$ describes the uncertainty in probability of ϕ . Thus, intuitively, this proposition corresponds to the fact that uncertainty in the formula increases as requirement for admittance of an evidence as support increases. That is, if more information is ignored by the analyst (due to stronger requirements) the uncertainty in probability of any formula ϕ increases.

6. Lower and upper bounds for classical probabilities

In this section, we show that Bel_{CP} , $\text{Bel}_{\mathbf{IC}}$ and $\text{Bel}_{\mathbf{C}}$ induce credal sets of classical probabilities on CL Lindenbaum algebra, and that, in general, it is not possible to find classical probabilities coherent with $\text{Bel}_{\mathbf{T}}$.

Let \mathbb{B} be the CL Lindenbaum algebra over \mathcal{L}_{BD} (Definition 2). Since, Prop is finite, \mathbb{B} is finite too. Let \mathbf{m} be a mass function on the canonical model, $\mathbf{X} \in \{\text{CP}, \mathbf{C}, \mathbf{IC}\}$, and $\text{Bel}_{\mathbf{X}}$ and $\text{Pl}_{\mathbf{X}}$ be the belief and plausibility functions on \mathcal{L}_{BD} corresponding to \mathbf{X} -support. Let $\text{bel}_{\mathbf{X}}, \text{pl}_{\mathbf{X}} : \mathbb{B} \rightarrow [0, 1]$ be as follows: for any $[\phi] \in \mathbb{B}$,

$$\text{bel}_{\mathbf{X}}([\phi]) = \text{Bel}_{\mathbf{X}}(\hat{\bigvee}[\phi]), \text{ and } \text{pl}_{\mathbf{X}}([\phi]) = \text{Pl}_{\mathbf{X}}(\hat{\bigwedge}[\phi]),$$

with $\hat{\bigvee}[\phi] := \bigvee\{\psi \in \mathcal{L}_{\text{BD}} \mid \psi \in [\phi] \text{ and } \psi \text{ is in iDNF}\}$, and $\hat{\bigwedge}[\phi] := \bigwedge\{\psi \in \mathcal{L}_{\text{BD}} \mid \psi \in [\phi] \text{ and } \psi \text{ is in iDNF}\}$. $[\phi] \subseteq \mathcal{L}_{\text{BD}}$ is an infinite set of formulas, but each formula in BD is equivalent to a formula in iDNF and there are finitely many formulas in iDNF. Therefore, $\hat{\bigvee}[\phi]$ is well-defined.

Proposition 16 The following equalities hold:

$$\begin{aligned} [\hat{\bigvee}[\phi]] &= [\phi] & [\hat{\bigwedge}[\phi]] &= [\phi] \\ [\hat{\bigvee}[\phi \vee \psi]] &= [\hat{\bigvee}([\phi] \vee [\psi])] = [\hat{\bigvee}[\phi] \vee \hat{\bigvee}[\psi]] \\ [\hat{\bigvee}[\phi \wedge \psi]] &= [\hat{\bigvee}([\phi] \wedge [\psi])] = [\hat{\bigvee}[\phi] \wedge \hat{\bigvee}[\psi]] \\ [\hat{\bigwedge}[\phi \vee \psi]] &= [\hat{\bigwedge}([\phi] \vee [\psi])] = [\hat{\bigwedge}[\phi] \vee \hat{\bigwedge}[\psi]] \\ [\hat{\bigwedge}[\phi \wedge \psi]] &= [\hat{\bigwedge}([\phi] \wedge [\psi])] = [\hat{\bigwedge}[\phi] \wedge \hat{\bigwedge}[\psi]] \\ [\neg\hat{\bigwedge}[\phi]] &= [\hat{\bigvee}\neg[\phi]] = [\hat{\bigvee}[\neg\phi]] \\ [\neg\hat{\bigvee}[\phi]] &= [\hat{\bigwedge}\neg[\phi]] = [\hat{\bigwedge}[\neg\phi]] \end{aligned}$$

Theorem 17 Let CM be the set of classical probability measures on the finite Boolean algebra \mathbb{B} , let $\mathbb{B}^* = \mathbb{B} \setminus \{\top_{\mathbb{B}}\}$, and for $\mathbf{X} \in \{\mathbf{IC}, \mathbf{C}, \mathbf{CP}\}$,

$$\mathcal{F}_{\mathbf{X}} := \{\mu \in \text{CM} \mid \forall a \in \mathbb{B}, \text{bel}_{\mathbf{X}}(a) \leq \mu(a)\}.$$

For $\mathbf{X} \in \{\mathbf{IC}, \mathbf{C}\}$ (resp. $\mathbf{X} = \mathbf{CP}$), $\text{bel}_{\mathbf{X}}$ and $\text{pl}_{\mathbf{X}}$ are non-normal (resp. normal) belief and plausibility functions on \mathbb{B} , and, for $a \in \mathbb{B}^*$ (resp. $a \in \mathbb{B}$), $\text{bel}_{\mathbf{X}}(a)$ and $\text{pl}_{\mathbf{X}}(a)$ provide optimal lower and upper bounds on $\mathcal{F}_{\mathbf{X}}$. In addition,

$$\emptyset \subsetneq \mathcal{F}_{\mathbf{CP}} \subseteq \mathcal{F}_{\mathbf{IC}} \subseteq \mathcal{F}_{\mathbf{C}}$$

Proof Let $\phi, \phi' \in \mathcal{L}_{\text{BD}}$. Note that for every $\phi' \in [\phi]$, we have $\text{Bel}_{\mathbf{X}}(\phi') \leq \text{bel}_{\mathbf{X}}([\phi])$ and $\text{pl}_{\mathbf{X}}([\phi]) \leq \text{Pl}_{\mathbf{X}}(\phi')$. Since, $\text{Bel}_{\mathbf{X}}$ and $\text{Pl}_{\mathbf{X}}$ are belief and plausibility assignments, Proposition 16 implies that $\text{bel}_{\mathbf{X}}$ and $\text{pl}_{\mathbf{X}}$ are monotone and satisfy equations (1) and (2) respectively. In addition, for $\phi \in \mathcal{L}_{\text{BD}}$, we have

$$\begin{aligned} \text{pl}_{\mathbf{X}}([\phi]) &= \text{Pl}_{\mathbf{X}}(\hat{\wedge}[\phi]) = 1 - \text{Bel}_{\mathbf{X}}(\neg \hat{\wedge}[\phi]) \\ &= 1 - \text{Bel}_{\mathbf{X}}(\hat{\vee}[\neg\phi]) = 1 - \text{bel}_{\mathbf{X}}(\neg[\phi]). \end{aligned}$$

Moreover, $\phi \in [\perp]$ implies $\text{Bel}_{\mathbf{X}}(\phi) = 0$, because $|\phi|^{\mathbf{T}} \cap \mathbf{IC} = |\phi|^{\mathbf{T}} \cap \mathbf{C} = \emptyset$ and $(\bigwedge_{l \in \mathcal{Q}_w} l) \not\models_{\text{CL}} \perp$ for every $w \in W_c$. In addition, $\text{bel}_{\mathbf{CP}}([\top]) = 1$, because $\emptyset \models_{\text{CL}} \top$. Therefore, $\text{bel}_{\mathbf{CP}}$ and $\text{pl}_{\mathbf{CP}}$ are normal belief and plausibility functions. However, for $\mathbf{X} \in \{\mathbf{IC}, \mathbf{C}\}$, $\text{bel}_{\mathbf{X}}$ and $\text{pl}_{\mathbf{X}}$ are non-normal belief and plausibility functions. Indeed, $\text{bel}_{\mathbf{X}}([\top])$ and $\text{pl}_{\mathbf{X}}([\perp])$ can be different than 1 and 0 respectively, because $\text{bel}_{\mathbf{X}}([\top]) = \text{bel}_{\mathbf{m}}(\mathbf{X}) \neq \text{bel}_{\mathbf{m}}(W_c)$.

For $\mathbf{X} = \mathbf{CP}$, since $\text{bel}_{\mathbf{CP}}$ is a normal belief function on a Boolean algebra, and $\text{pl}_{\mathbf{X}}$ is its associated plausibility function, $\mathcal{F}_{\mathbf{X}}$ is a non empty credal set and $\text{bel}_{\mathbf{X}}$ and $\text{pl}_{\mathbf{X}}$ are its lower and upper probabilities. Now, let us consider the case $\mathbf{X} \in \{\mathbf{IC}, \mathbf{C}\}$. Let $\mu \in \mathcal{F}_{\mathbf{X}}$. If $\text{bel}_{\mathbf{X}}(a) \leq \mu(a)$ for every $a \in \mathbb{B}$, then, for every $a \in \mathbb{B}$, we have

$$\mu(\neg a) = 1 - \mu(a) \leq 1 - \text{bel}_{\mathbf{X}}(a) = \text{pl}_{\mathbf{X}}(\neg a).$$

Thus, $\mathcal{F}_{\mathbf{X}}$ is the set of classical probability measures lying between $\text{bel}_{\mathbf{X}}$ and $\text{pl}_{\mathbf{X}}$ for every $a \in \mathbb{B}$.

We can describe $\mathcal{F}_{\mathbf{X}}$ constructively as follows. Let $\text{m}_{\text{bel}_{\mathbf{X}}} : \mathbb{B} \rightarrow [0, 1]$ be defined as follows: for $a \in \mathbb{B}$,

$$\text{m}_{\text{bel}_{\mathbf{X}}}(a) = \text{bel}_{\mathbf{X}}(a) - \sum_{b < a} \text{m}_{\text{bel}_{\mathbf{X}}}(b).$$

We have $\sum_{a \in \mathbb{B}} \text{m}_{\text{bel}_{\mathbf{X}}}(a) = \text{bel}_{\mathbf{X}}(\top_{\mathbb{B}}) = \text{bel}_{\mathbf{X}}([\top]) \leq 1$. Let At be the set of atoms of \mathbb{B} , $k = |\text{At}|$, and, for every $a \in \mathbb{B}$, let $\text{At}(a)$ be the set of atoms below a and $k_a = |\text{At}(a)|$. Let $\mathcal{M}_{\mathbf{X}}$ be the set of mass functions $\text{m} : \text{At} \rightarrow [0, 1]$ that can be constructed as follows.

1. Set $\text{m}(a) = 0$ for all $a \in \mathbb{B}$.
2. Let $\{\alpha_i\}_{i \in [1, k]}$ be a family of real numbers in $[0, 1]$ such that $\sum_{1 \leq i \leq k} \alpha_i = 1$. For $a_i \in \text{At}$,

$$\text{m}(a_i) \leftarrow \text{m}(a_i) + \alpha_i(1 - \text{bel}_{\mathbf{X}}(\top_{\mathbb{B}})).$$

3. Let $\{\beta_{i,a}\}_{i \in [1, k_a]}$ be a family of real numbers in $[0, 1]$ such that $\sum_{1 \leq i \leq k_a} \beta_{i,a} = 1$. For $a_i \in \text{At}(a)$,

$$\text{m}(a_i) \leftarrow \text{m}(a_i) + \beta_{i,a} \text{m}_{\text{bel}_{\mathbf{X}}}(a).$$

Notice that $\sum_{b \leq a} \text{m}_{\text{bel}_{\mathbf{X}}}(b) \leq \sum_{b \in \text{At}(a)} \text{m}(b)$. Thus, the probability measure μ_{m} induced by m satisfies $\text{bel}_{\mathbf{X}} \leq \mu_{\text{m}}$ and $\mu_{\text{m}} \in \mathcal{F}_{\mathbf{X}}$. On the other hand, if $\mu \in \mathcal{F}_{\mathbf{X}}$, its associated mass function m_{μ} satisfies: for any $a \in \mathbb{B}$, $\sum_{b \leq a} \text{m}_{\text{bel}_{\mathbf{X}}}(b) \leq \sum_{b \in \text{At}(a)} \text{m}_{\mu}(b)$ and $\sum_{a \in \text{At}} \text{m}_{\mu}(a) = 1$. Thus, m_{μ} can be constructed by the above method with the right choice of weights (α s and β s). Therefore, $\mathcal{F}_{\mathbf{X}}$ is the set of probability measures induced by the mass functions in $\mathcal{M}_{\mathbf{X}}$. ■

In the case of $\text{bel}_{\mathbf{T}}$, Theorem 17 does not hold in general. Consider a mass function m such that $\text{bel}_{\mathbf{m}}(\{w \mid w \in |p|^{\mathbf{B}} \cap |q|^{\mathbf{N}}\}) = 1$. Then the formula $(p \wedge \neg p) \vee (q \wedge \neg q)$ is classically equivalent to a contradiction, but we have $\text{bel}_{\mathbf{T}}((p \wedge \neg p) \vee (q \wedge \neg q)) = 1$. As $(p \wedge \neg p) \vee (q \wedge \neg q) \in [\perp]$, if some probability measure μ is above it, then we have $\text{bel}_{\mathbf{T}}((p \wedge \neg p) \vee (q \wedge \neg q)) = 1 \leq \mu[\perp]$ which is not possible for a classical probability measure. Therefore \mathbf{T} -support is not a notion of supporting evidence adequate to deduce classical probabilities from evidence formalised over a BD-model.

7. Conclusion

In this paper we provide a framework for applying DS-theory to sources that might be (self) contradictory. We describe how to encode different kind of evidence, and show via examples that DS-rule may behave unintuitively. We propose a variation of DS-rule that uses the algebraic structure of the truth values to describe more finely the available information and its nature (incomplete, consistent, contradictory). Finally, we show how one can extract a credal set of classical probability measures consistent with the evidence.

In the future, we intend to study the behaviour of more existing aggregation rules in the BD framework in order to understand exactly to what kind of situations they can be applied and how it compares to the classical framework. In addition, the semantics of BD offers more possibilities to define aggregation rules via new logical connectives. We wish to understand their meaning and mathematical properties.

Acknowledgments

The research of Marta Bílková was supported by the grant 22-01137S of the Czech Science Foundation. The research of Sabine Frittella and Daniil Kozhemiachenko was funded by the grant ANR JCJC 2019, project PRELAP (ANR-19-CE48-0006). This research is part of the MOSAIC project

financed by the European Union’s Marie Skłodowska-Curie grant No. 101007627. The research of Krishna Manoorkar is supported by the NWO grant KIVI.2019.001 awarded to Alessandra Palmigiano.

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