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Mean-square exponential stabilization of coupled hyperbolic systems with random parameters

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Abstract:
In this paper, we consider a system of two coupled scalar-valued hyperbolic partial differential equations (PDEs) with random parameters. We formulate a stability condition under which the classical backstepping controller (designed for a nominal system whose parameters are constant) stabilizes the system. More precisely, we guarantee closed-loop mean-square exponential stability under random system parameter perturbations, provided the nominal parameters are sufficiently close to the stochastic ones on average. The proof is based on a Lyapunov analysis, the Lyapunov functional candidate describing the contraction of L^2-norm of the system states. An illustrative traffic flow regulation example shows the viability and importance of the proposed result.

Keywords: PDE, backstepping, stochastic parameters, exponential stability, Lyapunov function

1. INTRODUCTION

Hyperbolic PDEs are frequently used in the literature to model physical phenomena involving matter transportation across space and propagation. Among the numerous examples of conservation and balance laws that hyperbolic systems can model, we can cite traffic flow (Yu and Krstic, 2019), heat exchangers (Xu and Sallet, 2002), fluids in open channels (Bastin and Coron, 2016), mechanical vibrations in drilling devices (Saldivar et al., 2016). This explains why the control and estimation of coupled hyperbolic PDEs is an active research topic, different types of control strategies having been developed in the literature (Litrico and Fromion, 2009; Woittennek et al., 2017; Bastin and Coron, 2016; Strecker and Aamo, 2017). Among them, we can emphasize the backstepping methodology (Krstic and Smyshlyaev, 2008) that enabled the design of explicit controllers first for scalar hyperbolic systems (Coron et al., 2013), then for general $n + m$ systems (Coron et al., 2017).

In all these contributions, the system parameters are assumed to be time-independent, and only a few results focus on hyperbolic systems with time-varying coefficients (Coron and Nguyen, 2021; Mokhtari and Ammar Khodja, 2022). However, when considering applications as freeway transportation systems, some parameters may be subject to abrupt changes due to external causes, e.g., the random flux at the entrance of the freeway (Colombo, 2003) or changes in drivers' behavior. This has motivated the stability analysis of switching hyperbolic systems (Amin et al., 2011) or Markov jump linear hyperbolic conservation laws (Zhang and Prieur, 2017). In this latter contribution, the authors considered stochastic velocities and showed mean-square exponential stability under appropriate conditions (balance between the dissipativity of the hyperbolic and the transition probability of the Markov process). The proof relied on a Lyapunov analysis.

In the present contribution, we consider a 2×2 hyperbolic system where all parameters are stochastic. More precisely, they are modeled by independent Markov processes with a finite number of states (Kolmanovsky and Maizenberg, 2001). The objective is to guarantee the mean-square closed-loop stability. To do so, we consider a backstepping controller designed for a nominal system with constant coefficients (Vazquez et al., 2011). We show that the closed loop system is mean-square exponentially stable, provided the nominal parameters are sufficiently close to the stochastic ones on average. The proposed approach follows the methodology presented in (Kong and Bresch-Pietri, 2022a,b) for the case of linear systems with random input delays. It relies on a Lyapunov analysis of the closed-loop system. The exponential stability is shown using the so-called technique of probabilistic delay averaging (Kolmanovsky and Maizenberg, 2001).

The paper is organized as follows. In Section 2, we formulate the problem under consideration, introduce the nominal system, present the control law, and define the real stochastic system. We formulate our stabilization result at the end of the section. In Section 3, we take advantage of the backstepping methodology to reformulate the stochastic system in a more amenable form. In Section 4, we present the Lyapunov analysis, which is used to prove the paper's main result. We apply our methodology to the

test case of traffic control in Section 5. Some concluding remarks end the paper in Section 6.

Notations: We denote $L^2([0,1],\mathbb{R})$ the space of real-valued square-integrable functions defined on $[0,1]$ with the standard L^2 norm, i.e., for any $f \in L^2([0,1],\mathbb{R})$, we have $||f||_{L^2} = \left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}}$. $\mathbb{E}(x)$ denotes the expectation of a random variable x. For a random signal $x(t)$, the conditional expectation of $x(t)$ at the instant t knowing that $x(s) = x_0$ at the instant $s \leq t$ is denoted $\mathbb{E}_{[s,x_0]}(x(t))$.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Nominal system

Let us consider the following nominal 2×2 linear hyperbolic system

$$\partial_t u_{\text{nom}}(t,x) + \lambda_0 \partial_x u_{\text{nom}}(t,x) = \sigma_0^+ v_{\text{nom}}(t,x), \quad (1)$$

$$\partial_t v_{\text{nom}}(t,x) - \mu_0 \partial_x v_{\text{nom}}(t,x) = \sigma_0^- u_{\text{nom}}(t,x), \quad (2)$$

with the boundary conditions

$$u_{\text{nom}}(t,0) = q_0 v_{\text{nom}}(t,0), \quad (3)$$

$$v_{\text{nom}}(t,1) = \rho_0 u_{\text{nom}}(t,1) + U(t), \quad (4)$$

where $(u_{\text{nom}}(t,x), v_{\text{nom}}(t,x))^T$ is the state of the system, the different arguments evolving in $\{(t,x) \text{ s.t. } t > 0,\ x \in [0,1]\}$. The nominal velocities are denoted $\lambda_0 > 0$ and $\mu_0 > 0$ are constant. The in-domain couplings σ_0^+ and σ_0^-, and the boundary couplings ρ_0 and q_0 are also assumed to be constant. The function $U(t)$ corresponds to the actuation, which can be chosen as desired. We assume $q_0 \neq 0$. The initial conditions u^0_{nom} and v^0_{nom} are assumed to belong in $L^2([0,1],\mathbb{R})$. System (1)-(4) is the more general form for a one-dimensional 2×2 hyperbolic linear system (without integral or boundary terms) (Vazquez et al., 2011). It appears in the linearization of Saint-Venant equations (Bastin and Coron, 2016), in the linearization of ARZ equations for trafic networks (Yu et al., 2020b; Espitia et al., 2022), in heat exchangers equations and other hyperbolic balance laws (Bastin and Coron, 2016).

System (1)-(4) may be unstable due to the in-domain couplings σ^+ and σ^-. The control input $U(t)$ may be designed to guarantee the exponential stability of the closed-loop system. We assume here that the boundary conditions are *dissipative* (i.e., $|\rho_0 q_0| < 1$) since such a condition is necessary to guarantee the delay-robustness of the closed-loop system (i.e., the closed-loop system is robust to the introduction of an arbitrarily small delay in the loop) (Auriol and Di Meglio, 2019; Auriol, 2018).

A stabilizing control law has been designed in (Coron et al., 2013; Vazquez et al., 2011; Auriol et al., 2018) using the backstepping approach (Krstic and Smyshlyaev, 2008). Consider the set of kernel equations

$$\lambda_0 \partial_x K^{uu}(x,y) + \lambda_0 \partial_y K^{uu}(x,y) = -\sigma_0^- K^{uv}(x,y), \quad (5)$$

$$\lambda_0 \partial_x K^{uv}(x,y) - \mu_0 \partial_y K^{uv}(x,y) = -\sigma_0^+ K^{uu}(x,y), \quad (6)$$

$$\mu_0 \partial_x K^{vu}(x,y) - \lambda_0 \partial_y K^{vu}(x,y) = \sigma_0^- K^{vv}(x,y), \quad (7)$$

$$\mu_0 \partial_x K^{vv}(x,y) + \mu_0 \partial_y K^{vv}(x,y) = \sigma_0^+ K^{vu}(x,y), \quad (8)$$

with the boundary conditions

$$K^{uu}(x,0) = \frac{\mu_0}{\lambda_0 q_0} K^{uv}(x,0),\ K^{uv}(x,x) = -\frac{\sigma_0^+}{\lambda_0 + \mu_0}, \quad (9)$$

$$K^{vv}(x,0) = \frac{\lambda_0 q_0}{\mu_0} K^{vu}(x,0),\ K^{vu}(x,x) = \frac{\sigma_0^-}{\lambda_0 + \mu_0}, \quad (10)$$

where the kernels $K^{\cdot\cdot}$ are defined on the triangular domain $\mathcal{T} = \{(x,y),\ 0 \leq y \leq x \leq 1\}$. The set of kernel equations (5)-(10) admits a unique (bounded) solution which is of class $\mathcal{C}^\infty$ on $\mathcal{T}$. An explicit expression of the kernels can be obtained using Marcum-Q functions (Vazquez and Krstic, 2014). Consider now the control law

$$U(t) = \int_0^1 (\rho_0 K^{uu}(1,y) - K^{vu}(1,y)) u_{nom}(t,y) dy + \int_0^1 (\rho_0 K^{uv}(1,y) - K^{vv}(1,y)) v_{nom}(t,y) dy. \quad (11)$$

Then the closed-loop system (1)-(4) is well-posed and exponentially converges to zero in the sense of the L^2-norm (Coron et al., 2013) (Auriol et al., 2018, Lemma 2).

2.2 Real system

We now consider the real stochastic 2×2 linear hyperbolic system

$$\partial_t u(t,x) + \lambda(t) \partial_x u(t,x) = \sigma^+(t) v(t,x), \quad (12)$$

$$\partial_t v(t,x) - \mu(t) \partial_x v(t,x) = \sigma^-(t) u(t,x), \quad (13)$$

with the boundary conditions

$$u(t,0) = q(t) v(t,0), \quad (14)$$

$$v(t,1) = \rho(t) u(t,1) + U(t), \quad (15)$$

where the state of the system is now $(u(t,x), v(t,x))^T$. The different parameters are now *random independent* variables. We denote $\mathfrak{S} = \{\lambda, \mu, \sigma^+, \sigma^-, q, \rho\}$ the set of random variables. Each random element X of the set $\mathfrak{S}$ is a Markov process with the following properties.

- (P1) $X(t) \in \{X_i, i \in \{1,\dots,r_X\}\}$, $r_X \in \mathbb{N}$ with $\underline{X} \leq X_1 < \dots < X_{r_X} \leq \bar{X}$.
- (P2) The transition probabilities $P^X_{ij}(t_1,t_2)$ qualify the probability to switch from X_i at time t_1 to X_j at time t_2 ($(i,j) \in \{1,\dots,r_X\}^2$, $0 \leq t_1 \leq t_2$). They satisfy
  (1) $P^X_{ij} : \mathbb{R}^2 \to [0,1]$ with $\sum_{j=1}^{r_X} P^X_{ij}(t_1,t_2) = 1$.
  (2) P^X_{ij} is a differentiable function which, for $s < t$ follows the Kolmogorov equation
  $$\partial_t P^X_{ij}(s,t) = -c^X_j(t) P^X_{ij}(s,t) + \sum_{k=1}^{r_X} P^X_{ik}(s,t) \tau^X_{kj}(t),$$
  $$P^X_{ii}(s,s) = 1, \text{ and } P^X_{ij}(s,s) = 0 \text{ for } i \neq j, \quad (16)$$
  where τ_{ij} and $c^X_j = \sum_{k=1}^{r_X} \tau^X_{jk}$ are nonnegative-valued functions such that for any t, $\tau^X_{ii}(t) = 0$. Moreover, the functions τ^X_{ik} are upper bounded by a constant $\tau^\star_X$.
- (P3) The realizations of X are right-continuous.

Moreover, we assume that for all $X \in \mathfrak{S}$, we have $\underline{X} \leq X_0 \leq \bar{X}$ (where X_0 is the nominal value given in the definition of system (1)-(4), e.g. $X_0 = \lambda_0$ if $X = \lambda$). We also assume that $\underline{\lambda} > 0$, $\underline{\mu} > 0$ and $|\bar{\rho}\bar{q}| < 1$. It is common to assume only a finite number of values in (P1) (Kolmanovsky and Maizenberg, 2001; Sadeghpour et al., 2019). Similarly, it is standard to assume Property (P3) for the modeling of continuous-time Markov chains. It is important to mention that the properties (P1) and

(P3), along with the Markov property, guarantee that P_{ij}^X satisfies the Kolmogorov Equation (16) for certain positive-valued functions τ_{ij}^X, c_j^X (Rausand and Hoyland, 2003; Ross, 2014). Thus Property (P2) only implies that the functions τ_{ij}^X are bounded, which is a mild modeling assumption. We emphasize that the parameter $\tau_{ij}^X \Delta t$ is approximately the probability of transition from X_i to X_j on the interval $[t, t+\Delta t)$. Moreover, $1 - c_j^X(t)\Delta t$ is the probability of staying at X_j during this time interval.

For each $X \in \mathfrak{S}$, we denote by $T_X = \{X_1, \dots, X_{r_X}\}$ the set of possible realizations for the variable X. Let $\mathfrak{T} = T_\lambda \times T_\mu \times T_{\sigma^+} \times T_{\sigma^-} \times T_q \times T_\rho$ and for any t, let $\delta(t) \in \mathbb{R}^6$ be defined by

$$\delta(t) = (\lambda(t), \mu(t), \sigma^+(t), \sigma^-(t), q(t), \rho(t)). \tag{17}$$

Since all the variables are independent, δ is a Markov process whose transition probabilities can be deduced from those of $\mathfrak{S}$, and with a finite number of states $r = r_\lambda r_\mu r_{\sigma^+} r_{\sigma^-} r_q r_\rho$. Finally, we denote by $\mathfrak{R}$ the Cartesian product of the sets $\{1, \dots, r_X\}$, $X \in \mathfrak{S}$. An element $j \in \mathfrak{R}$ is a multi-index composed of 6 indices (integers): $(j_\lambda, j_\mu, j_{\sigma^+}, j_{\sigma^-}, j_q, j_\rho)$, and we will say that $\delta(t) = \delta_j$ if $X(t) = X_{j_X}$ for any $X \in \mathfrak{S}$. Finally, we denote $\tau_{j\ell} = \Pi_{X \in \mathfrak{S}} \tau_{j_X \ell_X}^X$, and similarly, $P_{ij} = \Pi_{X \in \mathfrak{S}} P_{i_X j_X}$.

We now prove the well-posedness of our stochastic system.

Lemma 1. For any initial condition $(u^0, v^0) \in L^2[0,1]$ and for any initial states $\delta(0)$ for the stochastic parameters, the closed-loop system (12)-(13) with the control law (11) admits a unique solution (u, v) such that for any t,

$$\mathbb{E}_{[0,(u^0,v^0\delta(0))]}\{||(u(t,\cdot), v(t,\cdot)||_{L^2[0,1]}\} < \infty.$$

Proof. The proof can be easily adjusted from (Zhang and Prieur, 2017). Almost every sample path of our stochastic processes are right-continuous step functions with a finite number of jumps in any finite time interval. We can then define a sequence $\{t_k;\ k = 0, 1, \dots\}$ of stopping time such that $t_0, \lim_{k\to\infty} t_k = \infty$ and every $X \in \mathfrak{S}$ is constant on $t_k \le t < t_{k+1}$. We can then iteratively build the solution on each interval $[t_k, t_{k+1}]$ by applying (Bastin and Coron, 2016, Theorem A.4). The rest of the proof is analogous to (Zhang and Prieur, 2017, Proof of Proposition 1). □

2.3 Problem statement

The objective is this paper is to show that the nominal controller defined in equation (11) still stabilizes the stochastic system (12)-(15), provided the nominal parameters are sufficiently close to the stochastic ones on average. More precisely, we want to show the following sufficient condition for robust stabilization

Theorem 2. Consider the closed-loop system (12)-(15) with the control law (11). There exists a positive constant $\epsilon^\star$, such that if, for all time $t \ge 0$ and all $X \in \mathfrak{S}$,

$$\sum_{X \in \mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|) \le \epsilon^\star, \tag{18}$$

then the closed loop system is *mean-square exponentially stable*, that is, there exists $\kappa > 0$ and $\gamma > 0$ such that

$$\mathbb{E}_{[0,(w(0),\delta(0))]}(w(t)) \le \kappa e^{-\gamma t} w(0), \tag{19}$$

where

$$w(t) = \int_0^1 u^2(t,x) + v^2(t,x)dx. \tag{20}$$

Theorem 2 generalizes the deterministic robustness results stated in (Auriol and Di Meglio, 2020) to the case of stochastic parameters. The proof of Theorem 2 will be given in the next sections. It is inspired by (Kong and Bresch-Pietri, 2022b). First, the system is simplified using a backstepping transformation (Section 3). Then, the stability is shown using a Lyapunov analysis (Section 4).

3. BACKSTEPPING TRANSFORMATION

To prove Theorem 2, we first simplify the structure of the system using a backstepping transformation (Krstic and Smyshlyaev, 2008). Consider the integral change of coordinates

$$\alpha(t,x) = u(t,x) + \int_0^x K^{uu}(x,y)u(t,y) + K^{uv}(x,y)v(t,y)dy, \tag{21}$$

$$\beta(t,x) = v(t,x) + \int_0^x K^{vu}(x,y)u(t,y) + K^{vv}(x,y)v(t,y)dy, \tag{22}$$

where the kernels $K^{\cdot\cdot}$ are defined by (5)-(10). The transformation (21)-(22) is a Volterra transformation and is invertible (Yoshida, 1960). In particular, there exist bounded functions $L^{\cdot\cdot}$ defined on the triangular domain $\mathcal{T}$ such that

$$u(t,x) = \alpha(t,x) + \int_0^x L^{\alpha\alpha}(x,y)\alpha(t,y) + L^{\alpha\beta}(x,y)\beta(t,y)dy, \tag{23}$$

$$v(t,x) = \beta(t,x) + \int_0^x L^{\beta\alpha}(x,y)\alpha(t,y) + L^{\beta\beta}(x,y)\beta(t,y)dy. \tag{24}$$

The kernels L are explicitly defined in (Bou Saba et al., 2019). The states (α, β) and (u, v) have equivalent L^2 norms, i.e. there exist $m > 0$ and $M > 0$ such that

$$m||(u,v)||_{L^2} \le ||(\alpha,\beta)||_{L^2} \le M||(u,v)||_{L^2} \tag{25}$$

Consider now $U(t)$ as defined by equation (11). Differentiating equations (23)-(22) with respect to time and space and integrating by parts, we can show that the variables α and β are solutions to the following set of PDEs

$$\partial_t\alpha(t,x) + \lambda(t)\partial_x\alpha(t,x) = f_1(\delta(t))v(t,x) + f_2(\delta(t))\beta(t,0) + \int_0^x f_3(\delta(t),x,y)u(t,y) + f_4(\delta(t),x,y)v(t,y)dy \tag{26}$$

$$\partial_t\beta(t,x) - \mu(t)\partial_x\beta(t,x) = g_1(\delta(t))u(t,x) + g_2(\delta(t))\beta(t,0) + \int_0^x g_3(\delta(t),x,y)u(t,y) + g_4(\delta(t),x,y)v(t,y)dy, \tag{27}$$

with the boundary conditions

$$\alpha(t,0) = q(t)\beta(t,0), \tag{28}$$

$$\beta(t,1) = \rho(t)\alpha(t,1) + (\rho(t) - \rho_0)\int_0^1 K^{vu}(1,y)u(t,y)dy + (\rho(t) - \rho_0)\int_0^1 K^{vv}(1,y)v(t,y)dy \tag{29}$$

where the different functions are defined by

$$f_1(\delta(t)) = \sigma^+(t) - \sigma_0^+\frac{\lambda(t)+\mu(t)}{\lambda_0+\mu_0},\ g_1(\delta(t)) = \sigma^-(t) - \sigma_0^-\frac{\lambda(t)+\mu(t)}{\lambda_0+\mu_0},$$

$$f_2(\delta(t)) = (\frac{\lambda(t)q(t)}{\lambda_0 q_0}\mu_0 - \mu(t))K^{uv}(x,0),$$

$$g_2(\delta(t)) = (\lambda(t)q(t) - \frac{\mu(t)}{\mu_0}\lambda_0 q_0)K^{vu}(x,0),$$

$$f_3(\delta(t),x,y) = -\frac{\lambda(t)}{\lambda_0}\sigma_0^- K^{uv}(x,y) + \sigma^-(t)K^{uv}(x,y),$$

$$f_4(\delta(t),x,y) = (\lambda(t)-\lambda_0)\partial_x K^{uv}(x,y) - (\mu(t)-\mu_0)\partial_y K^{uv}(x,y) - K^{uu}(x,y)(\sigma_0^+ - \sigma^+(t)),$$

$$g_3(\delta(t),x,y) = (\lambda(t)-\lambda_0)\partial_y K^{vu}(x,y) - (\mu(t)-\mu_0)\partial_x K^{vu}(x,y) - K^{vu}(x,y)(\sigma_0^- - \sigma^-(t)),$$

$$g_4(\delta(t),x,y) = (\sigma^+(t) - \sigma_0^+\frac{\mu(t)}{\mu_0})K^{vu}(x,y).$$

Note that all the terms that depend on (u,v) in the target system (26)-(29) could be expressed in terms of (α,β) using the inverse transformation (23)-(24). However, this would make the computations more complex and is not required for the stability analysis. The target system (26)-(29) may appear much more complex than the original system (12)-(15). However, it is important to emphasize that all the terms that appear on the right-hand side of the different equations become *small* if the stochastic parameters are close enough to the nominal ones. More precisely, we have the following lemma

Lemma 3. There exists a constant $M_0 > 0$ such that for any realization $\delta(t) = \delta_j$ $(j \in \mathfrak{R})$ of the stochastic variable δ and for any $(x,y) \in \mathcal{T}$, for all $i \in \{1,2,3,4\}$, we have

$$|f_i(\delta_j)| < M_0 \sum_{X\in\mathfrak{S}} |X_0 - X_j|, \tag{30}$$

$$|g_i(\delta_j)| < M_0 \sum_{X\in\mathfrak{S}} |X_0 - X_j|, \tag{31}$$

Note that we omitted the variables (x,y) in the inequalities to ease the notations.

Proof. Consider the function $f_1(\delta_j)$. We have

$$f_1(\delta_j) = (\sigma^+_{j_{\sigma^+}} - \sigma_0^+) - \frac{\sigma_0^+}{\lambda_0+\mu_0}(\lambda_{j_\lambda} - \lambda_0) - \frac{\sigma_0^+}{\lambda_0+\mu_0}(\mu_{j_\mu} - \mu_0),$$

which implies

$$|f_1(\delta_j)| \le \max\{1, |\frac{\sigma_0^+}{\lambda_0+\mu_0}|\} \sum_{X\in\mathfrak{S}} |X_0 - X_j|. \tag{32}$$

Consider now the function $f_3(\delta_j,x,y)$. We obtain

$$\begin{aligned}|f_3(\delta_j,x,y)| &\le |\lambda_{j_\lambda} - \lambda_0|\frac{|\sigma_0^-|}{|\lambda_0|}\sup_{\mathcal{T}}|K^{uv}(x,y)| \\ &+ |\sigma_0^- - \sigma^-_{j_{\sigma^-}}|\sup_{\mathcal{T}}|K^{uv}(x,y)| \\ &\le (1+\frac{|\sigma_0^-|}{|\lambda_0|})\sup_{\mathcal{T}}|K^{uv}(x,y)| \sum_{X\in\mathfrak{S}}|X_0 - X_j|,\end{aligned}$$

where $\sup_{\mathcal{T}}|K^{uv}(x,y)|$ is well defined due to the boundedness of the kernels $K^{\cdot\cdot}$. The other inequalities can be shown in a similar way, using the boundedness of $\partial_x K^{\cdot\cdot}$ and $\partial_y K^{\cdot\cdot}$. □

Thus, the target system (26)-(29) is simpler in the sense that it simplifies the robustness analysis that will be carried out in the next section. With this new set of coordinates, we can now analyze the exponential stability of the closed-loop system.

4. LYAPUNOV ANALYSIS

Let us denote the state of the target system (26)-(29) as $z(t,\cdot) = (\alpha(t,\cdot),\beta(t,\cdot)) \in (L^2([0,1],\mathbb{R}))^2$. As the solution to (26)-(29) is unique (due to the unicity of the solution of (12)-(15)), $(z,\mathfrak{S})$ defines a continuous-time Markov process. Define the infinitesimal generator L (Kolmanovskii and Myshkis, 2013; Ross, 2014) acting on a functional $V : (L^2([0,1],\mathbb{R}))^2 \times \mathfrak{T} \to \mathbb{R}$ as

$$LV(z,\delta) = \limsup_{\Delta t\to 0^+}\frac{1}{\Delta t} \times \mathbb{E}_{[t,(z,\delta)]}(V(z(t+\Delta t),\delta(t+\Delta t)) - V(z,\delta)). \tag{33}$$

We define L_j, the infinitesimal generator of the Markov process (z,δ) obtained from the system (26)-(29) by fixing $\delta(t) = \delta_j$ $(j \in \mathfrak{R})$ as

$$L_jV(z) = \frac{dV}{dz}(z,\delta_j)h_j(z) + \sum_{\ell\in\mathfrak{R}}(V_\ell(z) - V_j(z))\tau_{j\ell}(t), \tag{34}$$

where $V_\ell(z) = V(z,\delta_\ell)$, and h_j is the operator corresponding to the dynamics of the target system (26)-(29) with the fixed value $\delta(t) = \delta_j$

$$h_j(z) = \begin{pmatrix} -\lambda_{j_\lambda}\partial_x\alpha(x) + f_1(\delta_j)v(t,x) + f_2(\delta_j)\beta(t,0) \\ + \int_0^x f_3(\delta_j,x,y)u(t,y) + f_4(\delta_j,x,y)v(t,y)dy \\ \mu_{j_\mu}\partial_x\alpha(x) + g_1(\delta_j)v(t,x) + g_2(\delta_j)\beta(t,0) \\ + \int_0^x g_3(\delta_j,x,y)u(t,y) + g_4(\delta_j,x,y)v(t,y)dy \end{pmatrix}$$

To shorten the computations, we denote in the sequel $V(t), LV(t), V_j(t)$ and $L_jV(t)$ instead of (respectively) $V(z(t),\delta(t)), LV(z(t),\delta(t)), V(z(t),\delta_j)$ and $L_j(V(z(t)))$. From now, we consider that $\delta(t=0) = \delta_i$ for some $i \in \mathfrak{R}$.

4.1 Derivation of the Lyapunov function

Consider the following Lyapunov functional candidate

$$V(z,\delta) = \int_0^1 \frac{e^{-\frac{\nu}{\lambda(t)}x}}{\lambda(t)}\alpha^2(t,x) + a\frac{e^{\frac{\nu}{\mu(t)}x}}{\mu(t)}\beta^2(t,x)dx, \tag{35}$$

with $a,\nu > 0$. This functional explicitly depend on δ through the velocities λ and μ. Since the velocities $\lambda(t)$ and $\mu(t)$ are upper and lower bounded, the functional V is equivalent to the L^2-norm of the state (α,β) (and consequently to the L^2-norm of the state (u,v) due to (25)), i.e. there exists k_u and k_ℓ such that

$$k_\ell||(\alpha,\beta)||^2_{L^2} \le V(t) \le k_u||(\alpha,\beta)||^2_{L^2} \tag{36}$$

We have the following lemma

Lemma 4. There exists $\eta > 0$, $M_1 > 0$ and $M_2 > 0$ such that the Lyapunov functional V satisfies

$$\sum_{j=1}^r P_{ij}(0,t)L_jV(t) \le -V(t)\big(\eta - M_2k(t) - (M_1 + M_2r\tau^\star)\sum_{X\in\mathfrak{S}}\mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|)\big), \tag{37}$$

where the function k is defined by

$$k(t) = \sum_{X\in\mathfrak{S}}\sum_{j=1}^r |X_j - X_0|(\partial_t P_{ij}(0,t) + c_jP_{ij}(0,t)).$$

Proof. Consider that $\delta = \delta_j$. Let us compute the first term of equation (34) $(\frac{dV_j}{dz}(z)h_j(z))$. Applying integration by parts, we obtain

$$\frac{dV_j}{dz}(z)h_j(z) = -\nu V_j(t) + \int_0^1 \frac{2}{\lambda_{j_\lambda}}e^{-\frac{\nu}{\lambda_{j_\lambda}}x}\alpha(t,x)[f_1(\delta_j) v(t,x) + f_2(\delta_j)\beta(t,0) + \int_0^x f_3(\delta_j,x,y)u(t,y) + f_4(\delta_j,x,y)$$

$$v(t,y)dy] + \frac{2}{\mu_{j_\mu}} e^{\frac{\nu}{\mu_{j_\mu}}x} \beta(t,x)[g_1(\delta_j)v(t,x) + g_2(\delta_j)\beta(t,0)$$
$$+ \int_0^x g_3(\delta_j,x,y)u(t,y) + g_4(\delta_j,x,y)v(t,y)dy]dx$$
$$+ (q_{j_q}^2 - a)\beta^2(t,0) - e^{-\frac{\nu}{\lambda_{j_\lambda}}}\alpha^2(t,1) + a e^{\frac{\nu}{\mu_{j_\mu}}}(\rho_{j_\rho}\alpha(t,1)$$
$$+ (\rho_{j_\rho} - \rho_0)\int_0^1 K^{vu}(1,y)u(t,y) + K^{vv}(1,y)v(t,y)dy)^2.$$

Notice first that there exists $\eta > 0$ such that for all j $-\nu V_j(t) \le -\eta V(t)$. Consider the term $\int_0^1 \frac{2}{\lambda_{j_\lambda}} e^{-\frac{\nu}{\lambda_{j_\lambda}}x}\alpha(t,x) f_1(\delta_j)v(t,x)dx$. Combining Young's inequality and Lemma 3, we obtain

$$\int_0^1 |\frac{2}{\lambda_{j_\lambda}} e^{-\frac{\nu}{\lambda_{j_\lambda}}x}\alpha(t,x)f_1(\delta_j)v(t,x)|dx$$
$$\le \frac{1}{\underline{\lambda}} M_0 \sum_{X\in\mathfrak{S}} |X_0 - X_j|(\int_0^1 v^2(t,x) + \alpha^2(t,x)dx)$$
$$\le \frac{M_0}{k_\ell \underline{\lambda}} \sum_{X\in\mathfrak{S}} |X_0 - X_j|(1 + \frac{1}{m})V(t)$$

where m and k_ℓ are defined in (25) and (36). Consider now the term $\int_0^1 \frac{2}{\lambda_{j_\lambda}} e^{-\frac{\nu}{\lambda_{j_\lambda}}}\alpha(t,x)f_2(\delta_j)\beta(t,0)dx$. Combining Young's inequality and Lemma 3, we obtain

$$\int_0^1 |\frac{2}{\lambda_{j_\lambda}} e^{-\frac{\nu}{\lambda_{j_\lambda}}}\alpha(t,x)f_2(\delta_j)\beta(t,0)|dx$$
$$\le \frac{M_0}{k_\ell \epsilon_0 \underline{\lambda}} \sum_{X\in\mathfrak{S}} |X_0 - X_j|V(t) + \frac{M_0}{\underline{\lambda}} \sum_{X\in\mathfrak{S}} |X_0 - \underline{X}|\epsilon_0 \beta^2(t,0),$$

where ϵ_0 can be chosen arbitrarily small. In what follows we denote $\epsilon_1 = \frac{M_0}{2\underline{\lambda}} \sum_{X\in\mathfrak{S}} |X_0 - \underline{X}|\epsilon_0$. Performing analogous computations on all the terms that appear in the expression of $\frac{dV_j}{dz}(z)h_j(z)$, we obtain

$$\frac{dV_j}{dz}(z)h_j(z) \le -\eta V(t) + M_1 \sum_{X\in\mathfrak{S}} |X_0 - X_j|V(t) + (\epsilon_1$$
$$+ q_{j_q}^2 - a)\beta^2(t,0) + \alpha^2(t,1)(a e^{\frac{\nu}{\mu_{j_\mu}}} \rho_{j_\rho}^2(1+\epsilon_2) - e^{-\frac{\nu}{\lambda_{j_\lambda}}}),$$

where the constant $\epsilon_2 > 0$ can be chosen arbitrarily small and where the constant M_1 depends on M_0, ϵ_1 and on the parameters of the system. Let us now choose ν, ϵ_1 and ϵ_2 such that

$$\epsilon_1 + q_{j_q}^2 - a < 0$$
$$a e^{\frac{\nu}{\mu_{j_\mu}}} \rho_{j_\rho}^2(1+\epsilon_2) - e^{-\frac{\nu}{\lambda_{j_\lambda}}} < 0.$$

These conditions are always feasible since $|\rho_{j_\rho} q_{j_q}| < 1$. Indeed, we choose ϵ_1 and ϵ_2 small enough such that $(q_{j_q}^2 - \epsilon_1)\rho_{j_\rho}^2(1+\epsilon_2) < 1$ and inject the first inequality to the second one to obtain $(q_{j_q}^2 - \epsilon_1)\rho_{j_\rho}^2 < e^{-\frac{\nu}{\lambda_{j_\lambda}} - \frac{\nu}{\mu_{j_\mu}}}$. It is then sufficient to choose ν small enough. The first equation gives a. Consequently, we obtain

$$\frac{dV_j}{dz}(z)h_j(z) \le -\eta V(t) + M_1 \sum_{X\in\mathfrak{S}} |X_0 - X_j|V(t). \quad (38)$$

Consider now the second term of $L_jV(z)$ that appear in (34):

$$\sum_{\ell=1}^r (V_\ell(z) - V_j(z))\tau_{j\ell} = \sum_{\ell=1}^r \tau_{j\ell} \int_0^1 (\frac{e^{-\frac{\nu}{\lambda_{\ell_\lambda}}x}}{\lambda_{\ell_\lambda}} - \frac{e^{-\frac{\nu}{\lambda_{j_\lambda}}x}}{\lambda_{j_\lambda}})$$
$$\alpha^2(t,x) + a(\frac{e^{\frac{\nu}{\mu_{\ell_\mu}}x}}{\mu_{\ell_\mu}} - \frac{e^{\frac{\nu}{\mu_{j_\mu}}x}}{\mu_{j_\mu}})\beta^2(t,x)dx.$$

Using the mean value theorem, on the functions $\lambda \mapsto \frac{e^{-\frac{\nu}{\lambda}x}}{\lambda}$ and $\mu \mapsto \frac{e^{\frac{\nu}{\mu}x}}{\mu}$, we obtain

$$\sum_{\ell=1}^r (V_\ell(z) - V_j(z))\tau_{j\ell} \le M_2 \sum_{\ell=1}^r \sum_{X\in\mathfrak{S}} \tau_{j\ell}|X_\ell - X_j|V(t),$$

where M_2 is defined by

$$M_2 = \frac{2\bar{\lambda}}{k_\ell}\frac{1}{\underline{\lambda}^2}(\frac{\nu}{\underline{\lambda}} + \bar{\lambda}) + \frac{2\bar{\mu}}{k_\ell}\frac{1}{\underline{\mu}^2}(\frac{\nu}{\underline{\mu}} + \bar{\mu})e^{\frac{\nu}{\underline{\mu}}}.$$

Combining this inequality with equation (38)

$$L_jV(t) \le -\eta V(t) + M_1 \sum_{X\in\mathfrak{S}} |X_0 - X_j|V(t)$$
$$+ M_2 \sum_{\ell=1}^r \sum_{X\in\mathfrak{S}} \tau_{j\ell}|X_\ell - X_j|V(t). \quad (39)$$

We now compute the quantity $\bar{L} = \sum_{j=1}^r P_{ij}(0,t)L_jV(t)$. Notice first that $\sum_{j=1}^r P_{ij}(0,t)\sum_{X\in\mathfrak{S}} |X_0 - X_j| = \mathbb{E}_{[0,\delta(0)]}(\sum_{X\in\mathfrak{S}} |X_0 - X(t)|) = \sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|)$, since all the variables are independent. Thus, applying the triangular inequality, the following inequality holds

$$\bar{L} \le -(\eta - M_1 \sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|))V(t)$$
$$+ M_2 \sum_{j=1}^r \sum_{\ell=1}^r P_{ij}(0,t) \sum_{X\in\mathfrak{S}} \tau_{j\ell}(|X_\ell - X_0| + |X_j - X_0|)V(t)$$
$$\le -(\eta - (M_1 + M_2 r\tau^\star) \sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|))V(t)$$
$$+ M_2 \sum_{j=1}^r \sum_{\ell=1}^r P_{ij}(0,t) \sum_{X\in\mathfrak{S}} \tau_{j\ell}|X_\ell - X_0|V(t)$$

Applying (16), we obtain

$$\bar{L} \le -(\eta - (M_1 + M_2 r\tau^\star) \sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|))V(t)$$
$$+ M_2 \sum_{X\in\mathfrak{S}} \sum_{j=1}^r |X_j - X_0|(\partial_t P_{ij}(0,t) + c_j P_{ij}(0,t))V(t).$$

This concludes the proof of Lemma 4. □

4.2 Proof of Theorem 2

We now have all the tools to prove Theorem 2. Let us denote $k_0(t) = \eta - (M_1 + M_2 r\tau^\star)\sum_{X\in\mathfrak{S}} \mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|) - M_2 k(t))$ and define the functional $Z(t)$ as

$$Z(t) = \exp(\int_0^t k_0(s)ds)V(t).$$

Using the expression of $k(t)$ in Lemma 4, we have

$$\int_0^t k(s)ds \le \sum_{X\in\mathfrak{S}} (\mathbb{E}_{[0,X(0)]}(|X_0 - X(t)|)$$
$$+ c^\star \int_0^t \mathbb{E}_{[0,X(0)]}(|X_0 - X(s)|)ds),$$

where $c^\star = r\tau^\star$. Consequently, choosing $\epsilon^\star$ (defined in the statement of Theorem 2 as

$$\epsilon^\star = \frac{\eta}{2(2M_2c^\star) + M_1} \quad (40)$$

we obtain

$$\mathbb{E}_{[0,(z,\delta)(0))]}(Z(t)) \geq \mathbb{E}_{[0,(z,\delta)(0))]}(\mathrm{e}^{-M_2\epsilon^\star+\frac{\eta}{2}t}V(t)). \quad (41)$$

In the meantime we have

$$\mathbb{E}_{[0,(z(0),\delta(0))]}(LZ(t)) = \mathrm{e}^{\int_0^t k_0(s)ds}\mathbb{E}_{[0,(z(0),\delta(0))]}(LV(t)).$$

Since $\mathbb{E}_{[0,(z(0),\delta(0))]}(LV(t)) = \mathbb{E}_{[0,(z(0),\delta(0))]}(\sum_{j=1}^r P_{ij}(0,t)$ $L_jV(t))$, we obtain using equation (37)

$$\mathbb{E}_{[0,(z(0),\delta(0))]}(LZ(t)) \leq \exp(\int_0^t k_0(s)ds)\mathbb{E}_{[0,(z(0),\delta(0))]}(k_0(t)$$

$$V(t) + \sum_{j=1}^r P_{ij}(0,t)L_jV(t)) \leq 0.$$

Therefore, according to Dynkin's formula (Dynkin, 2012, Theorem 5.1, p. 132), we obtain

$$\mathbb{E}_{[0,(z(0),\delta(0))]}(Z(t)) - Z(0) = \mathbb{E}_{[0,(z(0),\delta(0))]}(\int_0^t LZ(s)ds) \leq 0. \quad (42)$$

Consequently, defining $\gamma = \frac{\eta}{2}$, and combining equations (41) and (42), we obtain

$$\mathbb{E}_{[0,(z(0),\delta(0))]}(V(t)) \leq V(0)\mathrm{e}^{M_2\epsilon^\star}\mathrm{e}^{-\gamma t}. \quad (43)$$

We can easily conclude the proof of Theorem 2 using the fact that $V(t)$ is equivalent to the L^2-norm of the system, that is $w(t)$.

5. APPLICATION TO THE TRAFFIC CONGESTION REGULATION

In this section, we illustrate our theoretical results with the example of traffic congestion regulation under uncertain driver behaviour. The Aw–Rascle–Zhang (ARZ) model (Aw and Rascle, 2000; Zhang, 2002), consisting of second-order nonlinear partial differential equations (PDEs), can describe the macroscopic traffic dynamics. More precisely, considering a freeway segment of length L, the traffic flux q (number of vehicles per unit time which cross a given point on the road) and the traffic speed v verify (Yu and Krstic, 2019)

$$\partial_t q(t,x) + v\partial_x q(t,x) = \frac{q(\gamma p - v)}{v}\partial_x v(t,x) + \frac{q(v_f - p - v)}{\tau v}, \quad (44)$$

$$\partial_t v(t,x) - (\gamma p - v)\partial_x q(t,x) = \frac{v_f - p - v}{\tau}, \quad (45)$$

where the traffic pressure is defined by $p(t,x) = \frac{v_f}{\rho_m}(\frac{q}{v})^\gamma$. The parameter γ represents the overall drivers' property, reflecting their change of driving behavior to the increase of density, v_f is the maximum velocity, and τ is the relaxation time related to driving behavior (Dabiri and Kulcsar, 2022). In (Yu and Krstic, 2019), the system (44)-(45) is linearized around a steady-state $(q^\star, v^\star)$. The small deviations from the nominal profile are defined as $\tilde{q}(t,x) = q(t,x) - q^\star$ and $\tilde{v}(t,x) = v(t,x) - v^\star$. At the boundary, we consider a constant traffic flux $q^\star$ entering the domain (it can be obtained using mainline flux metering at the inlet): $q = q^\star$. At the other extremity, we have $v = \frac{v^\star}{q^\star}q$. Depending on the choice of parameters $(q^\star, v^\star)$ the traffic may be congested (Yu and Krstic, 2019). Then, it is possible to consider ramp metering as a boundary actuation to stabilize the oscillations of traffic flow and speed. We consider that we control the traffic upstream of the ramp metering (UORM). The objective is to stabilize the deviations $\tilde{q}(t,x)$ and $\tilde{v}(t,x)$. After linearization and a change of coordinates, these states verify equations, these states verify equations (1)-(4) where

$$\lambda_0 = v^\star,\ \mu_0 = \gamma p^\star - v^\star,\ q = 1 - \frac{\gamma p^\star}{v^\star},$$

$$\rho = \exp(-\frac{L}{\tau v^\star}),\ \sigma_0^+ = 0,\ \sigma_0^-(x) = -\frac{1}{\tau}\exp(-\frac{x}{\tau v^\star}),$$

For this linearized system, the controller (11) can be used to guarantee exponential stability. To obtain a more realistic description of the drivers' behavior, we now consider that the variable $\gamma(t)$ is stochastic. It follows a Markov process characterized by the properties (P1)-(P3). We will show in simulations that the closed-loop system is mean-square exponentially stable under condition (18). Note that this example cannot be seen as a direct application of Theorem 2 for two reasons:

(1) First, the parameter σ_0^- is spatially varying. Our methodology could easily be adjusted to overcome this limitation.
(2) Then, since μ_0 and q both depend on γ, they cannot be seen as independent variables. The computations we proposed could be adjusted but become more involved.
(3) Finally, the change of coordinates required to rewrite the linearized ARZ equations in the Riemann coordinates depends on γ and may add extra terms to the analysis. Again, the proposed computations could be adjusted to deal with this problem.

Despite these limitations, we chose to consider the proposed test case to emphasize possible extensions of our methodology. The length of the freeway segment is chosen as $L = 0.5$ km. The simulation time is $T = 600s$. The maximum speed limit is $v_m = 40$ ms^{-1}. The maximum density of the road is $\rho_m = 800$ vehicles per kilometer. The steady states are $(q^\star, v^\star) = (11640$ vehicles per hour$, 19.5$ km.h$^{-1})$. The relaxation time is $\tau = 60s$. We consider five different values for γ $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (0.1, 0.48, 0.52, 0.53, 1.5)$. The initial transition probabilities are taken as $(0.02, 0.32, 0.32, 0.32, 0.02)$, which means that the values are initially concentrated between $\gamma_2, \gamma_3, \gamma_4 \in [0.48, 0.53]$. We choose the reference value $\gamma_0 = 0.5$. We operate the simulation with the transition rates τ_{ij} defined as

$$\tau_{ij}(t) = \begin{cases} 0 & \text{if } i = j \\ 2\kappa & \text{if } i \in \{1,5\} \\ 0.1\kappa & \text{if } i \in \{2,3,4\}, j \in \{1,5\} \\ \kappa(1 + 2\cos(\epsilon(i+5j)t)^2 & \text{if } i,j \in \{2,3,4\}, i \neq j \end{cases}$$

where $\kappa = 10$ and $\epsilon = 10^{-3}$. The dynamic of the state probabilities $P_j^\gamma(t) = \mathbb{P}(\gamma(t) = \gamma_j)$ is pictured in Figure 1, and reflects the choice of transition rates: the mass of the probability function is concentrated on the three states $\gamma_2, \gamma_3, \gamma_4$ close to the nominal value (which take turns being the most probable state), and a residual mass (close to zero) is put evenly on the two remaining states γ_1 and γ_5. We have pictured in Figure 2 an example of a realization of the random parameter γ modeled by these probabilities. As expected, the process spends most of its

time around the nominal value $\gamma_0 = 0.5$, with frequent but short-lived jumps to the two extremal states γ_1 and γ_5.

We give in (3) the closed-loop simulation of traffic flow rate and velocity with the nominal control law (11) for a realization of the stochastic parameter $\gamma(t)$. The solver is identical to the one used in (Yu et al., 2020a, 2022). As expected, since the nominal value γ_0 is close to the different realisations, we have the exponential mean-square stability of the closed-loop system. Finally, we show in Figure 4 the Monte-Carlo simulation of $||(\tilde{q}, \tilde{v})||_{L^2}$ (100 trials) with the nominal control law (11) for the chosen stochastic parameter $\gamma(t)$. The illustrates the properties stated in Theorem 2.

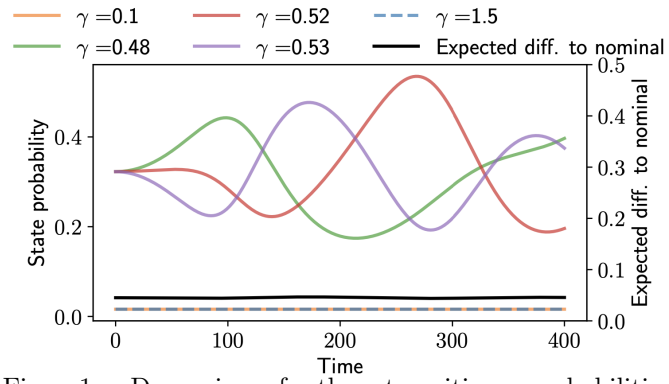

Fig. 1. Dynamic of the transition probabilities P_1, P_2, P_3, P_4, P_5 and expected difference to the nominal value.

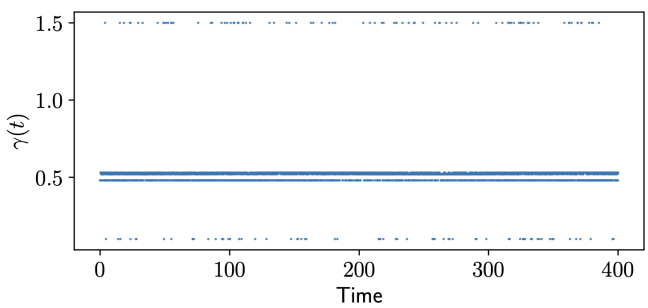

Fig. 2. Example of a realization of the random parameter $\gamma(t)$

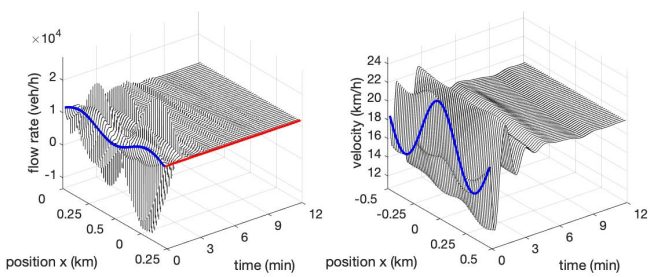

Fig. 3. Closed-loop simulation of traffic flow rate and velocity with the nominal control law (11) for a realization of the stochastic parameter $\gamma(t)$. The controlled flow rate evolution is highlighted in red.

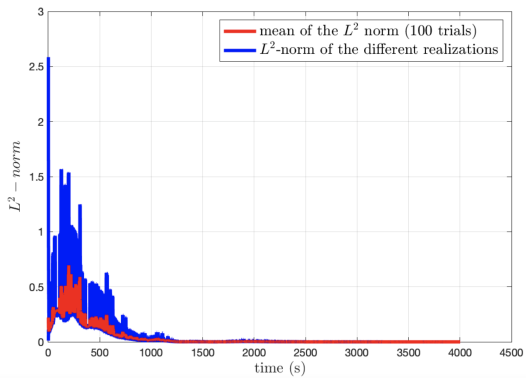

Fig. 4. Monte Carlo simulation of $||(\tilde{q}, \tilde{v})||_{L^2}$ (100 trials) with the nominal control law (11) for the chosen stochastic parameter $\gamma(t)$.

6. CONCLUSIONS

In this paper, we considered the stability of coupled and uncertain 2×2 hyperbolic system. Parametric uncertainties are modeled as independent Markov processes with a finite number of values. We showed that the classical backstepping controller proposed in (Vazquez et al., 2011) for nominal constant parameters guarantees the exponential mean-square stability of the closed-loop system provided that the nominal values are close enough on average to the real ones. This result generalizes the deterministic robustness results stated in (Auriol and Di Meglio, 2020). The proposed methodology is adjusted from (Kong and Bresch-Pietri, 2022a) and is based on a Lyapunov analysis. We illustrated our results on the test case of traffic control.

Future works will focus on the generalization of our approach to non-independent stochastic parameters and to a larger class of random variables (that may not be described by Markov processes) and random fields.

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