Estimation of the conditional tail moment for Weibull-type distributions

Yuri Goegebeur, Armelle Guillou, Jing Qin

Estimation of the conditional tail moment for Weibull-type distributions

Yuri Goegebeur[(1)], Armelle Guillou[(2,*)], Jing Qin[(1)]

[(1)] Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark
[(2)] Institut Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, 7 rue René Descartes, 67084 Strasbourg cedex, France
*Corresponding author: Armelle Guillou
E-mail: armelle.guillou@math.unistra.fr

Abstract

We consider the estimation of the conditional tail moment, defined for a positive random variable X as $\mathbb{E}(X^\beta | X > Q(1-p))$, where Q denotes the quantile function of X, $\beta > 0$ and $p \in (0,1)$, at extreme levels for the class of Weibull-type distributions. A two-step procedure is introduced where in the first stage one estimates the conditional tail moment at an intermediate level, followed by an extrapolation in the second stage. The asymptotic properties of the estimators introduced in the two stages are derived under suitable assumptions. The finite sample properties of the proposed estimator are examined with a simulation experiment. We conclude with two applications on real life data: wind speed measurements collected at an offshore wind farm and $PM_{2.5}$ air pollution data.
Keywords: Weibull-type distribution, extreme-value statistics, hazard rate function, empirical process.

1 Introduction

Quantifying and estimating the risk of extreme events is of interest in many scientific disciplines. Floods due to sea storms can entail massive losses, both in terms of human lives and economical, so the modelling of the distribution of wave heights and other sea state parameters is of crucial importance in the design of coastal protection. In environmental science, extreme levels of pollutants can have adverse effect on human health. Over the years several risk measures have been introduced, the most popular of these being the Value-at-Risk (also called return level) and the conditional tail expectation. We refer to, e.g., Jorion (2007), Singh et al. (2007), Coles (2013), Cooley (2013) and Salas and Obeysekera (2014). In this paper we consider the conditional tail moment (CTM), being a natural generalisation of the conditional tail expectation, and study its estimation for the class of Weibull-type distributions.

Let $\beta > 0$ and $p \in (0,1)$. We want to estimate

$$CTM_{\beta,p} := \mathbb{E}\left[X^{\beta}\middle| X > U_X(1/p)\right],$$

when the distribution of X is of Weibull-type, i.e., the distribution function satisfies

$$F_X(x) = 1 - e^{-x^{1/\theta}\ell_X(x)}, x > 0, \tag{1}$$

with θ a positive parameter and ℓ_X a slowly varying function at infinity, i.e., a positive measurable function such that

$$\lim_{t\to\infty} \frac{\ell_X(tx)}{\ell_X(t)} = 1 \quad \text{for all } x > 0,$$

and U_X the tail quantile function of X defined as $U_X(x) = \inf\{z : F_X(z) \geqslant 1 - 1/x\}, x > 1$. Specific interest is here in the extreme case, that is, the situation where p is very small, typically less than $1/n$, where n is the size of the random sample that is available for estimation. In this case, estimation is challenging as the conditional expectation in the definition of $CTM_{\beta,p}$ cannot simply be estimated by an empirical average. Indeed, for $p < 1/n$ the natural estimator for $U_X(1/p)$ is the largest observation in the sample, but then there are no further data to base the estimation of the conditional expectation upon.

The considered class of Weibull-type distributions forms a rich subclass of the Gumbel max-domain of attraction. The tail heaviness of Weibull-type distributions is clearly governed by the parameter θ, where larger values of θ correspond with a slower decay of the upper tail. Popular members of this family include the Weibull (corresponding to a constant function ℓ_X), normal, gamma and logistic distributions, to mention only a few. Loosely speaking, the upper tail decays exponentially fast, in contrast to the Pareto-type models which have a tail decay corresponding to a decreasing power function, and thus are heavier tailed. The Weibull-type distributions find important applications in areas like environmental science, climate science, hydrology, engineering and medicine. As the Weibull-type model depends only explicitly on the parameter θ, it is not unexpected that the estimation of this parameter received a lot of attention in the extreme-value literature. We refer to Broniatowski (1993), Beirlant et al. (1995), Girard (2004), Gardes and Girard (2005, 2008, 2016), Diebolt et al. (2008), Dierckx et al. (2009), Goegebeur et al. (2010), Goegebeur and Guillou (2011), Goegebeur et al. (2015), de Wet et al. (2016) and the references therein.

The estimation of the conditional tail moment is already studied for the context of Pareto-type distributions in Goegebeur et al. (2022a), who consider the classical setup of independent identically distributed (i.i.d.) random variables and in Goegebeur et al. (2022b) in an i.i.d. setting with random right censoring. El Methni et al. (2014) studied the estimation of the conditional tail moment in a regression setup where the conditional distribution of the response variable given the covariate is of Pareto-type. Estimation of the conditional tail moment for the class of Weibull-type distributions is to the best of our knowledge unexplored.

We organise this paper as follows. In Section 2 we provide as a first result an approximation of $CTM_{\beta,p}$ for $p \downarrow 0$, which will be used to derive the estimator for this risk measure under extrapolation. The proposed estimator depends on an estimator for the conditional tail moment in the intermediate case, i.e., for $p = k/n$, where $k \to \infty$ as the sample size $n \to \infty$ but in such a way that $k/n \to 0$. This intermediate case is studied in Section 3 where we introduce an estimator and derive its limiting distribution under suitable assumptions. Section 4 is devoted to the study of the asymptotic properties of the estimator for $CTM_{\beta,p}$ in the extreme case. The finite sample properties are examined by a simulation experiment in Section 5 while the practical applicability of the proposed method is illustrated in Section 6 on wind speed and air pollution data. Section 7 contains the proofs of the results.

2 Construction of the estimator

Let RV_ψ denote the class of regularly varying functions at infinity with index $\psi \in \mathbb{R}$, i.e., positive measurable functions f satisfying $f(tx)/f(t) \to x^\psi$, as $t \to \infty$, for all $x > 0$. Denote $H(x) = -\log \overline{F}_X(x)$, and hence $\overline{F}_X(x) = e^{-H(x)}$. In the case where F_X is of Weibull-type we have then $H \in RV_{1/\theta}$. Note that if H is differentiable with derivative H' then $H' = h$ where h is the hazard rate function.

We start to expand $CTM_{\beta,p}$ in case $p \downarrow 0$ in order to have an idea about how to estimate this risk measure under extrapolation, i.e., for $p < 1/n$, where n is the size of the sample on which the estimation is based.

Lemma 2.1 *Assume F_X satisfies (1), H is differentiable with derivative $h \in RV_{1/\theta-1}$. Then, as $p \downarrow 0$, we have*

$$\frac{CTM_{\beta,p}}{\left[U_X\left(\frac{1}{p}\right)\right]^\beta} \longrightarrow 1.$$

From this lemma, the first idea to estimate $CTM_{\beta,p}$ is to use an estimate of $\left[U_X\left(\frac{1}{p}\right)\right]^\beta$. Unfortunately, since by definition of the conditional tail moment we condition on the event $\{X > U_X(1/p)\}$, clearly such an estimate will underestimate the true value. An alternative method consists in using a two-step procedure as follows.

Let k be an intermediate sequence, i.e., a sequence such that $k \to \infty$ and $k/n \to 0$ as $n \to \infty$. Lemma 2.1 yields the approximation

$$\frac{CTM_{\beta,p}/\left[U_X\left(\frac{1}{p}\right)\right]^\beta}{CTM_{\beta,k/n}/\left[U_X\left(\frac{n}{k}\right)\right]^\beta} \sim 1,$$

from which we deduce that

$$CTM_{\beta,p} \sim \left(\frac{U_X(\frac{1}{p})}{U_X(\frac{n}{k})}\right)^\beta CTM_{\beta,k/n}.$$

Note that (1) is equivalent to assume that U_X satisfies

$$U_X(x) = (\log x)^\theta \, \ell_U(\log x),$$

where ℓ_U is also a slowly varying function at infinity. Thus, we deduce the following estimator for $CTM_{\beta,p}$

$$\widehat{CTM}_{\beta,p} := \left(\frac{\log(1/p)}{\log(n/k)}\right)^{\beta\,\widehat{\theta}_k} \overline{CTM}_{\beta,k/n}, \tag{2}$$

where $\widehat{\theta}_k$ is an estimator of θ and $\overline{CTM}_{\beta,k/n}$ an intermediate estimator of $CTM_{\beta,k/n}$.

3 Asymptotic properties of $\overline{CTM}_{\beta,k/n}$

Assume we have at our disposal a sample $X_1, \ldots, X_n$ of independent copies of the random variable X and denote by $X_{1,n} \leqslant \ldots \leqslant X_{n,n}$ the order statistics. We start with considering the estimation of the conditional tail moment in the intermediate case. In this situation, the natural estimator for $U_X(n/k)$ is then $X_{n-k,n}$, which is within the data range, and hence $CTM_{\beta,k/n}$ can be estimated empirically by the sample mean of the X_i^β for which $X_i > X_{n-k,n}$. We define

$$\overline{CTM}_{\beta,k/n} = \frac{1}{k}\sum_{i=1}^{n} X_i^\beta \mathbb{1}_{\{X_i > X_{n-k,n}\}},$$

and introduce

$$\overline{CTM}_{\beta,k/n}(x) := \frac{1}{k}\sum_{i=1}^{n} X_i^\beta \mathbb{1}_{\{X_i > U_X(\frac{n}{k}) + a_X(\frac{n}{k})\,x\}},$$

where

$$a_X(t) := \frac{1}{h(U_X(t))}.$$

Then, $\overline{CTM}_{\beta,k/n}$ can be rewritten as

$$\overline{CTM}_{\beta,k/n} = \overline{CTM}_{\beta,k/n}\left(\widehat{u}_{n,k}\right),$$

where $\widehat{u}_{n,k} := (X_{n-k,n} - U_X(n/k))/a_X(n/k)$.

The weak convergence of $\overline{CTM}_{\beta,k/n}$, after normalisation, is given in the following theorem.

Theorem 3.1 *Assume F_X satisfies (1), H is differentiable with derivative h which is ultimately monotone, and h is differentiable with derivative h' satisfying $xh'(x)/h(x) \to 1/\theta - 1$ as $x \to \infty$. Then, if $k \to \infty$ as $n \to \infty$ such that $k/n \to 0$ and $\sqrt{k}a_X(n/k)/U_X(n/k) \to 0$, we have*

$$\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\left(\frac{\overline{CTM}_{\beta,k/n}}{CTM_{\beta,k/n}} - 1\right) \xrightarrow{d} \Theta := \beta\int_0^\infty W\left(e^{-v}\right)\,dv,$$

where $W(z)$ is a zero centered Gaussian process with covariance function

$$\mathbb{E}\left\{W(z)\,W(\overline{z})\right\} = \min(z, \overline{z}).$$

Note that this theorem implies the convergence of the rescaled estimator of $CTM_{\beta,k/n}$ towards the $\mathcal{N}(0, 2\,\beta^2)$ distribution.

4 Asymptotic properties of $\widehat{CTM}_{\beta,p}$

In this section we study the asymptotic properties of the estimator $\widehat{CTM}_{\beta,p}$ given in (2). As estimator for the Weibull-tail coefficient θ we use the estimator proposed by Girard (2004), given by

$$\widehat{\theta}_k := \frac{\frac{1}{k}\sum_{i=1}^{k}\left(\log X_{n-i+1,n} - \log X_{n-k,n}\right)}{\frac{1}{k}\sum_{i=1}^{k}\left(\log\log\frac{n}{i} - \log\log\frac{n}{k}\right)}.$$

Below, we recall the limiting distribution of this estimator, properly normalised, in terms of a stochastic process, as established in Theorem 4.1 in Goegebeur et al. (2023).

Theorem 4.1 *Assume F_X satisfies (1), H is differentiable with derivative h which is ultimately monotone, and h is differentiable with derivative h' satisfying $xh'(x)/h(x) \to 1/\theta - 1$ as $x \to \infty$. Then, if $k \to \infty$ as $n \to \infty$ such that $\sqrt{k}a_X(n/k)/U_X(n/k) \to 0$ and $\sqrt{k}\,\frac{U_X(n/k)\ell'_X(U_X(n/k))}{\ell_X(U_X(n/k))} \to 0$, we have*

$$\sqrt{k}\left(\widehat{\theta}_k - \theta\right) \xrightarrow{d} \Gamma := \theta\left\{\int_0^\infty W\left(e^{-v}\right)\,dv - W\left(1\right)\right\}.$$

Note that the limiting distribution in Theorem 4.1 is $\mathcal{N}(0, \theta^2)$. As is clear from Theorem 3.1 and Theorem 4.1, the limiting behavior of $\overline{CTM}_{\beta,k/n}$ and $\widehat{\theta}_k$ involves the same stochastic process W. For more details we refer to Section 7 of the present paper and the proof of Theorem 4.1 in Goegebeur et al. (2023).

We can now state the main result of the paper, namely the weak convergence of $\widehat{CTM}_{\beta,p}$, properly normalised.

Theorem 4.2 *Assume F_X satisfies (1), H is differentiable with derivative h which is ultimately monotone, and h is differentiable with derivative h' satisfying $xh'(x)/h(x) \to 1/\theta - 1$ as $x \to \infty$. Then, if $k \to \infty$ as $n \to \infty$ such that $\sqrt{k}\frac{a_X(n/k)}{U_X(n/k)} \to 0$ and $\sqrt{k}\,\sup_{x \geqslant U_X(n/k)}\frac{x\ell'_X(x)}{\ell_X(x)} \to 0$, we have for p satisfying $p \leqslant \frac{k}{n}$ such that $\frac{\log 1/p}{\log n/k} \to \lambda \in (1,\infty)$ that*

$$\sqrt{k}\left(\frac{\widehat{CTM}_{\beta,p}}{CTM_{\beta,p}} - 1\right) \xrightarrow{d} \beta\,\log(\lambda)\,\Gamma.$$

From the result of Theorem 4.2, it is clear that the estimator $\widehat{CTM}_{\beta,p}$ inherits its limiting distribution, up to the factor $\beta\,\log(\lambda)$, from the estimator $\widehat{\theta}_k$ for the Weibull-tail coefficient. In particular, we obtain the $\mathcal{N}(0, (\theta\beta\log(\lambda))^2)$ distribution as limiting distribution for the rescaled estimator of $CTM_{\beta,p}$. As expected, more severe extrapolations, corresponding with larger values for λ, lead to an increase in the variance of the limiting distribution.

5 Simulation experiment

In this section we illustrate the finite sample behavior of the proposed estimator by means of a simulation experiment. The considered distributions are:

- Weibull distribution ($Weibull(\theta)$, $\theta > 0$), with distribution function as in (1) with $\ell_X = 1$. We set $\theta = 1$.
- Absolute normal distribution ($|N(\mu, \sigma^2)|$, $\mu \in \mathbb{R}$, $\sigma > 0$), i.e., the distribution of $|X|$ where $X \sim N(\mu, \sigma^2)$. For this distribution $\theta = 1/2$. We set $(\mu, \sigma^2) = (1.2, 1)$.
- Gamma distribution ($\Gamma(\alpha, \lambda)$, $\alpha, \lambda > 0$), for which $\theta = 1$. We set $(\alpha, \lambda) = (2, 1)$.
- Extended Weibull distribution ($EWeibull(\theta, \xi)$, $\theta > 1$, $\xi \in \mathbb{R}$) with survival function

$$1 - F_X(x) = r(x)e^{-x^{1/\theta}}, \quad x > 0,$$

  where $r \in RV_\xi$ (Klüppelberg and Villaseñor, 1993). We set $(\theta, \xi) = (1.5, -1)$.

From each distribution we simulate 500 datasets of sizes $n \in \{500, 2000\}$, and we consider estimation of $CTM_{\beta,p}$ with $p \in \{1/n, 1/(5n)\}$, and $\beta \in \{1, 2\}$.

In Figure 1 we show for the $Weibull(1)$ distribution the boxplots of $\widehat{CTM}_{1,p}/CTM_{1,p}$, computed over the 500 replications, as a function of k for $n = 500$ (left) and $n = 2000$ (right), and where the top row corresponds with $p = 1/n$ and the bottom row with $p = 1/(5n)$. Figures 2-4 are constructed analogously for the other distributions. From the simulations we observe the following:

- For all the distributions considered the estimation is good when k is small compared to n, which is in agreement with the theoretical condition $k/n \to 0$. With increasing k the estimators show typically an increase in bias. As for the bias we comment that estimation for Weibull-type distributions is practically challenging, with a bias that comes typically rather fast with increasing k. This was also already observed in the simpler context of the estimation of the Weibull-tail coefficient θ, see, e.g., Girard (2004), Gardes and Girard (2008), Goegebeur et al. (2010), to mention only a few, and obviously also manifests itself here.
- The estimation results improve with increasing sample size n.
- Decreasing p leads to increased variability of the estimates, which can be expected as smaller values of p lead to more severe extrapolations.
- Larger values of θ lead to more variable results. This is also in line with the expectations as θ determines the heaviness of the upper tail of the Weibull-type model.

Finally, in Figure 5 we illustrate the estimation of the second conditional tail moment, $CTM_{2,p}$, for the $|N(1.2, 1)|$ distribution. As expected, estimation of the second conditional tail moment is more challenging than that of the conditional tail expectation ($\beta = 1$), with larger variability of the estimates.

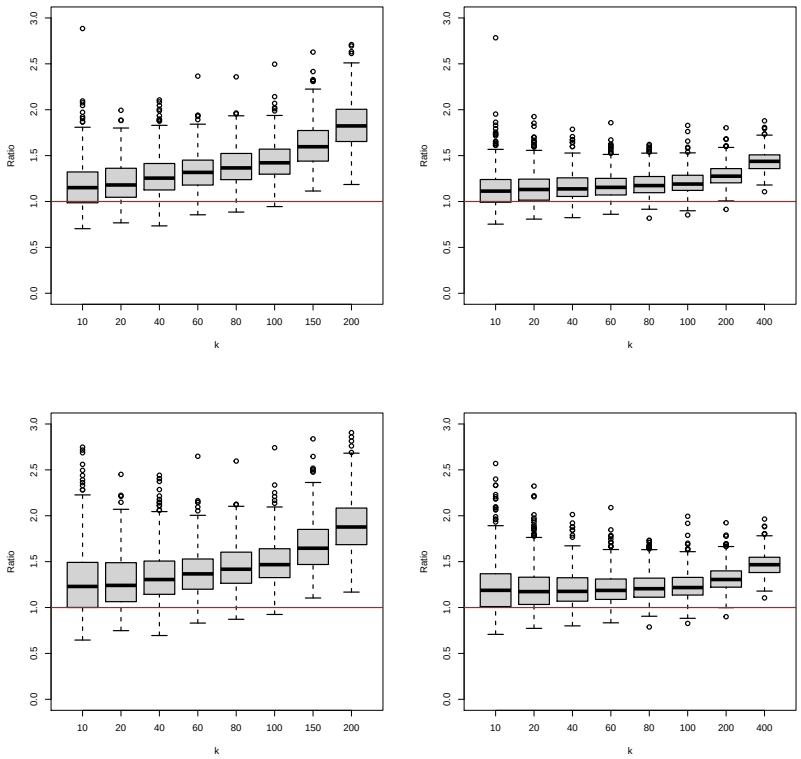

Figure 1: $Weibull(1)$ simulation. Boxplots of $\widehat{CTM}_{1,p}/CTM_{1,p}$ as a function of k for $n = 500$ (left) and $n = 2000$ (right) with $p = 1/n$ (top) and $p = 1/(5n)$ (bottom).

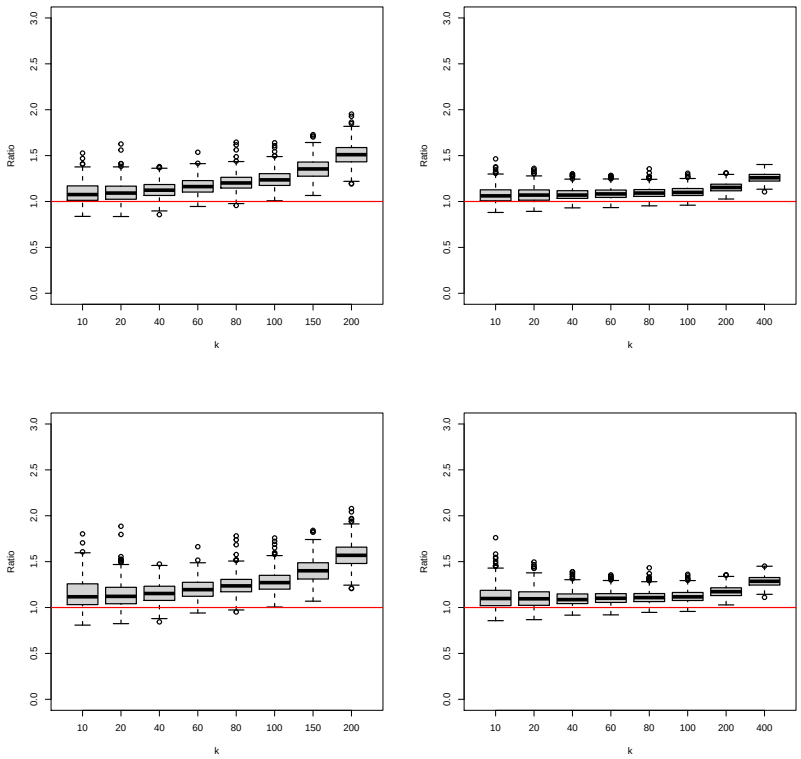

Figure 2: $|N(1.2,1)|$ simulation. Boxplots of $\widehat{CTM}_{1,p}/CTM_{1,p}$ as a function of k for $n = 500$ (left) and $n = 2000$ (right) with $p = 1/n$ (top) and $p = 1/(5n)$ (bottom).

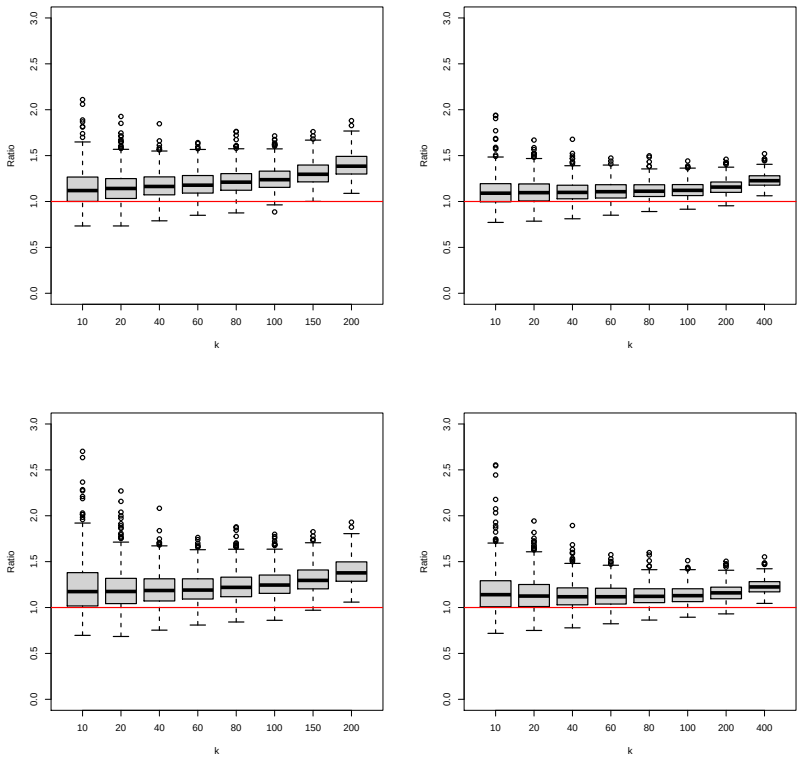

Figure 3: $\Gamma(2,1)$ simulation. Boxplots of $\widehat{CTM}_{1,p}/CTM_{1,p}$ as a function of k for $n = 500$ (left) and $n = 2000$ (right) with $p = 1/n$ (top) and $p = 1/(5n)$ (bottom).

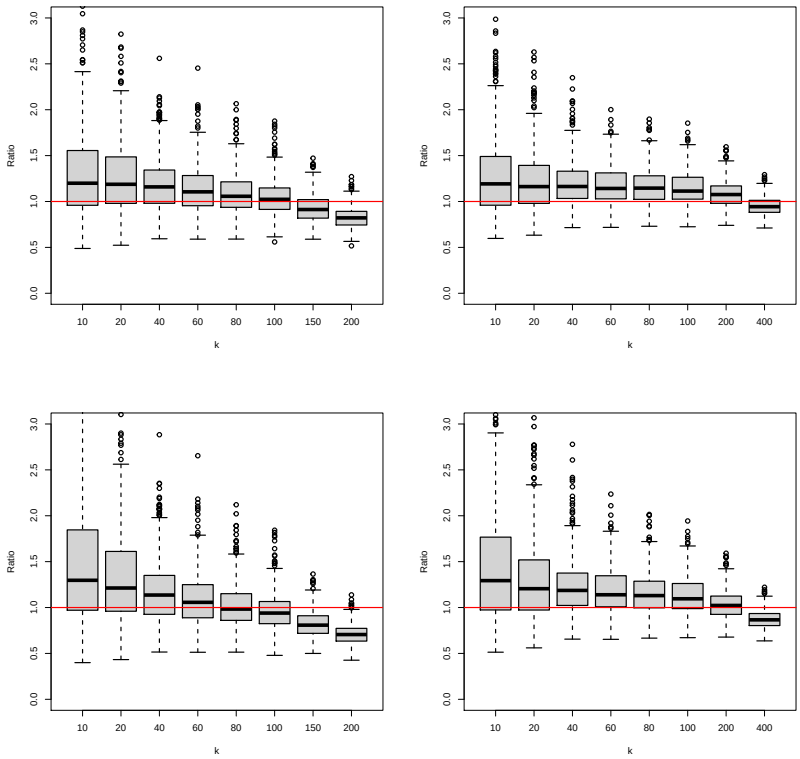

Figure 4: $EWeibull(1.5, -1)$ simulation. Boxplots of $\widehat{CTM}_{1,p}/CTM_{1,p}$ as a function of k for $n = 500$ (left) and $n = 2000$ (right) with $p = 1/n$ (top) and $p = 1/(5n)$ (bottom).

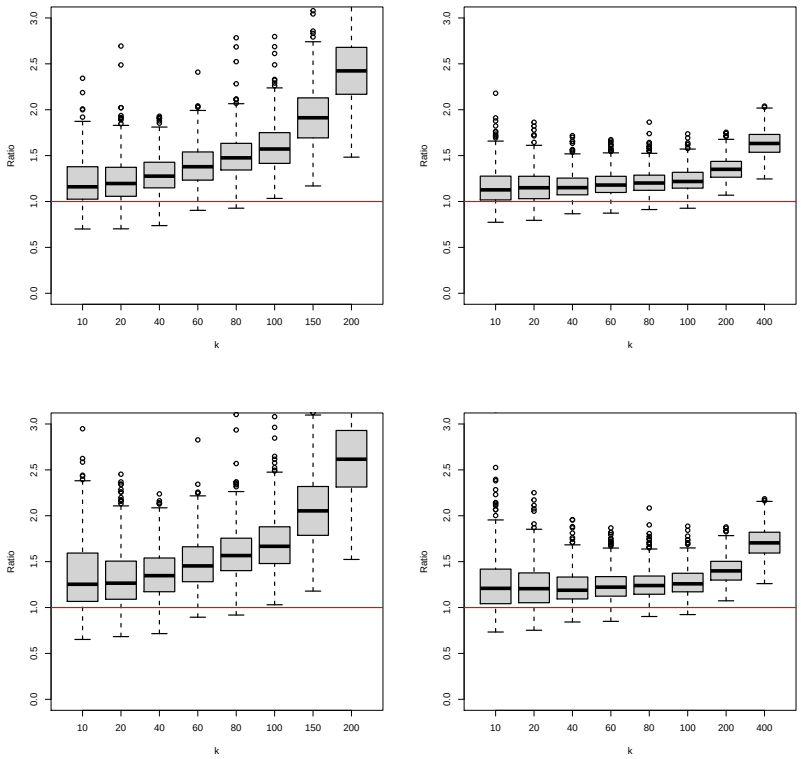

Figure 5: $|N(1.2, 1)|$ simulation. Boxplots of $\widehat{CTM}_{2,p}/CTM_{2,p}$ as a function of k for $n = 500$ (left) and $n = 2000$ (right) with $p = 1/n$ (top) and $p = 1/(5n)$ (bottom).

6 Real data analysis

6.1 Wind speeds at Anholt offshore wind farm

Following the worldwide economic development and population growth, global energy demand has also shown an important upward trend during the last decades. Traditional fossil fuels have many disadvantages: their reserves are limited and their use contributes to increased carbon dioxide levels in the atmosphere. On the other hand, renewable energy like wind power is inexhaustible and causes little pollution. Accurate modelling of the wind speed distribution is of crucial importance for wind energy output estimation, e.g., at an offshore wind farm. For a review of wind speed modelling approaches we refer to Shi et al. (2021).

Our interest is in an analysis of the upper tail of the wind speed distribution at the Anholt offshore wind farm, which is a Danish offshore wind power farm in the Kattegat, between Djursland and the island Anholt. We estimate the Weibull-tail index θ, the first conditional tail moment $CTM_{1,p}$ (also called the conditional tail expectation) and the conditional tail variance $V_p := CTM_{2,p} - CTM_{1,p}^2$, using a dataset that is publicly available at `https://orsted.com/en/what-we-do/renewable-energy-solutions/offshore-wind/offshore-wind-data`. In particular we use LIDAR measured wind speeds (in meters per second) at 100 meters altitude. The original dataset consists of a time series of 10 minutes measurements during the years 2013-2014, but in order to reduce temporal dependencies only one observation every three days is kept for the analysis, leading to $n = 242$. In order to evaluate the appropriateness of the Weibull-type model (1) we construct the Weibull QQ plot, which has coordinates

$$\left(\log\left(-\log\left(1 - \frac{i}{n+1} \right) \right), \log X_{i,n} \right), \quad i = 1, \ldots, n,$$

see Figure 6, top left panel. Clearly, the Weibull QQ plot becomes approximately linear in the largest observations, which supports the Weibull-type model. We refer to Goegebeur and Guillou (2010) for a discussion of the Weibull QQ plot. Next we estimate the Weibull-tail index θ, see Figure 6, top right, where we show $\widehat{\theta}_k$ as a function of k. This plot shows a stable horizontal part for k between, approximately, 30 and 60, indicating an estimate for θ of about 0.4. Finally, the bottom row of Figure 6 shows the estimates for $CTM_{1,p}$ (left) and for the conditional tail variance V_p (right) for $p = 1/n$ (solid line) and $p = 1/(5n)$ (dashed line) as a function of k. Again these plots show stability for k between 30 and 60 with estimates for $CTM_{1,1/n}$ and $CTM_{1,1/(5n)}$ of about 27 and 32 meters per second, respectively. Besides $CTM_{1,p}$, being an average above a high quantile, it is also relevant to estimate the variability in the upper tail, V_p. Again focusing on the stable k-region we have an estimate of V_p between 30-40 for $p = 1/n$ and $p = 1/(5n)$.

6.2 $PM_{2.5}$ air pollution

Air pollution is among the greatest environmental risks to human health. Of particular concern is the concentration of $PM_{2.5}$, referring to atmospheric particulate matter with a diameter of less than 2.5 micrometers. Since these particles are so small and light, they tend to stay longer in the air than heavier ones, increasing the chances of humans and animals inhaling them. $PM_{2.5}$ can travel deep in the respiratory tract, reaching the lungs and entering the blood stream. The

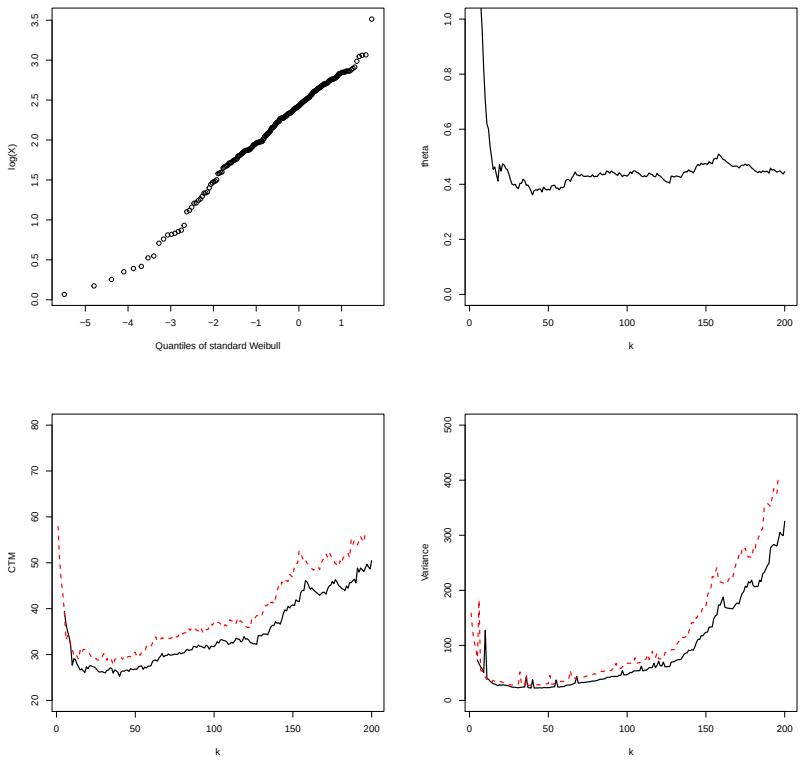

Figure 6: Anholt wind speed data. Weibull QQ plot (top left), $\widehat{\theta}_k$ as a function of k (top right), $\widehat{CTM}_{1,p}$ with $p = 1/n$ (solid line) and $p = 1/(5n)$ (dashed line) as a function of k (bottom left), and $\widehat{V}_p$ with $p = 1/n$ (solid line) and $p = 1/(5n)$ (dashed line) as a function of k (bottom right).

adverse health effects of $PM_{2.5}$ are well studied, see, e.g. Feng et al. (2016) and Li et al. (2017), and the references therein.

We apply the developed methodology to the $PM_{2.5}$ air pollution in the city of Slavonski Brod, Croatia, which is with an average of 28.0 micrograms per cubic meter ($\mu g/m^3$), measured during the last two years, amongst the most polluted cities in Europe. As a reference, note that the 2021 World Health Organisation's updated health-based guidelines for air quality recommend a maximum level of 5 $\mu g/m^3$ for long term exposure, in order to protect health. We use a dataset of daily average concentrations of $PM_{2.5}$ over the period 2015-2021, publicly available on the website of the European Environment Agency at
`https://discomap.eea.europa.eu/map/fme/AirQualityExport.htm`. We base the analysis on the data from the winter season (December-February), and in order to reduce the temporal dependence we only consider measurements that are three days spaced apart, leading to $n = 209$. The Weibull QQ plot shown in Figure 7, top left, becomes ultimately linear, supporting the assumption of an underlying Weibull-type distribution, and the plot of the Weibull-tail index estimates $\widehat{\theta}_k$ (Figure 7, top right) is stable for k between approximately 20 and 80, indicating an estimate of about 0.7-0.8. In the bottom row of Figure 7 we show $\widehat{CTM}_{1,p}$ (left panel) and $\widehat{V}_p$ (right panel) for $p = 1/n$ (solid line) and $p = 1/(5n)$ (dashed line) as a function of k. For these plots the stable region for k is from 20 to 50, approximately, with an estimate for $CTM_{1,p}$ of around 300 for $p = 1/n$ and 400 for $p = 1/(5n)$, and estimates for V_p around 10 000 and 15 000 for $p = 1/n$ and $p = 1/(5n)$, respectively.

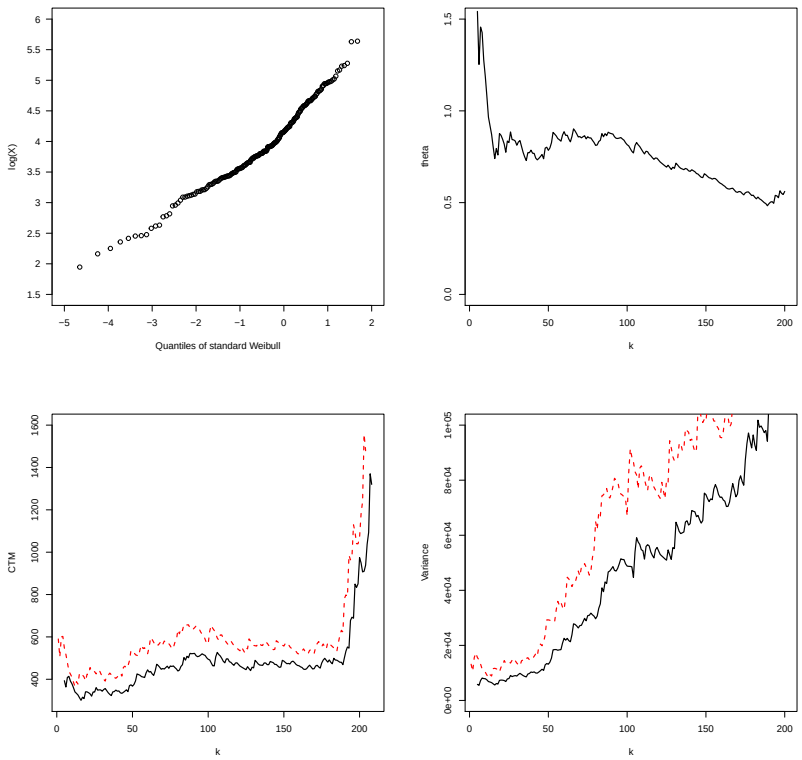

Figure 7: Air pollution data. Weibull QQ plot (top left), $\widehat{\theta}_k$ as a function of k (top right), $\widehat{CTM}_{1,p}$ with $p = 1/n$ (solid line) and $p = 1/(5n)$ (dashed line) as a function of k (bottom left), and $\widehat{V}_p$ with $p = 1/n$ (solid line) and $p = 1/(5n)$ (dashed line) as a function of k (bottom right).

7 Proofs

Remark that

$$
\begin{aligned}
&\overline{CTM}_{\beta,k/n}(x) \\
&= \beta\,\frac{1}{k}\sum_{i=1}^{n}\int_0^{X_i} u^{\beta-1}\,du\,\mathbb{1}_{\{X_i>U_X(\frac{n}{k})+a_X(\frac{n}{k})\,x\}} \\
&= \beta\int_0^{\infty} u^{\beta-1}\,\frac{1}{k}\sum_{i=1}^{n}\mathbb{1}_{\left\{X_i>\max\left(u,U_X(\frac{n}{k})+a_X\left(\frac{n}{k}\right)x\right)\right\}}du \\
&= \beta\int_0^{U_X(\frac{n}{k})+a_X(\frac{n}{k})\,x} u^{\beta-1}\,du\,\frac{1}{k}\sum_{i=1}^{n}\mathbb{1}_{\{X_i>U_X(\frac{n}{k})+a_X(\frac{n}{k})\,x\}} \\
&\quad +\beta\int_{U_X(\frac{n}{k})+a_X(\frac{n}{k})\,x}^{\infty} u^{\beta-1}\,\frac{1}{k}\sum_{i=1}^{n}\mathbb{1}_{\{X_i>u\}}\,du \\
&= \left[U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)x\right]^{\beta}\,\frac{1}{k}\sum_{i=1}^{n}\mathbb{1}_{\{X_i>U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)x\}} \\
&\quad +\beta\,a_X\left(\frac{n}{k}\right)\int_0^{\infty}\left[U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)(x+v)\right]^{\beta-1}\,\frac{1}{k}\sum_{i=1}^{n}\mathbb{1}_{\{X_i>U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)(x+v)\}}\,dv \\
&= \left[U_X\left(\frac{n}{k}\right)\right]^{\beta}\left\{\left[1+\frac{a_X\left(\frac{n}{k}\right)}{U_X\left(\frac{n}{k}\right)}\,x\right]^{\beta}\,T_n\left(\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)x\right)\right)\right. \\
&\quad \left.+\beta\,\frac{a_X\left(\frac{n}{k}\right)}{U_X\left(\frac{n}{k}\right)}\int_0^{\infty}\left[1+\frac{a_X\left(\frac{n}{k}\right)}{U_X\left(\frac{n}{k}\right)}(x+v)\right]^{\beta-1}\,T_n\left(\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)(x+v)\right)\right)\,dv\right\}\quad a.s.,
\end{aligned}
$$

where

$$T_n(x) := \frac{1}{k}\sum_{i=1}^{n}\mathbb{1}_{\{\overline{F}_X(X_i)<\frac{k}{n}\,x\}}.$$

The asymptotic behavior of $T_n(x)$, after proper normalisation, is given in Lemma 1 of Cai et al. (2015), according which, for any $\eta\in[0,1/2)$ and $T>0$ we have, for $n\to\infty$,

$$\sup_{x\in(0,T]}\left|\frac{\sqrt{k}[T_n(x)-x]-W(x)}{x^{\eta}}\right|\to 0\ a.s.. \tag{3}$$

Note that in the above result, all the involved processes are defined on the same probability space via the Skorohod construction. See also Einmahl et al. (2006) for convergence properties of related stochastic processes.

For completeness, we also include below three lemmas from Goegebeur et al. (2023), useful for our proofs. Note that the last one is actually a consequence of the proof of Lemma 5.4 in the aforementioned paper.

Lemma 7.1 *Let F_X satisfy (1), H be differentiable with derivative $h \in RV_{1/\theta-1}$, then, as $t \to \infty$*

$$\frac{\overline{F}_X\left(t + \frac{z}{h(t)}\right)}{\overline{F}_X(t)} \longrightarrow e^{-z},$$

uniformly for $z \geqslant -\delta$, for any $\delta > 0$.

Lemma 7.2 *If F_X satisfies (1) and H is differentiable with derivative $h \in RV_{1/\theta-1}$, then, for a sequence k such that $k \to \infty$ with $k/n \to 0$, we have*

$$\sqrt{k}\,\frac{X_{n-k,n} - U_X(\frac{n}{k})}{a_X(\frac{n}{k})} \xrightarrow{d} W(1).$$

Lemma 7.3 *Assume F_X satisfies (1), H is differentiable with derivative h which is ultimately monotone, and h is differentiable with derivative h' satisfying $xh'(x)/h(x) \to 1/\theta - 1$ as $x \to \infty$. Then, if $k \to \infty$ as $n \to \infty$ such that $k/n \to 0$ and $\sqrt{k}a_X(n/k)/U_X(n/k) \to 0$, we have, for any $\delta > 0$,*

$$\sup_{x\in[-\delta,\delta]} \sqrt{k} \int_0^\infty \left| \frac{n}{k}\overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)(x+v)\right) - e^{-(x+v)}\right| dv^\beta \to 0.$$

Proof of Lemma 2.1. We have

$$\begin{aligned}
CTM_{\beta,p} &= -\int_{U_X(1/p)}^\infty \frac{x^\beta}{\overline{F}_X(U_X(1/p))}\, d\overline{F}_X(x) \\
&= [U_X(1/p)]^\beta + \beta \int_{U_X(1/p)}^\infty x^{\beta-1} \frac{\overline{F}_X(x)}{\overline{F}_X(U_X(1/p))}\, dx \\
&= [U_X(1/p)]^\beta \left\{ 1 + \beta\, \frac{a_X(1/p)}{U_X(1/p)} \int_0^\infty \left[1 + z\, \frac{a_X(1/p)}{U_X(1/p)}\right]^{\beta-1} \frac{\overline{F}_X(U_X(1/p) + z\, a_X(1/p))}{\overline{F}_X(U_X(1/p))}\, dz \right\}. \quad (4)
\end{aligned}$$

Since $\frac{a_X(1/p)}{U_X(1/p)} \to 0$, to prove Lemma 2.1, it is sufficient to show that the integral in the right-hand side of (4) is finite. To this aim, by the mean value theorem, with $\xi \in (0,1)$, we have

$$\begin{aligned}
\frac{\overline{F}_X(U_X(1/p) + z\, a_X(1/p))}{\overline{F}_X(U_X(1/p))} &= e^{-H(U_X(1/p)+z\,a_X(1/p))+H(U_X(1/p))} \\
&= e^{-\frac{h(U_X(1/p)+\xi\, z\, a_X(1/p))}{h(U_X(1/p))}\, z},
\end{aligned}$$

from which, by the use of the Potter bounds (see Proposition B.1.9.5 in de Haan and Ferreira, 2006), for $0 < \delta_1 < 1, \delta_2 > 0$ and n large we have

$$\begin{aligned}
&\int_0^\infty \left[1 + z\, \frac{a_X(1/p)}{U_X(1/p)}\right]^{\beta-1} \frac{\overline{F}_X(U_X(1/p) + z\, a_X(1/p))}{\overline{F}_X(U_X(1/p))}\, dz \\
&\qquad \leqslant \int_0^\infty \left[1 + z\, \frac{a_X(1/p)}{U_X(1/p)}\right]^{\beta-1} e^{-(1-\delta_1)\,[1+\xi\,\frac{a_X(1/p)}{U_X(1/p)}\,z]^{\frac{1}{\theta}-1-\delta_2}\, z}\, dz.
\end{aligned}$$

If $\theta < 1$, take $0 < \delta_2 < 1/\theta - 1$, and obtain, for n large

$$\begin{aligned}
&\int_0^\infty \left[1 + z\,\frac{a_X(1/p)}{U_X(1/p)}\right]^{\beta-1} \frac{\overline{F}_X(U_X(1/p) + z\,a_X(1/p))}{\overline{F}_X(U_X(1/p))}\,dz \\
&\leqslant \int_0^\infty \left[1 + z\,\frac{a_X(1/p)}{U_X(1/p)}\right]^{\beta-1} e^{-(1-\delta_1)\,z}\,dz \\
&= \frac{U_X(1/p)}{a_X(1/p)}\,e^{(1-\delta_1)\,\frac{U_X(1/p)}{a_X(1/p)}} \int_1^\infty u^{\beta-1}\,e^{-(1-\delta_1)\,\frac{U_X(1/p)}{a_X(1/p)}\,u}\,du \\
&= \frac{1}{1-\delta_1}\,\frac{\Gamma\left(\beta, (1-\delta_1)\frac{U_X(1/p)}{a_X(1/p)}\right)}{\left[(1-\delta_1)\frac{U_X(1/p)}{a_X(1/p)}\right]^{\beta-1}\,e^{-(1-\delta_1)\,\frac{U_X(1/p)}{a_X(1/p)}}}, \qquad (5)
\end{aligned}$$

where $\Gamma(.,.)$ denotes the incomplete Gamma function, namely $\Gamma(s,z) := \int_z^\infty v^{s-1}\,e^{-v}\,dv$. For $p \downarrow 0$, the right-hand side in (5) is bounded.
Now, if $\theta \geqslant 1$, take $0 < \delta_2 < 1/\theta$, and for $\varepsilon, \Delta > 0$, we obtain for n large

$$\begin{aligned}
&\int_0^\infty \left[1 + z\,\frac{a_X(1/p)}{U_X(1/p)}\right]^{\beta-1} \frac{\overline{F}_X(U_X(1/p) + z\,a_X(1/p))}{\overline{F}_X(U_X(1/p))}\,dz \\
&\leqslant \int_0^\infty \left[\mathbb{1}_{\{\beta<1\}} + (1+\varepsilon\,z)^{\beta-1}\mathbb{1}_{\{\beta\geqslant 1\}}\right] e^{-(1-\delta_1)[1+\varepsilon\,z]^{\frac{1}{\theta}-1-\delta_2}\,z}\,dz \\
&= \int_0^\Delta \left[\mathbb{1}_{\{\beta<1\}} + (1+\varepsilon\,z)^{\beta-1}\mathbb{1}_{\{\beta\geqslant 1\}}\right] e^{-(1-\delta_1)[1+\varepsilon\,z]^{\frac{1}{\theta}-1-\delta_2}\,z}\,dz \\
&\quad + \int_\Delta^\infty \left[\mathbb{1}_{\{\beta<1\}} + (1+\varepsilon\,z)^{\beta-1}\mathbb{1}_{\{\beta\geqslant 1\}}\right] e^{-(1-\delta_1)[\varepsilon+\frac{1}{z}]^{\frac{1}{\theta}-1-\delta_2}\,z^{\frac{1}{\theta}-\delta_2}}\,dz \\
&\leqslant \left[\mathbb{1}_{\{\beta<1\}} + (1+\varepsilon\,\Delta)^{\beta-1}\mathbb{1}_{\{\beta\geqslant 1\}}\right] \int_0^\Delta e^{-(1-\delta_1)[1+\varepsilon\,\Delta]^{\frac{1}{\theta}-1-\delta_2}\,z}\,dz \\
&\quad + \int_\Delta^\infty \left[\mathbb{1}_{\{\beta<1\}} + (1+\varepsilon\,z)^{\beta-1}\mathbb{1}_{\{\beta\geqslant 1\}}\right] e^{-(1-\delta_1)[\varepsilon+\frac{1}{\Delta}]^{\frac{1}{\theta}-1-\delta_2}\,z^{\frac{1}{\theta}-\delta_2}}\,dz \\
&\leqslant \Delta\left[\mathbb{1}_{\{\beta<1\}} + (1+\varepsilon\,\Delta)^{\beta-1}\mathbb{1}_{\{\beta\geqslant 1\}}\right] \\
&\quad + \mathbb{1}_{\{\beta<1\}} \int_\Delta^\infty e^{-(1-\delta_1)[\varepsilon+\frac{1}{\Delta}]^{\frac{1}{\theta}-1-\delta_2}\,z^{\frac{1}{\theta}-\delta_2}}\,dz \\
&\quad + \mathbb{1}_{\{\beta\geqslant 1\}} \left(\varepsilon + \frac{1}{\Delta}\right)^{\beta-1} \int_\Delta^\infty z^{\beta-1}\,e^{-(1-\delta_1)[\varepsilon+\frac{1}{\Delta}]^{\frac{1}{\theta}-1-\delta_2}\,z^{\frac{1}{\theta}-\delta_2}}\,dz \\
&\leqslant \Delta\left[\mathbb{1}_{\{\beta<1\}} + (1+\varepsilon\,\Delta)^{\beta-1}\mathbb{1}_{\{\beta\geqslant 1\}}\right] \\
&\quad + \mathbb{1}_{\{\beta<1\}}\frac{\theta}{1-\theta\delta_2}\left[\frac{1}{(1-\delta_1)(\varepsilon+\frac{1}{\Delta})^{\frac{1}{\theta}-1-\delta_2}}\right]^{\frac{\theta}{1-\theta\delta_2}} \Gamma\left(\frac{\theta}{1-\theta\delta_2}\right) \\
&\quad + \mathbb{1}_{\{\beta\geqslant 1\}} \left(\varepsilon + \frac{1}{\Delta}\right)^{\beta-1} \frac{\theta}{1-\theta\delta_2}\left[\frac{1}{(1-\delta_1)(\varepsilon+\frac{1}{\Delta})^{\frac{1}{\theta}-1-\delta_2}}\right]^{\frac{\beta\theta}{1-\theta\delta_2}} \Gamma\left(\frac{\beta\theta}{1-\theta\delta_2}\right)
\end{aligned}$$

which is also bounded. This achieves the proof of Lemma 2.1.

Proof of Theorem 3.1. Remark that

$$\frac{\overline{CTM}_{\beta,k/n}}{CTM_{\beta,k/n}} - 1 = \frac{[U_X(n/k)]^\beta}{CTM_{\beta,k/n}} \left\{ \frac{\overline{CTM}_{\beta,k/n}(\widehat{u}_{n,k})}{[U_X(n/k)]^\beta} - \frac{CTM_{\beta,k/n}}{[U_X(n/k)]^\beta} \right\}. \tag{6}$$

The leading factor in the right-hand side of (6) tends to 1 by Lemma 2.1. Thus we only need to look at the term in the braces. First, we note that

$$T_n \left(\frac{n}{k} \overline{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) \widehat{u}_{n,k} \right) \right) = 1 \ a.s.,$$

from which we deduce that

$$
\begin{aligned}
&\frac{\overline{CTM}_{\beta,k/n}(\widehat{u}_{n,k})}{[U_X(n/k)]^\beta} - \frac{CTM_{\beta,k/n}}{[U_X(n/k)]^\beta} \\
= \;& \left\{ \left[\frac{X_{n-k,n}}{U_X(n/k)} \right]^\beta - 1 \right\} + \beta \, \frac{1}{\sqrt{k}} \, \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty W\left(e^{-v}\right) \, dv \\
&+ \beta \, \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} \left(\widehat{u}_{n,k} + v\right) \right]^{\beta-1} \\
&\qquad \times \left[T_n\left(\frac{n}{k} \, \overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right) \left(\widehat{u}_{n,k} + v\right) \right) \right) - \frac{n}{k} \, \overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right) \left(\widehat{u}_{n,k} + v\right) \right) \right. \\
&\qquad\qquad \left. - \frac{1}{\sqrt{k}} \, W\left(\frac{n}{k} \, \overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right) \left(\widehat{u}_{n,k} + v\right) \right) \right) \right] dv \\
&+ \beta \, \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} \left(\widehat{u}_{n,k} + v\right) \right]^{\beta-1} \\
&\qquad \times \left[\frac{n}{k} \, \overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right) \left(\widehat{u}_{n,k} + v\right) \right) - e^{-(\widehat{u}_{n,k}+v)} \right] dv \\
&+ \beta \, \frac{a_X(n/k)}{U_X(n/k)} \, e^{-\widehat{u}_{n,k}} \int_0^\infty \left[\left(1 + \frac{a_X(n/k)}{U_X(n/k)} \left(\widehat{u}_{n,k} + v\right) \right)^{\beta-1} - \left(1 + \frac{a_X(n/k)}{U_X(n/k)} \, v \right)^{\beta-1} \right] e^{-v} \, dv \\
&+ \beta \, \frac{a_X(n/k)}{U_X(n/k)} \left(e^{-\widehat{u}_{n,k}} - 1 \right) \int_0^\infty \left(1 + \frac{a_X(n/k)}{U_X(n/k)} \, v \right)^{\beta-1} e^{-v} \, dv \\
&+ \beta \, \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} \, v \right]^{\beta-1} \left[e^{-v} - \frac{n}{k} \, \overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right) \, v \right) \right] dv \\
&+ \beta \, \frac{1}{\sqrt{k}} \, \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} \left(\widehat{u}_{n,k} + v\right) \right]^{\beta-1} \\
&\qquad \times \left[W\left(\frac{n}{k} \, \overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right) \left(\widehat{u}_{n,k} + v\right) \right) \right) - W\left(e^{-(\widehat{u}_{n,k}+v)} \right) \right] dv \\
&+ \beta \, \frac{1}{\sqrt{k}} \, \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} \left(\widehat{u}_{n,k} + v\right) \right]^{\beta-1} \left[W\left(e^{-(\widehat{u}_{n,k}+v)} \right) - W\left(e^{-v} \right) \right] dv \\
&+ \beta \, \frac{1}{\sqrt{k}} \, \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[\left(1 + \frac{a_X(n/k)}{U_X(n/k)} \left(\widehat{u}_{n,k} + v\right) \right)^{\beta-1} - 1 \right] W\left(e^{-v} \right) \, dv \\
=: \;& \sum_{i=1}^{10} T_{i,n}.
\end{aligned}
$$

We will study all the terms separately.

Term $T_{1,n}$. According to Lemma 7.2, we have

$$
\sqrt{k} \, \frac{U_X(n/k)}{a_X(n/k)} \, T_{1,n} = \beta \, \sqrt{k} \, \frac{X_{n-k,n} - U_X(n/k)}{a_X(n/k)} \left(1 + o_{\mathbb{P}}(1)\right) = \beta \, W(1) + o_{\mathbb{P}}(1). \tag{7}
$$

Term $T_{2,n}$. Clearly

$$\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,T_{2,n} = \beta\int_0^\infty W\left(e^{-v}\right)\,dv. \tag{8}$$

Term $T_{3,n}$. Using Lemmas 7.1 and 7.2, with arbitrary large probability, for n large, $c>0$ and $0<\eta<1/2$, we have

$$\begin{aligned}\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,|T_{3,n}| \;\leqslant\; & \sup_{0<y\leqslant e^\delta+c}\frac{|\sqrt{k}[T_n(y)-y]-W(y)|}{y^\eta}\\ & \times\beta\int_0^\infty\left[1+\frac{a_X(n/k)}{U_X(n/k)}\,(\widehat{u}_{n,k}+v)\right]^{\beta-1}\left(\frac{\overline{F}_X(U_X(\frac{n}{k})+a_X(\frac{n}{k})\,(\widehat{u}_{n,k}+v))}{\overline{F}_X(U_X(\frac{n}{k}))}\right)^\eta dv.\end{aligned}$$

The supremum term is negligeable a.s. according to (3) and a treatment similar to that of the proof of Lemma 2.1 with Potter bounds ensures that the integral is bounded. This yields

$$\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,T_{3,n} = o_{\mathbb{P}}(1). \tag{9}$$

Term $T_{4,n}$. Let us define

$$I_n(x;\beta) = \sqrt{k}\int_0^\infty\left|\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)\,(x+v)\right)-e^{-(x+v)}\right|dv^\beta.$$

Using Lemma 7.2, with arbitrary large probability, for n large, $0<\varepsilon<1/\delta$ and any $\Delta>0$, we have

$$\begin{aligned}&\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,|T_{4,n}|\\ \leqslant\;& \mathbb{1}_{\{\beta<1\}}\,\beta\,(1-\delta\varepsilon)^{\beta-1}\sqrt{k}\int_0^\infty\left|\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)\,(\widehat{u}_{n,k}+v)\right)-e^{-(\widehat{u}_{n,k}+v)}\right|dv\\ &+\mathbb{1}_{\{\beta\geqslant1\}}\,\beta\,(1+\varepsilon(\delta+\Delta))^{\beta-1}\,\sqrt{k}\int_0^\Delta\left|\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)\,(\widehat{u}_{n,k}+v)\right)-e^{-(\widehat{u}_{n,k}+v)}\right|dv\\ &+\mathbb{1}_{\{\beta\geqslant1\}}\,\left(\varepsilon+\frac{1+\varepsilon\,\delta}{\Delta}\right)^{\beta-1}\sqrt{k}\int_\Delta^\infty\left|\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)\,(\widehat{u}_{n,k}+v)\right)-e^{-(\widehat{u}_{n,k}+v)}\right|dv^\beta\\ \leqslant\;& \left(\sup_{x\in[-\delta,\delta]}I_n(x;1)\right)\left\{\mathbb{1}_{\{\beta<1\}}\,\beta\,(1-\delta\varepsilon)^{\beta-1}+\mathbb{1}_{\{\beta\geqslant1\}}\,\beta\,(1+\varepsilon(\delta+\Delta))^{\beta-1}\right\}\\ &+\left(\sup_{x\in[-\delta,\delta]}I_n(x;\beta)\right)\,\mathbb{1}_{\{\beta\geqslant1\}}\,\left(\varepsilon+\frac{1+\varepsilon\,\delta}{\Delta}\right)^{\beta-1}.\end{aligned}$$

Thus, Lemma 7.3 implies that

$$\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,T_{4,n} = o_{\mathbb{P}}(1). \tag{10}$$

Term $T_{5,n}$. By Lemma 7.2, using the mean value theorem, with $\xi \in (0,1)$, we have with large probability, for n large and $0 < \varepsilon < 1/\delta$

$$\begin{aligned}
\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,|T_{5,n}| &\leqslant \beta\,|\beta-1|\,\frac{a_X(n/k)}{U_X(n/k)}\,\sqrt{k}\,\widehat{u}_{n,k}\int_0^\infty \left[1+\frac{a_X(n/k)}{U_X(n/k)}\,(v+\xi\,\widehat{u}_{n,k})\right]^{\beta-2} e^{-v}\,dv\,(1+o_{\mathbb{P}}(1)) \\
&\leqslant \beta\,|\beta-1|\,\frac{a_X(n/k)}{U_X(n/k)}\,\sqrt{k}\,\widehat{u}_{n,k}\,(1+o_{\mathbb{P}}(1)) \\
&\quad\times\left\{\mathbb{1}_{\{\beta<2\}}\,(1-\varepsilon\delta)^{\beta-2}\int_0^\infty e^{-v}\,dv + \mathbb{1}_{\{\beta\geqslant 2\}}\int_0^\infty [1+\varepsilon(v+\delta)]^{\beta-2}\,e^{-v}\,dv\right\} \\
&= o_{\mathbb{P}}(1), \qquad (11)
\end{aligned}$$

again by Lemma 7.2 and since $\frac{a_X(n/k)}{U_X(n/k)} \to 0$.
Term $T_{6,n}$. According to Lemma 7.2, we have

$$\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,T_{6,n} = -\beta\,\sqrt{k}\,\left(\frac{X_{n-k,n}-U_X(n/k)}{a_X(n/k)}\right) + o_{\mathbb{P}}(1) = -\beta\,W(1) + o_{\mathbb{P}}(1). \qquad (12)$$

Term $T_{7,n}$. It can be handled similarly as $T_{4,n}$. This yields

$$\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,T_{7,n} = o_{\mathbb{P}}(1). \qquad (13)$$

Term $T_{8,n}$. With arbitrary large probability, for n large, $0 < \varepsilon < 1/\delta$ and any $\Delta > 0$, we have

$$\begin{aligned}
\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,|T_{8,n}| &\leqslant \left\{\mathbb{1}_{\{\beta<1\}}\,\beta\,(1-\delta\varepsilon)^{\beta-1} + \mathbb{1}_{\{\beta\geqslant 1\}}\,\beta\,(1+\varepsilon(\delta+\Delta))^{\beta-1}\right\}\,\mathbb{W}_n(1) \\
&\quad + \mathbb{1}_{\{\beta\geqslant 1\}}\,\left(\varepsilon+\frac{1+\varepsilon\,\delta}{\Delta}\right)^{\beta-1}\,\mathbb{W}_n(\beta), \qquad (14)
\end{aligned}$$

where

$$\mathbb{W}_n(\beta) := \int_0^\infty \left|W\left(\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)\,(\widehat{u}_{n,k}+v)\right)\right) - W\left(e^{-(\widehat{u}_{n,k}+v)}\right)\right|\,dv^\beta.$$

For $T > 0$, we have

$$\begin{aligned}
\mathbb{W}_n(\beta) &\leqslant \int_T^\infty \left|W\left(e^{-(\widehat{u}_{n,k}+v)}\right)\right|\,dv^\beta + \int_T^\infty \left|W\left(\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)\,(\widehat{u}_{n,k}+v)\right)\right)\right|\,dv^\beta \\
&\quad + \int_0^T \left|W\left(\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right)+a_X\left(\frac{n}{k}\right)\,(\widehat{u}_{n,k}+v)\right)\right) - W\left(e^{-(\widehat{u}_{n,k}+v)}\right)\right|\,dv^\beta \\
&=: \mathbb{W}_n^{(1)}(\beta) + \mathbb{W}_n^{(2)}(\beta) + \mathbb{W}_n^{(3)}(\beta).
\end{aligned}$$

Then, for $a > 0$ arbitrary, we have

$$\mathbb{P}\left(\mathbb{W}_n(\beta) > a\right) \leqslant \mathbb{P}\left(\mathbb{W}_n^{(1)}(\beta) > \frac{a}{4}\right) + \mathbb{P}\left(\mathbb{W}_n^{(2)}(\beta) > \frac{a}{4}\right) + \mathbb{P}\left(\mathbb{W}_n^{(3)}(\beta) > \frac{a}{2}\right).$$

Clearly, for $\eta \in (0, 1/2)$, we have

$$\mathbb{P}\left(\mathbb{W}_n^{(1)}(\beta) > \frac{a}{4}\right) \leqslant \mathbb{P}\left(\sup_{0<y\leqslant e^{\delta}} \frac{|W(y)|}{y^{\eta}} > \frac{a\,\eta^{\beta}}{4\,\beta}\,\frac{e^{-\eta\delta}}{\Gamma(\beta, T\eta)}\right) + \mathbb{P}\left(|\widehat{u}_{n,k}| > \delta\right),$$

and similarly, by Lemma 7.1, $c > 0$ and n large

$$\begin{aligned}\mathbb{P}\left(\mathbb{W}_n^{(2)}(\beta) > \frac{a}{4}\right) \quad \leqslant \quad & \mathbb{P}\left(\sup_{0<y\leqslant e^{\delta}+c} \frac{|W(y)|}{y^{\eta}} \int_T^{\infty} \left[\frac{n}{k}\overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)(-\delta + v)\right)\right]^{\eta} dv^{\beta} > \frac{a}{4}\right) \\ & + \mathbb{P}\left(|\widehat{u}_{n,k}| > \delta\right).\end{aligned}$$

Following the lines of proof of Lemma 2.1 with the Potter bounds, the integral from the above display can be made arbitrary small by taking T large. Thus, combining Lemma 7.2 with Lemma 2 in Cai et al. (2015), and by choosing T large enough, for $\varepsilon > 0$, there exists n_1 such that for $n > n_1$ we have

$$\mathbb{P}\left(\mathbb{W}_n^{(1)}(\beta) > \frac{a}{4}\right) + \mathbb{P}\left(\mathbb{W}_n^{(2)}(\beta) > \frac{a}{4}\right) \leqslant \frac{\varepsilon}{2}.$$

With this specific value of T, we have

$$\begin{aligned}\mathbb{P}\left(\mathbb{W}_n^{(3)}(\beta) > \frac{a}{2}\right) \quad \leqslant \quad & \mathbb{P}\left(\sup_{y\in[-\delta,\delta+T]} \left|W\left(\frac{n}{k}\,\overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right) y\right)\right) - W\left(e^{-y}\right)\right| > \frac{a}{2\,T^{\beta}}\right) \\ & + \mathbb{P}\left(|\widehat{u}_{n,k}| > \delta\right).\end{aligned}$$

Lemmas 7.1 and 7.2 combined with the uniform continuity of W on compact sets yield, for $n > n_2$

$$\mathbb{P}\left(\mathbb{W}_n^{(3)}(\beta) > \frac{a}{2}\right) \leqslant \frac{\varepsilon}{2},$$

from which we deduce that, for $n > \max(n_1, n_2)$

$$\mathbb{P}\left(\mathbb{W}_n(\beta) > a\right) \leqslant \varepsilon. \tag{15}$$

Combining (14) and (15), we have

$$\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,T_{8,n} = o_{\mathbb{P}}(1). \tag{16}$$

Term $T_{9,n}$. It can be handled similarly as $T_{8,n}$, except for

$$\widetilde{\mathbb{W}}_n^{(3)}(\beta) := \int_0^T \left|W\left(e^{-(\widehat{u}_{n,k}+v)}\right) - W\left(e^{-v}\right)\right| dv^{\beta},$$

where we can use the bound

$$\mathbb{P}\left(\widetilde{\mathbb{W}}_n^{(3)}(\beta) > \frac{a}{2}\right) \leqslant \mathbb{P}\left(\sup_{v\in[0,T],|x|\leqslant k^{-1/4}} \left|W\left(e^{-(v+x)}\right) - W\left(e^{-v}\right)\right| > \frac{a}{2\,T^{\beta}}\right) + \mathbb{P}\left(|\widehat{u}_{n,k}| > k^{-\frac{1}{4}}\right).$$

By the uniform continuity of W on compact sets and Lemma 7.2, we have for $n > n_2$

$$\mathbb{P}\left(\widetilde{\mathbb{W}}_n^{(3)}(\beta) > \frac{a}{2}\right) \leqslant \frac{\varepsilon}{2},$$

from which we deduce that

$$\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,T_{9,n} = o_{\mathbb{P}}(1). \tag{17}$$

Term $T_{10,n}$. By the mean value theorem, with $\xi \in (0,1)$, we have with large probability, for n large, and $0 < \varepsilon < 1/\delta$ and $0 < \eta < 1/2$,

$$\begin{aligned}
\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\,|T_{10,n}| &= \left|\beta(\beta-1)\,\frac{a_X(n/k)}{U_X(n/k)}\int_0^\infty [\widehat{u}_{n,k}+v]\left[1+\xi\,\frac{a_X(n/k)}{U_X(n/k)}\,(\widehat{u}_{n,k}+v)\right]^{\beta-2} W\left(e^{-v}\right)\,dv\right| \\
&\leqslant \beta|\beta-1|\,\frac{a_X(n/k)}{U_X(n/k)} \\
&\quad\times\left\{\mathbb{1}_{\{\beta<2\}}\,[1-\delta\varepsilon]^{\beta-2}\int_0^\infty [\delta+v]\,\left|W\left(e^{-v}\right)\right|\,dv\right. \\
&\qquad\left.+\mathbb{1}_{\{\beta\geqslant 2\}}\int_0^\infty [\delta+v]\,[1+\varepsilon(\delta+v)]^{\beta-2}\left|W\left(e^{-v}\right)\right|\,dv\right\} \\
&\leqslant \beta|\beta-1|\,\frac{a_X(n/k)}{U_X(n/k)} \\
&\quad\times\left\{\mathbb{1}_{\{\beta<2\}}\,[1-\delta\varepsilon]^{\beta-2}\sup_{0<y\leqslant 1}\frac{|W(y)|}{y^\eta}\int_0^\infty [\delta+v]\,e^{-\eta\,v}\,dv\right. \\
&\qquad\left.+\mathbb{1}_{\{\beta\geqslant 2\}}\sup_{0<y\leqslant 1}\frac{|W(y)|}{y^\eta}\int_0^\infty [\delta+v]\,[1+\varepsilon(\delta+v)]^{\beta-2}\,e^{-\eta\,v}\,dv\right\} \\
&\leqslant C\,\frac{a_X(n/k)}{U_X(n/k)},
\end{aligned} \tag{18}$$

by Lemma 2 in Cai et al. (2015), and where C is a positive constant.
Combining (7)-(13) with (16)-(18), Theorem 3.1 follows.

Proof of Theorem 4.2. Consider the following decomposition

$$
\begin{aligned}
& \sqrt{k}\left(\frac{\widehat{CTM}_{\beta,p}}{CTM_{\beta,p}}-1\right) \\
= \; & \sqrt{k}\left(\underbrace{\left\{\left(\frac{\log 1/p}{\log n/k}\right)^{\beta[\widehat{\theta}_k-\theta]}\right\}}_{T_{11,n}}\underbrace{\left\{\frac{\overline{CTM}_{\beta,\frac{k}{n}}}{CTM_{\beta,\frac{k}{n}}}\right\}}_{T_{12,n}}\underbrace{\left\{\frac{(\log n/k)^{-\beta\,\theta}CTM_{\beta,\frac{k}{n}}}{(\log 1/p)^{-\beta\,\theta}CTM_{\beta,p}}\right\}}_{T_{13,n}}-1\right) \\
= \; & \sqrt{k}\left\{\left(\frac{\log 1/p}{\log n/k}\right)^{\beta[\widehat{\theta}_k-\theta]}-1\right\}T_{12,n}\,T_{13,n}+\sqrt{k}\left\{\frac{\overline{CTM}_{\beta,\frac{k}{n}}}{CTM_{\beta,\frac{k}{n}}}-1\right\}T_{13,n} \\
& +\sqrt{k}\left\{\frac{(\log n/k)^{-\beta\,\theta}CTM_{\beta,\frac{k}{n}}}{(\log 1/p)^{-\beta\,\theta}CTM_{\beta,p}}-1\right\} \\
= \; & \beta\,\log(\lambda)\,\sqrt{k}\,[\widehat{\theta}_k-\theta](1+o_{\mathbb{P}}(1))\,T_{12,n}\,T_{13,n}+\sqrt{k}\,\frac{U_X(n/k)}{a_X(n/k)}\left\{\frac{\overline{CTM}_{\beta,\frac{k}{n}}}{CTM_{\beta,\frac{k}{n}}}-1\right\}\frac{a_X(n/k)}{U_X(n/k)}\,T_{13,n} \\
& +\sqrt{k}\left\{\frac{CTM_{\beta,\frac{k}{n}}/[U_X(n/k)]^{\beta}}{CTM_{\beta,p}/[U_X(1/p)]^{\beta}}-1\right\}\left(\frac{U_X(n/k)}{U_X(1/p)}\right)^{\beta}\left(\frac{\log n/k}{\log 1/p}\right)^{-\beta\,\theta} \\
& +\sqrt{k}\left(\left(\frac{U_X(n/k)}{U_X(1/p)}\right)^{\beta}\left(\frac{\log n/k}{\log 1/p}\right)^{-\beta\,\theta}-1\right).
\end{aligned}
$$

According to (4), we have

$$
\begin{aligned}
\frac{CTM_{\beta,\frac{k}{n}}}{[U_X(n/k)]^{\beta}} &= 1+\beta\,\frac{a_X(n/k)}{U_X(n/k)}\left\{1+\int_0^{\infty}\left[\left(1+\frac{a_X(n/k)}{U_X(n/k)}\,z\right)^{\beta-1}-1\right\}e^{-z}\,dz\right. \\
&\quad \left.+\int_0^{\infty}\left(1+\frac{a_X(n/k)}{U_X(n/k)}\,z\right)^{\beta-1}\left(\frac{\overline{F}_X\left(U_X(n/k)+a_X(n/k)\,z\right)}{\overline{F}_X(U_X(n/k))}-e^{-z}\right)\,dz\right\} \\
&= 1+o\left(\frac{1}{\sqrt{k}}\right).
\end{aligned}
$$

A similar result can be obtained for $CTM_{\beta,\frac{1}{p}}/[U_X(1/p)]^{\beta}$. Thus, we have

$$
\frac{CTM_{\beta,\frac{k}{n}}/[U_X(n/k)]^{\beta}}{CTM_{\beta,p}/[U_X(1/p)]^{\beta}}-1 \;=\; o\left(\frac{1}{\sqrt{k}}\right).
$$

Now, the last term of the decomposition.

$$
\left(\frac{U_X(n/k)}{U_X(1/p)}\right)^{\beta}\left(\frac{\log n/k}{\log 1/p}\right)^{-\beta\,\theta}-1 \;=\; \left(\frac{\ell_U(\log n/k)}{\ell_U(\log 1/p)}\right)^{\beta}-1=\left(\frac{\ell_X(U_X(1/p))}{\ell_X(U_X(n/k))}\right)^{\beta\,\theta}-1,
$$

since $\ell_U(\log x) = [\ell_X(U_X(x))]^{-\theta}$.
Now, using the mean value theorem with $u_n \in [U_X(n/k); U_X(1/p)]$, we have, for $\delta_1, \delta_2 > 0$ and n large,

$$\begin{aligned}
\left|\frac{\ell_X(U_X(1/p))}{\ell_X(U_X(n/k))} - 1\right| &= \frac{|\ell'_X(u_n)|}{\ell_X(U_X(n/k))}\left[U_X\left(\frac{1}{p}\right) - U_X\left(\frac{n}{k}\right)\right] \\
&= \frac{u_n\,|\ell'_X(u_n)|}{\ell_X(u_n)}\,\frac{\ell_X(u_n)}{\ell_X(U_X(n/k))}\,\frac{U_X(n/k)}{u_n}\left(\frac{U_X(1/p)}{U_X(n/k)} - 1\right) \\
&\leqslant (1+\delta_1)\,\frac{u_n\,|\ell'_X(u_n)|}{\ell_X(u_n)}\left(\frac{u_n}{U_X(n/k)}\right)^{\delta_2}\left(\frac{U_X(1/p)}{U_X(n/k)} - 1\right) \\
&\leqslant (1+\delta_1)\sup_{x \geqslant U_X(n/k)}\frac{x\,|\ell'_X(x)|}{\ell_X(x)}\left(\frac{U_X(1/p)}{U_X(n/k)}\right)^{\delta_2}\left(\frac{U_X(1/p)}{U_X(n/k)} - 1\right) \\
&= o\left(\frac{1}{\sqrt{k}}\right),
\end{aligned}$$

since

$$\frac{U_X(1/p)}{U_X(n/k)} - 1 = \left[\left(\frac{\log 1/p}{\log n/k}\right)^{\theta} - 1\right]\frac{\ell_U(\log 1/p)}{\ell_U(\log n/k)} + \left[\frac{\ell_U(\log 1/p)}{\ell_U(\log n/k)} - 1\right] = O(1).$$

Combining Theorem 3.1 with Theorem 4.1, Theorem 4.2 follows.

Acknowledgement

The research of Armelle Guillou was supported by the French National Research Agency under the grant ANR-19-CE40-0013-01/ExtremReg project and by the CNRS International Research Network MaDeF.

References

Beirlant, J., Broniatowski, M., Teugels, J.L., Vynckier, P. (1995). The mean residual life function at great age: applications to tail estimation. *Journal of Statistical Planning and Inference*, **45**, 21–48.

Broniatowski, M. (1993). On the estimation of the Weibull tail coefficient. *Journal of Statistical Planning and Inference*, **35**, 349–366.

Cai, J.J., Einmahl, J.H.J., de Haan, L., Zhou, C. (2015). Estimation of the marginal expected shortfall: the mean when a related variable is extreme. *Journal of the Royal Statistical Society Series B*, **77**, 417–442.

Coles, S. (2013). *An introduction to the statistical modeling of extreme events.* Springer.

Cooley, D. (2013). Return periods and return levels under climate change. *Extremes in a changing climate: Detection, analysis and uncertainty*, Chapter 4, A. AghaKouchak, D. Easterling, and K. Hsu, eds., **65**, Springer, New York.

de Haan, L., Ferreira, A. (2006). *Extreme value theory, an introduction*. Springer.

de Wet, T., Goegebeur, Y., Guillou, A., Osmann, M. (2016). Kernel regression with Weibull-type tails. *Annals of the Institute of Statistical Mathematics*, **68**, 1135–1162.

Diebolt, J., Gardes, L., Girard, S., Guillou, A. (2008). Bias-reduced estimators of the Weibull tail-coefficient. *TEST*, **17**, 311–331.

Dierckx, G., Beirlant, J., de Waal, D., Guillou, A. (2009). A new estimation method for Weibull-type tails based on the mean excess function. *Journal of Statistical Planning and Inference*, **139**, 1905–1920.

Einmahl, J. H. J., de Haan, L., Li, D. (2006) Weighted approximations of tail copula processes with application to testing the bivariate extreme value condition. *The Annals of Statistics*, **34**, 1987–201.

El Methni, J., Gardes, L., Girard, S. (2014). Non-parametric estimation of extreme risk measures from conditional heavy-tailed distributions. *Scandinavian Journal of Statistics*, **41**, 988–1012.

Feng, S., Gao, D., Liao, F., Zhou, F., Wang, X. (2016). The health effects of ambient $PM_{2.5}$ and potential mechanisms. *Ecotoxicology and Environmental Safety*, **128**, 67–74.

Gardes, L., Girard, S. (2005). Estimating extreme quantiles of Weibull tail distributions. *Communications in Statistics – Theory and Methods*, **34**, 1065–1080.

Gardes, L., Girard, S. (2008). Estimation of the Weibull-tail coefficient with linear combination of upper order statistics. *Journal of Statistical Planning and Inference*, **138**, 1416–1427.

Gardes, L., Girard, S. (2016). On the estimation of the functional Weibull tail-coefficient. *Journal of Multivariate Analysis*, **146**, 29–45.

Girard, S. (2004). A Hill type estimator of the Weibull tail coefficient. *Communications in Statistics – Theory and Methods*, **33**, 205–234.

Goegebeur, Y., Beirlant, J., de Wet, T. (2010). Generalized kernel estimators for the Weibull tail coefficient. *Communications in Statistics – Theory and Methods*, **39**, 3695–3716.

Goegebeur, Y., Guillou, A. (2010). Goodness-of-fit testing for Weibull-type behavior. *Journal of Statistical Planning and Inference*, **140**, 1417–1436.

Goegebeur, Y., Guillou, A. (2011). A weighted mean excess function approach to the estimation of Weibull-type tails. *TEST*, **20**, 138–162.

Goegebeur, Y., Guillou, A., Pedersen, T., Qin, J., (2022a). Extreme-value based estimation of the conditional tail moment with application to reinsurance rating. *Insurance: Mathematics and Economics*, **107**, 102–122.

Goegebeur, Y., Guillou, A., Qin, J. (2022b). Conditional tail moment and reinsurance premium estimation under random right censoring, `https://hal.science/hal-03716114v2`.

Goegebeur, Y., Guillou, A., Qin, J. (2023). Estimation of marginal excess moments for Weibull-type distributions, `https://hal.science/hal-04012275`.

Goegebeur, Y., Guillou, A., Rietsch, T. (2015). Robust conditional Weibull-type estimation. *Annals of the Institute of Statistical Mathematics*, **67**, 479–514.

Jorion, P. (2007). *Value at risk: the new benchmark for managing financial risk.* McGraw-Hill, New York.

Klüppelberg, C., Villaseñor, J.A. (1993). Estimation of distribution tails – a semiparametric approach. *Deutschen Gesellschaft für Versicherungsmathematik*, **21**, 213–235.

Li, Z., Wen, Q., Zhang, R. (2017). Sources, health effects and control strategies of indoor fine particulate matter ($PM_{2.5}$): a review. *Science of The Total Environment*, **586**, 610–622.

Salas, J. D., Obeysekera, J. (2014). Revisiting the concepts of return period and risk for non-stationary hydrologic extreme events. *Journal of Hydrologic Engineering*, **19**, 554-568.

Shi, H., Dong, Z., Xiao, N., Huang, Q. (2021). Wind speed distributions used in wind energy assessment: a review. *Frontiers in Energy Research*, **9**, 769920.

Singh, V.P., Jain, S.K., Tyagi, A. (2007). *Risk and reliability analysis: A handbook for civil and environmental engineers.* Reston, VA: American Society of Civil Engineers.