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Estimation of the conditional tail moment for Weibull-type distributions

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Abstract

We consider the estimation of the conditional tail moment, defined for a positive random variable X as $\mathbb{E}(X^\beta | X > Q(1-p))$, where Q denotes the quantile function of X , $\beta > 0$ and $p \in (0, 1)$, at extreme levels for the class of Weibull-type distributions. A two-step procedure is introduced where in the first stage one estimates the conditional tail moment at an intermediate level, followed by an extrapolation in the second stage. The asymptotic properties of the estimators introduced in the two stages are derived under suitable assumptions.

Keywords: Weibull-type distribution, extreme-value statistics, hazard rate function, empirical process.

1 Introduction

Quantifying and estimating the risk of extreme events is of interest in many scientific disciplines. Floods due to sea storms can entail massive losses, both in terms of human lives and economical, so the modelling of the distribution of wave heights and other sea state parameters is of crucial importance in the design of coastal protection. In environmental science, extreme levels of pollutants can have adverse effect on human health. Over the years several risk measures have been introduced, the most popular of these being the Value-at-Risk (also called return level) and the conditional tail expectation. We refer to, e.g., Jorion (2007), Singh et al. (2007), Coles (2013), Cooley (2013) and Salas and Obeysekera (2014). In this paper we consider the conditional tail moment (*CTM*), being a natural generalisation of the conditional tail expectation, and study its estimation for the class of Weibull-type distributions.

Let $\beta > 0$ and $p \in (0, 1)$. We want to estimate

$$CTM_{\beta,p} := \mathbb{E} \left[X^\beta \middle| X > U_X(1/p) \right],$$

when the distribution of X is of Weibull-type, i.e., the distribution function satisfies

$$F_X(x) = 1 - e^{-x^{1/\theta} \ell_X(x)}, x > 0, \tag{1}$$

with θ a positive parameter and ℓ_X a slowly varying function at infinity, i.e., a positive measurable function such that

$$\lim_{t \rightarrow \infty} \frac{\ell_X(tx)}{\ell_X(t)} = 1 \quad \text{for all } x > 0,$$

and U_X the tail quantile function of X defined as $U_X(x) = \inf\{z : F_X(z) \geq 1 - 1/x\}$, $x > 1$. Specific interest is here in the extreme case, that is, the situation where p is very small, typically less than $1/n$, where n is the size of the random sample that is available for estimation. In this case, estimation is challenging as the conditional expectation in the definition of $CTM_{\beta,p}$ cannot simply be estimated by an empirical average. Indeed, for $p < 1/n$ the natural estimator for $U_X(1/p)$ is the largest observation in the sample, but then there are no further data to base the estimation of the conditional expectation upon.

The considered class of Weibull-type distributions forms a rich subclass of the Gumbel max-domain of attraction. The tail heaviness of Weibull-type distributions is clearly governed by the parameter θ , where larger values of θ correspond with a slower decay of the upper tail. Popular members of this family include the Weibull (corresponding to a constant function ℓ_X), normal, gamma and logistic distributions, to mention only a few. Loosely speaking, the upper tail decays exponentially fast, in contrast to the Pareto-type models which have a tail decay corresponding to a decreasing power function, and thus are heavier tailed. The Weibull-type distributions find important applications in areas like environmental science, climate science, hydrology, engineering and medicine. As the Weibull-type model depends only explicitly on the parameter θ , it is not unexpected that the estimation of this parameter received a lot of attention in the extreme-value literature. We refer to Broniatowski (1993), Beirlant et al. (1995), Girard (2004), Gardes and Girard (2005, 2008, 2016), Diebolt et al. (2008), Dierckx et al. (2009), Goegebeur et al. (2010), Goegebeur and Guillou (2011), Goegebeur et al. (2015), de Wet et al. (2016) and the references therein.

The estimation of the conditional tail moment is already studied for the context of Pareto-type distributions in Goegebeur et al. (2022a), who consider the classical setup of independent identically distributed (i.i.d.) random variables and in Goegebeur et al. (2022b) in an i.i.d. setting with random right censoring. El Methni et al. (2014) studied the estimation of the conditional tail moment in a regression setup where the conditional distribution of the response variable given the covariate is of Pareto-type. Estimation of the conditional tail moment for the class of Weibull-type distributions is to the best of our knowledge unexplored.

We organise this paper as follows. In Section 2 we provide as a first result an approximation of $CTM_{\beta,p}$ for $p \downarrow 0$, which will be used to derive the estimator for this risk measure under extrapolation. The proposed estimator depends on an estimator for the conditional tail moment in the intermediate case, i.e., for $p = k/n$, where $k \rightarrow \infty$ as the sample size $n \rightarrow \infty$ but in such a way that $k/n \rightarrow 0$. This intermediate case is studied in Section 3 where we introduce an estimator and derive its limiting distribution under suitable assumptions. Section 4 is devoted to the study of the asymptotic properties of the estimator for $CTM_{\beta,p}$ in the extreme case. Section 5 contains the proofs of the results.

2 Construction of the estimator

Let RV_ψ denote the class of regularly varying functions at infinity with index $\psi \in \mathbb{R}$, i.e., positive measurable functions f satisfying $f(tx)/f(t) \rightarrow x^\psi$, as $t \rightarrow \infty$, for all $x > 0$. Denote $H(x) = -\log \bar{F}_X(x)$, and hence $\bar{F}_X(x) = e^{-H(x)}$. In the case where F_X is of Weibull-type we have then $H \in RV_{1/\theta}$. Note that if H is differentiable with derivative H' then $H' = h$ where h is the hazard rate function.

We start to expand $CTM_{\beta,p}$ in case $p \downarrow 0$ in order to have an idea about how to estimate this risk measure under extrapolation, i.e., for $p < 1/n$, where n is the size of the sample on which the estimation is based.

Lemma 2.1 *Assume F_X satisfies (1), H is differentiable with derivative $h \in RV_{1/\theta-1}$. Then, as $p \downarrow 0$, we have*

$$\frac{CTM_{\beta,p}}{\left[U_X\left(\frac{1}{p}\right)\right]^\beta} \rightarrow 1.$$

From this lemma, the first idea to estimate $CTM_{\beta,p}$ is to use an estimate of $\left[U_X\left(\frac{1}{p}\right)\right]^\beta$. Unfortunately, since by definition of the conditional tail moment we condition on the event $\{X > U_X(1/p)\}$, clearly such an estimate will underestimate the true value. An alternative method consists in using a two-step procedure as follows.

Let k be an intermediate sequence, i.e., a sequence such that $k \rightarrow \infty$ and $k/n \rightarrow 0$ as $n \rightarrow \infty$. Lemma 2.1 yields the approximation

$$\frac{CTM_{\beta,p}/\left[U_X\left(\frac{1}{p}\right)\right]^\beta}{CTM_{\beta,k/n}/\left[U_X\left(\frac{n}{k}\right)\right]^\beta} \sim 1,$$

from which we deduce that

$$CTM_{\beta,p} \sim \left(\frac{U_X\left(\frac{1}{p}\right)}{U_X\left(\frac{n}{k}\right)}\right)^\beta CTM_{\beta,k/n}.$$

Note that (1) is equivalent to assume that U_X satisfies

$$U_X(x) = (\log x)^\theta \ell_U(\log x),$$

where ℓ_U is also a slowly varying function at infinity. Thus, we deduce the following estimator for $CTM_{\beta,p}$

$$\widehat{CTM}_{\beta,p} := \left(\frac{\log(1/p)}{\log(n/k)}\right)^{\beta \hat{\theta}_k} \overline{CTM}_{\beta,k/n}, \quad (2)$$

where $\hat{\theta}_k$ is an estimator of θ and $\overline{CTM}_{\beta,k/n}$ an intermediate estimator of $CTM_{\beta,k/n}$.

3 Asymptotic properties of $\overline{CTM}_{\beta,k/n}$

Assume we have at our disposal a sample X_1, \dots, X_n of independent copies of the random variable X and denote by $X_{1,n} \leq \dots \leq X_{n,n}$ the order statistics. We start with considering the estimation of the conditional tail moment in the intermediate case. In this situation, the natural estimator for $U_X(n/k)$ is then $X_{n-k,n}$, which is within the data range, and hence $CTM_{\beta,k/n}$ can be estimated empirically by the sample mean of the X_i for which $X_i > X_{n-k,n}$. We define

$$\overline{CTM}_{\beta,k/n} = \frac{1}{k} \sum_{i=1}^n X_i^\beta \mathbb{1}_{\{X_i > X_{n-k,n}\}},$$

and introduce

$$\overline{CTM}_{\beta,k/n}(x) := \frac{1}{k} \sum_{i=1}^n X_i^\beta \mathbb{1}_{\{X_i > U_X(\frac{n}{k}) + a_X(\frac{n}{k})x\}},$$

where

$$a_X(t) := \frac{1}{h(U_X(t))}.$$

Then, $\overline{CTM}_{\beta,k/n}$ can be rewritten as

$$\overline{CTM}_{\beta,k/n} = \overline{CTM}_{\beta,k/n}(\hat{u}_{n,k}),$$

where $\hat{u}_{n,k} := (X_{n-k,n} - U_X(n/k))/a_X(n/k)$.

The weak convergence of $\overline{CTM}_{\beta,k/n}$, after normalisation, is given in the following theorem.

Theorem 3.1 *Assume F_X satisfies (1), H is differentiable with derivative h which is ultimately monotone, and h is differentiable with derivative h' satisfying $xh'(x)/h(x) \rightarrow 1/\theta - 1$ as $x \rightarrow \infty$. Then, if $k \rightarrow \infty$ as $n \rightarrow \infty$ such that $k/n \rightarrow 0$ and $\sqrt{k}a_X(n/k)/U_X(n/k) \rightarrow 0$, we have*

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} \left(\frac{\overline{CTM}_{\beta,k/n}}{CTM_{\beta,k/n}} - 1 \right) \xrightarrow{d} \Theta := \beta \int_0^\infty W(e^{-v}) dv,$$

where $W(z)$ is a zero centered Gaussian process with covariance function

$$\mathbb{E}\{W(z)W(\bar{z})\} = \min(z, \bar{z}).$$

Note that this theorem implies the convergence of the rescaled estimator of $CTM_{\beta,k/n}$ towards the $\mathcal{N}(0, 2\beta^2)$ distribution.

4 Asymptotic properties of $\widehat{CTM}_{\beta,p}$

In this section we study the asymptotic properties of the estimator $\widehat{CTM}_{\beta,p}$ given in (2). As estimator for the Weibull-tail coefficient θ we use the estimator proposed by Girard (2004), given by

$$\widehat{\theta}_k := \frac{\frac{1}{k} \sum_{i=1}^k (\log X_{n-i+1,n} - \log X_{n-k,n})}{\frac{1}{k} \sum_{i=1}^k (\log \log \frac{n}{i} - \log \log \frac{n}{k})}.$$

Below, we recall the limiting distribution of this estimator, properly normalised, in terms of a stochastic process, as established in Theorem 4.1 in Goegebeur et al. (2023).

Theorem 4.1 *Assume F_X satisfies (1), H is differentiable with derivative h which is ultimately monotone, and h is differentiable with derivative h' satisfying $xh'(x)/h(x) \rightarrow 1/\theta - 1$ as $x \rightarrow \infty$. Then, if $k \rightarrow \infty$ as $n \rightarrow \infty$ such that $\sqrt{k}a_X(n/k)/U_X(n/k) \rightarrow 0$ and $\sqrt{k} \frac{U_X(n/k)\ell'_X(U_X(n/k))}{\ell_X(U_X(n/k))} \rightarrow 0$, we have*

$$\sqrt{k} (\widehat{\theta}_k - \theta) \xrightarrow{d} \Gamma := \theta \left\{ \int_0^\infty W(e^{-v}) dv - W(1) \right\}.$$

Note that the limiting distribution in Theorem 4.1 is $\mathcal{N}(0, \theta^2)$. As is clear from Theorem 3.1 and Theorem 4.1, the limiting behavior of $\widehat{CTM}_{\beta,k/n}$ and $\widehat{\theta}_k$ involves the same stochastic process W . For more details we refer to Section 7 of the present paper and the proof of Theorem 4.1 in Goegebeur et al. (2023).

We can now state the main result of the paper, namely the weak convergence of $\widehat{CTM}_{\beta,p}$, properly normalised.

Theorem 4.2 *Assume F_X satisfies (1), H is differentiable with derivative h which is ultimately monotone, and h is differentiable with derivative h' satisfying $xh'(x)/h(x) \rightarrow 1/\theta - 1$ as $x \rightarrow \infty$. Then, if $k \rightarrow \infty$ as $n \rightarrow \infty$ such that $\sqrt{k} \frac{a_X(n/k)}{U_X(n/k)} \rightarrow 0$ and $\sqrt{k} \sup_{x \geq U_X(n/k)} \frac{x\ell'_X(x)}{\ell_X(x)} \rightarrow 0$, we have for p satisfying $p \leq \frac{k}{n}$ such that $\frac{\log 1/p}{\log n/k} \rightarrow \lambda \in (1, \infty)$ that*

$$\sqrt{k} \left(\frac{\widehat{CTM}_{\beta,p}}{CTM_{\beta,p}} - 1 \right) \xrightarrow{d} \beta \log(\lambda) \Gamma.$$

From the result of Theorem 4.2, it is clear that the estimator $\widehat{CTM}_{\beta,p}$ inherits its limiting distribution, up to the factor $\beta \log(\lambda)$, from the estimator $\widehat{\theta}_k$ for the Weibull-tail coefficient. In particular, we obtain the $\mathcal{N}(0, (\theta\beta \log(\lambda))^2)$ distribution as limiting distribution for the rescaled estimator of $CTM_{\beta,p}$. As expected, more severe extrapolations, corresponding with larger values for λ , lead to an increase in the variance of the limiting distribution.

5 Proofs

Remark that

$$\begin{aligned}
& \overline{CTM}_{\beta,k/n}(x) \\
&= \beta \frac{1}{k} \sum_{i=1}^n \int_0^{X_i} u^{\beta-1} du \mathbb{1}_{\{X_i > U_X(\frac{n}{k}) + a_X(\frac{n}{k})x\}} \\
&= \beta \int_0^\infty u^{\beta-1} \frac{1}{k} \sum_{i=1}^n \mathbb{1}_{\{X_i > \max(u, U_X(\frac{n}{k}) + a_X(\frac{n}{k})x)\}} du \\
&= \beta \int_0^{U_X(\frac{n}{k}) + a_X(\frac{n}{k})x} u^{\beta-1} du \frac{1}{k} \sum_{i=1}^n \mathbb{1}_{\{X_i > U_X(\frac{n}{k}) + a_X(\frac{n}{k})x\}} \\
&\quad + \beta \int_{U_X(\frac{n}{k}) + a_X(\frac{n}{k})x}^\infty u^{\beta-1} \frac{1}{k} \sum_{i=1}^n \mathbb{1}_{\{X_i > u\}} du \\
&= \left[U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)x \right]^\beta \frac{1}{k} \sum_{i=1}^n \mathbb{1}_{\{X_i > U_X(\frac{n}{k}) + a_X(\frac{n}{k})x\}} \\
&\quad + \beta a_X\left(\frac{n}{k}\right) \int_0^\infty \left[U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)(x+v) \right]^{\beta-1} \frac{1}{k} \sum_{i=1}^n \mathbb{1}_{\{X_i > U_X(\frac{n}{k}) + a_X(\frac{n}{k})(x+v)\}} dv \\
&= \left[U_X\left(\frac{n}{k}\right) \right]^\beta \left\{ \left[1 + \frac{a_X(\frac{n}{k})}{U_X(\frac{n}{k})}x \right]^\beta T_n\left(\frac{n}{k}\overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)x\right)\right) \right. \\
&\quad \left. + \beta \frac{a_X(\frac{n}{k})}{U_X(\frac{n}{k})} \int_0^\infty \left[1 + \frac{a_X(\frac{n}{k})}{U_X(\frac{n}{k})}(x+v) \right]^{\beta-1} T_n\left(\frac{n}{k}\overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)(x+v)\right)\right) dv \right\} \text{ a.s.},
\end{aligned}$$

where

$$T_n(x) := \frac{1}{k} \sum_{i=1}^n \mathbb{1}_{\{\overline{F}_X(X_i) < \frac{k}{n}x\}}.$$

The asymptotic behavior of $T_n(x)$, after proper normalisation, is given in Lemma 1 of Cai et al. (2015), according which, for any $\eta \in [0, 1/2)$ and $T > 0$ we have, for $n \rightarrow \infty$,

$$\sup_{x \in (0, T]} \left| \frac{\sqrt{k}[T_n(x) - x] - W(x)}{x^\eta} \right| \rightarrow 0 \text{ a.s.} \quad (3)$$

Note that in the above result, all the involved processes are defined on the same probability space via the Skorohod construction. See also Einmahl et al. (2006) for convergence properties of related stochastic processes.

For completeness, we also include below three lemmas from Goegebeur et al. (2023), useful for our proofs. Note that the last one is actually a consequence of the proof of Lemma 5.4 in the aforementioned paper.

Lemma 5.1 Let F_X satisfy (1), H be differentiable with derivative $h \in RV_{1/\theta-1}$, then, as $t \rightarrow \infty$

$$\frac{\overline{F}_X\left(t + \frac{z}{h(t)}\right)}{\overline{F}_X(t)} \rightarrow e^{-z},$$

uniformly for $z \geq -\delta$, for any $\delta > 0$.

Lemma 5.2 If F_X satisfies (1) and H is differentiable with derivative $h \in RV_{1/\theta-1}$, then, for a sequence k such that $k \rightarrow \infty$ with $k/n \rightarrow 0$, we have

$$\sqrt{k} \frac{X_{n-k,n} - U_X\left(\frac{n}{k}\right)}{a_X\left(\frac{n}{k}\right)} \xrightarrow{d} W(1).$$

Lemma 5.3 Assume F_X satisfies (1), H is differentiable with derivative h which is ultimately monotone, and h is differentiable with derivative h' satisfying $xh'(x)/h(x) \rightarrow 1/\theta - 1$ as $x \rightarrow \infty$. Then, if $k \rightarrow \infty$ as $n \rightarrow \infty$ such that $k/n \rightarrow 0$ and $\sqrt{k}a_X(n/k)/U_X(n/k) \rightarrow 0$, we have, for any $\delta > 0$,

$$\sup_{x \in [-\delta, \delta]} \sqrt{k} \int_0^\infty \left| \frac{n}{k} \overline{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)(x+v)\right) - e^{-(x+v)} \right| dv^\beta \rightarrow 0.$$

Proof of Lemma 2.1. We have

$$\begin{aligned} CTM_{\beta,p} &= - \int_{U_X(1/p)}^\infty \frac{x^\beta}{\overline{F}_X(U_X(1/p))} d\overline{F}_X(x) \\ &= [U_X(1/p)]^\beta + \beta \int_{U_X(1/p)}^\infty x^{\beta-1} \frac{\overline{F}_X(x)}{\overline{F}_X(U_X(1/p))} dx \\ &= [U_X(1/p)]^\beta \left\{ 1 + \beta \frac{a_X(1/p)}{U_X(1/p)} \int_0^\infty \left[1 + z \frac{a_X(1/p)}{U_X(1/p)} \right]^{\beta-1} \frac{\overline{F}_X(U_X(1/p) + z a_X(1/p))}{\overline{F}_X(U_X(1/p))} dz \right\}. \end{aligned} \quad (4)$$

Since $\frac{a_X(1/p)}{U_X(1/p)} \rightarrow 0$, to prove Lemma 2.1, it is sufficient to show that the integral in the right-hand side of (4) is finite. To this aim, by the mean value theorem, with $\xi \in (0, 1)$, we have

$$\begin{aligned} \frac{\overline{F}_X(U_X(1/p) + z a_X(1/p))}{\overline{F}_X(U_X(1/p))} &= e^{-H(U_X(1/p) + z a_X(1/p)) + H(U_X(1/p))} \\ &= e^{-\frac{h(U_X(1/p) + \xi z a_X(1/p))}{h(U_X(1/p))} z}, \end{aligned}$$

from which, by the use of the Potter bounds (see Proposition B.1.9.5 in de Haan and Ferreira, 2006), for $0 < \delta_1 < 1, \delta_2 > 0$ and n large we have

$$\begin{aligned} \int_0^\infty \left[1 + z \frac{a_X(1/p)}{U_X(1/p)} \right]^{\beta-1} \frac{\overline{F}_X(U_X(1/p) + z a_X(1/p))}{\overline{F}_X(U_X(1/p))} dz \\ \leq \int_0^\infty \left[1 + z \frac{a_X(1/p)}{U_X(1/p)} \right]^{\beta-1} e^{-(1-\delta_1) \left[1 + \xi \frac{a_X(1/p)}{U_X(1/p)} z \right]^{\frac{1}{\theta} - 1 - \delta_2} z} dz. \end{aligned}$$

If $\theta < 1$, take $0 < \delta_2 < 1/\theta - 1$, and obtain, for n large

$$\begin{aligned}
& \int_0^\infty \left[1 + z \frac{a_X(1/p)}{U_X(1/p)} \right]^{\beta-1} \frac{\overline{F}_X(U_X(1/p) + z a_X(1/p))}{\overline{F}_X(U_X(1/p))} dz \\
& \leq \int_0^\infty \left[1 + z \frac{a_X(1/p)}{U_X(1/p)} \right]^{\beta-1} e^{-(1-\delta_1)z} dz \\
& = \frac{U_X(1/p)}{a_X(1/p)} e^{(1-\delta_1) \frac{U_X(1/p)}{a_X(1/p)}} \int_1^\infty u^{\beta-1} e^{-(1-\delta_1) \frac{U_X(1/p)}{a_X(1/p)} u} du \\
& = \frac{1}{1-\delta_1} \frac{\Gamma\left(\beta, (1-\delta_1) \frac{U_X(1/p)}{a_X(1/p)}\right)}{\left[(1-\delta_1) \frac{U_X(1/p)}{a_X(1/p)}\right]^{\beta-1} e^{-(1-\delta_1) \frac{U_X(1/p)}{a_X(1/p)}}}, \tag{5}
\end{aligned}$$

where $\Gamma(.,.)$ denotes the incomplete Gamma function, namely $\Gamma(s, z) := \int_z^\infty v^{s-1} e^{-v} dv$. For $p \downarrow 0$, the right-hand side in (5) is bounded.

Now, if $\theta \geq 1$, take $0 < \delta_2 < 1/\theta$, and for $\varepsilon, \Delta > 0$, we obtain for n large

$$\begin{aligned}
& \int_0^\infty \left[1 + z \frac{a_X(1/p)}{U_X(1/p)} \right]^{\beta-1} \frac{\overline{F}_X(U_X(1/p) + z a_X(1/p))}{\overline{F}_X(U_X(1/p))} dz \\
& \leq \int_0^\infty \left[\mathbb{1}_{\{\beta < 1\}} + (1 + \varepsilon z)^{\beta-1} \mathbb{1}_{\{\beta \geq 1\}} \right] e^{-(1-\delta_1)[1+\varepsilon z]^{\frac{1}{\theta}-1-\delta_2} z} dz \\
& = \int_0^\Delta \left[\mathbb{1}_{\{\beta < 1\}} + (1 + \varepsilon z)^{\beta-1} \mathbb{1}_{\{\beta \geq 1\}} \right] e^{-(1-\delta_1)[1+\varepsilon z]^{\frac{1}{\theta}-1-\delta_2} z} dz \\
& \quad + \int_\Delta^\infty \left[\mathbb{1}_{\{\beta < 1\}} + (1 + \varepsilon z)^{\beta-1} \mathbb{1}_{\{\beta \geq 1\}} \right] e^{-(1-\delta_1)[\varepsilon + \frac{1}{z}]^{\frac{1}{\theta}-1-\delta_2} z^{\frac{1}{\theta}-\delta_2}} dz \\
& \leq \left[\mathbb{1}_{\{\beta < 1\}} + (1 + \varepsilon \Delta)^{\beta-1} \mathbb{1}_{\{\beta \geq 1\}} \right] \int_0^\Delta e^{-(1-\delta_1)[1+\varepsilon \Delta]^{\frac{1}{\theta}-1-\delta_2} z} dz \\
& \quad + \int_\Delta^\infty \left[\mathbb{1}_{\{\beta < 1\}} + (1 + \varepsilon z)^{\beta-1} \mathbb{1}_{\{\beta \geq 1\}} \right] e^{-(1-\delta_1)[\varepsilon + \frac{1}{\Delta}]^{\frac{1}{\theta}-1-\delta_2} z^{\frac{1}{\theta}-\delta_2}} dz \\
& \leq \Delta \left[\mathbb{1}_{\{\beta < 1\}} + (1 + \varepsilon \Delta)^{\beta-1} \mathbb{1}_{\{\beta \geq 1\}} \right] \\
& \quad + \mathbb{1}_{\{\beta < 1\}} \int_\Delta^\infty e^{-(1-\delta_1)[\varepsilon + \frac{1}{\Delta}]^{\frac{1}{\theta}-1-\delta_2} z^{\frac{1}{\theta}-\delta_2}} dz \\
& \quad + \mathbb{1}_{\{\beta \geq 1\}} \left(\varepsilon + \frac{1}{\Delta} \right)^{\beta-1} \int_\Delta^\infty z^{\beta-1} e^{-(1-\delta_1)[\varepsilon + \frac{1}{\Delta}]^{\frac{1}{\theta}-1-\delta_2} z^{\frac{1}{\theta}-\delta_2}} dz \\
& \leq \Delta \left[\mathbb{1}_{\{\beta < 1\}} + (1 + \varepsilon \Delta)^{\beta-1} \mathbb{1}_{\{\beta \geq 1\}} \right] \\
& \quad + \mathbb{1}_{\{\beta < 1\}} \frac{\theta}{1-\theta\delta_2} \left[\frac{1}{(1-\delta_1)(\varepsilon + \frac{1}{\Delta})^{\frac{1}{\theta}-1-\delta_2}} \right]^{\frac{\theta}{1-\theta\delta_2}} \Gamma\left(\frac{\theta}{1-\theta\delta_2}\right) \\
& \quad + \mathbb{1}_{\{\beta \geq 1\}} \left(\varepsilon + \frac{1}{\Delta} \right)^{\beta-1} \frac{\theta}{1-\theta\delta_2} \left[\frac{1}{(1-\delta_1)(\varepsilon + \frac{1}{\Delta})^{\frac{1}{\theta}-1-\delta_2}} \right]^{\frac{\beta\theta}{1-\theta\delta_2}} \Gamma\left(\frac{\beta\theta}{1-\theta\delta_2}\right)
\end{aligned}$$

which is also bounded. This achieves the proof of Lemma 2.1.

Proof of Theorem 3.1. Remark that

$$\frac{\overline{CTM}_{\beta,k/n}}{CTM_{\beta,k/n}} - 1 = \frac{[U_X(n/k)]^\beta}{CTM_{\beta,k/n}} \left\{ \frac{\overline{CTM}_{\beta,k/n}(\hat{u}_{n,k})}{[U_X(n/k)]^\beta} - \frac{CTM_{\beta,k/n}}{[U_X(n/k)]^\beta} \right\}. \quad (6)$$

The leading factor in the right-hand side of (6) tends to 1 by Lemma 2.1. Thus we only need to look at the term in the braces. First, we note that

$$T_n \left(\frac{n}{k} \overline{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) \hat{u}_{n,k} \right) \right) = 1 \text{ a.s.},$$

from which we deduce that

$$\begin{aligned}
& \frac{\overline{CTM}_{\beta,k/n}(\hat{u}_{n,k})}{[U_X(n/k)]^\beta} - \frac{CTM_{\beta,k/n}}{[U_X(n/k)]^\beta} \\
= & \left\{ \left[\frac{X_{n-k,n}}{U_X(n/k)} \right]^\beta - 1 \right\} + \beta \frac{1}{\sqrt{k}} \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty W(e^{-v}) dv \\
& + \beta \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} (\hat{u}_{n,k} + v) \right]^{\beta-1} \\
& \quad \times \left[T_n \left(\frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) \right) - \frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) \right. \\
& \quad \left. - \frac{1}{\sqrt{k}} W \left(\frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) \right) \right] dv \\
& + \beta \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} (\hat{u}_{n,k} + v) \right]^{\beta-1} \\
& \quad \times \left[\frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) - e^{-(\hat{u}_{n,k}+v)} \right] dv \\
& + \beta \frac{a_X(n/k)}{U_X(n/k)} e^{-\hat{u}_{n,k}} \int_0^\infty \left[\left(1 + \frac{a_X(n/k)}{U_X(n/k)} (\hat{u}_{n,k} + v) \right)^{\beta-1} - \left(1 + \frac{a_X(n/k)}{U_X(n/k)} v \right)^{\beta-1} \right] e^{-v} dv \\
& + \beta \frac{a_X(n/k)}{U_X(n/k)} \left(e^{-\hat{u}_{n,k}} - 1 \right) \int_0^\infty \left(1 + \frac{a_X(n/k)}{U_X(n/k)} v \right)^{\beta-1} e^{-v} dv \\
& + \beta \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} v \right]^{\beta-1} \left[e^{-v} - \frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) v \right) \right] dv \\
& + \beta \frac{1}{\sqrt{k}} \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} (\hat{u}_{n,k} + v) \right]^{\beta-1} \\
& \quad \times \left[W \left(\frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) \right) - W \left(e^{-(\hat{u}_{n,k}+v)} \right) \right] dv \\
& + \beta \frac{1}{\sqrt{k}} \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} (\hat{u}_{n,k} + v) \right]^{\beta-1} \left[W \left(e^{-(\hat{u}_{n,k}+v)} \right) - W \left(e^{-v} \right) \right] dv \\
& + \beta \frac{1}{\sqrt{k}} \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty \left[\left(1 + \frac{a_X(n/k)}{U_X(n/k)} (\hat{u}_{n,k} + v) \right)^{\beta-1} - 1 \right] W \left(e^{-v} \right) dv \\
= & \sum_{i=1}^{10} T_{i,n}.
\end{aligned}$$

We will study all the terms separately.

Term $T_{1,n}$. According to Lemma 5.2, we have

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} T_{1,n} = \beta \sqrt{k} \frac{X_{n-k,n} - U_X(n/k)}{a_X(n/k)} (1 + o_{\mathbb{P}}(1)) = \beta W(1) + o_{\mathbb{P}}(1). \quad (7)$$

Term $T_{2,n}$. Clearly

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} T_{2,n} = \beta \int_0^\infty W(e^{-v}) dv. \quad (8)$$

Term $T_{3,n}$. Using Lemmas 5.1 and 5.2, with arbitrary large probability, for n large, $c > 0$ and $0 < \eta < 1/2$, we have

$$\begin{aligned} \sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} |T_{3,n}| &\leq \sup_{0 < y \leq e^\delta + c} \frac{|\sqrt{k}[T_n(y) - y] - W(y)|}{y^\eta} \\ &\quad \times \beta \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} (\hat{u}_{n,k} + v) \right]^{\beta-1} \left(\frac{\bar{F}_X(U_X(\frac{n}{k}) + a_X(\frac{n}{k}) (\hat{u}_{n,k} + v))}{\bar{F}_X(U_X(\frac{n}{k}))} \right)^\eta dv. \end{aligned}$$

The supremum term is negligible a.s. according to (3) and a treatment similar to that of the proof of Lemma 2.1 with Potter bounds ensures that the integral is bounded. This yields

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} T_{3,n} = o_{\mathbb{P}}(1). \quad (9)$$

Term $T_{4,n}$. Let us define

$$I_n(x; \beta) = \sqrt{k} \int_0^\infty \left| \frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (x + v) \right) - e^{-(x+v)} \right| dv^\beta.$$

Using Lemma 5.2, with arbitrary large probability, for n large, $0 < \varepsilon < 1/\delta$ and any $\Delta > 0$, we have

$$\begin{aligned} &\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} |T_{4,n}| \\ &\leq \mathbb{1}_{\{\beta < 1\}} \beta (1 - \delta\varepsilon)^{\beta-1} \sqrt{k} \int_0^\infty \left| \frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) - e^{-(\hat{u}_{n,k} + v)} \right| dv \\ &\quad + \mathbb{1}_{\{\beta \geq 1\}} \beta (1 + \varepsilon(\delta + \Delta))^{\beta-1} \sqrt{k} \int_0^\Delta \left| \frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) - e^{-(\hat{u}_{n,k} + v)} \right| dv \\ &\quad + \mathbb{1}_{\{\beta \geq 1\}} \left(\varepsilon + \frac{1 + \varepsilon\delta}{\Delta} \right)^{\beta-1} \sqrt{k} \int_\Delta^\infty \left| \frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) - e^{-(\hat{u}_{n,k} + v)} \right| dv^\beta \\ &\leq \left(\sup_{x \in [-\delta, \delta]} I_n(x; 1) \right) \left\{ \mathbb{1}_{\{\beta < 1\}} \beta (1 - \delta\varepsilon)^{\beta-1} + \mathbb{1}_{\{\beta \geq 1\}} \beta (1 + \varepsilon(\delta + \Delta))^{\beta-1} \right\} \\ &\quad + \left(\sup_{x \in [-\delta, \delta]} I_n(x; \beta) \right) \mathbb{1}_{\{\beta \geq 1\}} \left(\varepsilon + \frac{1 + \varepsilon\delta}{\Delta} \right)^{\beta-1}. \end{aligned}$$

Thus, Lemma 5.3 implies that

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} T_{4,n} = o_{\mathbb{P}}(1). \quad (10)$$

Term $T_{5,n}$. By Lemma 5.2, using the mean value theorem, with $\xi \in (0, 1)$, we have with large probability, for n large and $0 < \varepsilon < 1/\delta$

$$\begin{aligned}
\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} |T_{5,n}| &\leq \beta |\beta - 1| \frac{a_X(n/k)}{U_X(n/k)} \sqrt{k} \hat{u}_{n,k} \int_0^\infty \left[1 + \frac{a_X(n/k)}{U_X(n/k)} (v + \xi \hat{u}_{n,k}) \right]^{\beta-2} e^{-v} dv (1 + o_{\mathbb{P}}(1)) \\
&\leq \beta |\beta - 1| \frac{a_X(n/k)}{U_X(n/k)} \sqrt{k} \hat{u}_{n,k} (1 + o_{\mathbb{P}}(1)) \\
&\quad \times \left\{ \mathbb{1}_{\{\beta < 2\}} (1 - \varepsilon \delta)^{\beta-2} \int_0^\infty e^{-v} dv + \mathbb{1}_{\{\beta \geq 2\}} \int_0^\infty [1 + \varepsilon(v + \delta)]^{\beta-2} e^{-v} dv \right\} \\
&= o_{\mathbb{P}}(1),
\end{aligned} \tag{11}$$

again by Lemma 5.2 and since $\frac{a_X(n/k)}{U_X(n/k)} \rightarrow 0$.

Term $T_{6,n}$. According to Lemma 5.2, we have

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} T_{6,n} = -\beta \sqrt{k} \left(\frac{X_{n-k,n} - U_X(n/k)}{a_X(n/k)} \right) + o_{\mathbb{P}}(1) = -\beta W(1) + o_{\mathbb{P}}(1). \tag{12}$$

Term $T_{7,n}$. It can be handled similarly as $T_{4,n}$. This yields

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} T_{7,n} = o_{\mathbb{P}}(1). \tag{13}$$

Term $T_{8,n}$. With arbitrary large probability, for n large, $0 < \varepsilon < 1/\delta$ and any $\Delta > 0$, we have

$$\begin{aligned}
\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} |T_{8,n}| &\leq \left\{ \mathbb{1}_{\{\beta < 1\}} \beta (1 - \delta \varepsilon)^{\beta-1} + \mathbb{1}_{\{\beta \geq 1\}} \beta (1 + \varepsilon(\delta + \Delta))^{\beta-1} \right\} \mathbb{W}_n(1) \\
&\quad + \mathbb{1}_{\{\beta \geq 1\}} \left(\varepsilon + \frac{1 + \varepsilon \delta}{\Delta} \right)^{\beta-1} \mathbb{W}_n(\beta),
\end{aligned} \tag{14}$$

where

$$\mathbb{W}_n(\beta) := \int_0^\infty \left| W \left(\frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) \right) - W \left(e^{-(\hat{u}_{n,k} + v)} \right) \right| dv^\beta.$$

For $T > 0$, we have

$$\begin{aligned}
\mathbb{W}_n(\beta) &\leq \int_T^\infty \left| W \left(e^{-(\hat{u}_{n,k} + v)} \right) \right| dv^\beta + \int_T^\infty \left| W \left(\frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) \right) \right| dv^\beta \\
&\quad + \int_0^T \left| W \left(\frac{n}{k} \bar{F}_X \left(U_X \left(\frac{n}{k} \right) + a_X \left(\frac{n}{k} \right) (\hat{u}_{n,k} + v) \right) \right) - W \left(e^{-(\hat{u}_{n,k} + v)} \right) \right| dv^\beta \\
&=: \mathbb{W}_n^{(1)}(\beta) + \mathbb{W}_n^{(2)}(\beta) + \mathbb{W}_n^{(3)}(\beta).
\end{aligned}$$

Then, for $a > 0$ arbitrary, we have

$$\mathbb{P}(\mathbb{W}_n(\beta) > a) \leq \mathbb{P}\left(\mathbb{W}_n^{(1)}(\beta) > \frac{a}{4}\right) + \mathbb{P}\left(\mathbb{W}_n^{(2)}(\beta) > \frac{a}{4}\right) + \mathbb{P}\left(\mathbb{W}_n^{(3)}(\beta) > \frac{a}{4}\right).$$

Clearly, for $\eta \in (0, 1/2)$, we have

$$\mathbb{P}\left(\mathbb{W}_n^{(1)}(\beta) > \frac{a}{4}\right) \leq \mathbb{P}\left(\sup_{0 < y \leq e^\delta} \frac{|W(y)|}{y^\eta} > \frac{a\eta^\beta}{4\beta} \frac{e^{-\eta\delta}}{\Gamma(\beta, T\eta)}\right) + \mathbb{P}(|\hat{u}_{n,k}| > \delta),$$

and similarly, by Lemma 5.1, $c > 0$ and n large

$$\mathbb{P}\left(\mathbb{W}_n^{(2)}(\beta) > \frac{a}{4}\right) \leq \mathbb{P}\left(\sup_{0 < y \leq e^{\delta+c}} \frac{|W(y)|}{y^\eta} \int_T^\infty \left[\frac{n}{k} \bar{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)(-\delta+v)\right)\right]^\eta dv^\beta > \frac{a}{4}\right) + \mathbb{P}(|\hat{u}_{n,k}| > \delta).$$

Following the lines of proof of Lemma 2.1 with the Potter bounds, the integral from the above display can be made arbitrary small by taking T large. Thus, combining Lemma 5.2 with Lemma 2 in Cai et al. (2015), and by choosing T large enough, for $\varepsilon > 0$, there exists n_1 such that for $n > n_1$ we have

$$\mathbb{P}\left(\mathbb{W}_n^{(1)}(\beta) > \frac{a}{4}\right) + \mathbb{P}\left(\mathbb{W}_n^{(2)}(\beta) > \frac{a}{4}\right) \leq \frac{\varepsilon}{2}.$$

With this specific value of T , we have

$$\mathbb{P}\left(\mathbb{W}_n^{(3)}(\beta) > \frac{a}{2}\right) \leq \mathbb{P}\left(\sup_{y \in [-\delta, \delta+T]} \left|W\left(\frac{n}{k} \bar{F}_X\left(U_X\left(\frac{n}{k}\right) + a_X\left(\frac{n}{k}\right)y\right)\right) - W(e^{-y})\right| > \frac{a}{2T^\beta}\right) + \mathbb{P}(|\hat{u}_{n,k}| > \delta).$$

Lemmas 5.1 and 5.2 combined with the uniform continuity of W on compact sets yield, for $n > n_2$

$$\mathbb{P}\left(\mathbb{W}_n^{(3)}(\beta) > \frac{a}{2}\right) \leq \frac{\varepsilon}{2},$$

from which we deduce that, for $n > \max(n_1, n_2)$

$$\mathbb{P}(\mathbb{W}_n(\beta) > a) \leq \varepsilon. \tag{15}$$

Combining (14) and (15), we have

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} T_{8,n} = o_{\mathbb{P}}(1). \tag{16}$$

Term $T_{9,n}$. It can be handled similarly as $T_{8,n}$, except for

$$\widetilde{\mathbb{W}}_n^{(3)}(\beta) := \int_0^T \left|W\left(e^{-(\hat{u}_{n,k}+v)}\right) - W(e^{-v})\right| dv^\beta,$$

where we can use the bound

$$\mathbb{P}\left(\widetilde{\mathbb{W}}_n^{(3)}(\beta) > \frac{a}{2}\right) \leq \mathbb{P}\left(\sup_{v \in [0, T], |x| \leq k^{-1/4}} \left|W\left(e^{-(v+x)}\right) - W(e^{-v})\right| > \frac{a}{2T^\beta}\right) + \mathbb{P}\left(|\hat{u}_{n,k}| > k^{-1/4}\right).$$

By the uniform continuity of W on compact sets and Lemma 5.2, we have for $n > n_2$

$$\mathbb{P}\left(\widetilde{W}_n^{(3)}(\beta) > \frac{a}{2}\right) \leq \frac{\varepsilon}{2},$$

from which we deduce that

$$\sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} T_{9,n} = o_{\mathbb{P}}(1). \quad (17)$$

Term $T_{10,n}$. By the mean value theorem, with $\xi \in (0, 1)$, we have with large probability, for n large, and $0 < \varepsilon < 1/\delta$ and $0 < \eta < 1/2$,

$$\begin{aligned} \sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} |T_{10,n}| &= \left| \beta(\beta - 1) \frac{a_X(n/k)}{U_X(n/k)} \int_0^\infty [\widehat{u}_{n,k} + v] \left[1 + \xi \frac{a_X(n/k)}{U_X(n/k)} (\widehat{u}_{n,k} + v)\right]^{\beta-2} W(e^{-v}) dv \right| \\ &\leq \beta|\beta - 1| \frac{a_X(n/k)}{U_X(n/k)} \\ &\quad \times \left\{ \mathbb{1}_{\{\beta < 2\}} [1 - \delta\varepsilon]^{\beta-2} \int_0^\infty [\delta + v] |W(e^{-v})| dv \right. \\ &\quad \left. + \mathbb{1}_{\{\beta \geq 2\}} \int_0^\infty [\delta + v] [1 + \varepsilon(\delta + v)]^{\beta-2} |W(e^{-v})| dv \right\} \\ &\leq \beta|\beta - 1| \frac{a_X(n/k)}{U_X(n/k)} \\ &\quad \times \left\{ \mathbb{1}_{\{\beta < 2\}} [1 - \delta\varepsilon]^{\beta-2} \sup_{0 < y \leq 1} \frac{|W(y)|}{y^\eta} \int_0^\infty [\delta + v] e^{-\eta v} dv \right. \\ &\quad \left. + \mathbb{1}_{\{\beta \geq 2\}} \sup_{0 < y \leq 1} \frac{|W(y)|}{y^\eta} \int_0^\infty [\delta + v] [1 + \varepsilon(\delta + v)]^{\beta-2} e^{-\eta v} dv \right\} \\ &\leq C \frac{a_X(n/k)}{U_X(n/k)}, \end{aligned} \quad (18)$$

by Lemma 2 in Cai et al. (2015), and where C is a positive constant. Combining (7)-(13) with (16)-(18), Theorem 3.1 follows.

Proof of Theorem 4.2. Consider the following decomposition

$$\begin{aligned}
& \sqrt{k} \left(\frac{\widehat{CTM}_{\beta,p}}{CTM_{\beta,p}} - 1 \right) \\
&= \sqrt{k} \left(\underbrace{\left\{ \left(\frac{\log 1/p}{\log n/k} \right)^{\beta[\widehat{\theta}_k - \theta]} \right\}}_{T_{11,n}} \underbrace{\left\{ \frac{\widehat{CTM}_{\beta, \frac{k}{n}}}{CTM_{\beta, \frac{k}{n}}} \right\}}_{T_{12,n}} \underbrace{\left\{ \frac{(\log n/k)^{-\beta\theta} CTM_{\beta, \frac{k}{n}}}{(\log 1/p)^{-\beta\theta} CTM_{\beta,p}} \right\}}_{T_{13,n}} - 1 \right) \\
&= \sqrt{k} \left\{ \left(\frac{\log 1/p}{\log n/k} \right)^{\beta[\widehat{\theta}_k - \theta]} - 1 \right\} T_{12,n} T_{13,n} + \sqrt{k} \left\{ \frac{\widehat{CTM}_{\beta, \frac{k}{n}}}{CTM_{\beta, \frac{k}{n}}} - 1 \right\} T_{13,n} \\
&\quad + \sqrt{k} \left\{ \frac{(\log n/k)^{-\beta\theta} CTM_{\beta, \frac{k}{n}}}{(\log 1/p)^{-\beta\theta} CTM_{\beta,p}} - 1 \right\} \\
&= \beta \log(\lambda) \sqrt{k} [\widehat{\theta}_k - \theta] (1 + o_{\mathbb{P}}(1)) T_{12,n} T_{13,n} + \sqrt{k} \frac{U_X(n/k)}{a_X(n/k)} \left\{ \frac{\widehat{CTM}_{\beta, \frac{k}{n}}}{CTM_{\beta, \frac{k}{n}}} - 1 \right\} \frac{a_X(n/k)}{U_X(n/k)} T_{13,n} \\
&\quad + \sqrt{k} \left\{ \frac{CTM_{\beta, \frac{k}{n}}/[U_X(n/k)]^\beta}{CTM_{\beta,p}/[U_X(1/p)]^\beta} - 1 \right\} \left(\frac{U_X(n/k)}{U_X(1/p)} \right)^\beta \left(\frac{\log n/k}{\log 1/p} \right)^{-\beta\theta} \\
&\quad + \sqrt{k} \left(\left(\frac{U_X(n/k)}{U_X(1/p)} \right)^\beta \left(\frac{\log n/k}{\log 1/p} \right)^{-\beta\theta} - 1 \right).
\end{aligned}$$

According to (4), we have

$$\begin{aligned}
\frac{CTM_{\beta, \frac{k}{n}}}{[U_X(n/k)]^\beta} &= 1 + \beta \frac{a_X(n/k)}{U_X(n/k)} \left\{ 1 + \int_0^\infty \left[\left(1 + \frac{a_X(n/k)}{U_X(n/k)} z \right)^{\beta-1} - 1 \right] e^{-z} dz \right. \\
&\quad \left. + \int_0^\infty \left(1 + \frac{a_X(n/k)}{U_X(n/k)} z \right)^{\beta-1} \left(\frac{\overline{F}_X(U_X(n/k) + a_X(n/k)z)}{\overline{F}_X(U_X(n/k))} - e^{-z} \right) dz \right\} \\
&= 1 + o\left(\frac{1}{\sqrt{k}}\right).
\end{aligned}$$

A similar result can be obtained for $CTM_{\beta, \frac{1}{p}}/[U_X(1/p)]^\beta$. Thus, we have

$$\frac{CTM_{\beta, \frac{k}{n}}/[U_X(n/k)]^\beta}{CTM_{\beta,p}/[U_X(1/p)]^\beta} - 1 = o\left(\frac{1}{\sqrt{k}}\right).$$

Now, the last term of the decomposition.

$$\left(\frac{U_X(n/k)}{U_X(1/p)} \right)^\beta \left(\frac{\log n/k}{\log 1/p} \right)^{-\beta\theta} - 1 = \left(\frac{\ell_U(\log n/k)}{\ell_U(\log 1/p)} \right)^\beta - 1 = \left(\frac{\ell_X(U_X(1/p))}{\ell_X(U_X(n/k))} \right)^{\beta\theta} - 1,$$

since $\ell_U(\log x) = [\ell_X(U_X(x))]^{-\theta}$.

Now, using the mean value theorem with $u_n \in [U_X(n/k); U_X(1/p)]$, we have, for $\delta_1, \delta_2 > 0$ and n large,

$$\begin{aligned} \left| \frac{\ell_X(U_X(1/p))}{\ell_X(U_X(n/k))} - 1 \right| &= \frac{|\ell'_X(u_n)|}{\ell_X(U_X(n/k))} \left[U_X\left(\frac{1}{p}\right) - U_X\left(\frac{n}{k}\right) \right] \\ &= \frac{u_n |\ell'_X(u_n)|}{\ell_X(u_n)} \frac{\ell_X(u_n)}{\ell_X(U_X(n/k))} \frac{U_X(n/k)}{u_n} \left(\frac{U_X(1/p)}{U_X(n/k)} - 1 \right) \\ &\leq (1 + \delta_1) \frac{u_n |\ell'_X(u_n)|}{\ell_X(u_n)} \left(\frac{u_n}{U_X(n/k)} \right)^{\delta_2} \left(\frac{U_X(1/p)}{U_X(n/k)} - 1 \right) \\ &\leq (1 + \delta_1) \sup_{x \geq U_X(n/k)} \frac{x |\ell'_X(x)|}{\ell_X(x)} \left(\frac{U_X(1/p)}{U_X(n/k)} \right)^{\delta_2} \left(\frac{U_X(1/p)}{U_X(n/k)} - 1 \right) \\ &= o\left(\frac{1}{\sqrt{k}}\right), \end{aligned}$$

since

$$\frac{U_X(1/p)}{U_X(n/k)} - 1 = \left[\left(\frac{\log 1/p}{\log n/k} \right)^\theta - 1 \right] \frac{\ell_U(\log 1/p)}{\ell_U(\log n/k)} + \left[\frac{\ell_U(\log 1/p)}{\ell_U(\log n/k)} - 1 \right] = O(1).$$

Combining Theorem 3.1 with Theorem 4.1, Theorem 4.2 follows.

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