



## Verification of Square Lattices with Various Neighborhoods by Infinite Petri Nets

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Complex Systems Day, February 7, 2023, Nice

#### **Discrete Models for Complex Systems Team**

- The MDSC team aims at studying formal models for complex systems and their applications to various fields (mainly Biology, Mathematics and Computer Science)
- The team is organized into two themes:
  - Models for Complex Systems
  - Constraint programming and verification
- MDSC is mainly devoted to fundamental research which impacts the main current societal changes in the domain of:
  - Combinatorial optimization;
  - Complexity in dynamical models;
  - Computational complexity;
  - Concurrency;
  - Discrete dynamical systems;
  - Modeling for Complex Systems;
  - Problem solving;
  - Program verification;
  - Systems biology

Petri and Sleptsov nets, infinite and reenterable models

### **INTRODUCTION 1**

- Various types of switching lattice patterns [1], [2] are the basic structures of telecommunication networks and equipment, including supercomputers, clusters, network-on-chip, models for nanotechnology as well as biological and physical processes and systems.
- Cellular automata [3], [5] and Petri nets are well known formalisms of mathematical modeling, which have close intrinsic relation. A specially introduced class of infinite Petri nets [1], [2] is applied to describe the models in the present paper.
- Two basic principles of switching are considered: save-andforward and cut-through switching of packets, which can be combined.
- Using real-life boundary conditions [1], [3] allows us to specify switching packets between terminal devices connected to the boundary of the communication matrix, or to close the bounds to obtain the (hyper)torus surface.

[1] D. Zaitsev, I. D. Zaitsev, T. R. Shmeleva, "Infinite Petri Nets: Part 1, Modeling Square Grid Structures", Complex Systems, vol. 26, no. 2, 2017, pp.157-195.

[2] T. R. Shmeleva, D. A. Zaitsev, I. D. Zaitsev, "Infinite Petri Nets: Part 2, Modeling Triangular, Hexagonal, Hypercube and Hypertorus Structures", *Complex Systems*, vol. 26, no. 4, 2017, pp. 341-371.

[3] D. A. Zaitsev, "A generalized neighborhood for cellular automata", *Theoretical Computer Science*, vol. 666 2017 pp. 21-35
[5] A. Dennunzio, E.Formenti, D. Grinberg, L. Margara, "Dynamical behavior of additive cellular automata over finite abelian groups", *Theoretical Computer Science*, vol. 843, 2020, pp. 45–56.

#### **INTRODUCTION 2**

Most commonly used neighborhood types

Von Neumann neighborhood (a), Moore neighborhood (b)





b

#### **INTRODUCTION 3**

Basic principles of switching a) save-and-forward b) cut-through switching po:1-0.0 pil:1-0.0 • pol:1-0.0 pi:1-0.0 to 1 ti\_1,2 ti 1.4 tic:1 ti\_4,3 tic:1.4-0.0 po\_2 tic:2.1-0.0 tic:1.390.0 po:4-0.0 to 2 pi:2 pol\_2 • pol:4-0.0( pil:2 tic:2.4tf 1.4 pb\_2 pb 4 c.2.3-0.0tic:4.1-0. ti\_2,3 pi\_2 po:2 pi:4-0.0 ti 2.4 pol\_4 pol:2 ₂,₄ pil:4-0.0( ● ) tid 3.1-0.0 • • pb 3 tic:4.3-0.0 pol\_4 tic:3.2-0.0 ti 2,1 ti\_3,4 ti\_3,2 to\_3 po:3 pol:3 pil:3 po 3 c) combined switching node

## CONSTRUCTION OF SQUARE LATTICE MODELS WITH VON NEUMANN'S NEIGHBORHOOD tf-u p1ol p1il p1o 5 SO dol gil

#### PARAMETRIC EXPRESSION

According to the composition rules [1], the specification of an open lattice model is represented with (1).

$$\begin{pmatrix} \begin{pmatrix} tf_{u,v}^{i,j}: pi_{u}^{i,j} \to pf_{u,v}^{i,j} \\ tc_{u,v}^{i,j}: pf_{u,v}^{i,j}, pol_{v}^{i,j} \to po_{v}^{i,j}, pil_{u}^{i,j} \\ tb_{u,v}^{i,j}: pf_{u,v}^{i,j}, pbl^{i,j}, po_{v}^{i,j} \to po_{v}^{i,j}, pb_{v}^{i,j}, pil_{u}^{i,j} \end{pmatrix}, 1 \le v \le 4, \\ k \ge v \ne u \\ to_{u}^{i,j}: pb_{u}^{i,j}, pol_{u}^{i,j} \to po_{u}^{i,j}, pbl^{i,j} \\ to_{u}^{i,j}: pb_{u}^{i,j}, pol_{u}^{i,j} \to po_{u}^{i,j}, pbl^{i,j} \\ (tl_{1}^{i,j}: po_{1}^{i,j}, pil_{3}^{i-1,j} \to pi_{3}^{i-1,j}, pol_{1}^{i,j}), i \ne 1 \\ (tl_{4}^{i,j}: po_{3}^{i,j}, pil_{1}^{i+1,j} \to pi_{1}^{i+1,j}, pol_{3}^{i,j}), i \ne k \\ (tl_{2}^{i,j}: po_{2}^{i,j}, pil_{4}^{i,j+1} \to pi_{4}^{i,j+1}, pol_{2}^{i,j}), j \ne k \end{pmatrix}$$

Based on the parametric expression (1) that specifies communication square lattice with an arbitrary size k, we calculate place invariants [1] of infinite Petri nets and analyze properties [4] of an ideal communication protocol model.

[1] D. Zaitsev, I. D. Zaitsev, T. R. Shmeleva, "Infinite Petri Nets: Part 1, Modeling Square Grid Structures", Complex Systems, vol. 26, no. 2, 2017, pp.157-195.

[4] Shmeleva T.R., Zaitsev D.A., Groote J.F. "Verification of Hypertorus Communication Grids by Infinite Petri Nets and Process Algebra", IEEE/CAA Journal of Automatica Sinica, vol. 6, no. 3, 2019, pp. 733-742.

### **BASIC PETRI NET PROPERTIES**

The basic Petri net properties of communication systems:

**boundedness**, a bounded Petri net has a finite graph of reachable markings ;

conservativeness, conservativeness consists in preserving a weighted sum of tokens;

**consistency**, the transition invariant represents a count vector of the transition firing sequences leading to the initial marking – stationary repetitive; *liveness*, in a live Petri net, each transition can fire in a trace starting from any reachable marking.

#### SYSTEM OF EQUATIONS

An infinite system of linear algebraic equations for calculation of *place invariants* of the open lattice (1) is represented with (2).

$$\begin{pmatrix} \begin{pmatrix} tf_{u,v}^{i,j}: pi_{u}^{i,j} - pf_{u,v}^{i,j} = 0 \\ tc_{u,v}^{i,j}: pf_{u,v}^{i,j} + pol_{v}^{i,j} - po_{v}^{i,j} - pil_{u}^{i,j} = 0 \\ tb_{u,v}^{i,j}: pf_{u,v}^{i,j} + pbl_{v}^{i,j} + po_{v}^{i,j} - po_{v}^{i,j} - pb_{v}^{i,j} - pil_{u}^{i,j} = 0 \end{pmatrix}, 1 \le v \le 4, v \ne u \\ tb_{u,v}^{i,j}: pf_{u,v}^{i,j} + pol_{u}^{i,j} - po_{v}^{i,j} - pb_{v}^{i,j} - pil_{u}^{i,j} = 0 \\ to_{u}^{i,j}: pb_{u}^{i,j} + pol_{u}^{i,j} - po_{u}^{i,j} - pol_{1}^{i,j} = 0 \\ (tl_{1}^{i,j}: po_{1}^{i,j} + pil_{3}^{i-1,j} - pi_{3}^{i-1,j} - pol_{1}^{i,j} = 0), i \ne 1 \\ (tl_{4}^{i,j}: po_{4}^{i,j} + pil_{2}^{i,j-1} - pi_{3}^{i,j-1} - pol_{4}^{i,j} = 0), j \ne 1 \\ (tl_{3}^{i,j}: po_{3}^{i,j} + pil_{1}^{i+1,j} - pi_{1}^{i+1,j} - pol_{3}^{i,j} = 0), i \ne k \\ (tl_{2}^{i,j}: po_{2}^{i,j} + pil_{4}^{i,j+1} - pi_{4}^{i,j+1} - pol_{2}^{i,j} = 0), j \ne k \end{pmatrix}$$

### PARAMETRIC SOLUTION

For the open communication square lattice (1) of size k, the parametric solution (3) of system (2) was obtained.

$$\begin{pmatrix} (po_{u}^{i,j}, pol_{u}^{i,j}), 1 \le u \le 4, 1 \le i \le k, 1 \le j \le k \\ ((pb_{u}^{i,j}, 1 \le u \le 4), pbl^{i,j}), 1 \le i \le k, 1 \le j \le k \\ ((pf_{u,v}^{i,j}, 1 \le v \le 4, v \ne u), pi_{u}^{i,j}, pil_{u}^{i,j}), 1 \le u \le 4, 1 \le i \le k, 1 \le j \le k \\ ((pbl^{i,j}, ((pil_{u}^{i,j}, pol_{u}^{i,j}), 1 \le u \le 4), 1 \le i \le k, 1 \le j \le k) \\ ((pb_{u}^{i,j}, (pf_{u,v}^{i,j}, 1 \le v \le 4, v \ne u), pi_{u}^{i,j}, po_{u}^{i,j}), 1 \le u \le 4, 1 \le i \le k, 1 \le j \le k) \end{pmatrix}$$

**Lemma 1**. Each line of the sparse matrix (3) is a solution of equations system (2).

**Theorem 1**. The lattice model (1) is a *p*-invariant Petri net for an arbitrary natural number *k*.

According to the Lemmal and Theorem1, each row of (3) is a solution of system (2), therefore Petri net model is *p*-invariant, and consequently the model of lattice possesses boundedness and conservativeness properties for an arbitrary size *k*.

#### CONSTRUCTION OF SQUARE LATTICE MODELS WITH MOORE'S NEIGHBORHOOD 1



#### CONSTRUCTION OF SQUARE LATTICE MODELS WITH MOORE'S NEIGHBORHOOD 2



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#### TOTAL NUMBER OF PLACES IN SQUARE LATTICE AND SPARSE MATRIX

Number of places, transitions, and arcs in the open square lattice model with an arbitrary size k is represented in Table (Moore's neighb. 1).

Number	Number of	Number of
of places	transitions	arcs
$97 \cdot k^2$	$176 \cdot k^2 + 4 \cdot k \cdot (k-1) + 4 \cdot (k-1)^2$	$648 \cdot k^2 + 16 \cdot (2k-1) \cdot (k-1)$

Expressions of Table are applied in the model checking using process algebra [4], which is methodically complimentary to the investigation of rectangular lattice models based on Petri nets [1].

Model Checking Contest@Petri Nets, https://mcc.lip6.fr/models.php

Lattice size	Number of Places (Moore's neighborhoods 2)		
	In lattice	In sum of 1-16 and 17 lines (4)	In sum of 18 and 19 lines (4)
1×1	41	$41 = (2 \cdot 8 + 2 \cdot 8) + 9$	$41 = (2 \cdot 8 + 1) + (2 \cdot 8 + 8)$
2×2	164=41.4	$164 = 32 \cdot 4 + 9 \cdot 4$	$164 = 17 \cdot 4 + 24 \cdot 4$
	$41k^2$	$41k^2 = 32k^2 + 9k^2$	$41k^2 = 17k^2 + 24k^2$ • 13

#### REFERENCES

- 1. D. A. Zaitsev, I. D. Zaitsev, T. R. Shmeleva, "Infinite Petri Nets: Part 1, Modeling Square Grid Structures", <u>Complex Systems</u>, vol. 26, no. 2, 2017, pp.157-195.
- T. R. Shmeleva, D. A. Zaitsev, I. D. Zaitsev, "Infinite Petri Nets: Part 2, Modeling Triangular, Hexagonal, Hypercube and Hypertorus Structures", <u>Complex Systems</u>, vol. 26, no. 4, 2017, pp. 341-371.
- 3. D. A. Zaitsev, "A generalized neighborhood for cellular automata", *Theoretical Computer Science*, vol. 666, 2017, pp. 21-35.
- 4. Shmeleva T.R., Zaitsev D.A., Groote J.F. "Verification of Hypertorus Communication Grids by Infinite Petri Nets and Process Algebra", *IEEE/CAA Journal of Automatica Sinica*, vol. 6, no. 3, 2019, pp. 733-742.
- 5. A. Dennunzio, E.Formenti, D. Grinberg, L. Margara, "Dynamical behavior of additive cellular automata over finite abelian groups", *Theoretical Computer Science*, vol. 843, 2020, pp. 45–56.

## CONCLUSIONS

- The automated composition method for generating models of communication structures with Moore's neighboring can be used in a wide range of Petri nets applications, including telecommunications, biological systems, and programming.
- Infinite Petri net models of a square lattice with von Neumann's neighborhood with three types of devices: with the packets buffering, cut-through technology, combine cut-through and store-and-forward switching techniques are studied, its invariance proved.
- Infinite Petri net models of a square lattice with Moore's neighborhood with the packets buffering devices are studied, its invariance proved.
- Infinite Petri net models of a square lattice with Moore's neighborhood which combine cut-through and store-and-forward switching techniques have been composed.

#### FUTURE RESEARCH DIRECTION

- Comparative analysis of communication lattices with different type of switching nodes.
- Studying deadlocks of models of communication structures with Moore's neighboring and possible ways of liveness enforcing.
- Verification of square lattices with the Zaitsev neighborhood [3] by infinite Petri nets.

[3] D. A. Zaitsev, "A generalized neighborhood for cellular automata", Theoretical Computer Science, vol. 666, 2017, pp. 21-35.

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# Thank you for attention!

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