# Homotopy theory of Moore flows (III)

Philippe Gaucher

# HOMOTOPY THEORY OF MOORE FLOWS (III)

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Abstract. The previous paper of this series shows that the q-model categories of $\mathcal{G}$-multipointed $d$-spaces and of $\mathcal{G}$-flows are Quillen equivalent. In this paper, the same result is established by replacing the reparametrization category $\mathcal{G}$ by the reparametrization category $\mathcal{M}$. Unlike the case of $\mathcal{G}$, the execution paths of a cellular $\mathcal{M}$-multipointed $d$-space can have stop intervals. The technical tool to overcome this obstacle is the notion of globular naturalization. It is the globular analogue of Raussen's naturalization of a directed path in the geometric realization of a precubical set. The notion of globular naturalization working both for $\mathcal{G}$ and $\mathcal{M}$, the proof of the Quillen equivalence we obtain is valid for the two reparametrization categories. Together with the results of the first paper of this series, we then deduce that $\mathcal{G}$-multipointed $d$-spaces and $\mathcal{M}$-multipointed $d$-spaces have Quillen equivalent q-model structures.

## Contents



## 1. Introduction

**Presentation.** Multipointed $d$-spaces are a variant of Grandis' notion of $d$-space [17] which can be used for studying geometric models of concurrency [6]. They are used together with Moore flows in [15, Corollary 6.9] to give a model category interpretation of the fact that the space of nonconstant tame regular directed paths between two vertices in the geometric realization of a precubical set is homotopy equivalent to a CW-complex. This fact is explicitly proved in [28, Theorem 6.1 and Theorem 7.6] by constructing an explicit homotopy equivalence with the classifying space of a small category obtained from the precubical set, namely the small category of Ziemiański cube chains associated with the precubical set.

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It is proved in [11, Theorem 8.1] that multipointed $d$-spaces and Moore flows have Quillen equivalent q-model structures. The reparametrizations of execution paths allowed in [11] are the precomposition by the maps of the reparametrization category $\mathcal{G}$ in the sense of Proposition 2.3 which are the nondecreasing homeomorphisms between nontrivial segments of the real line. The advantage of this setting is that all execution paths of a cellular multipointed $d$-space are *regular* in the sense of [5, Definition 1.1], namely without *stop intervals*, i.e. without nontrivial intervals on which the path is constant (cf. Definition 4.10). It makes their geometry much simpler to understand. It is the primary reason for choosing $\mathcal{G}$ in the paper which introduced multipointed $d$-spaces [8]. The drawback of this setting is that it can be used only to study the homotopy theory of regular directed paths.

Grandis' notion of $d$-space allows more general reparametrizations of directed paths. It makes possible to deal with non-regular directed paths. The purpose of this paper is to prove the same Quillen equivalence between multipointed $d$-spaces and Moore flows by allowing the reparametrizations of execution paths by precomposing by all maps of the reparametrization category $\mathcal{M}$ in the sense of Proposition 2.4 which are the nondecreasing surjective maps between nontrivial segments of the real line. The technical obstacle to overcome is that the execution paths of a cellular multipointed $d$-space can now contain stop intervals. This implies that several crucial theorems belonging to the technical core of [11] are either false in this new setting or their proof must be modified (cf. Table 2).

By [23, Proposition 2.2], every nonconstant *Moore* path in a Hausdorff space has a regular reparametrization. Moreover, by [5, Proposition 3.8], the regular reparametrization of a given nonconstant Moore path in a Hausdorff space is unique up to a map of $\mathcal{G}$. In [24, Definition 2.14], Raussen introduces a notion of naturalization of a directed path. This means that any nonconstant *directed* path $\gamma$ in the geometric realization of a precubical set has a regular reparametrization called the naturalization $\underline{\mathrm{nat}}^{\square}(\gamma)$ which is morally more natural than the other ones. It is the unique reparametrization which makes the directed path a Moore composition of isometries for some Lawvere metric structure on the geometric realization of a precubical set. This idea is generalized to the setting of transverse sets in [14]. By Proposition 4.12, there is then a unique factorization $\gamma = \underline{\mathrm{nat}}^{\square}(\gamma)\eta$ where $\eta \in \mathcal{M}$.

The main technical tool of this paper is the introduction of a globular version of Raussen's idea in Proposition 4.13 and Definition 4.14. Every execution path $\gamma$ of a cellular multipointed $d$-space has also a regular reparametrization $\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma)$ which is morally more natural than the other ones. Again by Proposition 4.12, and since the underlying space of a cellular multipointed $d$-space is Hausdorff, there is then a unique factorization $\gamma = \underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma)\eta$ where $\eta \in \mathcal{M}$. It is the key point to adapt the technical core of [11].

Raussen's naturalization and the *globular naturalization* have in common the following property: the naturalization of a Moore composition is the Moore composition of the naturalizations. On the other hand, Raussen's naturalization and the globular naturalization do not behave in the same way with respect to continuous deformations. This point is explained in Corollary 4.25 and in the remark following it. Two directed paths in the geometric realization of a precubical set or of a transverse set which are dihomotopy equivalent relatively to the extremities have naturalizations of the same length. On the

contrary, the best that can be said in the globular case is that, on such a *compact* continuous path of directed paths, the natural length is bounded (actually, it takes finitely many values).

The main result of this paper can be stated as follows.

**Theorem.** *(Theorem 7.5 and Theorem 7.10) The inclusion functor $\mathcal{G} \subset \mathcal{M}$ induces a forgetful functor $\mathcal{M}\mathbf{dTop} \to \mathcal{G}\mathbf{dTop}$ from $\mathcal{M}$-multipointed d-spaces to $\mathcal{G}$-multipointed d-spaces. There is the commutative square of right Quillen equivalences between the four q-model structures*

$$\begin{array}{ccc} \mathcal{M}\mathbf{dTop} & \longrightarrow & \mathcal{G}\mathbf{dTop} \\ {\scriptstyle \mathbb{M}^{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \mathbb{M}^{\mathcal{G}}} \\ \mathcal{M}\mathbf{Flow} & \longrightarrow & \mathcal{G}\mathbf{Flow} \end{array}$$

*where $\mathcal{M}\mathbf{Flow}$ and $\mathcal{G}\mathbf{Flow}$ are the categories of $\mathcal{M}$-flows and $\mathcal{G}$-flows respectively. Finally, the unit maps and the counit maps of the two vertical adjunctions are isomorphisms on q-cofibrant objects.*

As a byproduct of some new tools introduced in this paper, it is also proved the following new fact which was not in [11], even for the case $\mathcal{P} = \mathcal{G}$:

**Theorem.** *(Theorem 6.8) Let $\mathcal{P}$ be either $\mathcal{G}$ or $\mathcal{M}$. The compact-open topology on the set of execution paths of a locally finite cellular $\mathcal{P}$-multipointed d-space is $\Delta$-generated. Therefore in this case, the space of execution paths is metrizable with the distance of the uniform convergence.*

Extending the results of [11] from $\mathcal{G}$ to $\mathcal{M}$ has another interest than making possible the homotopical study of non-regular directed paths. It enables us to find proofs which are independent of the choice of the reparametrization category $\mathcal{G}$ or $\mathcal{M}$: it was not the case for the proofs of [11]. In fact, the question raised by this work is to find a better definition of a reparametrization category than Definition 2.2. A good notion of a reparametrization category $\mathcal{P}$ should be a small category satisfying Definition 2.2 and additional properties so that it makes sense to talk about the $\mathcal{P}$-multipointed $d$-spaces (unless $\mathcal{P}$ is the final category). It is an open question which could be reworded as follows: is there any other "interesting" reparametrization category than the terminal category, $\mathcal{G}$ and $\mathcal{M}$ ?

Table 1 concludes the presentation. The left column is a reparametrization category and the right column is the corresponding type of tame nonconstant directed paths contained in the corresponding notion of flows (a *trace* is a directed path up to reparametrization, according to the terminology of e.g. [24]). The terminal category is mentioned in Table 1 for completeness.

**Outline of the paper.** Section 2 is a reminder about $\mathcal{P}$-flows for a reparametrization category $\mathcal{P}$ which is either $\mathcal{G}$ or $\mathcal{M}$ in this paper.

Section 3 adapts some results and constructions for $\mathcal{G}$-multipointed $d$-spaces proved in [11] to the case of $\mathcal{M}$-multipointed $d$-spaces.

Section 4 is the adaptation of [11, Section 5] to the case of cellular $\mathcal{M}$-multipointed $d$-spaces. The main results are the notion of globular naturalization of an execution path

| $\mathcal{P}$ | Type of directed paths |
|---|---|
| $\mathbf{1}$ | Trace |
| $\mathcal{G}$ | Regular |
| $\mathcal{M}$ | All |

TABLE 1. Type of directed paths

of a cellular $\mathcal{M}$-multipointed $d$-space (Proposition 4.13 and Definition 4.14). We then obtain, thanks to the notion of carrier, Theorem 4.17 which is a replacement for [11, Theorem 5.20] and Theorem 4.24 which is a replacement for [11, Theorem 5.19].

Section 5 is the adaptation of [11, Section 6]. A generalization of [11, Theorem 6.11] is proved in Theorem 5.9.

Section 6 is a digression which uses Theorem 4.21 and Theorem 5.9 to prove that the space of execution paths in the locally finite case is metrizable with the distance of the uniform convergence in Theorem 6.8.

Finally, Section 7 establishes the main theorem of the paper, namely Theorem 7.5 and Theorem 7.10.

**Erratum.** As explained in the corrected version of [10], the tenseur product of $\mathcal{P}$-spaces is not symmetric if $\mathcal{P}$ is $\mathcal{G}$ or $\mathcal{M}$. Therefore, the word symmetric must be removed everywhere from [11]. Besides, the terminology of *biclosed* semimonoidal structure should be used instead of the terminology of closed semimonoidal structure to describe the tensor product of $\mathcal{P}$-spaces.

**Prerequisites and notations.** We refer to [1] for locally presentable categories, to [26] for combinatorial model categories. We refer to [19, 20] for more general model categories. We refer to [21] and to [2, Chapter 6] for enriched categories. All enriched categories are topologically enriched categories: *the word topologically is therefore omitted*. A gold mine of examples and counterexamples in general topology can be found in [27]. A *cellular object* of a combinatorial model category is an object $X$ such that the canonical map $\varnothing \to X$ is a transfinite composition of pushouts of generating cofibrations.

The results of this paper rely heavily on the results of [11]. A self-contained paper would not help the reader much. The choice made for this work is to emphasize the differences between $\mathcal{G}$ and $\mathcal{M}$ instead of the similarities. Table 2 summarizes these differences. The left column is a list of theorems of [11]. The middle column gives the status of the statement for $\mathcal{P} = \mathcal{M}$. The right column gives the replacement in this paper: it consists of a statement which is modified if necessary and a new proof. In this paper, even if [11, Theorem 5.19] is still valid for $\mathcal{P} = \mathcal{M}$, it is replaced by Theorem 4.24 which is a much powerful statement both for the proof of Theorem 7.3 and to understand the difference between the globular naturalization and the cubical naturalization.

The category **Top** denotes the category of $\Delta$-*generated spaces* or of $\Delta$-*Hausdorff* $\Delta$-*generated spaces* (cf. [12, Section 2 and Appendix B]). The inclusion functor from the full subcategory of $\Delta$-generated spaces to the category of general topological spaces together with the continuous maps has a right adjoint called the $\Delta$-kelleyfication functor. The latter functor does not change the underlying set: it only adds open subsets. The category

| | Status for $\mathcal{M}$ | Replacement |
|---|---|---|
| [11, Theorem 3.9] | true | Theorem 3.16 |
| [11, Proposition 2.12] | true | Proposition 3.17 |
| [11, Theorem 5.7] | true | Theorem 4.8 |
| [11, Theorem 5.9] | false | Theorem 4.13 |
| [11, Theorem 5.20] | false | Theorem 4.17 |
| [11, Proposition 5.17] | false | Theorem 4.21 |
| [11, Theorem 5.18] | true | Corollary 4.22 |
| [11, Theorem 5.19] | true | Corollary 4.23 |
| [11, Proposition 6.3] | true | Proposition 5.4 |
| [11, Theorem 6.11] | true | Theorem 5.9 |
| [11, Theorem 7.2 and Theorem 7.3] | true | Theorem 7.3 |

TABLE 2. Main differences between $\mathcal{G}$ and $\mathcal{M}$

**Top** is locally presentable and cartesian closed. The internal hom $\mathbf{TOP}(X, Y)$ is given by taking the $\Delta$-kelleyfication of the compact-open topology on the set $\mathbf{Top}(X, Y)$. The category **Top** is equipped with its q-model structure. *A compact space is a quasicompact Hausdorff space (French convention).* All $\Delta$-generated spaces are sequential.

$\mathcal{K}^{op}$ denotes the opposite category of $\mathcal{K}$; $\mathrm{Obj}(\mathcal{K})$ is the class of objects of $\mathcal{K}$; $\mathcal{K}^I$ is the category of functors and natural transformations from a small category $I$ to $\mathcal{K}$; $\varnothing$ is the initial object, $\mathbf{1}$ is the final object, $\mathrm{Id}_X$ is the identity of $X$; $\mathcal{K}(X, Y)$ is the set of maps in a set-enriched, i.e. locally small, category $\mathcal{K}$; $\mathcal{K}(X, Y)$ is the space of maps in an enriched category $\mathcal{K}$. The underlying set of maps may be denoted by $\mathcal{K}_0(X, Y)$ if it is necessary to specify that we are considering the underlying set.

All Moore paths in this paper are nonconstant: see Definition 3.2.

## 2. Moore flow

2.1. **Notation.** The notations $\ell, \ell', \ell_i, L, \dots$ mean a strictly positive real number unless specified something else. $[\ell, \ell']$ denotes a segment: unless specified, it is always understood that $\ell < \ell'$.

2.2. **Definition.** [10, Definition 4.3] A *reparametrization category* $(\mathcal{P}, \otimes)$ is a small enriched semimonoidal category satisfying the following additional properties:

(1) The semimonoidal structure is strict, i.e. the associator is the identity.
(2) All spaces of maps $\mathcal{P}(\ell, \ell')$ for all objects $\ell$ and $\ell'$ of $\mathcal{P}$ are contractible.
(3) For all maps $\phi : \ell \to \ell'$ of $\mathcal{P}$, for all $\ell'_1, \ell'_2 \in \mathrm{Obj}(\mathcal{P})$ such that $\ell'_1 \otimes \ell'_2 = \ell'$, there exist two maps $\phi_1 : \ell_1 \to \ell'_1$ and $\phi_2 : \ell_2 \to \ell'_2$ of $\mathcal{P}$ such that $\phi = \phi_1 \otimes \phi_2 : \ell_1 \otimes \ell_2 \to \ell'_1 \otimes \ell'_2$ (which implies that $\ell_1 \otimes \ell_2 = \ell$).

The terminal category is a symmetric reparametrization category. It is not known whether there exist symmetric reparametrization categories not equivalent to the terminal category. Here are the two examples of reparametrization category used in this paper.

2.3. **Proposition.** *[10, Proposition 4.9] There exists a reparametrization category, denoted by $\mathcal{G}$, such that the semigroup of objects is the open interval $]0, +\infty[$ equipped with the addition and such that for every $\ell_1, \ell_2 > 0$, $\mathcal{G}(\ell_1, \ell_2)$ is the set of nondecreasing*

*homeomorphisms from* $[0,\ell_1]$ *to* $[0,\ell_2]$ *equipped with the* $\Delta$*-kelleyfication of the relative topology induced by the set inclusion* $\mathcal{G}(\ell_1,\ell_2)\subset\mathbf{TOP}([0,\ell_1],[0,\ell_2])$ *and such that for every* $\ell_1,\ell_2,\ell_3>0$*, the composition map* $\mathcal{G}(\ell_1,\ell_2)\times\mathcal{G}(\ell_2,\ell_3)\to\mathcal{G}(\ell_1,\ell_3)$ *is induced by the composition of continuous maps.*

2.4. **Proposition.** *[10, Proposition 4.11] There exists a reparametrization category, denoted by* $\mathcal{M}$*, such that the semigroup of objects is the open interval* $]0,+\infty[$ *equipped with the addition and such that for every* $\ell_1,\ell_2>0$*,* $\mathcal{M}(\ell_1,\ell_2)$ *is the set of nondecreasing surjective maps from* $[0,\ell_1]$ *to* $[0,\ell_2]$ *equipped with the* $\Delta$*-kelleyfication of the relative topology induced by the set inclusion* $\mathcal{M}(\ell_1,\ell_2)\subset\mathbf{TOP}([0,\ell_1],[0,\ell_2])$ *and such that for every* $\ell_1,\ell_2,\ell_3>0$*, the composition map* $\mathcal{M}(\ell_1,\ell_2)\times\mathcal{M}(\ell_2,\ell_3)\to\mathcal{M}(\ell_1,\ell_3)$ *is induced by the composition of continuous maps.*

2.5. **Notation.** A reparametrization category $\mathcal{P}$ which is either $\mathcal{G}$ or $\mathcal{M}$ is fixed for the rest of the paper.

2.6. **Proposition.** *The topology of* $\mathcal{P}(\ell_1,\ell_2)$ *is the compact-open topology. In particular, it is metrizable. A sequence* $(\phi_n)_{n\geqslant 0}$ *of* $\mathcal{P}(\ell_1,\ell_2)$ *converges to* $\phi\in\mathcal{P}(\ell_1,\ell_2)$ *if and only if it converges pointwise.*

*Proof.* It is mutatis mutandis the same argument as the one given for $\mathcal{P}=\mathcal{G}$ in [11, Proposition 2.5]. □

2.7. **Notation.** Let $\phi_i\in\mathcal{P}(\ell_i,\ell_i')$ for $n\geqslant 1$ and $1\leqslant i\leqslant n$. Then the map

$$\phi_1\otimes\ldots\otimes\phi_n:\sum_i\ell_i\longrightarrow\sum_i\ell_i'$$

denotes the nondecreasing surjective map defined by

$$(\phi_1\otimes\ldots\otimes\phi_n)(t)=\begin{cases}\phi_1(t) & \text{if } 0\leqslant t\leqslant\ell_1\\ \phi_2(t-\ell_1)+\ell_1' & \text{if } \ell_1\leqslant t\leqslant\ell_1+\ell_2\\ \ldots & \\ \phi_i(t-\sum_{j<i}\ell_j)+\sum_{j<i}\ell_j' & \text{if } \sum_{j<i}\ell_j\leqslant t\leqslant\sum_{j\leqslant i}\ell_j\\ \ldots & \\ \phi_n(t-\sum_{j<n}\ell_j)+\sum_{j<n}\ell_j' & \text{if } \sum_{j<n}\ell_j\leqslant t\leqslant\sum_{j\leqslant n}\ell_j.\end{cases}$$

2.8. **Proposition.** *Let* $\phi\in\mathcal{P}(\ell,\ell')$*. Let* $n\geqslant 1$*. Consider* $\ell_1',\ldots,\ell_n'>0$ *with* $n\geqslant 1$ *such that* $\ell_1'+\cdots+\ell_n'=\ell'$*. Then there exists a decomposition of* $\phi$ *of the form*

$$\phi=\phi_1\otimes\ldots\otimes\phi_n$$

*such that* $\phi_i\in\mathcal{P}(\ell_i,\ell_i')$ *for* $1\leqslant i\leqslant n$*. Moreover, if* $\mathcal{P}=\mathcal{G}$*, then this decomposition is unique.*

*Proof.* The case $n=1$ is trivial. The case $n=2$ comes from the fact $\mathcal{G}$ and $\mathcal{M}$ are reparametrization categories. We deduce the existence of the decomposition by induction on $n\geqslant 2$. The uniqueness when $\mathcal{P}=\mathcal{G}$ is [11, Proposition 3.2]. □

2.9. **Notation.** The enriched category of enriched presheaves from $\mathcal{P}$ to **Top** is denoted by $[\mathcal{P}^{op},\mathbf{Top}]$. The underlying set-enriched category of enriched maps of enriched presheaves

is denoted by $[\mathcal{P}^{op}, \mathbf{Top}]_0$. The objects of $[\mathcal{P}^{op}, \mathbf{Top}]_0$ are called the *$\mathcal{P}$-spaces*. Let

$$\mathbb{F}_\ell^{\mathcal{P}^{op}} U = \mathcal{P}(-, \ell) \times U \in [\mathcal{P}^{op}, \mathbf{Top}]_0$$

where $U$ is a topological space and where $\ell > 0$.

2.10. **Proposition.** *[9, Proposition 5.3 and Proposition 5.5] The category $[\mathcal{P}^{op}, \mathbf{Top}]_0$ is a full reflective and coreflective subcategory of $\mathbf{Top}^{\mathcal{P}_0^{op}}$. For every $\mathcal{P}$-space $F : \mathcal{P}^{op} \to \mathbf{Top}$, every $\ell > 0$ and every topological space $X$, we have the natural bijection of sets*

$$[\mathcal{P}^{op}, \mathbf{Top}]_0(\mathbb{F}_\ell^{\mathcal{P}^{op}} X, F) \cong \mathbf{Top}(X, F(\ell)).$$

2.11. **Theorem.** *([9, Proposition 5.1] and [10, Theorem 5.14]) The category $[\mathcal{P}^{op}, \mathbf{Top}]_0$ is locally presentable. Let $D$ and $E$ be two $\mathcal{P}$-spaces. Let*

$$D \otimes E = \int^{(\ell_1, \ell_2)} \mathcal{P}(-, \ell_1 + \ell_2) \times D(\ell_1) \times E(\ell_2).$$

*The pair $([\mathcal{P}^{op}, \mathbf{Top}]_0, \otimes)$ has the structure of a biclosed semimonoidal category.*

2.12. **Definition.** [10, Definition 6.2] A *$\mathcal{P}$-flow*, also called a *Moore flow* if there is no ambiguity on the choice of $\mathcal{P}$, is a small semicategory enriched over the biclosed semimonoidal category $([\mathcal{P}^{op}, \mathbf{Top}]_0, \otimes)$ of Theorem 2.11. The corresponding category is denoted by $\mathcal{P}\mathbf{Flow}$.

A $\mathcal{P}$-flow $X$ consists of a *set of states* $X^0$, for each pair $(\alpha, \beta)$ of states a $\mathcal{P}$-space $\mathbb{P}_{\alpha,\beta} X$ of $[\mathcal{P}^{op}, \mathbf{Top}]_0$ and for each triple $(\alpha, \beta, \gamma)$ of states an associative composition law

$$* : \mathbb{P}_{\alpha,\beta} X \otimes \mathbb{P}_{\beta,\gamma} X \to \mathbb{P}_{\alpha,\gamma} X.$$

A map of $\mathcal{P}$-flows $f$ from $X$ to $Y$ consists of a set map

$$f^0 : X^0 \to Y^0$$

(often denoted by $f$ as well if there is no possible confusion) together for each pair of states $(\alpha, \beta)$ of $X$ with a natural transformation

$$\mathbb{P} f : \mathbb{P}_{\alpha,\beta} X \longrightarrow \mathbb{P}_{f(\alpha), f(\beta)} Y$$

compatible with the composition law. The topological space $\mathbb{P}_{\alpha,\beta} X(\ell)$ is denoted by $\mathbb{P}^\ell_{\alpha,\beta} X$ and is called the space of *execution paths of length $\ell$*.

2.13. **Definition.** Let $X$ be a $\mathcal{P}$-flow. The *$\mathcal{P}$-space of execution paths* $\mathbb{P}X$ of $X$ is by definition the $\mathcal{P}$-space

$$\mathbb{P}X = \coprod_{(\alpha,\beta) \in X^0 \times X^0} \mathbb{P}_{\alpha,\beta} X.$$

It yields a well-defined functor $\mathbb{P} : \mathcal{P}\mathbf{Flow} \to [\mathcal{P}^{op}, \mathbf{Top}]_0$. The image of $\ell$ is denoted by $\mathbb{P}^\ell$. We therefore have the equality

$$\mathbb{P}^\ell X = \coprod_{(\alpha,\beta) \in X^0 \times X^0} \mathbb{P}^\ell_{\alpha,\beta} X.$$

The category $\mathcal{P}\mathbf{Flow}$ is locally presentable by [10, Theorem 6.11].

2.14. **Notation.** Let $D : \mathcal{P}^{op} \to \mathbf{Top}$ be a $\mathcal{P}$-space. We denote by $\mathrm{Glob}(D)$ the Moore flow defined as follows:

$$\mathrm{Glob}(D)^0 = \{0, 1\}$$
$$\mathbb{P}_{0,0}\mathrm{Glob}(D) = \mathbb{P}_{1,1}\mathrm{Glob}(D) = \mathbb{P}_{1,0}\mathrm{Glob}(D) = \varnothing$$
$$\mathbb{P}_{0,1}\mathrm{Glob}(D) = D.$$

There is no composition law. This construction yields a functor

$$\mathrm{Glob} : [\mathcal{P}^{op}, \mathbf{Top}]_0 \to \mathcal{P}\mathbf{Flow}.$$

2.15. **Remark.** The notation $\mathrm{Glob}(D)$ is not ambiguous since $D$ is always a $\mathcal{P}$-space with $\mathcal{P}$ being $\mathcal{G}$ or $\mathcal{M}$ and $\mathrm{Glob}(D)$ is then necessarily either a $\mathcal{G}$-flow or an $\mathcal{M}$-flow respectively.

By [9, Theorem 6.2], the category of $\mathcal{P}$-spaces $[\mathcal{P}^{op}, \mathbf{Top}]_0$ can be endowed with the projective model structure associated with the model structures $\mathbf{Top}_q$. It is called the projective q-model structure. It is combinatorial. The fibrations are the objectwise q-fibrations. The weak equivalences are the objectwise weak homotopy equivalences. All $\mathcal{P}$-spaces are fibrant for this model structure. By [10, Theorem 8.8, Theorem 8.9 and Theorem 8.16], the category of $\mathcal{P}$-flows can be endowed with a combinatorial model structure characterized as follows: 1) a map of $\mathcal{P}$-flows $f : X \to Y$ is a weak equivalence if and only if $f^0 : X^0 \to Y^0$ is a bijection and $\mathbb{P}f : \mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),f(\beta)}Y$ is a weak equivalence of the projective q-model structure of $\mathcal{P}$-flows; 2) a map of $\mathcal{P}$-flows $f : X \to Y$ is a fibration if and only if $\mathbb{P}f : \mathbb{P}_{\alpha,\beta}X \to \mathbb{P}_{f(\alpha),f(\beta)}Y$ is a fibration of the projective q-model structure of $\mathcal{P}$-flows, i.e. an objectwise q-fibration of topological spaces. All $\mathcal{P}$-flows are q-fibrant.

## 3. Multipointed d-space

3.1. **Definition.** Let $\gamma_1$ and $\gamma_2$ be two continuous maps from $[0, 1]$ to some topological space such that $\gamma_1(1) = \gamma_2(0)$. The composite defined by

$$(\gamma_1 *_N \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

is called the *normalized composition*. The normalized composition being not associative, a notation like $\gamma_1 *_N \cdots *_N \gamma_n$ will mean, by convention, that $*_N$ is applied from the left to the right.

3.2. **Definition.** Let $U$ be a topological space. A *(Moore) path* of $U$ consists in this paper of a *nonconstant* continuous map $\gamma : [0, \ell] \to U$ with $\ell > 0$. The real number $\ell$ is called the *length* of $\gamma$.

Let $\gamma_1 : [0, \ell_1] \to U$ and $\gamma_2 : [0, \ell_2] \to U$ be two paths of a topological space $U$ such that $\gamma_1(\ell_1) = \gamma_2(0)$. The *Moore composition* $\gamma_1 * \gamma_2 : [0, \ell_1 + \ell_2] \to U$ is the Moore path defined by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } t \in [0, \ell_1] \\ \gamma_2(t - \ell_1) & \text{for } t \in [\ell_1, \ell_1 + \ell_2]. \end{cases}$$

The Moore composition of Moore paths is strictly associative.

3.3. **Notation.** Let $\ell > 0$. Let $\mu_\ell : [0,\ell] \to [0,1]$ be the homeomorphism defined by $\mu_\ell(t) = t/\ell$.

3.4. **Definition.** A *$\mathcal{P}$-multipointed d-space $X$* or just *multipointed d-space $X$* if there is no ambiguity on the choice of $\mathcal{P}$ is a triple $(|X|, X^0, \mathbb{P}^{top}X)$ where

- The pair $(|X|, X^0)$ is a multipointed space. The space $|X|$ is called the *underlying space* of $X$ and the set $X^0$ the *set of states* of $X$.
- The set $\mathbb{P}^{top}X$ is a set of continous maps from $[0,1]$ to $|X|$ called the *execution paths*, satisfying the following axioms:
  - For any execution path $\gamma$, one has $\gamma(0), \gamma(1) \in X^0$.
  - Let $\gamma$ be an execution path of $X$. Then any composite $\gamma\phi$ with $\phi \in \mathcal{P}([0,1],[0,1])$ is an execution path of $X$.
  - Let $\gamma_1$ and $\gamma_2$ be two composable execution paths of $X$; then the normalized composition $\gamma_1 *_N \gamma_2$ is an execution path of $X$.

A map $f : X \to Y$ of $\mathcal{P}$-multipointed $d$-spaces is a map of multipointed spaces from $(|X|, X^0)$ to $(|Y|, Y^0)$ such that for any execution path $\gamma$ of $X$, the map $\mathbb{P}^{top}f : \gamma \mapsto f.\gamma$ is an execution path of $Y$.

3.5. **Notation.** The category of $\mathcal{P}$-multipointed $d$-spaces is denoted by $\mathcal{P}$**dTop**.

The category $\mathcal{P}$**dTop** is locally presentable by [8, Theorem 3.5]: the proof is written for the case $\mathcal{P} = \mathcal{G}$; it consists of finding an axiomatization of the category of multipointed $d$-spaces by a relational universal strict Horn theory without equality which contains a small set of axioms; it suffices to replace in the third axiom page 597 "$t$ is a strictly increasing reparametrization", i.e. $t \in \mathcal{G}(1,1)$ by $t \in \mathcal{M}(1,1)$ to obtain the proof for the case $\mathcal{P} = \mathcal{M}$.

The subset of execution paths from $\alpha$ to $\beta$ is the set of $\gamma \in \mathbb{P}^{top}X$ such that $\gamma(0) = \alpha$ and $\gamma(1) = \beta$; it is denoted by $\mathbb{P}^{top}_{\alpha,\beta}X$: $\alpha$ is called the *initial state* and $\beta$ the *final state* of such a $\gamma$. The set $\mathbb{P}^{top}_{\alpha,\beta}X$ is equipped with the $\Delta$-kelleyfication of the relative topology induced by the inclusion $\mathbb{P}^{top}_{\alpha,\beta}X \subset \mathbf{TOP}([0,1],|X|)$. It induces a functor

$$\mathbb{P}^{top} : \mathcal{P}\mathbf{dTop} \longrightarrow \mathbf{Top}.$$

Unless specified, the set $\mathbb{P}^{top}_{\alpha,\beta}X$ is always equipped with this topology.

3.6. **Notation.** The mapping $\mathbb{P}^{top}f$ will be often denoted by $f$ if there is no ambiguity.

The following examples play an important role in the sequel.

(1) Any set $E$ will be identified with the $\mathcal{P}$-multipointed $d$-space $(E, E, \varnothing)$.
(2) The *topological globe of $Z$ of length $\ell > 0$*, which is denoted by $\mathrm{Glob}^{\mathcal{P}}(Z)$, is the $\mathcal{P}$-multipointed $d$-space defined as follows
   - the underlying topological space is the quotient space
$$\frac{\{0,1\} \sqcup (Z \times [0,1])}{(z,0) = (z',0) = 0, (z,1) = (z',1) = 1}$$
   - the set of states is $\{0,1\}$
   - the set of execution paths is the set of continuous maps
$$\{\delta_z\phi \mid \phi \in \mathcal{P}(1,1), z \in Z\}$$

with $\delta_z(t) = (z, t)$. It is equal to the underlying set of the space $\mathcal{P}(1,1) \times Z$.

In particular, $\mathrm{Glob}^{\mathcal{P}}(\varnothing)$ is the $\mathcal{P}$-multipointed $d$-space $\{0,1\} = (\{0,1\}, \{0,1\}, \varnothing)$.

(3) The *directed segment* is the $\mathcal{P}$-multipointed $d$-space $\overrightarrow{I}^{\mathcal{P}} = \mathrm{Glob}^{\mathcal{P}}(\{0\})$.

3.7. **Remark.** The terminology *$\mathcal{P}$-multipointed $d$-space* is chosen because it is a variant of the notion of Grandis' $d$-space. The adjective *$\mathcal{P}$-multipointed* contains the information about the allowed reparametrizations and the constraints on the extremities for the execution paths. The set of execution paths of length 1 is denoted by $\mathbb{P}^{top}X$ for a $\mathcal{P}$-multipointed $d$-space $X$, and not $\mathbb{P}^{\mathcal{P}}X$, because there is no ambiguity on what it is. On the contrary, the notation $\mathrm{Glob}^{\mathcal{P}}(Z)$ for a given topological space $Z$ is used to specify that $\mathrm{Glob}^{\mathcal{P}}(Z)$ is a $\mathcal{P}$-multipointed $d$-space. The $\mathcal{G}$-multipointed $d$-space $\mathrm{Glob}^{\mathcal{G}}(Z)$ is not equal to the $\mathcal{M}$-multipointed $d$-space $\mathrm{Glob}^{\mathcal{M}}(Z)$ indeed.

3.8. **Proposition.** *Let $X$ be a $\mathcal{P}$-multipointed $d$-space. Let $(\alpha, \beta) \in X^0 \times X^0$. The following data assemble into a $\mathcal{P}$-space denoted by $\mathbb{P}^{\bullet}_{\alpha,\beta}X$:*

- $\mathbb{P}^{\ell}_{\alpha,\beta}X = \{\gamma\mu_\ell \mid \gamma \in \mathbb{P}^{top}_{\alpha,\beta}X\}$
- *For $\phi : \ell' \to \ell \in \mathcal{P}$ and $\gamma \in \mathbb{P}^{\ell}_{\alpha,\beta}X$, $\mathbb{P}^{\bullet}_{\alpha,\beta}X(\phi) = \gamma\phi$.*

*Proof.* Let $\gamma \in \mathbb{P}^{\ell}_{\alpha,\beta}X$. Then, by definition of $\mathbb{P}^{\ell}_{\alpha,\beta}X$, there exists a (unique) $\overline{\gamma} \in \mathbb{P}^{top}_{\alpha,\beta}X$ such that $\gamma = \overline{\gamma}\mu_\ell$. We obtain $\gamma\phi = \overline{\gamma}(\mu_\ell\phi\mu_{\ell'}^{-1})\mu_{\ell'}$. Since $\mu_\ell\phi\mu_{\ell'}^{-1} \in \mathcal{P}(1,1)$, we have $\overline{\gamma}(\mu_\ell\phi\mu_{\ell'}^{-1}) \in \mathbb{P}^{top}_{\alpha,\beta}X$ and therefore $\gamma\phi \in \mathbb{P}^{\ell'}_{\alpha,\beta}X$. □

3.9. **Notation.** Let $X$ be a $\mathcal{P}$-multipointed $d$-space. Let $(\alpha, \beta) \in X^0 \times X^0$. Let

$$\mathbb{P}_{\alpha,\beta}X = \varinjlim \mathbb{P}^{\bullet}_{\alpha,\beta}X.$$

The topological space $\mathbb{P}_{\alpha,\beta}X = \varinjlim \mathbb{P}^{\bullet}_{\alpha,\beta}X$ is the quotient of the topological space

$$\coprod_{\ell>0} \mathbb{P}^{\ell}_{\alpha,\beta}X$$

by the equivalence relation generated by the identifications $\gamma \sim \gamma' \Leftrightarrow \gamma\phi = \gamma\phi'$ with $\gamma \in \mathbb{P}^{\ell}_{\alpha,\beta}X$, $\gamma' \in \mathbb{P}^{\ell'}_{\alpha,\beta}X$, $\phi \in \mathcal{P}(\ell'', \ell)$ and $\phi' \in \mathcal{P}(\ell'', \ell')$.

3.10. **Definition.** ([5, Definition 1.2]) The above equivalence relation $\sim$ is called the *reparametrization equivalence*. The two Moore paths $\gamma$ and $\gamma'$ above are said *reparametrization equivalent*.

3.11. **Proposition.** *For every $u, v \in \mathcal{P}(1, \ell)$, there exist $\phi_u, \phi_v \in \mathcal{P}(1,1)$ such that $u\phi_u = v\phi_v$.*

*Proof.* Let $u, v \in \mathcal{P}(1, \ell)$. Then $\mu_\ell u$ and $\mu_\ell v$ belong to $\mathcal{P}(1,1)$. By [5, Proposition 2.19], when $\mathcal{P} = \mathcal{M}$, there exist $\phi_u, \phi_v \in \mathcal{P}(1,1)$ such that $\mu_\ell u\phi_u = \mu_\ell v\phi_v$. When $\mathcal{P} = \mathcal{G}$, the same statement holds with $\phi_u = \mathrm{Id}_1$ and $\phi_v = v^{-1}u$. In both cases, it implies that $u\phi_u = v\phi_v$, $\mu_\ell$ being invertible. □

3.12. **Proposition.** *Denote by $\mathcal{P}^1$ the full subcategory of $\mathcal{P}^{op}$ generated by 1. Let $X$ be a $\mathcal{P}$-multipointed $d$-space. Let $(\alpha, \beta) \in X^0 \times X^0$. The inclusion $j : \mathcal{P}^1 \subset \mathcal{P}^{op}$ is final. Consequently, the topological space $\mathbb{P}_{\alpha,\beta}X$ is also the quotient of $\mathbb{P}^{top}_{\alpha,\beta}X$ by the reparametrization equivalence.*

*Proof.* Consider the comma category $(j\downarrow\ell)$ for a fixed $\ell > 0$. It is nonempty because $\mathcal{P}(1,\ell)$ is nonempty. Let $u, v \in \mathcal{P}(1,\ell)$. Using Proposition 3.11, write $u\phi_u = v\phi_v$ for some $\phi_u, \phi_v \in \mathcal{P}(1,1)$. The equality $u\phi_u = v\phi_v$ means that there is the commutative diagram of $\mathcal{P}$

$$\begin{array}{ccccc} 1 & \xleftarrow{\phi_u} & 1 & \xrightarrow{\phi_v} & 1 \\ {\scriptstyle u}\downarrow & & \downarrow & & \downarrow{\scriptstyle v} \\ \ell & = & \ell & = & \ell \end{array}$$

In other terms, the comma category $(j\downarrow\ell)$ is connected. It implies that the inclusion functor $\mathcal{P}^1 \subset \mathcal{P}^{op}$ is final in the sense of [22, Section IX.3]. The proof is complete thanks to [22, Theorem IX.3.1]. □

3.13. **Definition.** Let $X$ be a $\mathcal{P}$-multipointed $d$-space. The space $\mathbb{P}^\ell_{\alpha,\beta}X$ is called the space of *execution paths of length* $\ell$ from $\alpha$ to $\beta$. Let

$$\mathbb{P}^\ell X = \coprod_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}^\ell_{\alpha,\beta}X.$$

A map of multipointed $d$-spaces $f : X \to Y$ induces for each $\ell > 0$ a continuous map $\mathbb{P}^\ell f : \mathbb{P}^\ell X \to \mathbb{P}^\ell Y$ by composition by $f$ (in fact by $|f|$). The space $\mathbb{P}^1 X$ is also denoted by $\mathbb{P}^{top}X$.

3.14. **Proposition.** *Let $X$ be a $\mathcal{P}$-multipointed $d$-space. Let $\gamma_1$ and $\gamma_2$ be two execution paths of $X$ with $\gamma_1(1) = \gamma_2(0)$. Let $\ell_1, \ell_2 > 0$. Then $(\gamma_1\mu_{\ell_1} * \gamma_2\mu_{\ell_1})\mu^{-1}_{\ell_1+\ell_2}$ is an execution path of $X$.*

*Proof.* It is mutatis mutandis the proof of [11, Proposition 4.10] after observing that $\mathcal{G}(1,1) \subset \mathcal{P}(1,1)$. □

3.15. **Proposition.** *Let $X$ be a multipointed $d$-space. Let $\ell_1, \ell_2 > 0$. The Moore composition of continuous maps yields a continuous maps $\mathbb{P}^{\ell_1}X \times \mathbb{P}^{\ell_2}X \to \mathbb{P}^{\ell_1+\ell_2}X$.*

*Proof.* It is a consequence of Definition 3.13 and Proposition 3.14 □

3.16. **Theorem.** *The mapping $\Omega : X \mapsto (|X|, X^0)$ induces a functor from $\mathcal{P}$***dTop** *to the category* **MTop** *of multipointed spaces which is topological and fibre-small. The $\Omega$-final structure is the set of finite Moore compositions of the form $(f_1\gamma_1) * \cdots * (f_n\gamma_n)$ such that $\gamma_i \in \mathbb{P}^{\ell_i}X_i$ for all $1 \leqslant i \leqslant n$ with $\sum_i \ell_i = 1$.*

Note that Theorem 3.16 holds both by working with $\Delta$-generated spaces and with $\Delta$-Hausdorff $\Delta$-generated spaces.

*Proof.* The first statement is proved for $\mathcal{P} = \mathcal{G}$ in [13, Proposition 6.5] using a description of the $\Omega$-initial structure which works for $\mathcal{P} = \mathcal{M}$ as well. The proof of the last statement is similar to the proof of [11, Theorem 3.9]. □

3.17. **Proposition.** *([11, Proposition 2.12] for $\mathcal{G}$ and $\mathcal{M}$) Let $Z$ be a topological space of* **Top**. *Then there is the homeomorphism*

$$\mathbb{P}^{top}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z) \cong \mathcal{P}(1,1) \times Z.$$

Because of the possible presence of stop intervals (see Definition 4.10) in the case $\mathcal{P} = \mathcal{M}$, the proof of Proposition 3.17 slightly differs from the proof of [11, Proposition 2.12]: the latter is valid only for the case $\mathcal{P} = \mathcal{G}$.

*Proof.* The set map $\Psi : \mathcal{P}(1,1) \times Z \to \mathbb{P}^{top}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z)$ defined by $\Psi(\phi, z) = \delta_z\phi$ is continuous because the mapping $(t, \phi, z) \mapsto (z, \phi(t))$ from $[0,1] \times \mathcal{P}(1,1) \times Z$ to $|\mathrm{Glob}^{\mathcal{P}}(Z)|$ is continuous. It is a bijection since, by definition of $\mathrm{Glob}^{\mathcal{P}}(Z)$, the underlying set of $\mathbb{P}^{top}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z)$ is equal to the underlying set of the space $\mathcal{P}(1,1) \times Z$. Consider the composite set map

$$\pi : \mathbb{P}^{top}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z) \longrightarrow \mathcal{P}(1,1) \times Z \longrightarrow Z$$

which takes $\delta_z\phi$ to $\mathrm{pr}_2(\Psi^{-1}(\delta_z\phi)) = z$ ($\mathrm{pr}_2$ is the projection on the second factor). Suppose that $\pi$ is not continuous. All involved topological spaces being sequential, there exist $z_\infty \in Z$, an open neighborhood $V$ of $z_\infty$ in $Z$, and a sequence $(\delta_{z_n}\phi_n)_{n\geqslant 0}$ which converges to the execution path $\delta_{z_\infty}\phi_\infty$ such that $z_n \in Z\backslash V$ for all $n \geqslant 0$. Choose $t_0 \in [0,1]$ such that $\phi_\infty(t_0) \in ]0,1[$. The convergence for the compact-open topology implies the pointwise convergence. Thus the sequence $(\delta_{z_n}\phi_n(t_0))_{n\geqslant 0}$ of $|\mathrm{Glob}^{\mathcal{P}}(Z)|$ converges to $\delta_{z_\infty}\phi_\infty(t_0)$. It implies that there exists $N \geqslant 0$ such that for all $n \geqslant N$, $(z_n, \phi_n(t_0)) \in |\mathrm{Glob}^{\mathcal{P}}(Z)|\backslash\{0,1\}$. By considering the image by the continuous projection (the left-hand term being equipped with the relative topology)

$$|\mathrm{Glob}^{\mathcal{P}}(Z)|\backslash\{0,1\} \longrightarrow Z$$

which is well-defined precisely because 0 and 1 are removed, we obtain that the sequence $(z_n)_{n\geqslant N}$ converges to $z_\infty$, and therefore that $z_\infty \in Z\backslash V$, the latter set being closed in $Z$: contradiction. This means that $\pi$ is continuous. The continuous map $Z \to \{0\}$ induces a continuous map

$$\begin{cases} \mathbb{P}^{top}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z) \longrightarrow \mathbb{P}^{top}_{0,1}\mathrm{Glob}^{\mathcal{P}}(\{0\}) \cong \mathcal{P}(1,1) \\ \gamma \mapsto p.\gamma, \end{cases}$$

where $p : |\mathrm{Glob}^{\mathcal{P}}(Z)| \to [0,1]$ is the projection map. Therefore the set map

$$\begin{cases} \Psi^{-1} : \mathbb{P}^{top}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z) \longrightarrow \mathcal{P}(1,1) \times Z \\ \gamma \mapsto (p.\gamma, \pi(\gamma)) \end{cases}$$

is continuous and $\Psi$ is a homeomorphism. □

3.18. **Corollary.** *Let $Z$ be a topological space of* **Top**. *Then there is the homeomorphism*

$$\mathbb{P}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z) \cong Z.$$

*Proof.* There are the homeomorphisms

$$\varinjlim_{\mathcal{P}^1}\Big(\mathcal{P}(1,1) \times Z\Big) \cong \Big(\varinjlim_{\mathcal{P}^1}\mathcal{P}(1,1)\Big) \times Z \cong Z,$$

the left-hand homeomorphism since **Top** is cartesian closed and the right-hand homeomorphism by Proposition 3.11. The proof is complete thanks to Proposition 3.17 and Proposition 3.12. □

3.19. **Theorem.** *Let $X$ be a $\mathcal{P}$-multipointed d-space. The following data assemble into a $\mathcal{P}$-flow denoted by $\mathbb{M}^{\mathcal{P}}(X)$:*

- *The set of states $X^0$ of $X$*
- *For all $\alpha, \beta \in X^0$, $\mathbb{P}_{\alpha,\beta}\mathbb{M}^{\mathcal{P}}(X) = \mathbb{P}^{\bullet}_{\alpha,\beta}X$.*
- *For all $\alpha, \beta, \gamma \in X^0$ and all real numbers $\ell, \ell' > 0$, the composition map is given by the map $* : \mathbb{P}^{\bullet}_{\alpha,\beta}X \otimes \mathbb{P}^{\bullet}_{\beta,\gamma}X \to \mathbb{P}^{\bullet}_{\alpha,\gamma}X$.*

*The mapping above induces a functor $\mathbb{M}^{\mathcal{P}} : \mathcal{P}\mathbf{dTop} \to \mathcal{P}\mathbf{Flow}$ which is a right adjoint.*

*Proof.* It is possible to construct a left adjoint by following step by step the method of [11, Appendix B]: the fact that all maps of $\mathcal{G}$ are invertible does not play any role at all. □

3.20. **Notation.** The left adjoint of $\mathbb{M}^{\mathcal{P}} : \mathcal{P}\mathbf{dTop} \to \mathcal{P}\mathbf{Flow}$ is denoted by

$$\mathbb{M}^{\mathcal{P}}_{!} : \mathcal{P}\mathbf{Flow} \to \mathcal{P}\mathbf{dTop}.$$

3.21. **Proposition.** *Let $X$ be a $\mathcal{P}$-multipointed $d$-space. Let $\ell > 0$ be a real number. Let $Z$ be a topological space. Then there is a bijection of sets*

$$\mathcal{P}\mathbf{dTop}(\mathrm{Glob}^{\mathcal{P}}(Z), X) \cong \coprod_{(\alpha,\beta)\in X^0\times X^0} \mathbf{Top}(Z, \mathbb{P}^{top}_{\alpha,\beta}X)$$

*which is natural with respect to $Z$ and $X$.*

*Proof.* It is mutatis mutandis the proof of [11, Proposition 4.13]. □

3.22. **Proposition.** *For all compact topological spaces $Z$, there are the natural isomorphisms $\mathbb{M}^{\mathcal{P}}(\mathrm{Glob}^{\mathcal{P}}(Z)) \cong \mathrm{Glob}(\mathbb{F}_1^{\mathcal{P}^{op}}(Z))$ and $\mathbb{M}^{\mathcal{P}}_{!}(\mathrm{Glob}(\mathbb{F}_1^{\mathcal{P}^{op}}(Z))) \cong \mathrm{Glob}^{\mathcal{P}}(Z)$.*

*Proof.* By definition of $\mathbb{M}^{\mathcal{P}}$ and by Proposition 3.17, the only nonempty path $\mathcal{P}$-space of $\mathbb{M}^{\mathcal{P}}(\mathrm{Glob}^{\mathcal{P}}(Z))$ is $\mathbb{P}_{0,1}\mathbb{M}^{\mathcal{P}}(\mathrm{Glob}^{\mathcal{P}}(Z)) = \mathcal{P}(-,1) \times Z$: we obtain the first isomorphism. For any $\mathcal{P}$-multipointed $d$-space $X$, there is the sequence of natural bijections

$$\begin{aligned}
\mathcal{P}\mathbf{dTop}\Big(\mathbb{M}^{\mathcal{P}}_{!}(\mathrm{Glob}(\mathbb{F}_1^{\mathcal{P}^{op}}(Z))), X\Big) &\cong \mathcal{P}\mathbf{Flow}\Big(\mathrm{Glob}(\mathbb{F}_1^{\mathcal{P}^{op}}(Z)), \mathbb{M}^{\mathcal{P}}X\Big) \\
&\cong \coprod_{(\alpha,\beta)\in X^0\times X^0} [\mathcal{P}^{op}, \mathbf{Top}]_0\Big(\mathbb{F}_1^{\mathcal{P}^{op}}(Z), \mathbb{P}_{\alpha,\beta}X\Big) \\
&\cong \coprod_{(\alpha,\beta)\in X^0\times X^0} \mathbf{Top}(Z, \mathbb{P}^1_{\alpha,\beta}X) \\
&\cong \mathcal{P}\mathbf{dTop}(\mathrm{Glob}^{\mathcal{P}}(Z), X),
\end{aligned}$$

the first bijection by adjunction, the second bijection by [10, Proposition 6.10], the third bijection by Proposition 2.10 and the last bijection by Proposition 3.21. The proof of the second isomorphism is then complete thanks to the Yoneda lemma. □

## 4. Globular naturalization and carrier

4.1. **Notation.** Let $n \geqslant 1$. Denote by $\mathbf{D}^n = \{b \in \mathbb{R}^n, |b| \leqslant 1\}$ the $n$-dimensional disk, and by $\mathbf{S}^{n-1} = \{b \in \mathbb{R}^n, |b| = 1\}$ the $(n-1)$-dimensional sphere. By convention, let $\mathbf{D}^0 = \{0\}$ and $\mathbf{S}^{-1} = \varnothing$.

The *q-model structure* of $\mathcal{P}$-multipointed $d$-spaces is the unique combinatorial model structure such that

$$\{\mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^n) \mid n \geqslant 0\} \cup \{C : \varnothing \to \{0\}, R : \{0,1\} \to \{0\}\}$$

is the set of generating cofibrations, the maps between globes being induced by the closed inclusions $\mathbf{S}^{n-1} \subset \mathbf{D}^n$, and such that

$$\{\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n+1}) \mid n \geqslant 0\}$$

is the set of generating trivial cofibrations, the maps between globes being induced by the closed inclusions $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. The weak equivalences are the maps of multipointed $d$-spaces $f : X \to Y$ inducing a bijection $f^0 : X^0 \cong Y^0$ and a weak homotopy equivalence $\mathbb{P}^{top} f : \mathbb{P}^{top} X \to \mathbb{P}^{\mathcal{P}} Y$ and the fibrations are the maps of multipointed $d$-spaces $f : X \to Y$ inducing a q-fibration $\mathbb{P}^{top} f : \mathbb{P}^{top} X \to \mathbb{P}^{top} Y$ of topological spaces. A construction of this model structure is given in [13, Theorem 6.16] for $\mathcal{P} = \mathcal{G}$. The argument works in the same way for $\mathcal{P} = \mathcal{M}$ because it relies on the use of the Quillen path object argument in [13, Theorem 6.14] applied to the right adjoint from $\mathcal{P}$-multipointed $d$-spaces to topological graphs which forgets the composition and the reparametrization of execution paths.

The space of execution paths of the cellular $\mathcal{P}$-multipointed $d$-spaces of the q-model structure are of particular interest. It is the purpose of this section to study them. Let $\lambda$ be an ordinal. We work with a colimit-preserving functor

$$X : \lambda \longrightarrow \mathcal{P}\mathbf{dTop}$$

such that

- The $\mathcal{P}$-multipointed $d$-space $X_0$ is a set, in other terms $X_0 = (X^0, X^0, \varnothing)$ for some set $X^0$.
- For all $\nu < \lambda$, there is a pushout diagram of multipointed $d$-spaces

$$\begin{array}{ccc} \mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^{n_\nu - 1}) & \xrightarrow{g_\nu} & X_\nu \\ \downarrow & & \downarrow \\ \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_\nu}) & \xrightarrow{\widehat{g_\nu}} & X_{\nu+1} \end{array}$$

  with $n_\nu \geqslant 0$.

4.2. **Notation.** Let $X_\lambda = \varinjlim_{\nu < \lambda} X_\nu$ be the transfinite composition.

4.3. **Definition.** A *cellular $\mathcal{P}$-multipointed $d$-space* is a $\mathcal{P}$-multipointed $d$-space of the form $X_\lambda$.

As already noticed in [11, Proposition 5.2], the underlying space $|X_\lambda|$ is a cellular topological space. Proposition 4.4 is required to be able to use [5, 23] for the Moore paths in $|X_\lambda|$.

4.4. **Proposition.** *The topological space $|X_\lambda|$ is Hausdorff. Let $K$ be a compact subspace of $|X_\lambda|$. Then $K$ intersects finitely many $c_\nu$.*

*Proof.* The space $X_0$ is normal, being discrete. Adding one cell preserves normality by [7, Proposition 1.1.2 (ii)]. Assume that $\nu \leqslant \lambda$ is a limit ordinal and that each $X_\mu$ for $\mu < \nu$ is normal. We prove that $X_\nu$ is normal by an argument similar to the one of [7, Proposition A.5.1 (iv)]. Each $X_\mu$ for $\mu \leqslant \lambda$ is $\Delta$-Hausdorff by [12, Proposition B.16], and therefore has closed points. Hence $|X_\lambda|$ is Hausdorff. The last assertion is [11, Proposition 5.5] whose proof mimicks [18, Proposition A.1]. □

For all $\nu \leqslant \lambda$, there is the equality $X^0_\nu = X^0$. Denote by

$$c_\nu = |\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_\nu})| \setminus |\mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^{n_\nu - 1})|$$

the $\nu$-th cell of $X_\lambda$. It is called a *globular cell*. Like in the usual setting of CW-complexes, $\widehat{g_\nu}$ induces a homeomorphism from $c_\nu$ to $\widehat{g_\nu}(c_\nu)$ equipped with the relative topology which will be therefore denoted in the same way. It also means that $\widehat{g_\nu}(c_\nu)$ equipped with the relative topology is $\Delta$-generated. The closure of $c_\nu$ in $|X_\lambda|$ is denoted by

$$\widehat{c_\nu} = \widehat{g_\nu}(|\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_\nu})|).$$

It is a compact closed subset of $|X_\lambda|$. The boundary of $c_\nu$ in $|X_\lambda|$ is denoted by

$$\partial c_\nu = \widehat{g_\nu}(|\mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^{n_\nu - 1})|).$$

The state $\widehat{g_\nu}(0) \in X^0$ ($\widehat{g_\nu}(1) \in X^0$ resp.) is called the *initial (final resp.) state* of $c_\nu$. Note that they are not necessarily distinct. The integer $n_\nu + 1$ is called the *dimension* of the globular cell $c_\nu$. It is denoted by $\dim c_\nu$. The states of $X^0$ are also called the *globular cells of dimension* 0.

4.5. **Proposition.** *[11, Proposition 5.2] The space $|X_\lambda|$ is a cellular space. It contains $X^0$ as a discrete closed subspace. For every $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \lambda$, the continuous map $|X_{\nu_1}| \to |X_{\nu_2}|$ is a q-cofibration of spaces, and in particular a closed $T_1$-inclusion.*

4.6. **Proposition.** *[11, Proposition 5.3] For all $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \lambda$, there is the equality*

$$\mathbb{P}^{top} X_{\nu_1} = \mathbb{P}^{top} X_{\nu_2} \cap \mathbf{TOP}([0,1], |X_{\nu_1}|).$$

4.7. **Theorem.** *The composite functor*

$$\lambda \xrightarrow{X} \mathcal{P}\mathbf{dTop} \xrightarrow{\mathbb{P}^{top}} \mathbf{Top}$$

*is colimit-preserving. In particular the continuous bijection*

$$\varinjlim (\mathbb{P}^{top}.X) \longrightarrow \mathbb{P}^{top} \varinjlim X$$

*is a homeomorphism. Moreover the topology of $\mathbb{P}^{top} \varinjlim X$ is the final topology.*

*Proof.* It is mutatis mutandis the proof of [11, Theorem 5.6] which relies on Proposition 4.6. □

4.8. **Theorem.** *The composite functor*

$$\lambda \xrightarrow{X} \mathcal{P}\mathbf{dTop} \xrightarrow{\mathbb{M}^{\mathcal{P}}} \mathcal{P}\mathbf{Flow}$$

*is colimit-preserving. In particular the natural map*

$$\varinjlim_{\nu < \lambda} \mathbb{M}^{\mathcal{P}}(X_\nu) \longrightarrow \mathbb{M}^{\mathcal{P}} X_\lambda$$

*is an isomorphism.*

*Proof.* The proof is mutatis mutandis the proof of [11, Theorem 5.7] whose proof relies on Proposition 4.5 and Theorem 4.7. □

4.9. **Notation.** Let $c_\nu$ be a globular cell of $X_\lambda$. For all $z \in \mathbf{D}^{n_\nu}$ and all $\phi \in \mathcal{P}(1,1)$, the composite $\widehat{g_\nu}\delta_z\phi$ is an execution path of $X_\lambda$. When there is no ambiguity on the globular cell $c_\nu$ or no need to mention it, the execution path $\widehat{g_\nu}\delta_z\phi$ of $X_\lambda$ will be denoted by $\delta_z\phi$

(which is strictly speaking an abuse of notations already made in [11] to not overload the notations).

4.10. **Definition.** [5, Definition 1.1] Let $\gamma : [0, L] \to |X_\lambda|$ be a Moore path. A *stop interval* of $\gamma$ is an interval $[a, b] \subset [0, L]$ with $a < b$ such that the restriction $\gamma \restriction_{[a,b]}$ is constant and such that $[a, b]$ is maximal for this property. The set of stop intervals of $\gamma$ is denoted by $\Delta_\gamma$. The path $\gamma$ is *regular* if $\Delta_\gamma = \varnothing$ (no stop interval) [1]. The Moore composition of two regular paths is regular.

The set $\Delta_\gamma$ contains only closed intervals since $|X_\lambda|$ is Hausdorff, Note that in the case $\mathcal{P} = \mathcal{G}$, all execution paths of $X_\lambda$ are regular by [11, Proposition 5.13]. In the case $\mathcal{P} = \mathcal{M}$, an execution path of the form $\widehat{g_\nu}\delta_z\phi$ with $\phi \in \mathcal{M}(1,1)\backslash\mathcal{G}(1,1)$ is not regular. Proposition 4.11 and Proposition 4.12 play a key role in the sequel.

4.11. **Proposition.** *Consider a globular cell $c_\nu$ of $X_\lambda$. Let $z \in \mathbf{D}^{n_\nu}\backslash\mathbf{S}^{n_\nu-1}$. The execution path $\widehat{g_\nu}\delta_z$ is regular.*

*Proof.* It is a consequence of the facts that $\widehat{g_\nu}$ induces a homeomorphism from $c_\nu$ to $\widehat{g_\nu}(c_\nu)$ and that $\delta_z$ is a regular execution path of $\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_\nu})$. □

4.12. **Proposition.** *Let $\gamma : [0, L] \to |X_\lambda|$ be a Moore path. Suppose that $\gamma$ is regular and that there exist $\eta_1, \eta_2 \in \mathcal{P}(\ell, L)$ such that $\gamma\eta_1 = \gamma\eta_2$. Then $\eta_1 = \eta_2$.*

*Proof.* Note that $\eta_1, \eta_2 \in \mathcal{P}(\ell, L) \subset \mathcal{M}(\ell, L)$. From $\gamma\eta_1 = \gamma\eta_2$, we obtain $\gamma\mu_L^{-1}\mu_L\eta_1\mu_\ell^{-1} = \gamma\mu_L^{-1}\mu_L\eta_2\mu_\ell^{-1}$. The Moore path $\gamma\mu_L^{-1} : [0, 1] \to |X_\lambda|$ is regular, $\mu_L^{-1}$ being a homeomorphism. Using [5, Lemma 3.9], we deduce that $\mu_L\eta_1\mu_\ell^{-1} = \mu_L\eta_2\mu_\ell^{-1}$ and therefore that $\eta_1 = \eta_2$. □

4.13. **Theorem.** *Let $\gamma$ be an execution path of $X_\lambda$. It can be decomposed as a Moore composition*

$$\gamma = (\widehat{g_{\nu_1}}\delta_{z_1}\phi_1\mu_{\ell_1}) * \cdots * (\widehat{g_{\nu_n}}\delta_{z_n}\phi_n\mu_{\ell_n})$$

*with $n \geqslant 1$, $\nu_i < \lambda$ and $z_i \in \mathbf{D}^{n_{\nu_i}}\backslash\mathbf{S}^{n_{\nu_i}-1}$ and $\phi_i \in \mathcal{P}(1,1)$ for all $i \in \{1, \ldots, n\}$ and $\ell_1 + \cdots + \ell_n = 1$. Consider a second decomposition*

$$\gamma = (\widehat{g_{\nu'_1}}\delta_{z'_1}\phi'_1\mu_{\ell'_1}) * \cdots * (\widehat{g_{\nu'_{n'}}}\delta_{z'_{n'}}\phi'_n\mu_{\ell'_{n'}}).$$

*Then $n = n'$, $\nu_i = \nu'_i$ and $z_i = z'_i$ for all $1 \leqslant i \leqslant n$ and*

$$(\phi_1\mu_{\ell_1}) \otimes \ldots \otimes (\phi_n\mu_{\ell_n}) = (\phi'_1\mu_{\ell'_1}) \otimes \ldots \otimes (\phi'_n\mu_{\ell'_n}) \in \mathcal{P}(1, n).$$

*Proof.* By Theorem 3.16, every execution path $\gamma$ from $\alpha$ to $\beta$ of $X_\lambda$ is of the form a Moore composition $\gamma = (\widehat{g_{\nu_1}}\delta_{z_1}\phi_1\mu_{\ell_1}) * \cdots * (\widehat{g_{\nu_n}}\delta_{z_n}\phi_n\mu_{\ell_n})$ with $n \geqslant 1$, $\nu_i < \lambda$ and $z_i \in \mathbf{D}^{n_{\nu_i}}\backslash\mathbf{S}^{n_{\nu_i}-1}$ and $\phi_i \in \mathcal{P}(1,1)$ for all $i \in \{1, \ldots, n\}$ and $\ell_1 + \cdots + \ell_n = 1$. Consider a second decomposition $\gamma = (\widehat{g_{\nu'_1}}\delta_{z'_1}\phi'_1\mu_{\ell'_1}) * \cdots * (\widehat{g_{\nu'_{n'}}}\delta_{z'_{n'}}\phi'_n\mu_{\ell'_{n'}})$. Then $\gamma$ is the Moore composition of a Moore path going from $\widehat{g_{\nu_1}}(0)$ to $\widehat{g_{\nu_1}}(1)$ in the globular cell $c_{\nu_1}$ followed by a Moore path going from $\widehat{g_{\nu_2}}(0)$ to $\widehat{g_{\nu_2}}(1)$ in the globular cell $c_{\nu_2}$ etc... until a Moore path going from $\widehat{g_{\nu_n}}(0)$ to $\widehat{g_{\nu_n}}(1)$ in the globular cell $c_{\nu_n}$. And $\gamma$ is also the Moore composition of a Moore path going from $\widehat{g_{\nu'_1}}(0)$ to $\widehat{g_{\nu'_1}}(1)$ in the globular cell $c_{\nu'_1}$ followed by a Moore

[1] Remember that by Definition 3.2, all Moore paths of this paper are nonconstant.

path going from $\widehat{g_{\nu'_2}}(0)$ to $\widehat{g_{\nu'_2}}(1)$ in the globular cell $c_{\nu'_2}$ etc... until a Moore path going from $\widehat{g_{\nu'_{n'}}}(0)$ to $\widehat{g_{\nu'_{n'}}}(1)$ in the globular cell $c_{\nu'_{n'}}$. From the set bijection

$$|X_\lambda| = X^0 \sqcup \coprod_{\nu<\lambda} c_\nu,$$

we deduce that $n = n'$, $\nu_i = \nu'_i$ and $z_i = z'_i$ for $1 \leqslant i \leqslant n$. By [11, Proposition 3.4], we also have

$$\Big((\widehat{g_{\nu_1}}\delta_{z_1}) * \cdots * (\widehat{g_{\nu_n}}\delta_{z_n})\Big)\Big((\phi_1\mu_{\ell_1}) \otimes \ldots \otimes (\phi_n\mu_{\ell_n})\Big)$$
$$= \Big((\widehat{g_{\nu'_1}}\delta_{z'_1}) * \cdots * (\widehat{g_{\nu'_{n'}}}\delta_{z'_{n'}})\Big)\Big((\phi'_1\mu_{\ell'_1}) \otimes \ldots \otimes (\phi'_n\mu_{\ell'_n})\Big).$$

Since $(\widehat{g_{\nu_1}}\delta_{z_1})*\cdots*(\widehat{g_{\nu_n}}\delta_{z_n}) = (\widehat{g_{\nu'_1}}\delta_{z'_1})*\cdots*(\widehat{g_{\nu'_{n'}}}\delta_{z'_{n'}})$ is a regular Moore path, being a Moore composition of regular Moore paths by Proposition 4.11, we deduce by Proposition 4.12 the equality

$$(\phi_1\mu_{\ell_1}) \otimes \ldots \otimes (\phi_n\mu_{\ell_n}) = (\phi'_1\mu_{\ell'_1}) \otimes \ldots \otimes (\phi'_n\mu_{\ell'_n}).$$

□

Theorem 4.13 is a modification of [11, Theorem 5.9] which is valid both for $\mathcal{G}$ and $\mathcal{M}$. Let $\Psi = (\phi_1\mu_{\ell_1}) \otimes \ldots \otimes (\phi_n\mu_{\ell_n}) = (\phi'_1\mu_{\ell'_1}) \otimes \ldots \otimes (\phi'_n\mu_{\ell'_n})$. If $\mathcal{P} = \mathcal{G}$, then $\ell_1 + \cdots + \ell_i = \Psi^{-1}(i)$ for $1 \leqslant i \leqslant n$. It implies that $\ell_i = \ell'_i$ for $1 \leqslant i \leqslant n$, and by Proposition 2.8, we deduce that $\phi_i = \phi'_i$ for $1 \leqslant i \leqslant n$, which implies [11, Theorem 5.9].

4.14. **Definition.** With the notations of Theorem 4.13. The regular Moore path

$$\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma) = (\widehat{g_{\nu_1}}\delta_{z_1}) * \cdots * (\widehat{g_{\nu_n}}\delta_{z_n})$$

is called the *(globular) naturalization* of $\gamma$. The sequence of globular cells

$$\mathrm{Carrier}(\gamma) = [c_{\nu_1}, \ldots, c_{\nu_n}]$$

is called the *carrier* of $\gamma$. The integer $n$ is called the *length* of the carrier. It is also called the *natural length* of $\gamma$.

The globular naturalization of the Moore composition of two execution paths is the Moore composition of the globular naturalizations.

4.15. **Proposition.** *Let $\gamma$ be an execution path of $X_\lambda$ of natural length $n$. Then the regular Moore path $\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma)$ is an execution path of $X_\lambda$ of length $n$.*

*Proof.* It is a consequence of Proposition 3.15. □

4.16. **Definition.** An execution path $\gamma$ of $X_\lambda$ is minimal [2] if $\gamma = \widehat{g_\nu}\delta_z\phi$ for some $\nu < \lambda$, some $z \in \mathbf{D}^{n_\nu}\backslash\mathbf{S}^{n_\nu-1}$ and some $\phi \in \mathcal{P}(1,1)$.

4.17. **Theorem.** *Let $0 \leqslant \nu < \lambda$. Then every execution path of $X_{\nu+1}$ can be written as a finite Moore composition*

$$(f_1\gamma_1\mu_{\ell_1}) * \cdots * (f_n\gamma_n\mu_{\ell_n})$$

*with $n \geqslant 1$ such that*

[2] It is not exactly the definition chosen in [11]. This one makes sense only for cellular $\mathcal{P}$-multipointed $d$-spaces.

*(1)* $\sum_i \ell_i = 1$.
*(2)* $f_i = f$ *and* $\gamma_i$ *is an execution path of* $X_\nu$ *or* $f_i = \widehat{g_\nu}$ *and* $\gamma_i = \delta_{z_i}\phi_i$ *with* $z_i \in \mathbf{D}^{n_\nu}\backslash\mathbf{S}^{n_\nu-1}$ *and some* $\phi_i \in \mathcal{P}(1,1)$.
*(3) for all* $1 \leqslant i < n$, *either* $f_i\gamma_i$ *or* $f_{i+1}\gamma_{i+1}$ *(or both) is (are) of the form* $\widehat{g_\nu}\delta_z\phi$ *for some* $z \in \mathbf{D}^{n_\nu}\backslash\mathbf{S}^{n_\nu-1}$ *and some* $\phi \in \mathcal{P}(1,1)$*: intuitively, there is no possible simplification using the Moore composition inside* $X_\nu$.

*If there is another finite Moore composition*

$$(f'_1\gamma'_1\mu_{\ell'_1}) * \cdots * (f'_n\gamma'_n\mu_{\ell'_{n'}})$$

*with* $n' \geqslant 1$ *satisfying the same properties, then* $n = n'$, *for all* $i \in \{1,\dots,n\}$ *one has* $f_i = f'_i$, $r_i = \underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma_i) = \underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma'_i)$ *and finally*

$$(\phi_1\mu_{\ell_1}) \otimes \ldots \otimes (\phi_n\mu_{\ell_n}) = (\phi'_1\mu_{\ell'_1}) \otimes \ldots \otimes (\phi'_n\mu_{\ell'_n})$$

*with* $\gamma_i = r_i\phi_i$ *and* $\gamma'_i = r_i\phi'_i$ *for all* $i \in \{1,\dots,n\}$.

*Proof.* The existence of the finite Moore composition is a consequence of Theorem 3.16. Let $i \in \{1,\dots,n\}$. If $f_i = f$, then $f_i$ is one-to-one by Proposition 4.5. Thus, the execution path $f\,\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma_i)$ is regular. Besides, $f\,\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma_i)$ is in this case the regular naturalization of $f\gamma_i$. If $f_i = \widehat{g_\nu}$ and $\gamma_i = \delta_{z_i}\phi_i$, then $f_i\delta_{z_i}$ is regular by Proposition 4.11. And by definition of the globular naturalization, $f_i\delta_{z_i} = \underline{\mathrm{nat}}^{\mathrm{gl}}(f_i\delta_{z_i})$. Therefore we obtain

$$(f_1\,\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma_1)) * \cdots * (f_n\,\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma_n)) = (f'_1\,\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma'_1)) * \cdots * (f'_n\,\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma'_n)) \in \mathbb{P}^n X_\lambda.$$

By definition of the Moore composition, it implies that $\underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma_i) = \underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma'_i)$ for all $i \in \{1,\dots,n\}$. The last equality is a consequence of Proposition 4.12. □

Theorem 4.17 is a replacement for [11, Theorem 5.20] which is valid both for $\mathcal{G}$ and $\mathcal{M}$. Let $\Psi = (\phi_1\mu_{\ell_1}) \otimes \ldots \otimes (\phi_n\mu_{\ell_n}) = (\phi'_1\mu_{\ell'_1}) \otimes \ldots \otimes (\phi'_n\mu_{\ell'_n})$. If $\mathcal{P} = \mathcal{G}$, then $\ell_1 + \cdots + \ell_i = \Psi^{-1}(i)$ for $1 \leqslant i \leqslant n$. It implies that $\ell_i = \ell'_i$ for $1 \leqslant i \leqslant n$, and by Proposition 2.8, we deduce that $\phi_i = \phi'_i$ for $1 \leqslant i \leqslant n$, which implies [11, Theorem 5.20].

4.18. **Definition.** Let $c_\nu$ be a globular cell of the cellular $\mathcal{P}$-multipointed $d$-space $X_\lambda$ with $\nu < \lambda$ and $\dim(c_\nu) \geqslant 1$. Let $0 < h < 1$. Let

$$\widehat{c_\nu}[h] = \left\{ \widehat{g_\nu}(z,h) \mid (z,h) \in |\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_\nu})| \right\}.$$

It is called an *achronal slice* of the globular cell $c_\nu$.

[11, Proposition 5.17] claims that, for any globular cell $c_\nu$ of any cellular $\mathcal{G}$-multipointed $d$-space $X_\lambda$ with $\dim(c_\nu) \geqslant 1$, there exists $b \in ]0,1[$ such that for all $h \in ]0,b]$, one has $\widehat{c_\nu}[h] \cap X^0 = \varnothing$. It implies that there exists $h \in ]0,1[$ such that $\widehat{c_\nu}[h] \cap X^0 = \varnothing$. In plain English, this means that there is an achronal slice of the globular cell $c_\nu$ which does not intersect $X^0$. It is the key fact to prove [11, Theorem 5.18], and then to deduce [11, Theorem 5.19]. Proposition 4.19 proves that [11, Proposition 5.17] is false for $\mathcal{P} = \mathcal{M}$.

4.19. **Proposition.** *There exists a cellular* $\mathcal{M}$*-multipointed* $d$*-space* $X_\lambda$ *and a globular cell* $c_\nu$ *with* $\nu < \lambda$ *and* $\dim(c_\nu) \geqslant 1$ *such that for all* $h \in ]0,1[$, $\widehat{c_\nu}[h] \cap X^0 \neq \varnothing$.

*Proof.* Consider the continuous map $\Psi : [-1,1] \times [0,1] \to [0,1]$ defined by

$$\Psi : (x,t) \mapsto \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \frac{2+x}{4} \\ 4t-2-x & \text{if } \frac{2+x}{4} \leqslant t \leqslant \frac{3+x}{4} \\ 1 & \text{if } \frac{3+x}{4} \leqslant t \leqslant 1 \end{cases}$$

The continuous map

$$f : ((x,y),t) \mapsto ((x,y),\Psi(x,t))$$

from $\mathbf{S}^1 \times [0,1]$ to $|\mathrm{Glob}^{\mathcal{M}}(\mathbf{S}^1)|$ induces a continuous map from $|\mathrm{Glob}^{\mathcal{M}}(\mathbf{S}^1)|$ to itself since $f((x,y),0) = ((x,y),\Psi(x,0)) = 0$ and $f((x,y),1) = ((x,y),\Psi(x,1)) = 1$ for all $(x,y) \in \mathbf{S}^1$ by definition of $\Psi$. An execution path of the form $\delta_{(x,y)}\phi$ with $\phi \in \mathcal{M}(1,1)$ is taken by this continuous map to the continuous path $\delta_{(x,y)}\Psi(x,\phi(t))$. The latter is an execution path of $|\mathrm{Glob}^{\mathcal{M}}(\mathbf{S}^1)|$ since $\Psi(x,-) \in \mathcal{M}(1,1)$ for all $x \in [-1,1]$. Consequently, we obtain a map of $\mathcal{M}$-multipointed $d$-spaces $f : \mathrm{Glob}^{\mathcal{M}}(\mathbf{S}^1) \to \mathrm{Glob}^{\mathcal{M}}(\mathbf{S}^1)$. Consider the $\mathcal{M}$-multipointed $d$-space $X$ defined by the pushout diagram of $\mathcal{M}$-multipointed $d$-spaces

$$\begin{array}{ccc} \mathrm{Glob}^{\mathcal{M}}(\mathbf{S}^1) & \xrightarrow{f} & \mathrm{Glob}^{\mathcal{M}}(\mathbf{S}^1) \\ \downarrow & & \downarrow \\ \mathrm{Glob}^{\mathcal{M}}(\mathbf{D}^2) & \xrightarrow{\widehat{f}} & X \end{array}$$

where the left vertical map is induced by the inclusion $\mathbf{S}^1 \subset \mathbf{D}^2$. The $\mathcal{M}$-multipointed space $X$ is cellular, $\mathrm{Glob}^{\mathcal{M}}(\mathbf{S}^1)$ being cellular. Let $h \in ]0,1[$. Consider the achronal slice

$$\widehat{f}[h] = \{\widehat{f}(z,h) \mid (z,h) \in |\mathrm{Glob}^{\mathcal{M}}(\mathbf{D}^2)|\}.$$

One has

$$\widehat{f}((0,1),h) = f((0,1),h) = ((0,1),\Psi(0,h)),$$

the first equality because the square above is commutative and the second equality by definition of $f$. This implies that $\widehat{f}((0,1),h) = 0$ when $h \leqslant 1/2$ by definition of $\Psi$. Similarly, there are the equalities

$$\widehat{f}((-1,0),h) = f((-1,0),h) = ((-1,0),\Psi(-1,h)).$$

This implies that $\widehat{f}((-1,0),h) = 1$ when $h \geqslant 1/2$ by definition of $\Psi$.

We deduce that for all $h \in ]0,1[$, $\widehat{f}[h] \cap Z^0 \neq \varnothing$. □

In this paper, we prove Theorem 4.21 instead. It enables us to deduce both [11, Theorem 5.18] and [11, Theorem 5.19] in a different way for the two reparametrization categories $\mathcal{G}$ and $\mathcal{M}$.

4.20. **Notation.** Let $(\alpha,\beta) \in X^0 \times X^0$. Denote by $(\mathbb{P}^{top}_{\alpha,\beta}X_\lambda)_{co}$ the set $\mathbb{P}^{top}_{\alpha,\beta}X_\lambda$ equipped with the compact-open topology.

4.21. **Theorem.** *(replacement for [11, Proposition 5.17]) Let $(\alpha,\beta) \in X^0 \times X^0$. Let $(\gamma_n)_{n\geqslant 0}$ be a sequence of $(\mathbb{P}^{top}_{\alpha,\beta}X_\lambda)_{co}$ which converges to $\gamma_\infty$. Then the set $\{\mathrm{Carrier}(\gamma_n) \mid n \geqslant 0\}$ is finite.*

*Proof.* Consider the one-point compactification $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ of the discrete space of integers $\mathbb{N}$. Note that $\overline{\mathbb{N}}$ is not $\Delta$-generated, its $\Delta$-kelleyfication being discrete. The

converging sequence $(\gamma_n)_{n\geqslant 0}$ gives rise to a continuous map

$$\psi : \overline{\mathbb{N}} \longrightarrow (\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co} \subset \mathbf{TOP}_{co}([0,1], |X_\lambda|)$$

where $\mathbf{TOP}_{co}([0,1], |X_\lambda|)$ is the set of continuous maps from $[0,1]$ to $|X_\lambda|$ equipped with the compact-open topology. Since $[0,1]$ is locally compact, it is exponential in the category of *general* topological spaces by [2, Proposition 7.1.5]. We obtain a continuous map

$$\widehat{\psi} : \overline{\mathbb{N}} \times_{gen} [0,1] \longrightarrow |X_\lambda|$$

where $\times_{gen}$ is the binary product in the category of general topological spaces. Since $\overline{\mathbb{N}} \times_{gen} [0,1]$ is compact by Tychonoff, the subset $\widehat{\psi}(\overline{\mathbb{N}} \times_{gen} [0,1])$ is compact and closed in $|X_\lambda|$, the latter being Hausdorff by Proposition 4.4. The subset $\widehat{\psi}(\overline{\mathbb{N}} \times_{gen} [0,1])$ therefore intersects finitely many globular cells $\{c_{\nu_j} \mid j \in J\}$ by Proposition 4.4. Suppose that the set $\{\text{Carrier}(\gamma_n) \mid n \geqslant 0\}$ is infinite. It implies that the sequence of lengths of $\text{Carrier}(\gamma_n)$ for $n \geqslant 0$ is not bounded, the set $J$ being finite. By extracting a subsequence, one can suppose that the sequence of lengths is strictly increasing. Each infinite sequence of $\{c_{\nu_j} \mid j \in J\}$ has a constant infinite subsequence since $J$ is finite. Therefore by a Cantor diagonalization argument, one can suppose that there exists a sequence $(c_{\nu_{j_n}})_{n\geqslant 0}$ of $\{c_{\nu_j} \mid j \in J\}$ such that for all $n \geqslant 0$, there is the equality

$$\text{Carrier}(\gamma_n) = [c_{\nu_{j_0}}, \ldots, c_{\nu_{j_{i_n}}}]$$

for some strictly increasing sequence of integers $(i_n)_{n\geqslant 0}$. This means that the execution path $\gamma_n$ is the composition of an execution path whose image is included in $\{\widehat{g_{\nu_{j_0}}}(0), \widehat{g_{\nu_{j_0}}}(1)\} \cup c_{\nu_{j_0}}$ from $\widehat{g_{\nu_{j_0}}}(0)$ to $\widehat{g_{\nu_{j_0}}}(1)$ followed by an execution path whose image is included in $\{\widehat{g_{\nu_{j_1}}}(0) \cup \widehat{g_{\nu_{j_1}}}(1)\} \cup c_{\nu_{j_1}}$ from $\widehat{g_{\nu_{j_0}}}(1) = \widehat{g_{\nu_{j_1}}}(0)$ to $\widehat{g_{\nu_{j_1}}}(1)$ etc... until an execution path whose image is included in $\{\widehat{g_{\nu_{j_{i_n}}}}(0) \cup \widehat{g_{\nu_{j_{i_n}}}}(1)\} \cup c_{\nu_{j_{i_n}}}$. The sequence of execution paths $(\gamma_n)_{n\geqslant 0}$ converges pointwise to $\gamma_\infty$ because it converges for the compact-open topology by hypothesis. The subset $\widehat{c_\nu}$ being closed in $|X_\lambda|$ for all $\nu < \lambda$, the sequence of globular cells $\text{Carrier}(\gamma_\infty)$ consists of a concatenation of sequences $S_{\nu_{j_n}}$ for $n \geqslant 0$ where either $S_{\nu_{j_n}} = [c_{\nu_{j_n}}]$ or $S_{\nu_{j_n}}$ is a nonempty finite sequence of globular cells intersecting $\partial c_{\nu_{j_n}}$. It implies that the sequence of globular cells $\text{Carrier}(\gamma_\infty)$ is infinite: contradiction. □

4.22. **Corollary.** *([11, Theorem 5.18] for $\mathcal{G}$ and $\mathcal{M}$) Let $\gamma_\infty$ be an execution path of $X_\lambda$. Let $\nu_0 < \lambda$. There exists an open neighborhood $\Omega$ of $\gamma_\infty$ in $\mathbb{P}^{top} X_\lambda$ such that for all execution paths $\gamma \in \Omega$, the number of copies of $c_{\nu_0}$ in the carrier of $\gamma$ cannot exceed the length of the carrier of $\gamma_\infty$.*

*Proof.* Let $\Omega_{\nu_0}$ be the set of execution paths $\gamma$ such that the number of copies of $c_{\nu_0}$ in the carrier of $\gamma$ does not exceed the length of the carrier of $\gamma_\infty$. Suppose that $\gamma_\infty$ is not in the interior of $\Omega_{\nu_0}$. Since $\mathbb{P}^{top} X_\lambda$ is sequential, being $\Delta$-generated, there exists a sequence $(\gamma_n)_{n\geqslant 0}$ of the complement of $\Omega_{\nu_0}$ converging to $\gamma_\infty$. By Theorem 4.21, the set $\{\text{Carrier}(\gamma_n) \mid n \geqslant 0\}$ is finite. Thus by extracting a subsequence, one can suppose that the sequence of carriers $(\text{Carrier}(\gamma_n))_{n\geqslant 0}$ is constant, write

$$\text{Carrier}(\gamma_n) = [c_{\nu_1}, \ldots, c_{\nu_N}]$$

for all $n \geqslant 0$. The integer $N$ is strictly greater than the length of Carrier$(\gamma_\infty)$ since $[c_{\nu_1}, \ldots, c_{\nu_N}]$ contains strictly more copies of $c_{\nu_0}$ than the length of Carrier$(\gamma_\infty)$ by definition of $\Omega_{\nu_0}$. The sequence $(\gamma_n)_{n\geqslant 0}$ converges also pointwise to $\gamma_\infty$. Thus, Carrier$(\gamma_\infty)$ consists of a concatenation of sequences $S_{\nu_n}$ for $1 \leqslant n \leqslant N$ where either $S_{\nu_n} = [c_{\nu_n}]$ or $S_{\nu_n}$ is a nonempty finite sequence of globular cells intersecting $\partial c_{\nu_n}$. This implies that the length of Carrier$(\gamma_\infty)$ is strictly greater than itself: contradiction. We deduce that $\gamma_\infty$ is in the interior of $\Omega_{\nu_0}$. Hence the existence of the open neighborhood. □

Corollary 4.22 proves the existence of an open neighborhood $\Omega$ in the $\Delta$-kelleyfication of the compact-open topology. The latter topology contains more open subsets than the compact-open topology. The proof of [11, Theorem 5.18] implies that $\Omega$ can even be taken in the compact-open topology when $\mathcal{P} = \mathcal{G}$.

4.23. **Corollary.** *([11, Theorem 5.19] for $\mathcal{G}$ and $\mathcal{M}$) Let $(\gamma_k)_{k\geqslant 0}$ be a sequence of execution paths of $X_\lambda$ which converges in $\mathbb{P}^{\mathcal{G}}X_\lambda$. Let $c_{\nu_0}$ be a globular cell of $X_\lambda$. Let $i_k$ be the number of times that $c_{\nu_0}$ appears in* Carrier$(\gamma_k)$*. Then the sequence of integers $(i_k)_{k\geqslant 0}$ is bounded.*

*Proof.* The sequence $(\gamma_k)_{k\geqslant 0}$ converges in $(\mathbb{P}^{\mathcal{G}}X_\lambda)_{co}$ as well because of the continuous map $\mathbb{P}^{\mathcal{G}}X_\lambda \to (\mathbb{P}^{\mathcal{G}}X_\lambda)_{co}$. Thus the set $\{\text{Carrier}(\gamma_n) \mid n \geqslant 0\}$ is finite by Theorem 4.21 and, therefore, the sequence of integers $(i_k)_{k\geqslant 0}$ is bounded. □

Theorem 4.24 is a much better statement that will be the replacement in this paper of [11, Theorem 5.19].

4.24. **Theorem.** *Let $(\alpha, \beta) \in X^0 \times X^0$. Let $\psi : [0,1] \to \mathbb{P}^{top}_{\alpha,\beta}X_\lambda$ be a continuous map. Then the set $\{\text{Carrier}(\psi(u)) \mid u \in [0,1]\}$ is finite.*

*Proof.* Suppose that the set $\{\text{Carrier}(\psi(u)) \mid u \in [0,1]\}$ is infinite. Then there exists a sequence $(t_n)_{n\geqslant 0}$ of $[0,1]$ such that

$$\forall m, n \geqslant 0, m \neq n \Rightarrow \text{Carrier}(\psi(t_m)) \neq \text{Carrier}(\psi(t_n)).$$

In particular, this means that $t_m = t_n$ implies $m = n$ for all $m, n \geqslant 0$. By extracting a subsequence, one can suppose that the sequence $(t_n)_{n\geqslant 0}$ converges to some $t_\infty \in [0,1]$. And the above condition ensures that the set of carriers $\{\text{Carrier}(\psi(t_n)) \mid n \geqslant 0\}$ is still infinite. Since $\psi$ is continuous, the sequence of execution paths $(\psi(t_n))_{n\geqslant 0}$ converges to $\psi(t_\infty)$ in $\mathbb{P}^{top}_{\alpha,\beta}X_\lambda$, and therefore in $(\mathbb{P}^{top}_{\alpha,\beta}X_\lambda)_{co}$ because of the continuous map $\mathbb{P}^{top}_{\alpha,\beta}X_\lambda \to (\mathbb{P}^{top}_{\alpha,\beta}X_\lambda)_{co}$. That contradicts Theorem 4.21. □

Theorem 4.24 enables us to understand the difference between Raussen's naturalization of [24, Definition 2.14] and the globular naturalization of Definition 4.14.

4.25. **Corollary.** *Let $(\alpha, \beta) \in X^0 \times X^0$. Let $\psi : [0,1] \to \mathbb{P}^{top}_{\alpha,\beta}X_\lambda$ be a continuous map. Then the set of natural lengths of $\psi(u)$ for $u$ running over $[0,1]$ is bounded.*

*Proof.* It is due to the fact that the set $\{\text{Carrier}(\psi(u)) \mid u \in [0,1]\}$ is finite by Theorem 4.24. □

The natural lengths of $\psi(u)$ for $u$ running over $[0,1]$ have no reason to be constant. Consider a pushout of $\mathcal{P}$-multipointed $d$-spaces

$$\begin{array}{ccc} \mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^0) & \longrightarrow & \overrightarrow{I}^{\mathcal{P}} * \overrightarrow{I}^{\mathcal{P}} \\ \downarrow & & \downarrow \\ \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^1) & \longrightarrow & X \end{array}$$

where $\overrightarrow{I}^{\mathcal{P}} * \overrightarrow{I}^{\mathcal{P}}$ means that the final state of the left copy of the directed segment is identified with the initial state of the right copy of the directed segment (see Notation 5.1) and where the top horizontal map (it is not unique) takes the initial (final resp.) state of $\mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^0)$ to the initial (final resp.) state of $\overrightarrow{I}^{\mathcal{P}} * \overrightarrow{I}^{\mathcal{P}}$. Then the natural length of an execution path of $X$ going from the initial to the final state is 2 on the boundary of $\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^1)$ and 1 inside $\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^1)$.

Corollary 4.25 shows the difference of behavior between the natural length in the globular setting and the length of the naturalization of a directed path between two vertices in the geometric realization of a precubical set. Indeed, the latter is constant by continuous deformation preserving the extremities [24, Section 2.2.1] [25, Proposition 2.2]. See also [15, Proposition 3.6].

## 5. Chain of globes

5.1. **Notation.** Let $Z_1, \ldots, Z_p$ be $p$ nonempty topological spaces with $p \geqslant 1$. Consider the $\mathcal{P}$-multipointed $d$-space

$$X = \mathrm{Glob}^{\mathcal{P}}(Z_1) * \cdots * \mathrm{Glob}^{\mathcal{P}}(Z_p)$$

with $p \geqslant 1$ where the $*$ means that the final state of a globe is identified with the initial state of the next one by reading from the left to the right. Let $\{\alpha_0, \alpha_1, \ldots, \alpha_p\}$ be the set of states such that the canonical map $\mathrm{Glob}^{\mathcal{P}}(Z_i) \to X$ takes the initial state 0 of $\mathrm{Glob}^{\mathcal{P}}(Z_i)$ to $\alpha_{i-1}$ and the final state 1 of $\mathrm{Glob}^{\mathcal{P}}(Z_i)$ to $\alpha_i$.

5.2. **Notation.** Each carrier $\underline{c} = [c_{\nu_1}, \ldots, c_{\nu_n}]$ gives rise to a map of $\mathcal{P}$-multipointed $d$-spaces from a chain of globes to $X_\lambda$

$$\widehat{g_{\underline{c}}} : \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_{\nu_1}}) * \cdots * \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_{\nu_n}}) \longrightarrow X_\lambda$$

by "concatenating" the attaching maps of the globular cells $c_{\nu_1}, \ldots, c_{\nu_n}$. Let $\alpha_{i-1}$ ($\alpha_i$ resp.) be the initial state (the final state resp.) of $\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_{\nu_i}})$ for $1 \leqslant i \leqslant n$ in $\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_{\nu_1}}) * \cdots * \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_{\nu_n}})$. It induces a continuous map

$$\mathbb{P}^{top}\widehat{g_{\underline{c}}} : X_{\underline{c}} = \mathbb{P}^{top}_{\alpha_0,\alpha_n}(\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_{\nu_1}}) * \cdots * \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n_{\nu_n}})) \longrightarrow \mathbb{P}^{top}X_\lambda.$$

As a consequence of the associativity of the semimonoidal structure on $\mathcal{P}$-spaces and of [10, Proposition 5.16], we have

5.3. **Proposition.** *Let $U_1, \ldots, U_p$ be $p$ topological spaces with $p \geqslant 1$. Let $\ell_1, \ldots, \ell_p > 0$. There is the natural isomorphism of $\mathcal{P}$-spaces*

$$\mathbb{F}^{\mathcal{P}^{op}}_{\ell_1} U_1 \otimes \ldots \otimes \mathbb{F}^{\mathcal{P}^{op}}_{\ell_p} U_p \cong \mathbb{F}^{\mathcal{P}^{op}}_{\ell_1+\cdots+\ell_p}(U_1 \times \ldots \times U_p).$$

5.4. **Proposition.** *([11, Proposition 6.3] for $\mathcal{G}$ and $\mathcal{M}$) Let $Z_1, \ldots, Z_p$ be $p$ topological spaces of* **Top** *with $p \geqslant 1$. Consider the $\mathcal{P}$-multipointed $d$-space $X = \mathrm{Glob}^{\mathcal{P}}(Z_1) * \cdots * \mathrm{Glob}^{\mathcal{P}}(Z_p)$ with $p \geqslant 1$. There is a homeomorphism*

$$\mathbb{P}^{top}_{\alpha_0,\alpha_p} X \cong \mathcal{P}(1,p) \times Z_1 \times \ldots \times Z_p.$$

The case $p = 1$ is treated in Proposition 3.17. The proof of Proposition 5.4 is a modified version of the proof of [11, Proposition 6.3], the latter working only for the case $\mathcal{P} = \mathcal{G}$. Like in the proof of Proposition 3.17, the verification of the continuity in one direction is different from the proof of [11, Proposition 6.3] because of the possible presence, in the case $\mathcal{P} = \mathcal{M}$, of stop intervals.

*Proof.* The Moore composition of paths induced a map of $\mathcal{P}$-spaces

$$\mathbb{P}^{\bullet}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z_1) \otimes \ldots \otimes \mathbb{P}^{\bullet}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z_p) \longrightarrow \mathbb{P}^{\bullet}_{\alpha_0,\alpha_p} X.$$

By Proposition 3.17, there is the isomorphism of $\mathcal{P}$-spaces

$$\mathbb{P}^{\bullet}_{0,1}\mathrm{Glob}^{\mathcal{P}}(Z) \cong \mathbb{F}_1^{\mathcal{P}^{op}} Z$$

for all topological spaces $Z$. We obtain a map of $\mathcal{P}$-spaces

$$\mathbb{F}_1^{\mathcal{P}^{op}} Z_1 \otimes \ldots \otimes \mathbb{F}_1^{\mathcal{P}^{op}} Z_p \longrightarrow \mathbb{P}^{\bullet}_{\alpha_0,\alpha_p} X.$$

By Proposition 5.3, and since $\mathbb{P}^1_{\alpha_0,\alpha_p} X = \mathbb{P}^{top}_{\alpha_0,\alpha_p} X$ by definition of the functor $\mathbb{P}^{\bullet}_{\alpha_0,\alpha_p} X$, we obtain a continuous map

$$\begin{cases} \Psi : & \mathcal{P}(1,p) \times Z_1 \times \ldots \times Z_p \longrightarrow \mathbb{P}^{top}_{\alpha_0,\alpha_p} X \\ & (\phi, z_1, \ldots, z_p) \mapsto (\delta_{z_1}\phi_1) * \cdots * (\delta_{z_p}\phi_p) \end{cases}$$

where $\phi_i \in \mathcal{P}(\ell_i, 1)$ with $\sum_i \ell_i = 1$ and $\phi = \phi_1 \otimes \ldots \otimes \phi_p$ being a decomposition given by the third axiom of reparametrization category. The map $\Psi$ is bijective by Theorem 4.13. The continuous maps $Z_i \to \{0\}$ for $1 \leqslant i \leqslant p$ induce by functoriality a map of $\mathcal{P}$-multipointed $d$-spaces $X \to \overrightarrow{I}^{\mathcal{P}} * \cdots * \overrightarrow{I}^{\mathcal{P}}$ ($p$ times) and then a continuous map

$$\begin{cases} k : & \mathbb{P}^{top}_{\alpha_0,\alpha_p} X \longrightarrow \mathbb{P}^{top}_{\alpha_0,\alpha_p}(\overrightarrow{I}^{\mathcal{P}} * \cdots * \overrightarrow{I}^{\mathcal{P}}) = \mathcal{P}(1,p) \\ & (\delta_{z_1}\phi_1) * \cdots * (\delta_{z_p}\phi_p) \mapsto (\delta_0\phi_1) * \cdots * (\delta_0\phi_p) = \phi_1 \otimes \ldots \otimes \phi_p. \end{cases}$$

Consider the set map

$$\begin{cases} \overline{k} : & \mathbb{P}^{top}_{\alpha_0,\alpha_p} X \longrightarrow Z_1 \times \ldots \times Z_p \\ & (\delta_{z_1}\phi_1) * \cdots * (\delta_{z_p}\phi_p) \mapsto (z_1, \ldots, z_p). \end{cases}$$

Let $i \in \{1, \ldots, p\}$. Suppose that the composite set map $\mathrm{pr}_i\, \overline{k} : \mathbb{P}^{top}_{\alpha_0,\alpha_p} \to Z_i$ is not continuous where $\mathrm{pr}_i$ is the projection on the $i$-th factor. All involved topological spaces being sequential, there exist $z_i^{\infty} \in Z_i$, an open neighborhood $V$ of $z_i^{\infty}$ in $Z_i$, and a sequence $((\delta_{z_1^n}\phi_1^n) * \cdots * (\delta_{z_p^n}\phi_p^n))_{n\geqslant 0}$ which converges to $(\delta_{z_1^{\infty}}\phi_1^{\infty}) * \cdots * (\delta_{z_p^{\infty}}\phi_p^{\infty})$ such that $z_i^n \in Z_i \backslash V$ for all $n \geqslant 0$. Let $\phi^n = \phi_1^n \otimes \ldots \otimes \phi_p^n$ for $n \geqslant 0$ and $\phi^{\infty} = \phi_1^{\infty} \otimes \ldots \otimes \phi_p^{\infty}$. Choose $t_0 \in [0,1]$ such that $\phi^{\infty}(t_0) \in ]i-1, i[$. The sequence

$$\Big(\big((\delta_{z_1^n}\phi_1^n) * \cdots * (\delta_{z_p^n}\phi_p^n)\big)(t_0)\Big)_{n\geqslant 0}$$

converges to

$$((\delta_{z_1^{\infty}}\phi_1^{\infty}) * \cdots * (\delta_{z_p^{\infty}}\phi_p^{\infty}))(t_0) = (\delta_{z_1^{\infty}} * \cdots * \delta_{z_p^{\infty}})(\phi^{\infty}(t_0)).$$

By continuity of the map $k : \mathbb{P}^{top}_{\alpha_0,\alpha_p} X \to \mathcal{P}(1,p)$, the sequence $(\phi^n(t_0))_{n\geqslant 0}$ of $[0,p]$ converges to $\phi^\infty(t_0) \in ]i-1,i[$. It implies that there exists $N \geqslant 0$ such that for all $n \geqslant N$, $\phi^n(t_0) \in ]i-1,i[$. We obtain that the sequence $((z_i^n, \phi^n(t_0)-i+1))_{n\geqslant N}$ converges to $(z_i^\infty, \phi^\infty(t_0)-i+1)$ in $|\mathrm{Glob}^{\mathcal{P}}(Z_i)|\backslash\{0,1\}$. By considering the well-defined projection (the left-hand term being equipped with the relative topology)

$$|\mathrm{Glob}^{\mathcal{P}}(Z_i)|\backslash\{0,1\} \longrightarrow Z_i,$$

we obtain that the sequence $(z_i^n)_{n\geqslant N}$ converges to $z_i^\infty$, and therefore that $z_i^\infty \in Z_i\backslash V$, the latter set being closed in $Z_i$: contradiction.

This means that the composite set map $\mathrm{pr}_i\, \overline{k}$ is continuous for all $i \in \{1,\ldots,p\}$, and therefore, by the universal property of the product, that the set map $\overline{k} : \mathbb{P}^{top}_{\alpha_0,\alpha_p} X \to Z_1 \times \ldots \times Z_p$ is continuous. It implies that the set map

$$\Psi^{-1} = (k, \overline{k}) : (\delta_{z_1}\phi_1) * \cdots * (\delta_{z_p}\phi_p) \mapsto (\phi_1 \otimes \ldots \otimes \phi_p, z_1, \ldots, z_p).$$

is continuous and that $\Psi$ is a homeomorphism. □

5.5. **Corollary.** *Let $Z_1,\ldots,Z_p$ be $p$ topological spaces of* **Top** *with $p \geqslant 1$. Consider the $\mathcal{P}$-multipointed d-space $X = \mathrm{Glob}^{\mathcal{P}}(Z_1) * \cdots * \mathrm{Glob}^{\mathcal{P}}(Z_p)$ with $p \geqslant 1$. There is a homeomorphism*

$$\mathbb{P}_{\alpha_0,\alpha_p} X \cong Z_1 \times \ldots \times Z_p.$$

*Proof.* There are the homeomorphisms

$$\varinjlim_{\mathcal{P}^1} \Big(\mathcal{P}(1,p) \times Z_1 \times \ldots \times Z_p\Big) \cong \Big(\varinjlim_{\mathcal{P}^1} \mathcal{P}(1,p)\Big) \times Z_1 \times \ldots \times Z_p \cong Z_1 \times \ldots \times Z_p,$$

the left-hand homeomorphism since **Top** is cartesian closed and the right-hand homeomorphism by Proposition 3.11. The proof is complete thanks to Proposition 5.4 and Proposition 3.12. □

5.6. **Lemma.** *[11, Lemma 6.10] Let $U_1,\ldots,U_p$ be $p$ first-countable $\Delta$-Hausdorff $\Delta$-generated spaces with $p \geqslant 1$. Let $(u_n^i)_{n\geqslant 0}$ be a sequence of $U_i$ for $1 \leqslant i \leqslant p$ which converges to $u_\infty^i \in U_i$. Then the sequence $((u_n^1,\ldots,u_n^p))_{n\geqslant 0}$ converges to $(u_\infty^1,\ldots,u_\infty^p) \in U_1 \times \ldots \times U_p$ for the product calculated in* **Top**.

5.7. **Notation.** Let $\underline{c}$ be the carrier of some execution path of $X_\lambda$. Using the identification provided by the homeomorphism of Proposition 5.4, we can use the notation

$$(\mathbb{P}^{top}\widehat{g_{\underline{c}}})(\phi, z^1, \ldots, z^n) = (\widehat{g_{\nu_1}}\delta_{z^1} * \cdots * \widehat{g_{\nu_n}}\delta_{z^n})\phi.$$

Lemma 5.8 is implicitly used in [11, Theorem 6.11 and Theorem 7.3] and in [16, Theorem 7.7].

5.8. **Lemma.** *Let $X$ be a sequential topological space. Let $x_\infty \in X$. Let $(x_n)_{n\geqslant 0}$ be a sequence such that $x_\infty$ is a limit point of all subsequences. Then the sequence $(x_n)_{n\geqslant 0}$ converges to $x_\infty$.*

*Proof.* Otherwise, consider an open neighborhood $V$ of $x_\infty$ such that for all $n \geqslant N$, $x_n \notin V$ for some $N \geqslant 0$. This means that $x_\infty \in X\backslash V$, the subset $X\backslash V$ being sequentially closed in $X$: contradiction. □

5.9. **Theorem.** *Let $\underline{c}$ be the carrier of some execution path of $X_\lambda$.*

*(1) Consider a sequence $(\gamma_k)_{k\geqslant 0}$ of the image of $\mathbb{P}^{top}\widehat{g_{\underline{c}}}$ which converges pointwise to $\gamma_\infty$ in $\mathbb{P}^{top}X_\lambda$. Let*

$$\gamma_k = (\mathbb{P}^{top}\widehat{g_{\underline{c}}})(\phi_k, z_k^1, \ldots, z_k^n)$$

*with $\phi_k \in \mathcal{P}(1,n)$ and $z_k^i \in \mathbf{D}^{n_{\nu_i}}$ for $1 \leqslant i \leqslant n$ and $k \geqslant 0$. Then there exist $\phi_\infty \in \mathcal{P}(1,n)$ and $z_\infty^i \in \mathbf{D}^{n_{\nu_i}}$ for $1 \leqslant i \leqslant n$ such that*

$$\gamma_\infty = (\mathbb{P}^{top}\widehat{g_{\underline{c}}})(\phi_\infty, z_\infty^1, \ldots, z_\infty^n)$$

*and such that $(\phi_\infty, z_\infty^1, \ldots, z_\infty^n)$ is a limit point of the sequence $((\phi_k, z_k^1, \ldots, z_k^n))_{k\geqslant 0}$.*
*(2) The image of $\mathbb{P}^{top}\widehat{g_{\underline{c}}}$ is closed in $\mathbb{P}^{top}X_\lambda$.*

*Proof.* The case $\mathcal{P} = \mathcal{G}$ is treated in [11, Theorem 6.11]. Let us suppose that $\mathcal{P} = \mathcal{M}$. The proof is similar but simpler because it is not necessary to verify anymore that some limit execution paths are regular.

(1) By a Cantor diagonalization argument, we can suppose that the sequence $(z_k^i)_{k\geqslant 0}$ converges to $z_\infty^i \in \mathbf{D}^{n_{\nu_i}}$ for each $1 \leqslant i \leqslant n$ and that the sequence $(\phi_k(r))_{k\geqslant 0}$ converges to a real number denoted by $\phi_\infty(r) \in [0,n]$ for all rational numbers $r \in [0,1]\cap\mathbb{Q}$. Since the sequence of execution paths $(\gamma_k)_{k\geqslant 0}$ converges pointwise to $\gamma_\infty$, we obtain

$$\gamma_\infty(r) = (\widehat{g_{\nu_1}}\delta_{z_\infty^1} * \cdots * \widehat{g_{\nu_n}}\delta_{z_\infty^n})(\phi_\infty(r))$$

for all $r \in [0,1]\cap\mathbb{Q}$. For $r_1 < r_2 \in [0,1]\cap\mathbb{Q}$, $\phi_k(r_1) \leqslant \phi_k(r_2)$ for all $k \geqslant 0$. Therefore by passing to the limit, we obtain $\phi_\infty(r_1) \leqslant \phi_\infty(r_2)$. Note that $\phi_\infty(0) = 0$ and $\phi_\infty(1) = n$ since $0, 1 \in \mathbb{Q}$. For $t \in ]0,1[$, let us extend the definition of $\phi_\infty$ as follows:

$$\phi_\infty(t) = \sup\{\phi_\infty(r) \mid r \in ]0,t]\cap\mathbb{Q}\}.$$

By continuity, we deduce that

$$\gamma_\infty(t) = (\widehat{g_{\nu_1}}\delta_{z_\infty^1} * \cdots * \widehat{g_{\nu_n}}\delta_{z_\infty^n})(\phi_\infty(t))$$

for all $t \in [0,1]$. It is easy to see that the set map $\phi_\infty : [0,1] \to [0,n]$ is nondecreasing and that it preserves extremities. By definition of the Moore composition, there exist $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = 1$ such that for all $1 \leqslant i \leqslant n$,

$$\forall t \in [t_{i-1}, t_i], \gamma_\infty(t) = \widehat{g_{\nu_i}}(z_\infty^i, \phi_\infty(t) - i + 1).$$

It implies that the restriction of $\phi_\infty$ to $[t_{i-1}, t_i]$ is surjective. We deduce that the nondecreasing set map $\phi_\infty : [0,1] \to [0,n]$ is surjective, and therefore that $\phi_\infty \in \mathcal{M}(1,n)$. Let $t \in [0,1]\backslash\mathbb{Q}$. The sequence $(\phi_k(t))_{k\geqslant 0}$ has at least one limit point $\ell$. There exists a subsequence of $(\phi_k(t))_{k\geqslant 0}$ which converges to $\ell$. We obtain: $\forall r \in [0,t]\cap\mathbb{Q}, \forall r' \in [t,1]\cap\mathbb{Q}, \phi_\infty(r) \leqslant \ell \leqslant \phi_\infty(r')$. Since $\phi_\infty \in \mathcal{M}(1,n)$ and by density of $\mathbb{Q}$, we deduce that $\ell = \phi_\infty(t)$ necessarily. Using Lemma 5.8, we deduce that the sequence $(\phi_k)_{k\geqslant 0}$ converges pointwise to $\phi_\infty$. Using Proposition 2.6, we deduce that $(\phi_k)_{k\geqslant 0}$ converges uniformly to $\phi_\infty$. We deduce that $(\phi_\infty, z_\infty^1, \ldots, z_\infty^n)$ is a limit point of the sequence $((\phi_k, z_k^1, \ldots, z_k^n))_{k\geqslant 0}$ in $\mathcal{M}(1,n) \times \mathbf{D}^{n_{\nu_1}} \times \ldots \times \mathbf{D}^{n_{\nu_n}}$ by Proposition 2.6 and Lemma 5.6.

(2) Let $(\mathbb{P}^{top}\widehat{g_{\underline{c}}}(\Gamma_n))_{n\geqslant 0}$ be a sequence of $(\mathbb{P}^{top}\widehat{g_{\underline{c}}})(X_{\underline{c}})$ which converges in $\mathbb{P}^{top}X_\lambda$. The limit $\gamma_\infty \in \mathbb{P}^{top}X_\lambda$ of the sequence of execution paths $(\mathbb{P}^{top}\widehat{g_{\underline{c}}}(\Gamma_n))_{n\geqslant 0}$ is also a pointwise limit. We can suppose by extracting a subsequence that the sequence $(\Gamma_n)_{n\geqslant 0}$ of $X_{\underline{c}}$ converges in $X_{\underline{c}}$. Thus, by continuity of $\mathbb{P}^{top}\widehat{g_{\underline{c}}}$, we obtain $\gamma_\infty = (\mathbb{P}^{top}\widehat{g_{\underline{c}}})(\Gamma_\infty)$ for some

$\Gamma_\infty \in X_{\underline{c}}$. We deduce that $\mathbb{P}^{top}\widehat{g_{\underline{c}}}(X_{\underline{c}})$ is sequentially closed in $\mathbb{P}^{top}X_\lambda$. Since $\mathbb{P}^{top}X_\lambda$ is sequential, being a $\Delta$-generated space, the proof is complete. □

As a corollary of Theorem 5.9, we obtain:

5.10. **Corollary.** *Suppose that $X_\lambda$ is a finite cellular $\mathcal{P}$-multipointed d-space, i.e. $X^0$ is finite and $\lambda$ is a finite ordinal. If $X_\lambda$ has no loops, then the topology of $\mathbb{P}^{top}X_\lambda$ is the topology of the pointwise convergence which is therefore $\Delta$-generated.*

*Proof.* It is mutatis mutandis the proof of [11, Corollary 6.12]. □

## 6. Locally finite cellular multipointed d-space

We want to give an application of Theorem 4.21 and Theorem 5.9 before addressing the main subject of this paper. The reading of this section is not necessary to understand Section 7. A cellular $\mathcal{P}$-multipointed $d$-space $X_\lambda$ is fixed.

6.1. **Definition.** The cellular $\mathcal{P}$-multipointed $d$-space space $X_\lambda$ is *locally finite* if for all $\nu < \lambda$, the set $\{\nu' < \lambda \mid \widehat{c_{\nu'}} \cap c_\nu \neq \varnothing\}$ is finite and each state meets a finite number of $\widehat{c_\nu}$. In other terms, the underlying topological space $|X_\lambda|$, which is cellular by Proposition 4.5, is locally finite.

Lemma 6.2 is a consequence of [3, Proposition 3.4] and [3, Proposition 3.10]. It can be easily proved without using diffeological spaces.

6.2. **Lemma.** *Every $\Delta$-generated space is locally path-connected.*

*Proof.* Let $U$ be an open subset of a $\Delta$-generated space $X$. Then $U$ equipped with the relative topology is $\Delta$-generated by [12, Proposition 2.4]. Therefore $U$ equipped with the relative topology is homeomorphic to the disjoint sum of its path-connected components by [8, Proposition 2.8]. Thus $X$ is locally path-connected. □

6.3. **Definition.** A topological space $X$ is *weakly locally path-connected* if for every $x \in X$ and every neighborhood $W$ of $x$, there exists a path-connected neighborhood (not necessarily open) $W'$ of $x$ such that $W' \subset W$.

6.4. **Lemma.** *(well-known) Every weakly locally path-connected space is locally path-connected.*

*Proof.* Let $W$ be a neighborhood of $x \in X$. Then there exists a path-connected neighborhood $W'$ of $x$ such that $W' \subset W$. This means that $W'$ is included in the path-connected component $C$ of $x$ in $W$. Therefore $x$ is in the interior of $C$. Thus $C$ is open and $X$ is locally path-connected. □

6.5. **Proposition.** *Let $\lambda$ be an ordinal. Let $Z : \lambda \to \mathbf{Top}$ be a colimit-preserving functor such that $Z_\lambda$ is cellular for the q-model structure of* **Top**. *If the cellular space $Z_\lambda$ is locally finite, then the topological space $Z_\lambda$ is metrizable.*

*Sketch of proof.* The technique used in [7] to reorganize and regroup the cells in a CW-complex using the notion of star of a subset [7, Example 2] works in the same way for cellular topological spaces, even when $\lambda$ is not countable. Assume first that $Z_\lambda$ is path-connected. By [7, Proposition 1.5.12], the ordinal $\lambda$ is countable, $Z_\lambda$ being locally finite.

Using [7, Proposition 1.5.13], the cells are reorganized so that $\lambda = \aleph_0$ and so that each $Z_n$ for $n$ finite is a finite cellular topological space (i.e. built using finitely many cells). Moreover, for all $n \geqslant 0$, the space $Z_n$ is contained in the interior $\overset{\circ}{Z}_{n+1}$ of $Z_{n+1}$ for the topology of $Z_\lambda$ and there is the equality

$$\bigcup_{n\geqslant 0} \overset{\circ}{Z}_n = \bigcup_{n\geqslant 0} Z_n = Z_\lambda.$$

Using [7, Theorem 1.5.16], we deduce that $Z_\lambda$ is metrizable and that it can be embedded in the Hilbert cube equipped with the $\ell^2$ metric. This means that the metric of $Z_\lambda$ is bounded, namely by the constant $\pi/\sqrt{6}$ which does not depend on $Z_\lambda$. In the general case, the $\Delta$-generated space $Z_\lambda$ is homeomorphic to the disjoint sum of its path-connected components by [8, Proposition 2.8]. Thus, the metric on each path-connected component being bounded by $\pi/\sqrt{6}$, the disjoint sum is metrizable. □

In fact, we could prove the equivalence for cellular topological spaces of the conditions locally finite, metrizable, locally compact and first-countable as it is done in [7, Proposition 1.5.10 and Proposition 1.5.17] for CW-complexes.

6.6. **Corollary.** *Assume $X_\lambda$ locally finite. Let $(\alpha, \beta) \in X^0 \times X^0$. Then the topological space $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ is metrizable, and therefore sequential and first-countable.*

*Proof.* By Proposition 6.5, the topological space $|X_\lambda|$ is metrizable, $X_\lambda$ being locally finite by hypothesis. The space $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ is therefore metrizable by [18, Proposition A.13]. □

6.7. **Proposition.** *Assume $X_\lambda$ locally finite. Let $(\alpha, \beta) \in X^0 \times X^0$. The space $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ is locally path-connected.*

*Proof.* By Lemma 6.4, it suffices to prove that $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ is weakly locally path-connected. Consider an execution path $\gamma_\infty$ of $\mathbb{P}^{top}_{\alpha,\beta} X_\lambda$. Let $\Omega$ be an open subset of $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ containing $\gamma_\infty$. Then $\Omega$ is an open subset of $\mathbb{P}^{top}_{\alpha,\beta} X_\lambda$ since the $\Delta$-kelleyfication adds open subsets. Let $\mathcal{T}$ be the set of all carriers of all execution paths of $\mathbb{P}^{top}_{\alpha,\beta} X_\lambda$. Consequently, for each carrier $\underline{c} \in \mathcal{T}$ and for each $\Gamma \in (\mathbb{P}^{top}\widehat{g}_{\underline{c}})^{-1}(\gamma_\infty)$, there exists an open neighborhood $\Omega_\Gamma$ of $\Gamma$ such that $(\mathbb{P}^{top}\widehat{g}_{\underline{c}})(\Omega_\Gamma) \subset \Omega$. By Lemma 6.2, we can suppose that $\Omega_\Gamma$ is path-connected. Consider

$$U = \bigcup_{\underline{c}\in\mathcal{T}} \bigcup_{\Gamma\in(\mathbb{P}^{top}\widehat{g}_{\underline{c}})^{-1}(\gamma_\infty)} (\mathbb{P}^{top}\widehat{g}_{\underline{c}})(\Omega_\Gamma).$$

Then $U$ is path-connected and $U \subset \Omega$. Suppose that $\gamma_\infty$ is not in the interior of $U$ in $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$. The space $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ being sequential by Corollary 6.6, there exists a sequence $(\gamma_n)_{n\geqslant 0}$ of execution paths not belonging to $U$ converging to $\gamma_\infty$ in $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$. Since the set $\{\mathrm{Carrier}(\gamma_n) \mid n \geqslant 0\}$ is finite by Theorem 4.21, we can always suppose that the sequence of carriers $(\mathrm{Carrier}(\gamma_n))_{n\geqslant 0}$ is constant and e.g. equal to some $\underline{c} \in \mathcal{T}$ by extracting a subsequence. Therefore we can write $\gamma_n = (\mathbb{P}^{top}\widehat{g}_{\underline{c}})(\Gamma_n)$ with $\Gamma_n \in X_{\underline{c}}$ (see Notation 5.2). The sequence of execution paths $(\gamma_n)_{n\geqslant 0}$ converges pointwise to $\gamma_\infty$. Thus, by Theorem 5.9, we can suppose that the sequence $(\Gamma_n)_{n\geqslant 0}$ converges to $\Gamma_\infty \in X_{\underline{c}}$ after extracting a subsequence again. By continuity, we obtain the equality $\gamma_\infty = (\mathbb{P}^{top}\widehat{g}_{\underline{c}})(\Gamma_\infty)$. There exists $N \geqslant 0$ such that for all $n \geqslant N$, $\Gamma_n \in \Omega_{\Gamma_\infty}$, i.e. $\gamma_n = (\mathbb{P}^{top}\widehat{g}_{\underline{c}})(\Gamma_n) \in U$ for all $n \geqslant N$. Contradiction. Thus $\gamma_\infty$ is in the interior of $U$. □

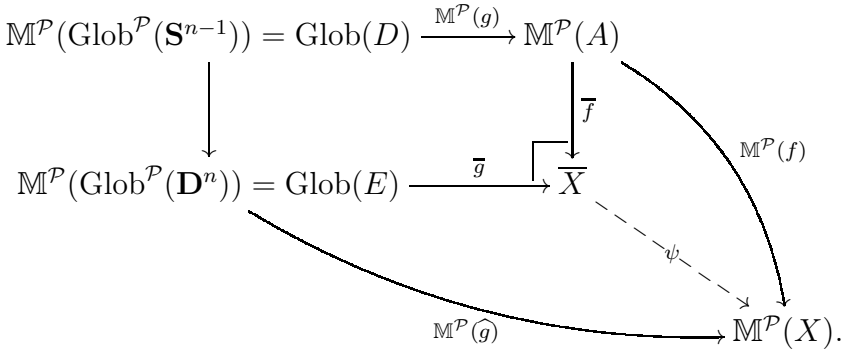



FIGURE 1. Definition of $\overline{X}$

6.8. **Theorem.** *Assume $X_\lambda$ locally finite. Let $(\alpha,\beta) \in X^0 \times X^0$. The topological space $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ equipped with the compact-open topology is $\Delta$-generated. The topological space*

$$\mathbb{P}^{top} X_\lambda = \coprod_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}^{top}_{\alpha,\beta} X_\lambda$$

*is metrizable with the distance of the uniform convergence. The underlying topology is the compact-open topology.*

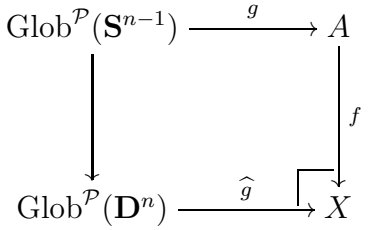
*Proof.* By Corollary 6.6, the topological space $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ is first-countable. The space $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ is locally path-connected by Proposition 6.7. Using [3, Proposition 3.11], we deduce that $(\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$ is $\Delta$-generated. The set of all execution paths equipped with the compact-open topology $(\mathbb{P}^{top} X_\lambda)_{co}$ satisfies

$$(\mathbb{P}^{top} X_\lambda)_{co} \cong \coprod_{(\alpha,\beta)\in X^0\times X^0} (\mathbb{P}^{top}_{\alpha,\beta} X_\lambda)_{co}$$

because $X^0$ is a discrete subspace of $|X_\lambda|$. Hence $(\mathbb{P}^{top} X_\lambda)_{co}$ is $\Delta$-generated and metrizable by the distance of the uniform convergence. □

## 7. Multipointed d-space and Moore flow

Consider a pushout diagram of $\mathcal{P}$-multipointed $d$-spaces




with $n \geqslant 0$ and $A$ cellular. Note that $A^0 = X^0$. Let $D = \mathbb{F}_1^{\mathcal{P}^{op}} \mathbf{S}^{n-1}$ and $E = \mathbb{F}_1^{\mathcal{P}^{op}} \mathbf{D}^n$. Consider the $\mathcal{P}$-flow $\overline{X}$ defined by the pushout diagram of Figure 1 where the two equalities

$$\mathbb{M}^{\mathcal{P}}(\mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^{n-1})) = \mathrm{Glob}(D)$$
$$\mathbb{M}^{\mathcal{P}}(\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n})) = \mathrm{Glob}(E)$$

come from Proposition 3.22 and where the map $\psi$ is induced by the universal property of the pushout.

The $\mathcal{P}$-space of execution paths of the Moore flow $\overline{X}$ can be calculated by introducing a diagram of $\mathcal{P}$-spaces $\mathcal{D}^f$ over a Reedy category $\mathcal{P}^{g(0),g(1)}(A^0)$ whose definition is recalled now. It was introduced for the first time in [12, Section 3]. Let $S$ be a nonempty set. Let $\mathcal{P}^{u,v}(S)$ [3] be the small category defined by generators and relations as follows:

- $u, v \in S$ ($u$ and $v$ may be equal).
- The objects are the tuples of the form

  $$\underline{m} = ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n))$$

  with $n \geqslant 1$, $u_0, \dots, u_n \in S$, $\epsilon_1, \dots, \epsilon_n \in \{0, 1\}$ and

  $$\forall i \text{ such that } 1 \leqslant i \leqslant n, \epsilon_i = 1 \Rightarrow (u_{i-1}, u_i) = (u, v).$$

- There is an arrow

  $$c_{n+1} : (\underline{m}, (x, 0, y), (y, 0, z), \underline{n}) \to (\underline{m}, (x, 0, z), \underline{n})$$

  for every tuple $\underline{m} = ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n))$ with $n \geqslant 1$ and every tuple $\underline{n} = ((u'_0, \epsilon'_1, u'_1), (u'_1, \epsilon'_2, u'_2), \dots, (u'_{n'-1}, \epsilon'_{n'}, u'_{n'}))$ with $n' \geqslant 1$. It is called a *composition map*.
- There is an arrow

  $$I_{n+1} : (\underline{m}, (u, 0, v), \underline{n}) \to (\underline{m}, (u, 1, v), \underline{n})$$

  for every tuple $\underline{m} = ((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n))$ with $n \geqslant 1$ and every tuple $\underline{n} = ((u'_0, \epsilon'_1, u'_1), (u'_1, \epsilon'_2, u'_2), \dots, (u'_{n'-1}, \epsilon'_{n'}, u'_{n'}))$ with $n' \geqslant 1$. It is called an *inclusion map*.
- There are the relations (group A) $c_i.c_j = c_{j-1}.c_i$ if $i < j$ (which means since $c_i$ and $c_j$ may correspond to several maps that if $c_i$ and $c_j$ are composable, then there exist $c_{j-1}$ and $c_i$ composable satisfying the equality).
- There are the relations (group B) $I_i.I_j = I_j.I_i$ if $i \neq j$. By definition of these maps, $I_i$ is never composable with itself.
- There are the relations (group C)

  $$c_i.I_j = \begin{cases} I_{j-1}.c_i & \text{if } j \geqslant i+2 \\ I_j.c_i & \text{if } j \leqslant i-1. \end{cases}$$

  By definition of these maps, $c_i$ and $I_i$ are never composable as well as $c_i$ and $I_{i+1}$.

By [12, Proposition 3.7], there exists a structure of Reedy category on $\mathcal{P}^{u,v}(S)$ with the $\mathbb{N}$-valued degree map defined by

$$d((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \dots, (u_{n-1}, \epsilon_n, u_n)) = n + \sum_i \epsilon_i.$$

The maps raising the degree are the inclusion maps. The maps decreasing the degree are the composition maps.

[3] The use of the letter $\mathcal{P}$ here has nothing to do with the reparametrization category $\mathcal{P}$. It is a bit unfortunate but I prefer to not change the notation.

Let $T$ be the $\mathcal{P}$-space defined by the pushout diagram of $[\mathcal{P}^{op}, \mathbf{Top}]_0$

$$\begin{array}{ccc} D & \xrightarrow{\mathbb{PM}^{\mathcal{P}}(g)} & \mathbb{P}_{g(0),g(1)}\mathbb{M}^{\mathcal{P}}(A) \\ \downarrow & & \downarrow{\scriptstyle \mathbb{P}\overline{f}} \\ E & \xrightarrow{\mathbb{P}\overline{g}} & T. \end{array}$$

Consider the diagram of spaces $\mathcal{D}^f : \mathcal{P}^{g(0),g(1)}(A^0) \to [\mathcal{P}^{op}, \mathbf{Top}]_0$ defined as follows:

$$\mathcal{D}^f((u_0, \epsilon_1, u_1), (u_1, \epsilon_2, u_2), \ldots, (u_{n-1}, \epsilon_n, u_n)) = Z_{u_0,u_1} \otimes Z_{u_1,u_2} \otimes \ldots \otimes Z_{u_{n-1},u_n}$$

with

$$Z_{u_{i-1},u_i} = \begin{cases} \mathbb{P}_{u_{i-1},u_i}\mathbb{M}^{\mathcal{P}}(A) & \text{if } \epsilon_i = 0 \\ T & \text{if } \epsilon_i = 1 \end{cases}$$

In the case $\epsilon_i = 1$, $(u_{i-1}, u_i) = (g(0), g(1))$ by definition of $\mathcal{P}^{g(0),g(1)}(A^0)$. The inclusion maps $I_i's$ are induced by the map $\mathbb{P}\overline{f} : \mathbb{P}_{g(0),g(1)}\mathbb{M}^{\mathcal{P}}(A) \to T$. The composition maps $c_i's$ are induced by the compositions of paths of the Moore flow $\mathbb{M}^{\mathcal{P}}(A)$.

7.1. **Theorem.** *[10, Theorem 9.7] We obtain a well-defined diagram of $\mathcal{P}$-spaces*

$$\mathcal{D}^f : \mathcal{P}^{g(0),g(1)}(A^0) \to [\mathcal{P}^{op}, \mathbf{Top}]_0.$$

*There is the isomorphism of $\mathcal{P}$-spaces $\varinjlim \mathcal{D}^f \cong \mathbb{P}\overline{X}$.*

By the universal property of the pushout, we obtain a canonical map of $\mathcal{P}$-spaces

$$\mathbb{P}\psi : \varinjlim \mathcal{D}^f \longrightarrow \mathbb{PM}^{\mathcal{P}}X.$$

7.2. **Definition.** Let $\underline{x}$ be an element of some vertex of the diagram of spaces $\mathcal{D}^f$. We say that $\underline{x} \in \mathcal{D}^f(\underline{n})$ is *simplified* if

$$d(\underline{n}) = \min\left\{d(\underline{m}) \mid \exists \underline{m} \in \mathrm{Obj}(\mathcal{P}^{g(0),g(1)}(A^0)) \text{ and } \exists \underline{y} \in \mathcal{D}^f(\underline{m}), \underline{y} = \underline{x} \in \varinjlim \mathcal{D}^f\right\}.$$

7.3. **Theorem.** *Under the hypotheses and the notations of this section. The map of $\mathcal{P}$-spaces*

$$\mathbb{P}\psi : \varinjlim \mathcal{D}^f \longrightarrow \mathbb{PM}^{\mathcal{P}}(X)$$

*is an isomorphism.*

*Proof.* The structure of the proof is the same as the one of the proofs of [11, Theorem 7.2 and Theorem 7.3]. At first it must be proved that the map $\mathbb{P}\psi$ is an objectwise bijection. The role of [11, Theorem 5.20] is played by Theorem 4.17. Then it must be proved that the map $\mathbb{P}\psi$ is an objectwise homeomorphism. The roles of [11, Theorem 5.19] and [11, Theorem 6.11] are played by Theorem 4.24 and Theorem 5.9 respectively.

The map $\psi$ of Figure 1 is obtained by the universal property of the pushout. Thus, it is bijective on states. It then suffices to prove that the map

$$\mathbb{P}^1\psi : \varinjlim \mathcal{D}^f(1) \to \mathbb{P}^1\mathbb{M}^{\mathcal{P}}(X) = \mathbb{P}^{top}X$$

is a homeomorphism since $\mathcal{G} \subset \mathcal{P}$. By Theorem 4.17, every execution path of $X$ can be written as a finite Moore composition $(f_1\gamma_1\mu_{\ell_1}) * \cdots * (f_n\gamma_n\mu_{\ell_n})$ with $n \geqslant 1$ such that $\ell_1 + \cdots + \ell_n = 1$ and such that $f_i = f$ and $\gamma_i$ is an execution path of $A$ or $f_i = \widehat{g}$ and

$\gamma_i = \delta_{z_i}\phi_i$ with $z_i \in \mathbf{D}^n\backslash\mathbf{S}^{n-1}$ and some $\phi_i \in \mathcal{P}(1,1)$. Let $\overline{f_i} = \overline{g}$ if $f_i = \widehat{g}$ and $\overline{f_i} = \overline{f}$ if $f_i = f$ for $i \in \{1,\dots,n\}$. It gives rise to the execution path $\mathbb{P}\overline{f_1}(\gamma_1\mu_{\ell_1}) * \cdots * \mathbb{P}\overline{f_n}(\gamma_n\mu_{\ell_n})$ of the Moore flow $\overline{X}$. By the commutativity of the diagram of Figure 1, we obtain the equality

$$(f_1\gamma_1\mu_{\ell_1}) * \cdots * (f_n\gamma_n\mu_{\ell_n}) = (\mathbb{P}^1\psi)\Big(\mathbb{P}\overline{f_1}(\gamma_1\mu_{\ell_1}) * \cdots * \mathbb{P}\overline{f_n}(\gamma_n\mu_{\ell_n})\Big).$$

This means that the map of Moore flows $\psi : \overline{X} \to \mathbb{M}^{\mathcal{P}}(X)$ induces a surjective continuous map from $\mathbb{P}^1\overline{X}$ to $\mathbb{P}^{top}X$. In other terms, the map $\mathbb{P}^1\psi$ is a surjection.

Let $\underline{n} = (u_0,\epsilon_1,u_1),(u_1,\epsilon_2,u_2),\dots,(u_{n-1},\epsilon_n,u_n))$. By [10, Corollary 5.13], the topological space $(Z_{u_0,u_1} \otimes Z_{u_1,u_2} \otimes \dots \otimes Z_{u_{n-1},u_n})(1)$ is the quotient of

$$\coprod_{(\ell_1,\dots,\ell_n)} \mathcal{P}(1,\ell_1+\cdots+\ell_n) \times Z_{u_0,u_1}(\ell_1) \times \dots \times Z_{u_{n-1},u_n}(\ell_n)$$

by the equivalence relation generated by the identifications

$$(\phi, x_1\phi_1,\dots,x_p\phi_p) \sim ((\phi_1 \otimes \dots \otimes \phi_p)\phi, x_1,\dots,x_p)$$

for $\phi \in \mathcal{P}(1,\ell_1+\cdots+\ell_p)$, $\phi_i \in \mathcal{P}(\ell_i,\ell_i')$ and $x_i \in Z_{u_{i-1},u_i}(\ell_i')$ for $1 \leqslant i \leqslant n$. Assume that

$$(\phi,\gamma_1,\dots,\gamma_n) \in \mathcal{P}(1,\ell_1+\cdots+\ell_n) \times Z_{u_0,u_1}(\ell_1) \times \dots \times Z_{u_{n-1},u_n}(\ell_n)$$

is a representative of $\underline{x}$ in $\mathcal{D}^f(\underline{n})$ with $\underline{x}$ simplified. Then

$$\mathbb{P}^1\psi(\underline{x}) = \Big((f_1\gamma_1) * \dots (f_n\gamma_n)\Big)\phi$$

with $f_i = f$ if $\epsilon_i = 0$ and $f_i = \widehat{g}$ if $\epsilon_i = 1$. Using Proposition 2.8, write $\phi = \phi_1 \otimes \dots \otimes \phi_n$ with $\phi_i : \ell_i' \to \ell_i$ for $1 \leqslant i \leqslant n$ for some $\ell_1',\dots,\ell_n'$ such that $\ell_1'+\cdots+\ell_n' = 1$. Then one has

$$(\phi,\gamma_1,\dots,\gamma_n) \sim (\mathrm{Id}_1,\gamma_1\phi_1,\dots,\gamma_n\phi_n)$$

in $\mathcal{D}^f(\underline{n})$ and therefore

$$\mathbb{P}^1\psi(\underline{x}) = (f_1\gamma_1\phi_1) * \dots (f_n\gamma_n\phi_n).$$

Recall that

$$d((u_0,\epsilon_1,u_1),(u_1,\epsilon_2,u_2),\dots,(u_{n-1},\epsilon_n,u_n)) = n + \sum_i \epsilon_i.$$

As already explained in the proof of [11, Theorem 7.2] with a lot of details in the case $\mathcal{P} = \mathcal{G}$, since $\underline{x}$ is simplified by hypothesis, it is impossible to have $\epsilon_i = \epsilon_{i+1} = 0$ for some $1 \leqslant i < n$. There is a composition map starting from $\underline{n}$ in $\mathcal{P}^{g(0),g(1)}(A^0)$ otherwise, which identifies $\underline{x}$ to some $\underline{y} \in \mathcal{D}^f(\underline{m})$ in $\varinjlim \mathcal{D}^f(1)$ with $d(\underline{m}) < d(\underline{n})$, and it is a contradiction. As also already seen in the proof of [11, Theorem 7.2] in the case $\mathcal{P} = \mathcal{G}$, if $\epsilon_i = 1$, then $\gamma_i = \overline{g}\delta_{z_i}\psi_i$ with $z_i \in \mathbf{D}^n\backslash\mathbf{S}^{n-1}$ and $\psi_i \in \mathcal{P}(\ell_i,1)$. Indeed, if $z_i \in \mathbf{S}^{n-1}$, then there is an inclusion map whose image contains $\underline{x}$, which means also that $\underline{x}$ is identified to some $\underline{y} \in \mathcal{D}^f(\underline{m})$ in $\varinjlim \mathcal{D}^f(1)$ with $d(\underline{m}) < d(\underline{n})$, which contradicts the fact that $\underline{x}$ is simplified.

This means that the finite Moore composition $(f_1\gamma_1\phi_1) * \dots (f_n\gamma_n\phi_n)$ is one of the finite Moore compositions given by Theorem 4.17. Consider another simplified element $\underline{x}'$ in $\mathcal{D}^f(\underline{n}')$ such that $\mathbb{P}^1\psi(\underline{x}) = \mathbb{P}^1\psi(\underline{x}')$. It gives rise to another finite Moore composition $(f_1'\gamma_1'\phi_1') * \dots (f_{n'}'\gamma_n'\phi_{n'})$ as the ones given by Theorem 4.17. Using Theorem 4.17, we deduce that $n = n'$ and that

$$\text{(R)} \qquad \forall 1 \leqslant i \leqslant n, r_i = \underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma_i) = \underline{\mathrm{nat}}^{\mathrm{gl}}(\gamma_i'), \gamma_i = r_i\eta_i, \gamma_i' = r_i\eta_i'$$

and

$$(\mathrm{P}) \qquad (\eta_1\phi_1)\otimes\ldots\otimes(\eta_n\phi_n)=(\eta'_1\phi'_1)\otimes\ldots\otimes(\eta'_n\phi'_n).$$

We then obtain in $(Z_{u_0,u_1}\otimes Z_{u_1,u_2}\otimes\ldots\otimes Z_{u_{n-1},u_n})(1)$ the following sequence of identifications:

$$\begin{aligned}(\phi,\gamma_1,\ldots,\gamma_n)&\sim((\eta_1\otimes...\otimes\eta_n)\phi,r_1,\ldots,r_n)\\&=((\eta'_1\otimes...\otimes\eta'_n)\phi',r_1,\ldots,r_n)\\&\sim(\phi',\gamma'_1,\ldots,\gamma'_n),\end{aligned}$$

the first and third identifications by (R) and [10, Corollary 5.13] and the equality by (P). This means that $\underline{x}=\underline{x}'$ in $\varinjlim\mathcal{D}^f(1)$ and, therefore, that the map $\mathbb{P}^1\psi$ is one-to-one.

At this point, it is proved that the map $\mathbb{P}^1\psi:\varinjlim\mathcal{D}^f(1)\to\mathbb{P}^{top}X$ is a continuous bijection with $\varinjlim\mathcal{D}^f(1)$ equipped with the final topology. If we work with the category of $\Delta$-Hausdorff $\Delta$-generated spaces, we deduce that $\varinjlim\mathcal{D}^f(1)$ equipped with the final topology is $\Delta$-Hausdorff as well, the space $\mathbb{P}^{top}X$ being $\Delta$-Hausdorff. So whether we work with $\Delta$-Hausdorff or not $\Delta$-generated spaces, the topology of $\varinjlim\mathcal{D}^f(1)$ is always the final topology.

By [11, Corollary 2.3], we must now prove that for all set maps $\xi:[0,1]\to\varinjlim\mathcal{D}^f(1)$, if the composite map $(\mathbb{P}^1\psi)\xi:[0,1]\to\mathbb{P}^{top}X$ is continuous, then the set map $\xi:[0,1]\to\varinjlim\mathcal{D}^f(1)$ is continuous as well. By Theorem 4.24, the set of carriers

$$\mathcal{T}=\{\mathrm{Carrier}((\mathbb{P}^1\psi)\xi(u))\mid u\in[0,1]\}$$

is finite. For each carrier $\underline{c}\in\mathcal{T}$, let

$$U_{\underline{c}}=\{u\in[0,1]\mid\mathrm{Carrier}(\overline{\xi}(u))=\underline{c}\}.$$

Consider the closure $\widehat{U_{\underline{c}}}$ of $U_{\underline{c}}$ in $[0,1]$. We obtain a finite covering of $[0,1]$ by the closed subsets $\widehat{U_{\underline{c}}}$ for $\underline{c}$ running over $\mathcal{T}$. Each $\widehat{U_{\underline{c}}}$ is compact, metrizable and therefore sequential. Note that $\widehat{U_{\underline{c}}}$ has no reason to be $\Delta$-generated: it could be e.g. the Cantor set which is not $\Delta$-generated because it is not homeomorphic to the disjoint sum of its path-connected components. Fix the carrier $\underline{c}$.

The end of the proof is the *third reduction* and *sequential continuity* sections of the proof of [11, Theorem 7.3] with the use of Theorem 5.9 instead of [11, Theorem 6.11]. The argument is sketched for the ease of the reader. It suffices to prove that the restriction

$$\xi:\widehat{U_{\underline{c}}}\longrightarrow\varinjlim\mathcal{D}^f(1)$$

is sequentially continuous to complete the proof. Let $(u_n)_{n\geqslant0}$ be a sequence of $\widehat{U_{\underline{c}}}$ which converges to $u_\infty$. Then the sequence of execution paths $(\mathbb{P}^1\psi(\xi(u_n)))_{n\geqslant0}$ converges to $\mathbb{P}^1\psi(\xi(u_\infty))$, and therefore, it converges pointwise. All execution paths $\mathbb{P}^1\psi(\xi(u_n))$ for $n\geqslant0$ and $\mathbb{P}^1\psi(\xi(u_\infty))$ belong to the image of $\mathbb{P}^{top}\widehat{g}_{\underline{c}}$ (see Notation 5.2), this image being closed in $\mathbb{P}^{top}X_\lambda$ by Theorem 5.9. Besides, each subsequence of $(\mathbb{P}^1\psi(\xi(u_n)))_{n\geqslant0}$ has a limit point by Theorem 5.9. This limit point is unique since

$$\mathbb{P}^1\psi:\varinjlim\mathcal{D}^f(1)\longrightarrow\mathbb{P}^{top}X$$

is a bijection. The proof is complete thanks to Lemma 5.8. □

7.4. **Corollary.** *Suppose that $A$ is a cellular $\mathcal{P}$-multipointed d-space. Consider a pushout diagram of $\mathcal{P}$-multipointed d-spaces*

$$\begin{array}{ccc} \mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^{n-1}) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n}) & \longrightarrow & X \end{array}$$

*with $n \geqslant 0$. Then there is the pushout diagram of Moore flows*

$$\begin{array}{ccc} \mathbb{M}^{\mathcal{P}}(\mathrm{Glob}^{\mathcal{P}}(\mathbf{S}^{n-1})) = \mathrm{Glob}(\mathbb{F}_1^{\mathcal{P}^{op}}\mathbf{S}^{n-1}) & \longrightarrow & \mathbb{M}^{\mathcal{P}}(A) \\ \downarrow & & \downarrow \\ \mathbb{M}^{\mathcal{P}}(\mathrm{Glob}^{\mathcal{P}}(\mathbf{D}^{n})) = \mathrm{Glob}(\mathbb{F}_1^{\mathcal{P}^{op}}\mathbf{D}^{n}) & \longrightarrow & \mathbb{M}^{\mathcal{P}}(X). \end{array}$$

7.5. **Theorem.** *Consider the adjunction $\mathbb{M}_!^{\mathcal{P}} \dashv \mathbb{M}^{\mathcal{P}}$ between $\mathcal{P}$-multipointed d-spaces and $\mathcal{P}$-flows. Then the unit map and the counit map induce isomorphisms on q-cofibrant objects. This adjunction is a Quillen equivalence between the q-model structures of $\mathcal{P}$-multipointed d-spaces and of $\mathcal{P}$-flows.*

*Proof.* From Corollary 7.4 and Theorem 4.8, we deduce that the unit map and the counit map are isomorphisms on cellular objects, and then, on q-cofibrant objects since the retract of an isomorphism is an isomorphism. From this fact and the fact that all objects are q-fibrant, we deduce that the Quillen adjunction is a Quillen equivalence. See the proofs of [11, Theorem 7.6, Corollary 7.9 and Theorem 8.1] for further details. □

7.6. **Definition.** The category of small topologically enriched semicategories is isomorphic to the category of **1**-flows. This category is denoted by **Flow** and its objects are called *flows* (without using the prefixes Moore or **1**).

7.7. **Notation.** Let $\mathcal{C}$ be a small category. The *constant diagram functor* is denoted by

$$\Delta_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathbf{Top}.$$

By [10, Proposition 10.5], the constant diagonal functor induces a functor denoted by

$$\mathbb{M} : \mathbf{Flow} \longrightarrow \mathcal{P}\mathbf{Flow}$$

such that $\mathbb{M}(X)^0 = X^0$ and such that $\mathbb{P}_{\alpha,\beta}\mathbb{M}(X) = \Delta_{\mathcal{P}^{op}}(\mathbb{P}_{\alpha,\beta}X)$ for all $(\alpha, \beta) \in X^0 \times X^0$. We refer to [10, Proposition 10.5] for further details.

7.8. **Corollary.** *([11, Theorem 8.14] for $\mathcal{G}$ and $\mathcal{M}$) Let $X$ be a $\mathcal{P}$-multipointed d-space. There exists a flow $\mathrm{cat}(X)$ with $\mathrm{cat}(X)^0 = X^0$, $\mathbb{P}_{\alpha,\beta}\mathrm{cat}(X) = \mathbb{P}_{\alpha,\beta}X$ (see Notation 3.9) and the composition law $* : \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \to \mathbb{P}_{\alpha,\gamma}X$ is for every triple $(\alpha, \beta, \gamma) \in$*

$X^0 \times X^0 \times X^0$ *the unique map making the following diagram commutative:*

$$\begin{array}{ccc} \mathbb{P}^{top}_{\alpha,\beta}X \times \mathbb{P}^{top}_{\beta,\gamma}X & \xrightarrow{*_N} & \mathbb{P}^{top}_{\alpha,\gamma}X \\ \downarrow{\scriptstyle [-]_{\alpha,\beta}\times[-]_{\beta,\gamma}} & & \downarrow{\scriptstyle [-]_{\alpha,\gamma}} \\ \mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X & \longrightarrow & \mathbb{P}_{\alpha,\gamma}X \end{array}$$

*where* $*_N$ *is the normalized composition (cf. Definition 3.1). The mapping* $X \mapsto \mathrm{cat}(X)$ *induces a functor from* $\mathcal{P}\mathbf{dTop}$ *to* $\mathbf{Flow}$*. It takes q-cofibrant* $\mathcal{P}$*-multipointed d-spaces to q-cofibrant flows. Its total left derived functor in the sense of [4] induces an equivalence of categories between the homotopy categories of the q-model structures.*

*Proof.* The existence of the functor $\mathrm{cat} : \mathcal{P}\mathbf{dTop} \to \mathbf{Flow}$ is proved in [8, Theorem 7.2] for $\mathcal{P} = \mathcal{G}$. The last part is mutatis mutandis the proof of [11, Theorem 8.14] by replacing $\mathcal{G}$ by $\mathcal{P}$ and by using Theorem 7.5. We recall the definition of the functors in what follows for the ease of the reader:

$$(\mathbf{L}cat) : \mathcal{P}\mathbf{dTop} \xrightarrow{(-)^{cof}} \mathcal{P}\mathbf{dTop} \xrightarrow{cat} \mathbf{Flow}$$

$$(\mathbf{L}cat)^{-1} : \mathbf{Flow} \xrightarrow{\mathbb{M}} \mathcal{P}\mathbf{Flow} \xrightarrow{(-)^{cof}} \mathcal{P}\mathbf{Flow} \xrightarrow{\mathbb{M}^{\mathcal{P}}_{!}} \mathcal{P}\mathbf{dTop}$$

where $(-)^{cof}$ is a q-cofibrant replacement functor. □

Proposition 7.9 should have been put in [10] as an application of the results of the latter paper: it is an omission. It is used in Theorem 7.10. The inclusion functor $i : \mathcal{G} \subset \mathcal{M}$ induces an enriched functor

$$i^* : [\mathcal{M}^{op}, \mathbf{Top}] \longrightarrow [\mathcal{G}^{op}, \mathbf{Top}]$$

from $\mathcal{M}$-spaces to $\mathcal{G}$-spaces. It is a right adjoint between the underlying categories, the left adjoint being the enriched left Kan extension along $i$ given by the formula

$$\mathrm{Lan}_i(D) = \int^{\ell} \mathcal{M}(-, i(\ell)) \times D(\ell).$$

7.9. **Proposition.** *The functor* $i^* : [\mathcal{M}^{op}, \mathbf{Top}] \to [\mathcal{G}^{op}, \mathbf{Top}]$ *induces a functor*

$$i^* : \mathcal{M}\mathbf{Flow} \longrightarrow \mathcal{G}\mathbf{Flow}$$

*which is a right Quillen equivalence between the q-model structures of* $\mathcal{M}\mathbf{Flow}$ *and* $\mathcal{G}\mathbf{Flow}$*.*

*Proof.* By [10, Section 6], a $\mathcal{P}$-flow consists of a set of states $X^0$, for each pair $(\alpha, \beta)$ of states a $\mathcal{P}$-space $\mathbb{P}_{\alpha,\beta}X$ of $[\mathcal{P}^{op}, \mathbf{Top}]_0$ and for each triple $(\alpha, \beta, \gamma)$ of states an associative composition law $* : \mathbb{P}^{\ell_1}_{\alpha,\beta}X \times \mathbb{P}^{\ell_2}_{\beta,\gamma}X \to \mathbb{P}^{\ell_1+\ell_2}_{\alpha,\gamma}X$ which is natural with respect to $(\ell_1, \ell_2)$ in an obvious way. From a $\mathcal{M}$-flow $D$, we therefore obtain a $\mathcal{G}$-flow $i^*(D)$ with $D^0 = i^*(D)^0$ and $\mathbb{P}_{\alpha,\beta}i^*(D) = i^*(\mathbb{P}_{\alpha,\beta}D)$. By the explicit calculation of limits in $\mathcal{M}\mathbf{Flow}$ and in $\mathcal{G}\mathbf{Flow}$ made in [10, Theorem 6.8], and since limits are calculated objectwise in $[\mathcal{M}^{op}, \mathbf{Top}]_0$ and $[\mathcal{G}^{op}, \mathbf{Top}]_0$ by [9, Proposition 5.3], the functor $i^* : \mathcal{M}\mathbf{Flow} \to \mathcal{G}\mathbf{Flow}$ is limit-preserving. By [10, Theorem 6.13], the $\mathcal{P}$-space of execution paths functor $\mathbb{P} : \mathcal{P}\mathbf{Flow} \to [\mathcal{P}^{op}, \mathbf{Top}]_0$ of Definition 2.13 is a right adjoint for any reparametrization category $\mathcal{P}$ [4].

[4]Note that this fact holds because we work with locally presentable categories: see [12, Theorem 5.10].

Therefore it is accessible by [1, Theorem 1.66]. Since colimits are calculated objectwise in $[\mathcal{M}^{op}, \mathbf{Top}]_0$ and $[\mathcal{G}^{op}, \mathbf{Top}]_0$ by [9, Proposition 5.3], the functor $i^* : \mathcal{M}\mathbf{Flow} \to \mathcal{G}\mathbf{Flow}$ is then accessible. Therefore it is a right adjoint by [1, Theorem 1.66]. The functor $i^* : \mathcal{M}\mathbf{Flow} \to \mathcal{G}\mathbf{Flow}$ preserves q-fibrations and trivial q-fibrations by definition of the q-model structures. Consequently, it is a right Quillen adjoint. Thus the commutative diagram of right adjoints

$$\begin{array}{ccc} \mathbf{Top} & = & \mathbf{Top} \\ \downarrow{\scriptstyle \Delta_{\mathcal{M}^{op}}} & & \downarrow{\scriptstyle \Delta_{\mathcal{G}^{op}}} \\ [\mathcal{M}^{op}, \mathbf{Top}] & \xrightarrow{i^*} & [\mathcal{G}^{op}, \mathbf{Top}] \end{array}$$

gives rise by [10, Proposition 10.7] to the commutative diagram of right Quillen adjoints

$$\begin{array}{ccc} \mathbf{Flow} & = & \mathbf{Flow} \\ \downarrow{\scriptstyle \mathbb{M}} & & \downarrow{\scriptstyle \mathbb{M}} \\ \mathcal{M}\mathbf{Flow} & \xrightarrow{i^*} & \mathcal{G}\mathbf{Flow} \end{array}$$

where **Flow** is equipped with its q-model structure. By [10, Theorem 10.9], the two vertical right Quillen adjoints are right Quillen equivalences. The proof is complete thanks to the two-out-of-three property. □

We conclude with the following comparison theorem:

7.10. **Theorem.** *The inclusion functor $i : \mathcal{G} \subset \mathcal{M}$ induces a functor*

$$j : \mathcal{M}\mathbf{dTop} \longrightarrow \mathcal{G}\mathbf{dTop}.$$

*There is the commutative square of right Quillen equivalences between the four q-model structures*

$$\begin{array}{ccc} \mathcal{M}\mathbf{dTop} & \xrightarrow{j} & \mathcal{G}\mathbf{dTop} \\ \downarrow{\scriptstyle \mathbb{M}^{\mathcal{M}}} & & \downarrow{\scriptstyle \mathbb{M}^{\mathcal{G}}} \\ \mathcal{M}\mathbf{Flow} & \xrightarrow{i^*} & \mathcal{G}\mathbf{Flow} \end{array}$$

*Proof.* It is easy to see that the diagram is commutative: each functor is a forgetful functor indeed. The forgetful functor $\Omega : \mathcal{P}\mathbf{dTop} \to \mathbf{MTop}$ from $\mathcal{P}$-multipointed $d$-spaces to multipointed spaces being topological by Theorem 3.16 for $\mathcal{P}$ equal to $\mathcal{G}$ or $\mathcal{M}$, the functor $j : \mathcal{M}\mathbf{dTop} \to \mathcal{G}\mathbf{dTop}$ is limit-preserving and finitely accessible: *finitely* because a multipointed $d$-space is equipped with a *set* of execution paths and because the $\Omega$-final structure is given by the *finite* Moore compositions by Theorem 3.16. By [1, Theorem 1.66], the functor $j : \mathcal{M}\mathbf{dTop} \to \mathcal{G}\mathbf{dTop}$ is therefore a right adjoint. It takes (trivial resp.) q-fibrations to (trivial resp.) q-fibrations by definition of them. Thus it is a right Quillen adjoint. The two vertical functors are right Quillen equivalences by Theorem 7.5. The bottom horizontal functor is a right Quillen equivalence by Proposition 7.9. The proof is complete thanks to the two-out-of-three property. □

7.11. **Notation.** Write $\mathcal{F}_{\mathcal{G}}^{\mathcal{M}} : \mathcal{G}\mathbf{dTop} \to \mathcal{M}\mathbf{dTop}$ for the left adjoint of the inclusion functor $j : \mathcal{M}\mathbf{dTop} \subset \mathcal{G}\mathbf{dTop}$.

The unit of the adjunction

$$X \longrightarrow j(\mathcal{F}_{\mathcal{G}}^{\mathcal{M}}(X))$$

preserves the underlying space and the set of states. It induces a map from the space of execution paths of $X$ to its closure under the reparametrization by all maps of $\mathcal{M}$. It is a weak homotopy equivalence when $X$ is a q-cofibrant $\mathcal{G}$-multipointed $d$-space by Theorem 7.10: this assertion is also a consequence of Corollary 7.4 and Theorem 4.8 and of the fact that

$$\mathcal{F}_{\mathcal{G}}^{\mathcal{M}}(\mathrm{Glob}^{\mathcal{G}}(Z)) = \mathrm{Glob}^{\mathcal{M}}(Z)$$

for all topological spaces $Z$. The counit map

$$\mathcal{F}_{\mathcal{G}}^{\mathcal{M}}(j(Y)) \xrightarrow{\cong} Y$$

is an isomorphism for all $\mathcal{M}$-multipointed $d$-spaces $Y$ by definition of $\mathcal{F}_{\mathcal{G}}^{\mathcal{M}}$. By Theorem 7.10, we deduce that $\mathcal{F}_{\mathcal{G}}^{\mathcal{M}}(j(Y)^{cof})$ is a q-cofibrant replacement of $Y$ in $\mathcal{M}\mathbf{dTop}$ where $j(Y)^{cof}$ is a q-cofibrant replacement of $j(Y)$ in $\mathcal{G}\mathbf{dTop}$. The latter fact can be proved directly by obtaining a q-cofibrant replacement by the small object argument.

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