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# On Global Asymptotic Stability of Heterogeneous Modular Networks with Three Time-Scales

Anes Lazri   Elena Panteley   Antonio Loría

**Abstract**—We analyse the collective behaviour of relatively large networks with the characteristic that nodes may be compartmentalised in modules. These are groups of systems that may be regarded as subnetworks in which each group of nodes achieves consensus with a certain rapidity. The dynamics of such networks exhibits, at least, two time-scales, for which singular perturbations methods may be used to assess the overall behaviour. In this paper, we demonstrate that there are actually three natural time-scales and, accordingly, three interconnected dynamical systems with distinct speeds of convergence coexist. Our main statement establishes conditions for global asymptotic stability of the origin in such networked systems. In particular, we focus on bilinear heterogeneous systems and provide an illustrative example concerning chaotic oscillators.

## I. INTRODUCTION

Modular networks are, generally speaking, networks of a large number of nodes that are compartmentalised in groups of nodes called modules. These are sparsely connected among themselves and the nodes contained within are densely interconnected [1], [2], [3]. Because of their complexity and large dimension, they are difficult to analyse; one approach consists in considering many agents in a module as a single node. This approach has been used both for undirected [4], [5], [6] and directed networks [7], [8]. This view is specially fit in networks such that agents within a module reach consensus relatively fast, before achieving consensus with nodes of other modules. Or, in other words, modules that have reached their own individual consensus find agreement among themselves at a slower pace [6], [9]. This is specially true if the coupling gain within a module is considerably larger than the one amongst modules.

Such considerations lead naturally to multi-time scale models that can be studied using singular-perturbation theory [10], using the inverse of the coupling gain as singular parameter [11]. In the latter, is used to define a two-time scale modelling of a heterogeneous network ; the authors show that the emergent dynamics correspond to a slow subsystem while the synchronisation errors form a fast subsystem. Singular perturbation is also used in [12] to show that, for interconnected linear systems with switching interconnection topology and linear coupling, if the coupling gain is sufficiently high, the synchronised behaviour can be approximated by a reduced order switching system.

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Using time-scale separation for sparse dynamic networks goes back to [13], where the authors focus on the consensus problem for linear systems and develop a two-time scale model of the network, allowing to approximate the overall behaviour of a clustered network. This idea is extended in [4], where a nonlinear network of internally dense and externally sparse interconnections is studied. The densely connected nodes in these areas synchronise in the fast time scale, and behave as aggregate nodes that dominate the slow dynamics of the network. A fundamental assumption is that the interaction network consists of sparsely connected modules of densely connected agents.

In addition to control problems, analysis problems have also been addressed for systems with three different time scales. In [14] the authors study the synchronisation of *linear* systems interconnected over a modular network. In order to explain the three-time scales behaviour, the authors use transformations taking into account the spectral properties of the network Laplacian.

In order to explicit the dynamics of variables evolving on different time scales, the authors generally resort to numerous transformations. In [15], the authors propose a projection-based model reduction method for multi-agent systems defined on a graph. The proposed method allows to gather many nodes of a graph into one and gives a maximum bound of the  $H_2$ -norm comparing the input-output behaviour of the original system and the reduced system, for simple integrators. Moreover, many projections by eigenvalue assignment can also be found in the literature. In [16], based on the eigenvalues of the Laplacian, the authors propose a method of order reduction allowing to obtain a reduced model of the same structure (the coupling remains diffusive) and thus to keep the synchronisation properties for a network with linear dynamics.

In this paper, we present a three-time scales modelling of undirected modular networks based on the values of the interconnection gains within clusters and between clusters, but also on the intensity of interconnections within clusters. The goal of this modelling is to facilitate the demonstration of conditions allowing to obtain the global stability of the origin for the dynamics of such networks.

The remainder of this paper is organised as follows. In section II we lay down our main hypotheses; in Section III we describe develop the three-timescales network model; in Section IV we present our main statement; numerical simulation results are provided in Section IV; and we conclude with some remarks in Section VI.

## II. PROBLEM FORMULATION

We consider the problem of stabilising a (large) group of  $N$  heterogeneous dynamical systems interacting over a network under the following

*Standing assumption:* The network's graph is connected and undirected and the network's topology is invariant.

It is further assumed that the network comprises *modules*, which consist in subgroups of nodes that are densely interconnected and the modules themselves are sparsely connected—see Figure 1. Such types of networks are common, *e.g.*, in large-scale power networks [13]. In the latter, subnetworks concentrated in smaller geographical regions comprise many more links than there are connecting one region to another.

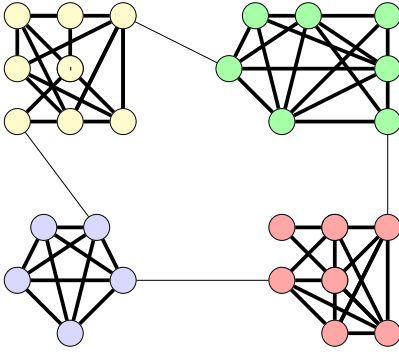


Fig. 1. Schematic representation of an undirected modular network composed of  $N$  nodes and  $m$  modules; the nodes within one module are densely connected while the modules are sparsely connected among themselves

Under the standing assumption, the network's topology is captured by the properties of the so-called Laplacian matrix  $L \in \mathbb{R}^{N \times N}$ , which possesses a unique zero eigenvalue and all the others have positive real part. Furthermore, in view of the modular structure of the network the Laplacian may be decomposed into two matrices that we refer to as internal and external Laplacians and are defined, respectively, as

$$L^I = \text{blockdiag}[L_1^I, L_2^I, \dots, L_m^I], \quad L_k^I \in \mathbb{R}^{N_k \times N_k}, \quad k \leq m$$

and  $L^E = L - L^I$ . The entries of  $L_k^I$  and  $L$  are respectively given by

$$\ell_{i,j}^{Ik} = \begin{cases} -a_{ij}^{Ik}, & i \neq j \\ \sum_{\substack{\ell=1 \\ \ell \neq i}}^{N_k} a_{i\ell}^{Ik}, & i = j \end{cases}, \quad \ell_{i,j} = \begin{cases} -a_{ij}, & i \neq j \\ \sum_{\substack{\ell=1 \\ \ell \neq i}}^N a_{i\ell}, & i = j \end{cases}$$

where  $a_{i,j}^{Ik} > 0$  if there is a link between the nodes belonging to the  $k$ th module, labelled  $(k, i)$  and  $(k, j)$  and  $a_{i,j}^{Ik} = 0$  otherwise. For the Laplacian  $L$ , as is customary,  $a_{i,j} > 0$  if there is a link between the  $i$ th and  $j$ th nodes within the entire network, *i.e.*, for all  $i, j \leq N$  and  $a_{i,j} = 0$  otherwise.

*Assumption 1:* For each agent the influence between the agents of the same cluster is higher than the influence among clusters, *i.e.*,  $\sigma^E |L^E| < \sigma^I |L^I|$ , where  $|\cdot|$  denotes the induced  $L_2$  norm. •

We address the problem of stability analysis and set-point stabilisation of the origin for such networks of nonlinear systems with dynamics given by

$$\dot{x}_{k,i} = A_{k,i}(x_{k,i})x_{k,i} + u_{k,i}, \quad k \leq m, \quad i \leq N_k, \quad (1)$$

where  $m$  denotes the number of modules in the network,  $N_k$  denotes the number of nodes in the  $k$ th module, for each  $(k, i)$ ,  $x_{k,i} \in \mathbb{R}^{n_x}$  denotes the state of the  $i$ th node within the  $k$ th module,  $u_{k,i}$  denotes its control input, and  $A_{k,i}(x_{k,i}) \in \mathbb{R}^{n_x \times n_x}$ . In addition, we state the following.

*Assumption 2:* For each  $(k, i)$ ,  $A_{k,i}$  is continuously differentiable and of linear growth, that is, there exist  $\beta_1, \beta_2$ , such that

$$|A_{k,i}(x_{k,i})| \leq \beta_1 + \beta_2 |x_{k,i}| \quad \forall x_{k,i} \in \mathbb{R}^{n_x}. \quad (2)$$

In addition, the systems's solutions are bounded. •

*Remark 1:* The class of systems that we consider includes a variety of physical systems, notably nonlinear oscillators, such as bilinear chaotic systems. The assumption on semi-passivity is imposed to ensure that the solutions are ultimately bounded [17]. •

Now, because of the modular structure of the network, we consider that it is driven by fully distributed consensus control laws of the form

$$u_{k,i} = u_{k,i}^I + u_{k,i}^E, \quad (3a)$$

$$u_{k,i}^I = -\sigma^I \sum_{j=1}^{N_k} a_{ij}^{Ik} (x_{k,i} - x_{k,j}) \quad (3b)$$

$$u_{k,i}^E = -\sigma^E \sum_{\substack{\ell \neq k \\ \ell=1}}^m \sum_{j=1}^{N_\ell} a_{ij}^{E\ell} (x_{k,i} - x_{\ell,j}) \quad (3c)$$

where  $\sigma^I$  is the interconnection gain between the agents within one cluster and  $\sigma^E$  corresponds to the interconnection gain among clusters, and  $a_{ij}^{E\ell}$  denotes each link between the  $i$ th node within the  $k$ th module and its neighbours in *other* modules, *i.e.*, the  $j$ th neighbour within the  $\ell$ th module for all  $j \leq N_\ell$  and  $\ell \leq m$ , with  $\ell \neq k$ . We have that  $a_{ij}^{E\ell} > 0$  if such link exists and  $a_{ij}^{E\ell} = 0$  otherwise.

In multi-variable compact form, these control laws become  $u^I = -\sigma^I [L^I \otimes I_{n_x}]x$  and  $u^E = -\sigma^E [L^E \otimes I_{n_x}]x$  and  $u = u^I + u^E$ , where  $u := [u_{1,1}^\top \dots u_{1,N_1}^\top \dots u_{m,1}^\top \dots u_{m,N_m}^\top]$ . So, defining  $x := [x_{1,1}^\top \dots x_{1,N_1}^\top \dots x_{m,1}^\top \dots x_{m,N_m}^\top]$ , and  $A(x) := \text{blockdiag}[A_{1,1}(x_{1,1}), A_{1,2}(x_{1,2}), \dots, A_{m,N_m}(x_{m,N_m})]$ , we write the closed-loop dynamics

$$\dot{x} = A(x)x - \sigma^I [L^I \otimes I_{n_x}]x - \sigma^E [L^E \otimes I_{n_x}]x. \quad (4)$$

After [17], the collective behaviour of a connected network of nonlinear heterogeneous systems, with associated Laplacian matrix  $L$ , may be completely assessed by the study of an “average” dynamical system with state  $x_s = [v_\ell^\top \otimes I_n]x$ , where  $v_\ell$  corresponds to the left eigen-vector associated to the unique zero eigenvalue of  $L$ , and dynamics

$$\dot{x}_s = F_s(x_s), \quad (5)$$

and the *synchronisation* errors  $e$ , which are defined relative to the trajectories of (5), *i.e.*,  $e := x - [\mathbf{1}_N \otimes I_n]x_s$ . Then, we say that the systems tend to *dynamic consensus* if  $e \rightarrow 0$ . That is, if all systems tend to mutually synchronise and, in doing so, their behaviour approaches asymptotically that of (5). This means, in particular, that the origin is globally asymptotically stable for the networked system (4) if, both, the average and synchronisation error dynamics are globally asymptotically stable at the origin  $\{(x_s, e) = (0, 0)\}$ .

The rest of the paper, is devoted to analysing the stability of the origin  $\{x = 0\}$ , for Eq. (4). Our main statement is that for sufficiently large coupling gains  $\sigma^I$  and  $\sigma^E$ , the origin is globally asymptotically stable. The analysis relies on the fact that by virtue of an appropriate change of coordinates inspired from [17], Eq. (4) may be written in singular-perturbations form [10]. As a matter of fact, we demonstrate that *three* different scales of time co-exist corresponding to three underlying dynamics: the fastest corresponds to what we call the intra-cluster synchronisation errors, a moderately paced dynamics corresponds to that of the average motion for each module, and a slow dynamics pertains to that of the average of average motions. These dynamical models are described in detail next.

### III. TIME-SCALE SEPARATION

In view of the modularity of networks such as the one depicted in Figure 1 and the fact that nodes within each cluster synchronise “rapidly”, it results sensible to reduce the network’s dimension, roughly, by considering that each module behaves as one node. To that end, as in [15], we identify the network’s modules and label them  $C_k$ . The set of such modules forms a so-called partition that we denote by  $\pi = \{(C_1), (C_2), \dots, (C_m)\}$ , and to this partition corresponds a characteristic matrix  $P(\pi) \in \mathbb{R}^{N^m}$  defined by

$$P(\pi) = \text{blockdiag}_{k \leq m} \{\mathbf{1}_{N_k}\} \quad (6)$$

that may be employed to define new coordinates that lead to a model reduction of the network [15].

In this paper, we also rely on coordinate transformations, albeit different from the one used in [15], to exhibit the triple time-scale separation mentioned above, we perform certain changes of variable, following [17].

#### A. Intra-modular dynamics

Consider, for the time-being, the  $k$ th module which, itself, consists in a connected (sub)network with Laplacian  $L_k^I$  and containing  $N_k$  nodes. This matrix contains a unique zero eigenvalue with associated left eigenvector  $v_{\ell k} := \frac{1}{N_k} \mathbf{1}_{N_k}$ , with  $\mathbf{1}_{N_k} := [1 \ 1 \ \dots \ 1]^\top$ , and  $N_k - 1$  others with positive real part. Moreover, it admits the Jordan’s decomposition

$$L_k^I = V_k \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_k^I \end{bmatrix} V_k^\top \quad (7)$$

where  $\Lambda_k^I \in \mathbb{R}^{(N_k-1) \times (N_k-1)}$  is positive definite and  $V_k \in \mathbb{R}^{N_k \times N_k}$  is an invertible transformation matrix defined as  $V_k = [\mathbf{1}_{N_k} \ Q_k]$ . That is, its first column corresponds to the

right eigenvector of  $L_k^I$  and has other  $N_k - 1$  linearly independent columns gathered in the matrix  $Q_k \in \mathbb{R}^{N_k \times (N_k-1)}$ , which admits the left inverse  $Q_k^\dagger := (Q_k^\top Q_k)^{-1} Q_k^\top$ . Thus, we introduce the new variables  $\zeta_k \in \mathbb{R}^{n_x}$  and  $\xi_k \in \mathbb{R}^{n_x N_k - 1}$

$$\begin{bmatrix} \zeta_k \\ \xi_k \end{bmatrix} = \begin{bmatrix} [v_{\ell k}^\top \otimes I_{n_x}] \\ [Q_k^\dagger \otimes I_{n_x}] \end{bmatrix} \bar{x}_k,$$

where  $\bar{x}_k := [x_{k,1}^\top \ x_{k,2}^\top \ \dots \ x_{k,N_k}^\top]^\top$  denotes the vector of states corresponding to all the nodes in the  $k$ th module. The state variable  $\zeta_k$  may be regarded as the weighted average of the  $k$ th module’s nodes’ states while  $\xi_k$  is a projection of the synchronisation errors,  $e_k \in \mathbb{R}^{n_x N_k}$ , that is,

$$e_k := \bar{x}_k - [\mathbf{1}_{N_k} \otimes I_{n_x}] \zeta_k = [Q_k \otimes I_{n_x}] \xi_k. \quad (8)$$

Hence,  $\xi_k = 0$  if and only if the nodes in the  $k$ th cluster synchronise with the dynamics of the corresponding averaged system.

After the previous observations we have that all the nodes states in the  $k$ th module,  $x_{k,i}$ , converge to zero if and only if so do  $\xi_k$  and  $\zeta_k$ . But since the modules are not isolated, we must study the dynamics of  $\xi_k$  and  $\zeta_k$  for all the modules. To that end, we introduce the variables  $\zeta \in \mathbb{R}^m$ ,  $\zeta := [\zeta_1 \ \dots \ \zeta_m]$  and  $\xi \in \mathbb{R}^{N_k - 1 m}$ ,  $\xi := [\xi_1^\top \ \dots \ \xi_m^\top]^\top$  defined by

$$\begin{bmatrix} \zeta \\ \xi \end{bmatrix} = \begin{bmatrix} [(P^\top P)^{-1} P^\top] \otimes I_{n_x} \\ [(Q^\top \ Q)^{-1} Q^\top] \otimes I_{n_x} \end{bmatrix} x = \begin{bmatrix} P^\dagger \\ Q^\dagger \end{bmatrix} x, \quad (9)$$

where  $P$  is defined in (6) and  $Q = \text{blockdiag}_{k \leq m} \{Q_k\}$

Hence, differentiating on both sides of (9) we obtain

$$\dot{\zeta} = f_1(\zeta, \xi) \quad (10a)$$

$$\dot{\xi} = -\sigma^I (\Lambda^I \otimes I_{n_x}) \xi + f_2(\zeta, \xi), \quad (10b)$$

for which we used  $L^I P = (P^\top P)^{-1} P^\top L^I = 0$  and  $\Lambda^I \in \mathbb{R}^{(N-m) \times (N-m)}$  is defined as  $\Lambda^I := \text{blockdiag}_{k \leq m} \{\Lambda_k^I\}$ ,

$$f_1(\zeta, \xi) = P^\dagger A (\bar{P} \zeta + \bar{Q} \xi) [\bar{P} \zeta + \bar{Q} \xi] - \sigma^E P^\dagger [L^E \otimes I_{n_x}] [\bar{P} \zeta + \bar{Q} \xi], \quad (11)$$

$$f_2(\zeta, \xi) = Q^\dagger A (\bar{P} \zeta + \bar{Q} \xi) [\bar{P} \zeta + \bar{Q} \xi] - \sigma^E Q^\dagger [L^E \otimes I_{n_x}] [\bar{P} \zeta + \bar{Q} \xi],$$

$\bar{P} := (P \otimes I_{n_x})$ , and  $\bar{Q} := (Q \otimes I_{n_x})$ .

From Eqs. (10) and the expressions above, we see the following. Considering  $f_\xi$  as composed of perturbing terms depending on  $\zeta$  and on “output injection” terms depending on  $\xi$ , one may infer that the  $-\Lambda^I$  being Hurwitz and the coupling gain  $\sigma^I$  being relatively large, such that  $\sigma^I \gg \sigma^E$ ,  $\xi \rightarrow 0$ . Then, on the synchronisation manifold  $\{\xi = 0\}$ , all the nodes behave as their respective average dynamics, comprised in  $\dot{\zeta} = f_1(\zeta, 0)$ . This rationale is made more precise in Section IV.

#### B. Inter-modular dynamics

Now we investigate the dynamics of (10), which roughly corresponds to the network of averaged modules. In other words, we consider the modules as nodes in a reduced network of  $m$  elements—cf. [16],[18], [19]. Then, to asses

the behaviour of this reduced network we apply, again, a similar change of variable as above. To that end, we start by observing that the reduced-order network is also connected and has the associated Laplacian

$$\bar{L}^E = \begin{bmatrix} \frac{a_{11}^E}{N_1} & -\frac{a_{12}^E}{N_1} & \dots & -\frac{a_{1m}^E}{N_1} \\ \vdots & \ddots & \dots & \vdots \\ \vdots & \dots & \ddots & \vdots \\ \frac{a_{m1}^E}{N_m} & \frac{a_{m2}^E}{N_m} & \dots & \frac{a_{mm}^E}{N_m} \end{bmatrix}. \quad (12)$$

After [15, Lemma 3], the eigenvalues of  $\bar{L}^E$  satisfy  $\lambda_1(\bar{L}^E) = 0$  and  $\lambda_i(\bar{L}^E) > 0, \forall i \in \{2, 3, \dots, m\}$ . Therefore, as  $L$  and  $L^I, L^E$  admits a Jordan decomposition; that is, there exists a nonsingular matrix  $W \in \mathbb{R}^{m \times m}$  such that

$$\bar{L}^E = W \begin{bmatrix} 0 & 0 \\ 0 & \Lambda^E \end{bmatrix} W^{-1}, \quad (13)$$

where  $\Lambda^E \in \mathbb{R}^{(m-1) \times (m-1)}$  is the diagonal matrix defined by the eigenvalues of  $\bar{L}^E$  with positive real part, that is,  $\Lambda^E := \text{diag}_{i \in \{2, 3, \dots, m\}} \{\lambda_i(\bar{L}^E)\}$ . As a matter of fact,

$$W = [\mathbf{1}_m \ W_1], \quad (14)$$

$$W^{-1} = \begin{bmatrix} v_\ell^\top \\ W_1^\dagger \end{bmatrix}, \quad W_1^\dagger = (W_1^\top W_1)^{-1} W_1^\top.$$

Then, as before, following [17], we introduce the new variables

$$\begin{bmatrix} x_e \\ \eta \end{bmatrix} = \begin{bmatrix} v_\ell^\top \otimes I_{n_x} \\ W_1^\dagger \otimes I_{n_x} \end{bmatrix} \zeta; \quad (15)$$

the variable  $x_e \in \mathbb{R}^{n_x}$  corresponds to the ‘‘weighted average of averages  $\zeta_k$ ’’, while  $\eta$  is a projection of all the synchronisation errors  $\zeta_k - x_e$ . That is, the vector of synchronisation errors corresponds to

$$e_\eta := \zeta - [\mathbf{1}_m \otimes I_{n_x}] x_e = [W_1 \otimes I_{n_x}] \eta. \quad (16)$$

That is, when  $\eta = 0$  all the states of the reduced network (of modules) reach consensus with the  $x_e$ -system. To analyse the behaviour of  $e_\eta$  and  $x_e$ , we differentiate on both sides of (15). We obtain

$$\dot{x}_e = f_e(x_e, \eta, \xi) \quad (17a)$$

$$\dot{\eta} = -\sigma^E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, \xi), \quad (17b)$$

where

$$f_e(x_e, \eta, \xi) = [v_\ell^\top \otimes I_{n_x}] f_1 \left( [W \otimes I_{n_x}] \begin{bmatrix} x_e \\ \eta \end{bmatrix}, \xi \right)$$

$$f_\eta(x_e, \eta, \xi) = [W_1^\dagger \otimes I_{n_x}] f_1 \left( [W \otimes I_{n_x}] \begin{bmatrix} x_e \\ \eta \end{bmatrix}, \xi \right) + \sigma^E \bar{\Lambda}^E \eta,$$

and  $\Lambda^E \in \mathbb{R}^{(m-1) \times (m-1)}$  is defined as  $\bar{\Lambda}^E = \Lambda^E \otimes I_{n_x}$ . We remark that in the argument of  $f_1$  above, we used (16) and

$$\begin{bmatrix} v_\ell^\top \otimes I_{n_x} \\ W_1^\dagger \otimes I_{n_x} \end{bmatrix}^{-1} = [W \otimes I_{n_x}]. \quad (18)$$

### C. Multi-timescale overall network dynamics

Define  $\epsilon_E := 1/\sigma^E$  and  $\epsilon_I := 1/\sigma^I$ , which satisfy, by assumption,  $\epsilon_I \ll \epsilon_E \ll 1$ . After the previous computations, we see that

$$\dot{x}_e = f_e(x_e, \eta, \xi) \quad (19a)$$

$$\epsilon_E \dot{\eta} = -\bar{\Lambda}^E \eta + \epsilon_E f_\eta(x_e, \eta, \xi) \quad (19b)$$

$$\epsilon_I \dot{\xi} = -\bar{\Lambda}^I \xi + \epsilon_I f_\xi(x_e, \eta, \xi), \quad (19c)$$

where  $f_\eta$  is defined above and

$$f_\xi(x_e, \eta, \xi) = f_2 \left( [W \otimes I_{n_x}] \begin{bmatrix} x_e \\ \eta \end{bmatrix}, \xi \right).$$

The system (19) is in standard singular perturbation form [10], albeit with three time-scales. As explained at the end of Section II, there co-exist three dynamical systems: Eq. (19a), which is the slowest, corresponds to the weighted average of all nodes' states; Eq. (19b), which is moderately fast, corresponds to the projection of the inter-cluster synchronisation errors; and Eq. (19c), which is the fastest, corresponds to a projection of the intra-cluster synchronisation errors.

Our main statement, which is presented in the next section, establishes global asymptotic stability of the origin  $\{(x_e, \eta, \xi) = (0, 0, 0)\}$  for (19), which is equivalent to global asymptotic stability of the origin  $\{x = 0\}$  for (4).

## IV. MAIN RESULT

*Proposition 1:* Consider the networked system (4) under the Standing Assumption and Hypotheses 1 and 2. In addition, assume that for the system  $\dot{x}_e = f_e(x_e, 0, 0)$ , the origin  $x_e = 0$  is globally asymptotically stable and that there exists a continuously differentiable Lyapunov function  $V_e : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$  and a class  $\mathcal{K}_\infty$  function  $\alpha(x_e) = \sqrt{q_1 |x_e|^2}$ ,  $q_1 > 0$  such that

$$\frac{\partial V_e}{\partial x_e^\top} f_e(x_e, 0, 0) \leq -\alpha(x_e)^2 \quad (20)$$

and  $c_1 > 0$  such that,

$$\left| \frac{\partial V_e}{\partial x_e^\top} \right| \leq c_1 \alpha(x_e). \quad (21)$$

Then, there exists  $\sigma^{E^*} > 0$  and  $\sigma^{I^*} > 0$  such that, for all  $\sigma^I > \sigma^{I^*}$  and  $\sigma^E > \sigma^{E^*}$ , the origin of (4) is GAS.  $\square$

*Sketch of proof:* To analyse the stability of the origin for (19), we recursively use Lemma 1 from the Appendix. We use the lemma one first time with  $x := [x_e^\top \ \eta^\top]^\top$ ,  $z := \xi$ ,  $\epsilon := \epsilon_I$ ,

$$f(x, z) := \begin{bmatrix} f_e(x_e, \eta, \xi) \\ -\sigma^E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, \xi) \end{bmatrix},$$

$g_1(z) := -\bar{\Lambda}^I \xi$ , and  $g_2(x, z) := f_\xi(x_e, \eta, \xi)$  to conclude the main statement of global asymptotic stability for  $(x, z) = (0, 0)$ , that is, for  $\{(x_e, \eta, \xi) = (0, 0, 0)\}$ . To that end, according to Lemma 1 it must be established that both the reduced model  $\dot{x} = f(x, 0)$ , that is,

$$\dot{x}_e = f_e(x_e, \eta, 0) \quad (22a)$$

$$\dot{\eta} = -\sigma^E \bar{\Lambda}^E \eta + f_\eta(x_e, \eta, 0), \quad (22b)$$

and the boundary-layer system

$$\frac{d\xi}{d\tau_I} = -\bar{\Lambda}^I \xi, \quad \tau_I := t/\epsilon_I \quad (23)$$

which is expressed in a new time scale, are both globally asymptotically stable. For the latter, this is evident since  $-\bar{\Lambda}^I$  is Hurwitz and for (22) we apply Lemma 1 once more.

We consider now the reduced model (22) expressed in its standard singular-perturbation form (19a)-(19b). In this case, the boundary-layer system is

$$\frac{d\eta}{d\tau_E} = -\bar{\Lambda}^E \eta, \quad \tau_E := t/\epsilon_E \quad (24)$$

and the reduced model is  $\dot{x}_e = f_e(x_e, 0, 0)$ , where  $f_e(x_e, 0, 0)$  corresponds to  $[v_\ell^\top \otimes I_{n_x}] f_1(\zeta, \xi)$  with  $f_1$  as defined in (11) with  $\zeta = [v_\ell^\top \otimes I_{n_x}] x_e$  and  $\xi = 0$ . That is,  $\dot{x}_e = f_e(x_e, 0, 0)$  is equivalent to

$$\dot{x}_e = A_e(x_e) x_e, \quad (25)$$

with  $A_e(x_e) = [v_\ell^\top \otimes I_{n_x}] P^\dagger A \left( \bar{P} [v_\ell \otimes I_{n_x}] x_e \right) \bar{P} [v_\ell \otimes I_{n_x}]$ .

*Remark 2:* It is important to emphasise that the matrix  $A_e$  above satisfies Assumption 2 that is, the averaged dynamics (25) inherits the characteristics of the individual nodes—cf. [17]. In particular, it is of linear growth in  $x_e$  and, in view of the definition of  $P$ —see (6),  $A_e$  corresponds to an average of all the  $A_{k,i}$ s. More precisely,

$$A_e = \sum_{k=1}^m \frac{1}{m N_k} \sum_{i=1}^{N_k} A_{k,i} \left( \frac{x_e}{\sqrt{m}} \right). \quad (26)$$

## V. SIMULATIONS RESULTS

To illustrate our main result, we performed numerical simulations, using Matlab, for a modular network of  $N = 12$  agents, compartmentalised into  $m = 4$  modules with  $N_k = 3 \forall k \in \{1, 2, 3, 4\}$  and interconnected over the undirected graph showed in Figure 2 below

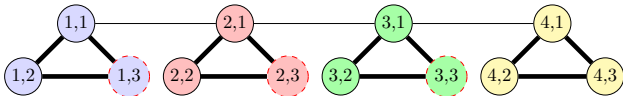


Fig. 2. Graph representing a modular network of twelve nodes organised into four modules: nine different Lorenz oscillators and three stable systems represented by red-circles

All the systems are of dimension three, with state  $x_{k,i} := [x_{k,i} \ y_{k,i} \ z_{k,i}]^\top$ ; in the numerical example nine of the individual systems correspond to Lorenz oscillators

$$\begin{bmatrix} \dot{x}_{k,i} \\ \dot{y}_{k,i} \\ \dot{z}_{k,i} \end{bmatrix} = \underbrace{\begin{bmatrix} -\sigma_{k,i} & \sigma_{k,i} & 0 \\ \rho_{k,i} & -1 & -x_{k,i} \\ 0 & x_{k,i} & -\beta_{k,i} \end{bmatrix}}_{A_{k,i}(x_{k,i})} \begin{bmatrix} x_{k,i} \\ y_{k,i} \\ z_{k,i} \end{bmatrix}, \quad (27)$$

for all  $(k, i) \in \{(1, 1) (1, 2) (2, 1) (2, 2) (3, 1) (3, 2) (4, 1) (4, 2) (4, 3)\}$  and three stable dynamical systems with dynamics  $\dot{x}_{k,i} = -\alpha x_{k,i}$  with  $\alpha = 15$  and  $(k, i) \in$

$\{(1, 3) (2, 3) (3, 3)\}$ . Note that for these systems, being either (chaotic) oscillators or exponentially stable, the solutions are bounded. Furthermore, it is clear from (27) that  $A_{k,i}$  satisfies (2), so Assumption (2) holds. Furthermore, the overall average dynamics (25) yields  $\dot{x}_e = A_e(x_e) x_e$ , with

$$A_e(x_e) := \frac{-1}{m N_k} \sum_{k=1}^m \sum_{i=1}^{N_k} \begin{bmatrix} \alpha + 2\sigma_{k,i} & -\sigma_{k,i} & 0 \\ -\rho_{k,i} & \alpha + 1 & x_{k,i} \\ 0 & -x_{k,i} & \alpha + \beta_{k,i} \end{bmatrix},$$

so the Lyapunov function  $V_e(x_e) = \frac{1}{2}|x_e|^2$  satisfies  $\dot{V}_e(x_e) = x_e^\top [A_e(x_e) + A_e(x_e)^\top] x_e \leq -\frac{\alpha}{2}|x_e|^2$  for sufficiently large values of  $\alpha$ . In other words, if the poles of the three stable systems are sufficiently far to the right of the origin in the complex plane, the whole network is globally asymptotically stable. The chaotic behaviour of the Lorenz oscillators is “tamed”.

In Figure 3 we depict the trajectories of the system (19) in singular-perturbation form.

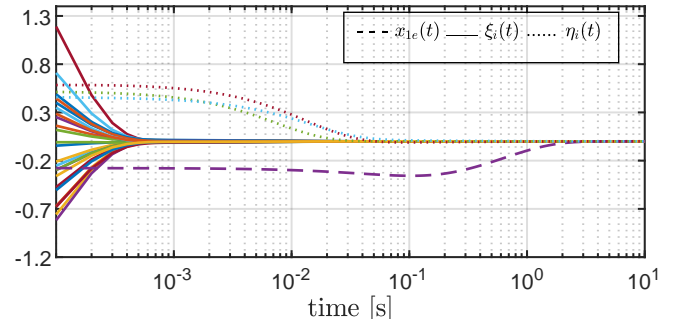


Fig. 3. Trajectories of the singularly-perturbed system (19) in logarithmic time scale for better appreciation. The thinner solid lines represent (modulo a projection) the synchronisation errors of the individual nodes relative to the modules’ averages, all converging to zero. The dotted thicker lines represent the synchronisation errors of each module relative to the modules’ average. The latter, which also corresponds to the trajectories of the overall average system, is depicted by the dashed thick magenta line. In this simulation we used  $\sigma^I = 3000$ ,  $\sigma^E = 100$

The three-time scales is also appreciated in the behaviour of the individual systems’ trajectories, which achieve consensus within each module, then the modules achieve consensus and finally, all of the systems, before converging to zero. See Figure 4.

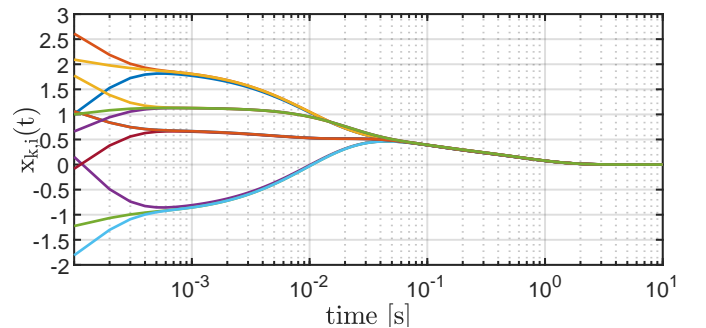


Fig. 4. Trajectories of the nine chaotic oscillators and the three asymptotically stable systems in closed-loop, *i.e.*, the solutions of (4), with  $\sigma^I = 3000$  and  $\sigma^E = 100$

Then, we executed a second simulation test using equal values for the interconnection inter- and intra-modular gains. In this case, the network is still asymptotically stable at the origin, but only two time scales appear. The results are showed in Figure (5).

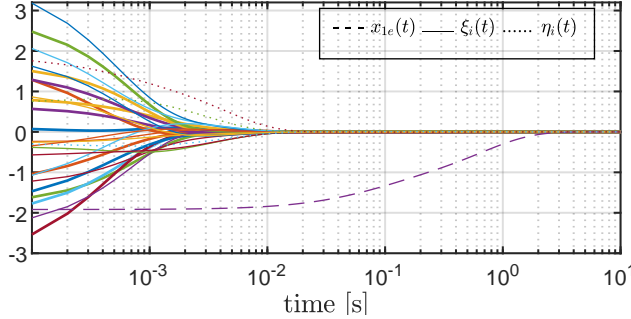


Fig. 5. Trajectories of the singularly-perturbed system (19) using the same nomenclature as in Fig. 3 above. In this case, however,  $\sigma^I = \sigma^E = 500$ , hence only two time-scales appear

## VI. CONCLUSION

Through meaningful changes of coordinates, we exhibit three different dynamics embeded in a large networked system in which groups of rapidly synchronising agents interact. We demonstrate that these interconnected systems may be regarded as that of the collective dynamics within modules, among modules and among all the agents. These systems evolve in three different time-scales, which leads naturally to a system that may be analysed via singular-perturbation theory. A standing assumption for this time separation is that the interconnection gains, within and among modules are high. For nonlinear systems, however, rich very different behaviours may arise by varying the coupling intensity. Current research is focused on extending the class of systems as well as investigating the behaviour of the network in under relatively low values of the coupling gains.

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## APPENDIX: RECALL ON SINGULAR-PERTURBATION THEORY

*Lemma 1 (corollary of Theorem 11.3 in [10]):* Consider a nonlinear autonomous singular perturbed system, where  $f$  and  $g$  are continuously differentiable functions.

$$\dot{x} = f(x, z) \quad (28a)$$

$$\epsilon \dot{z} = g_1(z) + \epsilon g_2(x, z), \quad (28b)$$

Assume further that the system is semi-passive, let  $h^*$  be the associated root of  $g_1(z) = 0$  defined on  $D_x \in \mathbb{R}^n$  and  $y = z - h^*$ . Assume that there exists Lyapunov functions  $V(x)$  and  $W(z)$  such that the following is satisfied on  $D_x$

$$\frac{\partial V}{\partial x} f(x, h^*) \leq -\alpha_1 \phi_1(x)^2 \quad (29a)$$

$$\frac{\partial W}{\partial y} g_1(y + h^*) \leq -\alpha_2 \phi_2(y)^2 \quad (29b)$$

$$\frac{\partial V}{\partial x} [f(x, y + h^*) - f(x, h^*)] \leq \beta_1 \phi_1(x) \phi_2(y) \quad (29c)$$

Also,  $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$  are real positive constants and  $\phi_1, \phi_2$  are positive definite functions. Then,  $\exists \epsilon^* > 0$  such that  $\forall \epsilon < \epsilon^*$  the origin of (28) is asymptotically stable.  $\square$