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# Bipartite Formation over Undirected Signed Networks with Collision Avoidance

Pelin Şekercioglu Ioannis Sarras Antonio Loría Elena Panteley Julien Marzat

**Abstract**— We address the problem of bipartite formation control, with collision avoidance, for double integrators with limited sensing ranges. We assume that the systems are interconnected over an undirected, signed, and balanced network. Then, in order to ensure that the proximity constraints are satisfied, we design a barrier-Lyapunov-function-based control law that guarantees connectivity maintenance for cooperative agents, and inter-agent collision avoidance. Relying on the edge-based agreement, we establish asymptotic stability of the bipartite formation control for signed networks. Finally, we illustrate our theoretical results via numerical simulations.

**Index Terms**— Formation consensus, non-holonomic constraints, Lyapunov stability

## I. INTRODUCTION

In spite of the abundant literature on consensus and synchronization of multi-agent systems [1], most of the available works on distributed control pertain to the case of cooperative agents. However, there are scenarios in which not all the agents cooperate, but some compete with each other. This is the case, *e.g.*, in Robotics, in the context of herding control [2], [3]. In these *cooperation networks* [4], competitive interactions are represented by negative weights on the edges and cooperative ones by non-negative weights. In view of this, the classical consensus goal (stabilizing over a common equilibrium point) is unachievable. Instead, the general attainable goal for this kind of networked systems is *multi-partite consensus*, also called *fragmentation* [4]. More precisely, for structurally balanced undirected networks, the achievable goal is *bipartite consensus* [5], in which all agents converge to the same state in modulus, but with opposite signs. There are various studies on bipartite consensus control, *e.g.*, for single or double-integrators [6] and linear high-order dynamics [7].

For autonomous vehicles, in addition to the convergence towards a desired position/trajectory, a secondary objective is to guarantee inter-agent collisions, while maintaining the information exchange amongst cooperative agents. These objectives are encoded as inter-agent constraints and are often handled by artificial potential (navigation or barrier) functions—see

*e.g.*, [8], [9] and [10]. A barrier Lyapunov function [10] is a type of Lyapunov function that is used to ensure that the multi-agent system satisfies certain safety constraints, and to make sure that the system does not enter a certain unsafe region. Yet, there are few works in the literature that focus on ensuring that the inter-agent constraints of a system are respected in a scenario where agents may be in competition. The connectivity-constrained multi-swarm herding is solved in [2] using a mixed integer quadratically constrained program, non-cooperative herding is achieved for single-integrator dynamics in [3], while preventing some agents to escape from a protected zone using control barrier functions. Yet, the control law in [2], [3] are based on optimisation methods and the system is modeled by a traditional cooperative network. Moreover, in [3], only a two agents case is considered. In [11] collision avoidance and connectivity maintenance for bipartite flocking is achieved with artificial potential functions. The statements rely on LaSalle’s invariance principle. However, in [11], a minimal safety distance between agents is not guaranteed.

In this letter, we study the bipartite formation problem for structurally balanced undirected signed networks of second-order systems, under relative distance constraints. We propose a bipartite formation controller that prevents the vehicles both, from colliding and from separating beyond sensing ranges. Our control design and analysis rely on the edge-based formulation for signed networks [12], which allows to recast the problem into one of stabilization of the origin in error coordinates. On the other hand, in contrast to all the references mentioned previously, we use barrier Lyapunov functions and we base our control law on the gradient of a barrier Lyapunov function for signed networks.

Thus, relative to the existing literature, we contribute with a control law that ensures the bipartite formation for structurally balanced signed graphs with guaranteed connectivity maintenance for cooperative agents and inter-agent collision avoidance by respecting a minimal safety distance between any two agent. Our results apply to double integrators and we establish asymptotic stability of the consensus manifold using Lyapunov’s direct method, based on the edge-representation. To the best of our knowledge, this has never been done for signed graphs.

The remainder of this paper is organized as follows. In Section II, we present the model and the problem statement adopting the edge-based representation for signed networks. The main results are presented in Section III. In Section IV, we illustrate our numerical simulations, and we conclude the paper with some closing remarks in Section V.

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## II. MODEL AND PROBLEM FORMULATION

### A. Coopetition Networks

Consider a group of  $N$  dynamical systems modeled by

$$\dot{\bar{x}}_i = v_i, \quad \bar{x}_i, v_i \in \mathbb{R} \quad (1a)$$

$$\dot{v}_i = u_i, \quad u_i \in \mathbb{R}, \quad i \in \mathcal{I}_N, \quad (1b)$$

where  $\mathcal{I}_N := \{1, 2, \dots, N\}$ ,  $\bar{x}_i = x_i - d_i$ , where  $x_i \in \mathbb{R}$  is the position of the  $i$ th agent with respect to a global frame, and  $d_i$  is the relative displacement of the  $i$ th agent from the center of a formation.

A commonly pursued initial problem in the literature, whose solution can serve as a basis to tackle more complex missions for such systems, is to gather in formation around a non-specified consensual set-point. In this letter we focus our attention on the case in which neighboring agents may be *cooperative* or *competitive*. More precisely, we assume that the agents interact over an undirected signed graph  $\mathcal{G}$  and the set of all vertices is  $\mathcal{V} := \{\nu_1, \nu_2, \dots, \nu_N\}$ . Then, the interaction of cooperative agents is expressed by  $a_{ij} = 1$  and, if it is competitive, by  $a_{ij} = -1$ . In this case, formation consensus is impossible, but the systems may achieve *bipartite consensus* [5]. That is, either  $\bar{x}_i \rightarrow x_c$  or  $\bar{x}_i \rightarrow -x_c$  or, more precisely,

$$\lim_{t \rightarrow \infty} [\bar{x}_i - \text{sgn}(a_{ij})\bar{x}_j] = 0 \quad (2a)$$

$$\lim_{t \rightarrow \infty} v_i = 0 \quad \forall i, j \leq N. \quad (2b)$$

In the absence of constraints, bipartite consensus for (1) is achieved via the distributed control law

$$u_i = -k_1 \sum_{j=1}^N |a_{ij}| [\bar{x}_i - \text{sgn}(a_{ij})\bar{x}_j] - k_2 \sum_{j=1}^N |a_{ij}| [v_i - \text{sgn}(a_{ij})v_j] - k_3 v_i, \quad (3)$$

where  $k_1, k_2 > 0$ ,  $k_3 \geq 0$ , if and only if  $\mathcal{G}$  contains a spanning tree and is structurally balanced [6]. A signed graph is *structurally balanced* if it may be split into two disjoint sets of vertices  $\mathcal{V}_1$  and  $\mathcal{V}_2$ , where  $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ ,  $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$  such that for every  $i, j \in \mathcal{V}_p, p \in \{1, 2\}$ , if  $a_{ij} \geq 0$ , while for every  $i \in \mathcal{V}_p, j \in \mathcal{V}_q$ , with  $p, q \in \{1, 2\}, p \neq q$ , if  $a_{ij} \leq 0$ . It is *structurally unbalanced*, otherwise [5].

In [6] it is established that the distributed control law in (3) guarantees that the synchronization error  $\bar{e}_k := [\bar{e}_{x_k} \ \bar{e}_{v_k}]^\top$ , defined as

$$\bar{e}_{x_k} := \bar{x}_i - \text{sgn}(a_{ij})\bar{x}_j, \quad \bar{e}_{v_k} := v_i - \text{sgn}(a_{ij})v_j, \quad (4)$$

where  $k \in \mathcal{I}_M := \{1, 2, \dots, M\}$ , converges to zero, thereby making two sets of vertices  $\mathcal{V}_1$  and  $\mathcal{V}_2$  converge to the same state in module but opposite in signs. For a structurally balanced network, that is. However, in [6] neither proximity constraints nor collision-avoidance are considered.

In realistic application scenarios, however, agents have limited communication range and must keep a minimum distance from each other in order to prevent inter-agent collisions. Yet, the control law in (3) does not guarantee bipartite consensus under these constraints.

Mathematically, the agent  $i$  staying in the sensing zone of agent  $j$ , where  $i, j \in \mathcal{V}_l, l = \{1, 2\}$ , is expressed as  $|x_i - x_j| < R_k$  for all  $i, j \in \mathcal{V}_1$ , and keeping a minimum distance to keep to avoid collisions between agents means that  $|x_i - x_j| > \Delta_k$ , where  $\Delta_k$  denotes the minimum distance to keep between agents  $i$  and  $j$ , so that there are no collisions. For the multi-agent system, the latter is defined as a set  $\mathcal{I}$  consisting of two inter-agent constraint sets. The first pertains to the connectivity constraints among cooperative agents, and is defined as

$$\mathcal{I}_r := \{x_i \in \mathbb{R} : |\delta_k| < R_k, \forall i, j \in \mathcal{V}_l, l = \{1, 2\}\}. \quad (5)$$

The second one is defined by the collision-avoidance constraints, *i.e.*,

$$\mathcal{I}_c := \{\delta_k \in \mathbb{R} : \Delta_k < |\delta_k|, \forall k \in \mathcal{I}_M\}, \quad (6)$$

where  $\delta_k := x_i - x_j$ .

*Remark 1:* Connectivity constraints for competitive agents are not imposed since it is assumed that these have different objectives and, therefore, they do not need to stay close to each other. •

Then, for a pair of cooperative agents, the synchronization errors defined in (4) take the form

$$\bar{e}_{x_k} = \bar{x}_i - \bar{x}_j = \delta_k - \bar{d}_k, \quad i, j \in \mathcal{V}_l, l = \{1, 2\}, \quad (7)$$

while, for two competitive agents  $x_i$  and  $x_j$ , the error becomes

$$\bar{e}_{x_k} = \bar{x}_i + \bar{x}_j = \delta_k - \bar{d}_k + 2x_j, \quad i \in \mathcal{V}_1, j \in \mathcal{V}_2, \quad (8)$$

with  $\bar{d}_k = d_i - \text{sgn}(a_{ij})d_j$ .

Thus, the bipartite-consensus objective for (1) is equivalent to making  $x_i - \text{sgn}(a_{ij})x_j \rightarrow d_i - \text{sgn}(a_{ij})d_j$ , or equivalently  $\bar{e}_k \rightarrow 0$  and  $v_i \rightarrow 0$ . In that light, we remark that the synchronization errors correspond to edge states [13], which are particularly well-suited to multi-agent systems under state constraints, such as (5) and (6). Hence, in this paper, we study the bipartite consensus problem with constraints in edge-based representation for signed networks [12]. This representation has the advantage to recast the consensus problem into that of stability of the origin in error coordinates  $\bar{e}$ . Thus, before presenting our main results, we recall some facts about edge-based signed graphs, according to [5] and [12] and we present some new statements that are useful to establish our main results.

### B. The edge-based formulation for signed graphs

The elements of the Laplacian of a signed graph,  $L_s$ , are

$$\ell_{s_{ij}} = \begin{cases} \sum_{h \in \mathcal{I}_N} |a_{ih}| & i = j \\ -a_{ij} & i \neq j. \end{cases} \quad (9)$$

The following definition introduces the incidence matrix of a structurally balanced signed graph.

*Definition 1:* Consider a structurally balanced undirected signed network  $\mathcal{G}$  that contains  $N$  nodes and  $M$  edges. The

incidence matrix  $E_s \in \mathbb{R}^{N \times M}$  of  $\mathcal{G}$  is defined as

$$[E_s]_{ik} := \begin{cases} +1, & \text{if } v_i \text{ is the initial node of the edge } e_k; \\ -1, & \text{if } v_i, v_j \in \mathcal{V}_l, l = \{1, 2\} \text{ and } v_i \text{ is the} \\ & \text{terminal node of the edge } e_k; \\ +1, & \text{if } v_i \in \mathcal{V}_1, v_j \in \mathcal{V}_2 \text{ and } v_i \text{ is the} \\ & \text{terminal node of the edge } e_k; \\ 0, & \text{otherwise,} \end{cases}$$

where  $e_k = v_i v_j$ ,  $k \in \mathcal{I}_M$ ,  $i, j \in \mathcal{I}_N$  are arbitrarily oriented edges and  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are the two disjoint sets of vertices. •

Using the incidence matrix we may express the synchronization errors in (4), in the vector forms

$$\bar{e}_x = E_s^\top \bar{x}, \quad \bar{e}_v = E_s^\top v. \quad (10)$$

The incidence matrix is also useful to factorize the node and edge Laplacians, as established next.

*Claim 1:* The Laplacian matrix  $L_s$  and the edge Laplacian matrix  $L_{e_s}$  of a structurally-balanced graph  $\mathcal{G}$  satisfy

$$L_s = E_s E_s^\top, \quad L_{e_s} = E_s^\top E_s. \quad (11)$$

*Proof of Claim 1:* As the signed graph is structurally balanced, we apply the *gauge transformation* on  $L_s$ , which consists in pre- and post-multiplying  $L_s$  by the matrix  $D = \text{diag}(\sigma)$ , where  $\sigma = [\sigma_1 \dots \sigma_N]$ , with  $\sigma_i = \{\pm 1\}$ ,  $i \in \mathcal{I}_N$  to obtain the Laplacian matrix  $L$  of an unsigned network. Then again, we apply the *edge gauge transformation* on the incidence matrix  $E_s$  of the signed network, using the matrix  $D = \text{diag}(\sigma)$  and  $D_e = \text{diag}(\sigma_e)$ , where  $\sigma_e = [\sigma_{e_1} \dots \sigma_{e_M}]$ , with  $\sigma_{e_i} = 1$  if  $v_i \in \mathcal{V}_1$  and  $\sigma_{e_i} = -1$  if  $v_i \in \mathcal{V}_2$  with  $v_i$  being the initial node of the edge [12, Lemma 4], to obtain the incidence matrix  $E$  of an unsigned network.

Now, from the definition of the Laplacian matrix  $L = EE^\top$  and the edge Laplacian matrix  $L_e = E^\top E$  of an unsigned network, and by applying the edge gauge transformation on the incidence matrix  $E$  of an unsigned network from [12, Lemma 4], we have  $L = DE_s D_e (DE_s D_e)^\top = DE_s E_s^\top D$  and  $L_e = (DE_s D_e)^\top DE_s D_e = D_e E_s^\top E_s D_e$  as  $D_e D_e^\top = I_{M \times M}$  and  $D^\top D = I_{N \times N}$ . Then, the definitions of  $L_s$  and  $L_{e_s}$  in (11) follow. •

Another utility of the incidence matrix, as defined in Definition 1, is that for structurally balanced signed graphs containing a spanning tree (sufficient and necessary condition for bipartite consensus), it allows to distinguish the state-variables related to an underlying-tree graph  $\mathcal{G}_t$ , from the rest of states, corresponding to the graph  $\mathcal{G}_c := \mathcal{G} \setminus \mathcal{G}_t$ . Then, the consensus problem may be addressed as that of the stabilization of the origin for a reduced-order system. To better see this, let the incidence matrix  $E_s$  be partitioned as

$$E_s = [E_{t_s} \quad E_{c_s}], \quad (12)$$

where  $E_{t_s} \in \mathbb{R}^{N \times N-1}$  is the incidence matrix representing the edges of the spanning tree of and  $E_{c_s} \in \mathbb{R}^{N \times M - (N-1)}$  is the incidence matrix representing the remaining edges. In the Proposition 1 below we define  $E_s$  in terms of  $E_{t_s}$ . Then, after (10) and (12) we have

$$\bar{e}_x = [(E_{t_s}^\top \bar{x})^\top \quad (E_{c_s}^\top \bar{x})^\top]^\top, \quad \bar{e}_v = [(E_{t_s}^\top v)^\top \quad (E_{c_s}^\top v)^\top]^\top,$$

so we may define  $\bar{e}_x =: [\bar{e}_{x_t}^\top \quad \bar{e}_{x_c}^\top]^\top$  and  $\bar{e}_v =: [\bar{e}_{v_t}^\top \quad \bar{e}_{v_c}^\top]^\top$ . The indices  $t$  and  $c$  refer, respectively, to states of the graphs  $\mathcal{G}_t$

and  $\mathcal{G}_c$ . The Proposition 1 also establishes a relation between the consensus errors  $\bar{e}_x$  and the spanning-tree errors  $\bar{e}_{x_t}$ , from which it follows that the objective (2) is attained if  $\bar{e}_{x_t} \rightarrow 0$  and  $\bar{e}_{v_t} \rightarrow 0$ .

*Proposition 1:* For a structurally balanced signed graph, there exists a matrix  $R_s$  such that

$$E_s = E_{t_s} R_s, \quad (13)$$

where

$$R_s := [I_{N-1} \quad T_s], \quad T_s := (E_{t_s}^\top E_{t_s})^{-1} E_{t_s}^\top E_{c_s}. \quad (14)$$

□

*Proof:* Applying the edge-gauge transformation and using the partition of  $E_s$  as in (12) we express the incidence matrix of an unsigned graph  $E$  as

$$E = D[E_{t_s} \quad E_{c_s}]D_e = [DE_{t_s}D_{e_t} \quad DE_{c_s}D_{e_c}], \quad (15)$$

where  $D_e = \text{diag}([\sigma_{e_t}, \sigma_{e_c}])$  with  $\sigma_{e_{t_j}} = \{\pm 1\}$  and  $\sigma_{e_{c_l}} = \{\pm 1\}$  for  $j < N-1$  and  $l \leq M-N+1$ ,  $D_{e_t} = \text{diag}(\sigma_{e_t})$  and  $D_{e_c} = \text{diag}(\sigma_{e_c})$  are the parts of the edge-gauge transformation matrix corresponding to the spanning tree and the remaining edges, respectively. For an unsigned network, the columns of the incidence matrix representing the remaining edges,  $E_{c_s}$ , are linearly dependent on the columns of the incidence matrix representing the spanning tree,  $E_{t_s}$ , and this can be expressed using a matrix  $T$  by  $E_{c_s} = E_{t_s} T$  [13, Theorem 4.3]. So, replacing  $E_{c_s}$  and  $E_{t_s}$  by the expressions in (15) and left multiplying it by  $(DE_{t_s}D_{e_t})^\top$ , we obtain

$$T = D_{e_t} (E_{t_s}^\top E_{t_s})^{-1} E_{t_s}^\top E_{c_s} D_{e_c}.$$

Then, we define  $T_s := D_{e_t} T D_{e_c}$  and  $R_s := [I_{N-1} \quad T_s]$ , and the statement in (13) follows. ■

After Proposition 1, we have

$$\bar{e}_x = (E_{t_s} R_s)^\top \bar{x} = R_s^\top \bar{e}_{x_t}, \quad (16a)$$

$$\bar{e}_v = (E_{t_s} R_s)^\top v = R_s^\top \bar{e}_{v_t}, \quad (16b)$$

so the bipartite consensus objective is attained if  $\bar{e}_{x_t} \rightarrow 0$  and  $\bar{e}_{v_t} \rightarrow 0$ , as stated above. We show how next.

### III. MAIN RESULTS

#### A. Barrier-Lyapunov-Function-based controller

A *barrier Lyapunov function* (BLF) [10], [13], [14] is defined as follows.

*Definition 2:* Consider the system  $\dot{x} = f(x)$  and let  $\mathcal{I}$  be an open set containing the origin. A BLF is a positive definite function  $W : \mathcal{I} \rightarrow \mathbb{R}_{\geq 0}$ ,  $x \mapsto W(x)$ , that is  $\mathcal{C}^1$ , satisfies  $\dot{W}(x) \leq 0$ , and has the property that  $W(x) \rightarrow \infty$ ,  $\nabla W(x) \rightarrow \infty$  as  $x \rightarrow \partial \mathcal{I}$ . •

Now, from (7) and (8), the constraints in (5) and (6) in terms of the errors in (4) are given by the sets

$$\mathcal{I}_r := \{\bar{e}_{x_k} \in \mathbb{R} : |\bar{e}_{x_k} + \alpha_k| < R_k, \forall k \in \mathcal{I}_m\}, \quad (17a)$$

$$\mathcal{I}_c := \{\bar{e}_{x_k} \in \mathbb{R} : \Delta_k < |\bar{e}_{x_k} + \alpha_k|, \forall k \in \mathcal{I}_M\}, \quad (17b)$$

where  $\mathcal{I} = \mathcal{I}_r \cup \mathcal{I}_c$  and  $\mathcal{I}_m \subset \mathcal{I}_M$  consists of the index of  $m$  cooperative edges, such that  $m < M$ , which are the edges with strictly positive weights, and  $\alpha_k$  is defined as

$$\alpha_k := \begin{cases} \bar{d}_k & \text{if } i, j \in \mathcal{V}_l, l = \{1, 2\} \\ \delta_k - \bar{e}_{x_k} & \text{if } i \in \mathcal{V}_1, j \in \mathcal{V}_2. \end{cases} \quad (18)$$

Then, for each  $k \in \mathcal{I}_M$ , we define a BLF  $W_k : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ , as

$$W_k(\bar{e}_{x_k}) = \frac{1}{2} [|\bar{e}_{x_k}|^2 + B_k(\bar{e}_{x_k})], \quad (19)$$

where  $B_k(\bar{e}_{x_k})$  is the sum of two functions satisfying Definition 2, each of them encoding the constraints in (17), respectively, *i.e.*,

$$B_k(\bar{e}_{x_k}) = \frac{1}{2}(1 + \sigma_k)B_{r_k}(\bar{e}_{x_k}) + B_{c_k}(\bar{e}_{x_k}). \quad (20)$$

In (20)  $\sigma_k = 1$  if  $k \in \mathcal{I}_m$  if the edge is cooperative, and  $\sigma_k = -1$  otherwise. Moreover,  $B_k(\bar{e}_{x_k})$  is non-negative and satisfies  $B_k(0) = 0$  and  $B_k(\bar{e}_{x_k}) \rightarrow \infty$  as  $|\bar{e}_{x_k}| \rightarrow \Delta_k$  for all  $k \leq M$  and as  $|\bar{e}_{x_k}| \rightarrow R_k$  for  $k \in \mathcal{I}_m$ . However, in view of the constraints defined in (17b), the barrier function has to be modified so that the solution lies in the interior of the constraint sets in (17) and that the convergence of the system to the desired point is ensured. For this purpose, we use the concept of the *gradient recentered barrier function* [15]. Let  $\widehat{W}_k : (\Delta_k, \infty) \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  for competitive and  $\widehat{W}_k : (\Delta_k, R_k) \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  for cooperative agents defined as

$$\widehat{W}_k(\alpha_k, \bar{e}_{x_k}) := W_k(\bar{e}_{x_k} + \alpha_k) - W_k(\alpha_k) - \frac{\partial W_k}{\partial s}(\alpha_k) \bar{e}_{x_k} \quad (21)$$

which satisfies  $\widehat{W}_k(\alpha_k, 0) = 0$ ,  $\nabla_{\bar{e}_{x_k}} \widehat{W}_k(\alpha_k, 0) = 0$ , where  $\nabla_{\bar{e}_{x_k}} \widehat{W}_k = \frac{\partial \widehat{W}_k}{\partial \bar{e}_{x_k}}$ , and  $\widehat{W}_k(\alpha_k, \bar{e}_{x_k}) \rightarrow \infty$  as  $|\delta_k| \rightarrow \Delta_k$  for  $k \in \mathcal{I}_M$ , and as  $|\delta_k| \rightarrow R_k$  for all  $\sigma_k = 1$ . Moreover,  $\widehat{W}_k(\alpha_k, \bar{e}_{x_k})$  satisfies  $\frac{\kappa_1}{2} \bar{e}_{x_k}^2 \leq \widehat{W}_k(\alpha_k, \bar{e}_{x_k}) \leq \kappa_2 [\nabla_{\bar{e}_{x_k}} \widehat{W}_k]^2$ .

Now we introduce the BLF-gradient-based bipartite-consensus control law for the system (1), given by

$$u_i := -k_1 \sum_{k=1}^M [E_s]_{ik} \nabla_{\bar{e}_{x_k}} \widehat{W}_k - k_2 \sum_{k=1}^M [E_s]_{ik} \bar{e}_{v_k} - k_3 v_i - k_1 \sum_{k=1}^M [\mathbb{B}]_{ik} \nabla_{\alpha_k} \widehat{W}_k, \quad (22)$$

which, upon defining  $\bar{W}(\alpha, \bar{e}_x) = \sum_{k=1}^M \widehat{W}_k(\alpha_k, \bar{e}_{x_k})$  and

$$\mathbb{B} = (1 + \beta)(\beta I + E_{t_s} L_{e_{t_s}}^{-1} E_{t_s}^\top)^{-1} (E - E_s), \quad \beta > 0, \quad (23)$$

we write in the vector form as

$$u = -k_1 E_s \nabla_{\bar{e}_x} \bar{W} - k_2 E_s \bar{e}_v - k_3 v - k_1 \mathbb{B} \nabla_{\alpha} \bar{W}. \quad (24)$$

*Remark 2:* In [14] the author proposes a consensus control law for multi-agent systems under proximity constraints, but in the simpler scenario in which all the agents are cooperative. In [14] the control law corresponds to

$$u'_i = -k_1 \sum_{k=1}^M [E]_{ik} \nabla_{\bar{e}_{x_k}} W_k - k_3 v_i, \quad (25)$$

which is a particular case of  $u_i$  in (22).

It is also important to stress that the last term on the right-hand side of (22)—see also (23), is added after a Lyapunov control redesign, *i.e.*, to render negative semidefinite the derivative of a Lyapunov function for the closed-loop system—see the proof of Proposition 2. •

## B. Asymptotic Stability

In accordance with Section II, we analyze the stability of the system (1) in closed loop with the bipartite formation control law (24), expressed in terms of the errors corresponding to the spanning-tree subgraph,  $\bar{e}_t$ . To that end, we introduce the function  $\tilde{W}$  as  $\tilde{W}(\alpha, \bar{e}_{x_t}) = \bar{W}(\alpha, R_s^\top \bar{e}_{x_t})$ , where  $\bar{W}$  is defined above (24). Now, after (16a),

$$\nabla_{\bar{e}_{x_t}} \tilde{W} = \frac{\partial \tilde{W}(\alpha, \bar{e}_{x_t})^\top}{\partial \bar{e}_x} \frac{\partial \bar{e}_x}{\partial \bar{e}_{x_t}} = \nabla_{\bar{e}_x} \bar{W}^\top R_s^\top \quad (26)$$

and, after (13) and (26), the control law in (24) becomes

$$u = -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{B} \nabla_{\alpha} \tilde{W}. \quad (27)$$

Then, differentiating on both sides of (10) and using  $\dot{v} = u$  and (27) we obtain, in spanning-tree coordinates

$$\dot{\bar{e}}_{x_t} = \bar{e}_{v_t} \quad (28a)$$

$$\dot{\bar{e}}_{v_t} = -k_1 L_{e_{t_s}} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 L_{e_{t_s}} R_s R_s^\top \bar{e}_{v_t} - k_3 \bar{e}_{v_t} - k_1 E_{t_s}^\top \mathbb{B} \nabla_{\alpha} \tilde{W} \quad (28b)$$

*Remark 3:*  $E_{t_s}^\top E_{t_s} = L_{e_{t_s}}$  corresponds to the edge Laplacian of a spanning tree and has  $N - 1$  edges. With the edge-gauge transformation we obtain the edge Laplacian matrix corresponding to a spanning tree of an unsigned network, *i.e.*,  $L_{e_t} = D_e L_{e_{t_s}} D_e$ . Also, we have  $D_e = D_e^\top = D_e^{-1}$  and  $D_e L_{e_{t_s}} D_e$  is a similarity transformation, so the spectra of  $L_{e_t}$  and  $L_{e_{t_s}}$  coincide. Consequently, as  $L_{e_t}$  has the  $N - 1$  non-zero eigenvalues of the Laplacian matrix  $L$ , so does  $L_{e_{t_s}}$ . •

*Remark 4:*  $R_s R_s^\top$  is a symmetric positive definite matrix as  $\bar{e}_{x_t}^\top R_s R_s^\top \bar{e}_{x_t} = \bar{e}_x^\top \bar{e}_x > 0$ . •

*Proposition 2:* Consider the system system (1) in closed loop with the distributed control law (27), under the conditions of Proposition 1 and assume that the resulting network contains an underlying spanning tree. Let  $L_{e_{t_s}}$  denote the edge Laplacian corresponding to that spanning-tree. Then, the origin for the closed-loop system is asymptotically stable for any initial conditions respecting the constraints in (17), and  $|\alpha_k(0)| > \Delta_k$  for all  $k \in \mathcal{I}_M$ . In addition, the constraints hold for all  $t$ . □

*Remark 5:* The condition on  $\alpha_k(0)$  is not restrictive. For cooperative agents,  $\alpha_k = \bar{d}_k$ , so  $|\alpha_k(0)| > \Delta_k$  means that the formation must respect the collision-avoidance constraints (the formation must be “safe”). For competitive agents,  $|\alpha_k(0)| > \Delta_k$  means that  $|-2x_j(0) + \bar{d}_k| > \Delta_k$ , which restricts the initial conditions in *absolute* coordinates, *i.e.*, with respect to a fixed frame. However, in the scenarii considered in this letter, the measurements are relative (edge coordinates). That is, absolute positions are irrelevant. So,  $x_j(0)$  may be conveniently defined by replacing the origin of the fixed frame if needed. •

*Proof:* First, we express the constraints in (17) in terms of the spanning-tree coordinates. Let  $\mathcal{I}_t = \mathcal{I}_{r_t} \cup \mathcal{I}_{c_t}$ , with

$$\mathcal{I}_{r_t} := \{\bar{e}_{x_{t_k}} \in \mathbb{R} : |r_{s_k}^\top \bar{e}_{x_{t_k}} + \alpha_k| < R_k, \forall k \in \mathcal{I}_m\}, \quad (29)$$

$$\mathcal{I}_{c_t} := \{\bar{e}_{x_{t_k}} \in \mathbb{R} : \Delta_k < |r_{s_k}^\top \bar{e}_{x_{t_k}} + \alpha_k|, \forall k \in \mathcal{I}_M\}. \quad (30)$$

where  $r_{s_k}$  is the  $k$ th column of  $R_s$ .

Next, consider the Lyapunov function candidate

$$V(\alpha, \bar{e}_t, v) = (1 + \beta)k_1 \tilde{W}(\alpha, \bar{e}_{x_t}) + \frac{1}{2} \left[ \bar{e}_{v_t}^\top L_{e_{t_s}}^{-1} \bar{e}_{v_t} + \beta |v|^2 \right]. \quad (31)$$

We note that  $V$  in (31) is positive definite in the state variables  $\bar{e}_{x_t}$  and  $\bar{e}_{v_t}$ , uniformly in  $\alpha$  and radially unbounded in  $\bar{e}_{v_t}$ . Also, by construction,  $V(\alpha, \bar{e}_t, v) \rightarrow \infty$ , for any fixed  $\bar{e}_{v_t}$  and  $v$ , as  $\bar{e}_{x_t} \rightarrow \partial \mathcal{I}_t$ , where  $\partial \mathcal{I}_t$  denotes the boundary of  $\mathcal{I}_t$ .

Next, we compute the total derivative of  $V$ . To that end, we note that from the definition of  $\alpha$  in (18), we have  $\dot{\alpha} = (E - E_s)^\top v$  and for (1b) and (27), we have

$$\dot{v} = -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} - k_3 v - k_1 \mathbb{B} \nabla_\alpha \tilde{W}. \quad (32)$$

Then, we have

$$\begin{aligned} \dot{V}(\alpha, \bar{e}_t, v) &= (1 + \beta)k_1 \nabla_{e_{x_t}} \tilde{W}^\top \bar{e}_{v_t} + (1 + \beta)k_1 \nabla_\alpha \tilde{W}^\top \dot{\alpha} \\ &\quad + \bar{e}_{v_t}^\top L_{e_{t_s}}^{-1} \left[ -k_1 L_{e_{t_s}} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 L_{e_{t_s}} R_s R_s^\top \bar{e}_{v_t} \right. \\ &\quad \quad \quad \left. - k_3 \bar{e}_{v_t} - k_1 E_{t_s}^\top \mathbb{B} \nabla_\alpha \tilde{W} \right] \\ &\quad + \beta v^\top \left[ -k_1 E_{t_s} \nabla_{\bar{e}_{x_t}} \tilde{W} - k_2 E_{t_s} R_s R_s^\top \bar{e}_{v_t} \right. \\ &\quad \quad \quad \left. - k_3 v - k_1 \mathbb{B} \nabla_\alpha \tilde{W} \right]. \quad (33) \end{aligned}$$

Hence,

$$\begin{aligned} \dot{V}(\alpha, \bar{e}_t, v) &= (1 + \beta)k_1 \nabla_\alpha \tilde{W}^\top (E - E_s)^\top v - k_3 \beta |v|^2 \\ &\quad - k_1 v^\top [E_{t_s} L_{e_{t_s}}^{-1} E_{t_s}^\top + \beta I] \mathbb{B} \nabla_\alpha \tilde{W} \\ &\quad - \bar{e}_{v_t}^\top [(1 + \beta)k_2 R_s R_s^\top + k_3 L_{e_{t_s}}^{-1}] \bar{e}_{v_t}. \quad (34) \end{aligned}$$

However, in view of (23), we have  $[E_{t_s} L_{e_{t_s}}^{-1} E_{t_s}^\top + \beta I] \mathbb{B} = (1 + \beta)(E - E_s)^\top$ . Thus,

$$\begin{aligned} \dot{V}(\alpha, \bar{e}_t, v) &= -\bar{e}_{v_t}^\top \left[ (1 + \beta)k_2 R_s R_s^\top + k_3 L_{e_{t_s}}^{-1} \right] \bar{e}_{v_t} \\ &\quad - k_3 \beta |v|^2, \end{aligned}$$

which is negative semidefinite.

Next, we use Barbashin-Krasovskii's theorem. To that end, we note that on the set  $\{(\bar{e}, v) : \dot{V} = 0\}$  we have  $v = 0$  and  $\bar{e}_{v_t} = 0$ . Therefore,  $\dot{\alpha} = 0$ , which means that  $\alpha$  is constant. In turn, after (28b), we have

$$k_1 L_{e_{t_s}} \nabla_{\bar{e}_{x_t}} \tilde{W} = -k_1 E_{t_s}^\top \mathbb{B} \nabla_\alpha \tilde{W}. \quad (35)$$

On the other hand, after (21), we have

$$\nabla_\alpha \tilde{W} = \nabla_{\bar{e}_{x_t}} \tilde{W} - \frac{\partial}{\partial \alpha} \left\{ \frac{\partial W}{\partial \alpha}(\alpha) \right\} \bar{e}_{x_t}, \quad (36)$$

but since we have  $\alpha \equiv \text{const}$  on  $\{\dot{V} = 0\}$ , it follows that the last term of the right hand-side of (36) equals to zero, so (35) holds if and only if  $-k_1 [L_{e_{t_s}} + E_{t_s}^\top \mathbb{B}] \nabla_{\bar{e}_{x_t}} \tilde{W} = 0$ . The latter holds since  $[L_{e_{t_s}} + E_{t_s}^\top \mathbb{B}]$  is full rank and  $\nabla_{\bar{e}_{x_t}} \tilde{W}$  vanishes only at zero. It follows that the only solution of (28) that may

remain in  $\{(\bar{e}, v) : \dot{V} = 0\}$  for all  $t$ , is the origin. Asymptotic stability in the large, on the domain of definition of  $V$  follows.

Next we demonstrate inter-agent collision avoidance. From (29), we remark that  $\bar{e}_{x_t} \in \mathcal{I}_t$  implies  $\bar{e}_x \in \mathcal{I}$ . Then, we proceed by contradiction to show that  $\mathcal{I}_t$  is forward invariant. Assume that there exist a  $T > 0$  such that  $\bar{e}_{x_t}(T) \notin \mathcal{I}_t$ . It means that  $|\bar{e}_{x_{t_k}} + \alpha_k| \rightarrow \Delta_k$ ,  $k \in \mathcal{I}_M$  or  $|\bar{e}_{x_{t_k}} + \alpha_k| \rightarrow R_k$ ,  $k \in \mathcal{I}_m$  for at least one  $k \leq M$ , which makes  $\tilde{W}_k(\alpha_k, \bar{e}_{x_{t_k}}) \rightarrow \infty$ , so  $V(\alpha, \bar{e}_t, v) \rightarrow \infty$  as  $t \rightarrow T$ . However, the latter contradicts the fact that  $\dot{V}(\alpha, \bar{e}_t, v) \leq 0$ . Inter-agent collision avoidance follows.

We now show that the set  $\mathcal{I}$  corresponds to the domain of attraction for the closed-loop system (28), by showing that all solutions starting in  $\mathcal{I}_t$  converge to the origin. For any  $\epsilon_1 \in (0, R_k)$  and  $\epsilon_2 \in (0, \Delta_k)$ , consider subsets  $\mathcal{I}_{\epsilon_{r_t}} \subset \mathcal{I}_{r_t}$  and  $\mathcal{I}_{\epsilon_{c_t}} \subset \mathcal{I}_{c_t}$  defined as  $\mathcal{I}_{\epsilon_{r_t}} := \{\bar{e}_{x_k} \in \mathbb{R} : |\bar{e}_{x_k} + \alpha_k| < R_k - \epsilon_1, \forall k \in \mathcal{I}_m\}$  and  $\mathcal{I}_{\epsilon_{c_t}} := \{\bar{e}_{x_k} \in \mathbb{R} : \Delta_k + \epsilon_2 < |\bar{e}_{x_k} + \alpha_k|, \forall k \in \mathcal{I}_M\}$  with  $\mathcal{I}_{\epsilon_{r_t}} \cup \mathcal{I}_{\epsilon_{c_t}} = \mathcal{I}_{\epsilon_t}$ . From the definition of the  $\tilde{W}(\alpha, \bar{e}_{x_t})$ ,  $V(\alpha, \bar{e}_t, v)$  is positive definite for all  $\bar{e}_{x_t} \in \mathcal{I}_{\epsilon_t}$  and  $\bar{e}_{v_t} \in \mathbb{R}$  and satisfies  $a|\bar{e}_{x_t}|^2 + b|\bar{e}_{v_t}|^2 + c|v|^2 \leq V(\alpha, \bar{e}_t, v) \leq h(|\bar{e}_{x_t}|) + d|\bar{e}_{v_t}|^2 + f|v|^2$  with  $a, b, c, d, f > 0$  and  $h$  is a strictly increasing everywhere in  $\mathcal{I}_{\epsilon_t}$ . This means  $V(\alpha, \bar{e}_t, v) \rightarrow 0$  as  $\bar{e}_t \rightarrow 0$  and  $v \rightarrow 0$ . Therefore, we have that for all trajectories for the closed-loop system starting in  $\mathcal{I}_{\epsilon_t}$ , the origin is asymptotically stable. As  $\epsilon_1$  and  $\epsilon_2$  are arbitrarily small, taking  $\epsilon_1 \rightarrow 0$  and  $\epsilon_2 \rightarrow 0$ , we establish asymptotic stability of the origin of (28) for all trajectories starting in  $\mathcal{I}_t$ . Thus, bipartite consensus is achieved with inter-agent collision avoidance and connectivity. ■

#### IV. SIMULATION RESULTS

We provide some numerical examples to show the performance of our control law in (24) with  $k_1 = 1$ ,  $k_2 = 1.2$ ,  $k_3 = 1$  and the barrier Lyapunov function in (21), with

$$B_{r_k}(s) = \ln \left( \frac{R_k^2}{R_k^2 - |s|^2} \right), \quad B_{c_k}(s) = \ln \left( \frac{|s|^2}{|s|^2 - \Delta_k^2} \right). \quad (37)$$

The simulation consists of an undirected signed network of 6 agents and 7 edges as the one depicted on Figure 1, subject to inter-agent collision avoidance and connectivity maintenance restrictions. We define the orientation of the edges as follows:  $e_1 = x_1 + x_2$ ,  $e_2 = x_1 - x_3$ ,  $e_3 = x_1 + x_4$ ,  $e_4 = x_2 + x_5$ ,  $e_5 = x_2 - x_6$ ,  $e_6 = x_3 + x_4$  and  $e_7 = x_5 + x_6$ . According to Definition 1, the incidence matrix corresponding to the graph is

$$E_s = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

The network can be split the agents into two disjoint subgroups, such as  $\mathcal{V}_1 = \{x_1, x_3, x_5\}$  and  $\mathcal{V}_2 = \{x_2, x_4, x_6\}$  so the network is structurally balanced. From the decomposition in (12), edges

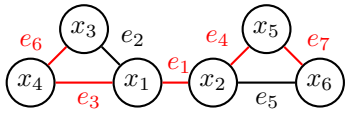


Fig. 1. An undirected signed network of 6 agents. The black lines represent cooperative edges and the red line represents the competitive edge.

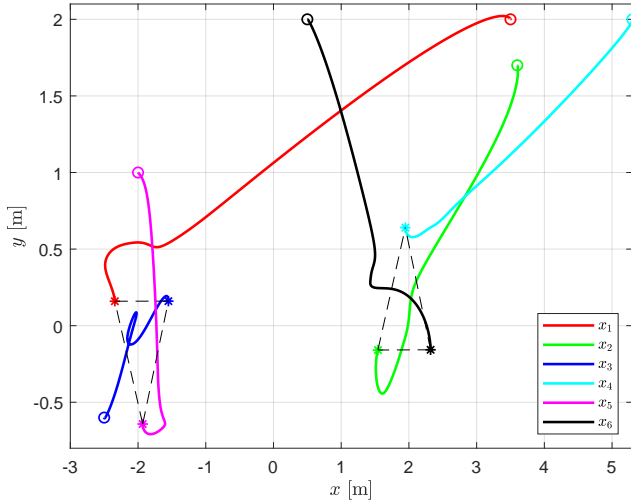


Fig. 2. Bipartite formation of system (1) with control input (24).

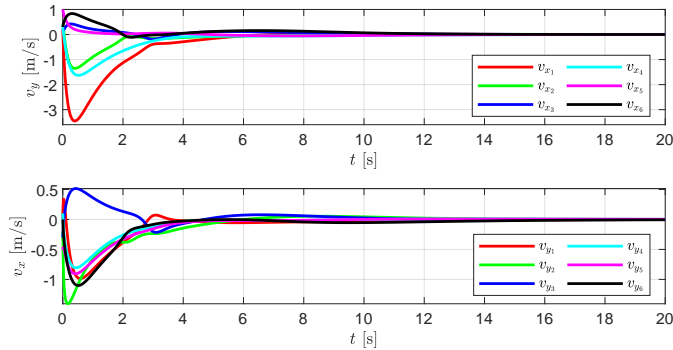


Fig. 3. Bipartite formation of system (1) with control input (24) on velocity.

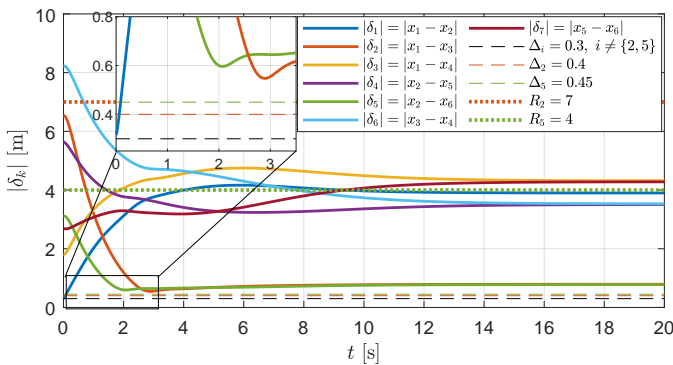


Fig. 4. Trajectories of the norm of the inter-agent distances with control input (24). The dashed and dotted lines are the distance constraints for agents.

$e_i$ ,  $i \leq 5$  correspond to  $\mathcal{G}_t$  and the remaining edges  $e_5$  and  $e_6$  correspond to  $\mathcal{G}_c$ . The respective agents' initial states are  $x(0) = [3.5, 3.6, -4.5, 5.3, -2, 0]^T$ ,  $y(0) = [2, 1.7, -2.6, 2, 1, 2]^T$ ,  $v_x(0) = [0.2, 0.4, 0.1, 0.3, 1, 0.3]^T$ ,  $v_y(0) = [0, -0.2, -0.3, 0.1, -0.5, 0]^T$  and the relative

displacements are  $d_x = [-0.4, -0.4, 0.4, 0, 0, 0.4]^T$  and  $d_y = [0.4, -0.4, 0.4, 0.4, -0.4, -0.4]^T$ . The constraints sets are  $\Delta = [0.3, 0.4, 0.3, 0.3, 0.45, 0.3, 0.3]^T$  and  $R = [7, 4]^T$ . The paths of each agent up to bipartite formation is depicted on Figure 2. The agents reach the desired formation around two symmetric consensus points. Furthermore, it is clear from Figure 4 that the inter-agent collision avoidance constraints in (6) are always respected.

## V. CONCLUSIONS

We presented a BLF-based control law for structurally balanced undirected signed networks to address the problem of bipartite formation control for double integrators. Our control law ensures inter-agent collision avoidance for any two agents and connectivity maintenance for cooperative agents. Via a change of coordinates, we established asymptotic stability of the system using Lyapunov's direct method. Further research is focused on extending these results to structurally unbalanced and directed signed networks.

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