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Stability of Perfectly Matched Layers for Maxwell's Equations in Rectangular Solids

Laurence Halpern ^{*} Jeffrey Rauch [†]

Abstract

Perfectly matched layers are an essential tool designed in the nineties for the computation of electromagnetic waves. The method replaces the Maxwell equations by a larger system, and introduce variable absorption coefficients that are nonvanishing near the boundary of the computational box. Classic absorbing conditions are imposed at the boundary. Well posedness of the resulting initial boundary value problem is proved here for the first time. The analysis proceeds by Laplace transform in time on smoothed domains. There we design boundary conditions for a non selfadjoint Helmholtz system. Estimates uniform in the smoothing are proved using carefully constructed test functions. One estimate is of energy type with less positivity than usual. A second follows Jerison-Kenig-Mitrea from elliptic problems in Lipschitz domains.

Keywords. Perfectly matched layers, Maxwell's equations, absorbing boundary conditions, trihedral corner, stability.

AMS Subject Classification. 35L50, 35L53, 35J25, 35Q61, 35B30, 78M20.

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1 Introduction

This paper proves stability of algorithms that compute approximate solutions of Maxwell's equations

$$\partial_t E - \operatorname{curl} B = -j, \quad \partial_t B + \operatorname{curl} E = 0, \quad (1.1)$$

on the unbounded domain $\mathbb{R}_t \times \mathbb{R}^3$. The algorithms use a bounded computational domain $\mathcal{Q} \subset \mathbb{R}^3$. The values of the fields are sought on a smaller domain of interest denoted \mathcal{Q}_I . Perfectly matched layers are employed in $\mathcal{Q} \setminus \mathcal{Q}_I$.

The current density j and charge density ρ supported in $\{t \geq 0\} \times \mathcal{Q}_I$ are given and satisfy the conservation identity

$$\partial_t \rho = -\operatorname{div} j. \quad (1.2)$$

Taking the divergence of (1.1) implies that,

$$\partial_t(\operatorname{div} E) + \operatorname{div} j = \partial_t(\operatorname{div} B) = 0.$$

Since E, B, j are supported in $t \geq 0$, it follows that E and B satisfy

$$\operatorname{div} E = \rho, \quad \text{and}, \quad \operatorname{div} B = 0. \quad (1.3)$$

The fundamental difficulty posed by the bounded computational domain is that the boundary of the domain, $\partial\mathcal{Q}$, is not physical. Waves in \mathbb{R}^3 simply pass through it undisturbed. A numerical algorithm needs to mimic that invisibility. This is often done by introducing a boundary condition chosen to approximately reproduce this transparency. Such conditions let waves leave \mathcal{Q} so that solutions viewed in \mathcal{Q} seem to lose energy. The conditions are called *absorbing boundary conditions*. In dimension $d > 1$ there is invariably some reflection that pollutes the approximation. A remedy with extremely high computational cost is to enlarge \mathcal{Q} so that the reflections do not have time to come back. A second remedy is to modify the equations in a layer about \mathcal{Q}_I to absorb waves. The aim is that waves that reach $\partial\mathcal{Q}$ are weak so that their reflections are small. However even very cleverly constructed layers themselves reflect and can create errors analogous to those from the boundary conditions. In the early nineties perfectly reflectionless layers were constructed.

1.1 The PML strategy

Assumption 1.1 \mathcal{Q} is a rectangular solid, and, there is a rectangular solid \mathcal{Q}_I with $\overline{\mathcal{Q}_I} \subset \mathcal{Q}$ called the **domain of interest**. Assume that the charge density ρ and current density j are supported in $[0, \infty[\times \overline{\mathcal{Q}_I}$ and satisfy (1.2).

The PML strategy introduces a new larger system of equations in \mathcal{Q} . A linear combination of the new unknowns yields the approximate solution in \mathcal{Q}_I . There are three goals.

- In $\mathcal{Q} \setminus \overline{\mathcal{Q}_I}$ solutions of the augmented system are damped.
- There is *no reflection at all* from the layers in $\mathcal{Q} \setminus \overline{\mathcal{Q}_I}$.
- They are easy to implement.

Such perfectly matched layers for Maxwell's equations were constructed by Bérenger [6, 7]. The existence of such layers is truly remarkable. After Bérenger, closely related PML were introduced [30, 10, 37, 22, 34, 38]. They all have at their core the stretched equations studied below. Our analysis of the stretched equations suffices to analyse them all (see §9).

The combination of perfectly matched layers for Maxwell's equations with standard absorbing boundary conditions on $\partial\mathcal{Q}$ is amazingly efficient. It is deservedly one of the most widely used algorithms in computational physics. As a result of this massive experience, the behavior of the algorithm in practice is well understood. Three desirable properties are observed.

- **Stability.** The computed solutions are bounded on bounded time intervals.
- **Absorbing layers.** The computed waves decrease as they pass through the regions with nonvanishing absorptions (introduced in §1.3).
- **Uniform stability.** The computed solutions are bounded in $\{t \geq 0\}$.

There is an enormous gap between what is understood in practice and what is proved. In this paper we prove the stability. Solutions with perfectly matched layers and absorbing boundary conditions at $\partial\mathcal{Q}$ exist, are uniquely determined, and remain bounded on bounded intervals of time. The other two properties are outstanding open problems.

The stability problem has remained unsolved for more than thirty years for several reasons.

- The computational domain has trihedral corners. The boundary conditions change when one passes from one face of the rectangular solid to

another. Such hyperbolic problems are little developed.

- The PML involve absorption coefficients $\sigma_j(x_j)$ that vanish identically on \mathcal{Q}_I . These variable coefficients defeat exact solution strategies.
- The PML introduce auxiliary variables. The augmented systems are usually degenerate in one way or another. They are plagued by characteristic boundaries, loss of ellipticity, weak hyperbolicity, etc.

To formulate precise theorems introduce some definitions.

Definition 1.1 Define a \mathbb{C}^6 valued function u and a 6×6 system of differential operators \mathcal{A} by,

$$u := (E, B), \quad \mathcal{A}(\partial) := \begin{pmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{pmatrix}, \quad \partial = (\partial_1, \partial_2, \partial_3).$$

Equation (1.1) is equivalent to

$$\partial_t u + \mathcal{A}(\partial)u = f, \quad f := (-j, 0). \quad (1.4)$$

Define real antisymmetric 3×3 matrices A_k and 6×6 matrices \mathcal{A}_k ,

$$\begin{aligned} A_1 &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, & A_2 &:= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ A_3 &:= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathcal{A}_k &:= \begin{pmatrix} 0 & A_k \\ -A_k & 0 \end{pmatrix}. \end{aligned}$$

Then $\mathcal{A}(\partial) = \sum \mathcal{A}_k \partial_k$. Define

$$\mathcal{A}(\xi) := \sum_k \xi_k \mathcal{A}_k = \begin{pmatrix} 0 & -\xi^\wedge \\ \xi^\wedge & 0 \end{pmatrix}. \quad (1.5)$$

Definition 1.2 Denote by $\mathcal{Q} = \mathcal{Q}(L_1, L_2, L_3)$ the rectangular solid

$$\mathcal{Q} := \{x \in \mathbb{R}^3 : |x_j| < L_j/2, \quad j = 1, 2, 3\}.$$

The boundary of \mathcal{Q} has six open faces Γ_k with $1 \leq k \leq 6$. For $i = 1, 2, 3$,

$$\begin{aligned} \Gamma_i &:= \{x_i = -L_i/2, \text{ and, } |x_j| < L_j/2 \text{ for } j \neq i\}, \\ \Gamma_{i+3} &:= \{x_i = L_i/2, \text{ and, } |x_j| < L_j/2 \text{ for } j \neq i\}. \end{aligned}$$

For a point $x \in \Gamma_k$, $\nu(x)$ denotes the outward unit normal to \mathcal{Q} at x .

The singular subset $\mathcal{S} \subset \partial\mathcal{Q}$ consists of the points of the boundary so that $|x_j| = L_j/2$ for more than one value of j .

1.2 Energy flux and absorbing boundary conditions

Recall the energy law for real solutions of Maxwell's equations.

Definition 1.3 For vectors v and w in \mathbb{C}^d the dot product is defined as $v \cdot w := \sum_k v_k w_k$ the sum of the products of the components. The \mathbb{C}^d scalar product equal to $v \cdot \bar{w}$ is denoted (v, w) .

Remark 1.1 If vectors $v(\tau)$ and $w(\tau)$ depend holomorphically on τ then $v(\tau) \cdot w(\tau)$ depends holomorphically on τ . The \mathbb{C}^d scalar product need not be holomorphic.

Suppose that E, B satisfies Maxwell's equations with currents vanishing on an open set Ω . The energy density is defined to be $(|E|^2 + |B|^2)/2$. Compute three equivalent expressions for the flux of energy through an element of surface $d\Sigma$ with unit outward normal ν .

Start with

$$\partial_t(|E|^2 + |B|^2)/2 = (E \cdot E_t + B \cdot B_t) = (E, B) \cdot (E, B)_t.$$

Maxwell's equations yield

$$E \cdot E_t + B \cdot B_t = E \cdot \text{curl } B - B \cdot \text{curl } E.$$

The vector identity $\text{div}(u \wedge v) = v \cdot \text{curl } u - u \cdot \text{curl } v$ yields Poynting's identity,

$$\partial_t(|E|^2 + |B|^2)/2 + \text{div}(E \wedge B) = 0.$$

Integrating and using the Divergence Theorem yields

$$\partial_t \int_{\Omega} (|E|^2 + |B|^2)/2 \, dx + \int_{\partial\Omega} (E \wedge B) \cdot \nu \, d\Sigma = 0$$

The outward energy flux is equal to

$$(E \wedge B) \cdot \nu \, d\Sigma. \tag{1.6}$$

Compute the flux using the matricial form of the Maxwell operator. With $u = (E, B)$,

$$u \cdot (u_t + \sum \mathcal{A}_k \partial_k u) = 0.$$

Since the \mathcal{A}_k are real and symmetric this yields the conservation law

$$\partial_t |u|^2 + \sum_k \partial_k (u \cdot \mathcal{A}_k u) = 0.$$

Integrating over Ω yields

$$\partial_t \int_{\Omega} |u|^2 dx + \int_{\partial\Omega} u \cdot (\mathcal{A}(\nu)u) d\Sigma = 0$$

Definition 1.4 For a hermitian matrix A , $\mathcal{E}^{\pm}(A)$ denotes the spectral subspaces corresponding to strictly positive and strictly negative eigenvalues. Denote by $\pi^{\pm}(A)$ the orthogonal projection on those spaces. Denote the nullspace as $\mathcal{E}^0(A)$ with projector $\pi^0(A)$.

Lemma 2.1 shows that the eigenvalues of $\mathcal{A}(\nu)$ are ± 1 and 0. Therefore

$$u \cdot (\mathcal{A}(\nu)u) = \|\pi^+(\mathcal{A}(\nu))u\|^2 - \|\pi^-(\mathcal{A}(\nu))u\|^2.$$

The outward flux of energy is equal to

$$\frac{1}{2} u \cdot (\mathcal{A}(\nu)u) d\Sigma = \frac{1}{2} \left(\|\pi^+(\mathcal{A}(\nu))u\|^2 - \|\pi^-(\mathcal{A}(\nu))u\|^2 \right) d\Sigma. \quad (1.7)$$

The boundary of the computational domain \mathcal{Q} is an artificial boundary. A natural choice of boundary condition is to require that $\pi^-(\mathcal{A}(\nu))(E, B) = 0$. This says that the incoming part of the energy flux from (1.7) vanishes at $\partial\mathcal{Q}$.

Definition 1.5 If $\mathcal{O} \subset \mathbb{R}^3$ is an embedded 2-manifold, $\omega \in \mathcal{O}$, ν is a unit normal vector to \mathcal{O} at ω , and $V \in T_{\omega}(\mathcal{O})$, the tangential component of V , denoted V_{tan} , is defined as $V - (V \cdot \nu)\nu$.

Example 2.4 shows that $\pi^-(\mathcal{A}(\nu))(E, B) = 0$ if and only if $E_{tan} = B_{tan} \wedge \nu$. With this boundary condition on $\cup\Gamma_k$, the outward energy flux density is non negative. For *sufficiently regular solutions* this implies that the energy in \mathcal{Q} is non increasing. The theory of symmetric positive boundary value problems of Friedrichs [14, 21, 31] shows that the initial boundary value problem for Maxwell's equations has weak solutions and that strong solutions are unique. The fact that the boundary is irregular prevents one from using mollifiers to prove the uniqueness of weak solutions and therefore that the initial value problem is well posed.

The uniqueness in Part 1 of [16] shows that for each τ the Laplace transform of the solution belongs to $H^{1/2}(\mathcal{Q})$. A capacity argument shows that $H^{1/2}$ regularity suffices for each τ to justify the energy identity. The $H^{1/2}$ regularity in the Maxwell context relies on results of Jerison and Kenig and was

proved by Costabel [12]. We rediscovered that result and give the correct attribution here. Recently Nicaise and Tomezyk [27] prove that for convex polyhedra (hence \mathcal{Q}), the Laplace transform of a solution of Maxwell's equations with the ABC on $\partial\mathcal{Q}$ belongs to $H^1(\mathcal{Q})$. With this regularity the justification of uniqueness is straight forward.

1.3 The split and stretched equations

Assumption 1.2 *The split equations involve non negative absorption coefficients $\sigma_k \in C^\infty(\mathbb{R})$. Assume that $\sigma_k(x_k)$ vanishes for all $x \in \overline{\mathcal{Q}_I}$ and that for all $\ell \in \mathbb{N}$, the derivative $\sigma_k^\ell(\cdot) \in L^\infty(\mathbb{R})$.*

Remark 1.2 *The absorption $\sigma_k(x_k)$ is typically strictly positive on a neighborhood of $\Gamma_k \cup \Gamma_{k+3}$. As a function of x_k , σ_k usually increases from value zero in \mathcal{Q}_I to a maximum at $\partial\mathcal{Q}$. The strips where $\sigma_k > 0$ are called **absorbing layers**.*

Bérenger's splitting introduces $\tilde{U} := (U^1, U^2, U^3)$ with U^k taking values in \mathbb{R}^6 . The split equations imposed on \tilde{U} are

$$(\partial_t + \sigma_k(x_k))U^k + \mathcal{A}_k \partial_k (U^1 + U^2 + U^3) = f_k, \quad k = 1, 2, 3. \quad (1.8)$$

The k^{th} equation has the ∂_k derivatives from Maxwell's system. The source terms f_k are chosen so that $f_1 + f_2 + f_3 = (-j, 0) := f$. The most symmetrical choice is $f_k = f/3$. Another common choice is $f_1 = f$ and $f_2 = f_3 = 0$. To cover both treat

$$f_k := c_k f, \quad f = (-j, 0), \quad c_k \geq 0, \quad c_1 + c_2 + c_3 = 1. \quad (1.9)$$

Our method yields the same conclusion for each of these choices with estimates uniform for c_k as in (1.9).

Definition 1.6 *The system (1.8) with (1.9) is called the **split equations**.*

*For a solution of the split equations, the **associated electric and magnetic fields** are defined by $(E, B) := U^1 + U^2 + U^3$.*

The restriction of E, B to \mathcal{Q}_I is the output of Bérenger's method.

The analysis of the pure initial value problem on \mathbb{R}^{1+d} for the split equations corresponds to a computational domain $\mathcal{Q} = \mathbb{R}^3$. It is *much* simpler than

the mixed initial boundary value problem with boundary conditions on $\partial\mathcal{Q}$ with its edges and corners.

The first proofs of stability for Bérenger's system on \mathbb{R}^3 differentiated the equations to construct a large system for a highly heterogeneous mix of derivatives. The system has constant coefficient principal part that is symmetrized by a Fourier multiplier $S(D)$, see [23, 22, 29, 24, 15].

The split equations are not a symmetric hyperbolic system. The estimates in all of the above papers involve subtle constructions. Though the layers are called absorbing there is no known simple energy decay law for Bérenger's split Maxwell system. For systems whose characteristic polynomial has factor $\tau^2 - q(\xi, \xi)$ with real positive definite anisotropic quadratic q with axes of symmetry not aligned with the coordinate axes, the analogous layers for the split equations are amplifying [15, 3]. For the anisotropic Maxwell system even with axes aligned with the coordinate axes, the layers are amplifying [4].

The solutions of (1.8) are supported in $t \geq 0$. If the solutions grow no faster than e^{Mt} , then their Laplace transforms are holomorphic in $\text{Re } \tau > M$. The Laplace transform, indicated with a hat, of (1.8) is

$$(\tau + \sigma_k(x_k))\widehat{U}^k + \mathcal{A}_k \partial_k (\widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3) = \widehat{f}_k, \quad k = 1, 2, 3. \quad (1.10)$$

Definition 1.7 *Define the stretched derivatives*

$$\widetilde{\partial}_k := \frac{\tau}{\tau + \sigma_k(x_k)} \partial_k, \quad k = 1, 2, 3, \quad \widetilde{\partial} := (\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3).$$

The corresponding vector operators are,

$$\begin{aligned} \widetilde{\text{grad}} &:= (\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3), & \widetilde{\text{div}} &:= (\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3) \cdot, \\ \widetilde{\text{curl}} &:= (\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3) \wedge, & \widetilde{\Delta} &:= \widetilde{\text{div}} \widetilde{\text{grad}}. \end{aligned}$$

Multiply the k^{th} equation of (1.10) by $\tau/(\tau + \sigma_k(x_k))$. The sum of the three resulting equations shows that $(\widehat{E}, \widehat{B})$ satisfies

$$(\tau + \sum_k \mathcal{A}_k \widetilde{\partial}_k)(\widehat{E}, \widehat{B}) = \widehat{f}. \quad (1.11)$$

Equivalently, using Assumption 1.1 and Assumption 1.2,

$$\tau \widehat{E} - \widetilde{\text{curl}} \widehat{B} = -\widehat{j} \sum_k \frac{\tau c_k}{\tau + \sigma_k(x_k)} = -\widehat{j}, \quad \tau \widehat{B} + \widetilde{\text{curl}} \widehat{E} = 0. \quad (1.12)$$

Definition 1.8 *The equivalent systems (1.11) and (1.12) satisfied by \widehat{E}, \widehat{B} are called the **stretched Maxwell equations** or simply the **stretched equations**.*

Applying $\widetilde{\text{div}}$ to the stretched Maxwell equations and using the continuity equation in Assumption 1.1 yields the stretched divergence identities

$$\tau \widetilde{\text{div}} \widehat{E} = -\widetilde{\text{div}} \widehat{j} = \tau \widehat{\rho}, \quad \tau \widetilde{\text{div}} \widehat{B} = 0. \quad (1.13)$$

When $\tau \in]0, \infty[$, the stretched Maxwell equations are real and symmetric in the sense of Friedrichs [14], that is, the matrices multiplying ∂_k are real and symmetric. For τ real and large the stretched system is positive in his sense. However, when τ is not real the coefficients are not hermitian symmetric. The split system (1.8) and (1.10) is never symmetric and never hermitian. If the absorptions are not identically equal to zero, the stretched Maxwell system is hermitian for and only for real τ .

Definition 1.9 *For $\tau \in \mathbb{C} \setminus 0$ the coordinate stretchings $X_k(\tau, x_k)$ are the solutions of ordinary differential equation initial value problems*

$$\frac{\partial X_k}{\partial x_k} = \frac{\tau + \sigma_k(x_k)}{\tau}, \quad X_k(\tau, 0) = 0. \quad (1.14)$$

Remark 1.3 *For real $\tau > 0$, $\partial X_k / \partial x_k > 0$ so $x \mapsto X(\tau, x) := (X_1, X_2, X_3)$ is a diffeomorphism from \mathbb{R}^3 onto itself.*

Compute for $\tau \in]0, \infty[$,

$$\frac{\partial}{\partial x_j} = \sum_k \frac{\partial X_k}{\partial x_j} \frac{\partial}{\partial X_k} = \frac{\tau + \sigma_j(x_j)}{\tau} \frac{\partial}{\partial X_j}, \quad \frac{\tau}{\tau + \sigma_j(x_j)} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial X_j}. \quad (1.15)$$

For τ complex, X is complex valued so is no longer a τ -dependent change of variable on \mathbb{R}^3 .

Stretched vector operators satisfy the familiar identities,

$$\begin{aligned} \widetilde{\text{div}} \widetilde{\text{curl}} &= 0, & \widetilde{\text{div}}(\Phi \wedge \Psi) &= (\widetilde{\text{curl}} \Phi) \cdot \Psi - (\widetilde{\text{curl}} \Psi) \cdot \Phi, \\ \widetilde{\Delta} &= \widetilde{\text{grad}} \widetilde{\text{div}} - \widetilde{\text{curl}} \widetilde{\text{curl}}, & \widetilde{\text{curl}} \widetilde{\text{grad}} &= 0. \end{aligned} \quad (1.16)$$

Proof. For τ real, the stretched version follows from the unstretched version by coordinate stretching.

The terms in the stretched identities are holomorphic in $\{\text{Re } \tau > 0\}$. Since the identities hold on the real axes, they follow by analytic continuation for all $\text{Re } \tau > 0$. \square

1.4 Main Theorem for the stretched equations

Consider (1.11). At a boundary point in Γ_j the normal matrix is equal to

$$\sum \nu_k \frac{\tau}{\tau + \sigma_k(x_k)} \mathcal{A}_k = \frac{\tau}{\tau + \sigma_j(x_j)} \nu_j \mathcal{A}_j, \quad \nu_j = \pm 1.$$

When $\tau > 0$ this is a strictly positive scalar multiple of the normal matrix for the Maxwell system. The energy flux argument shows that the boundary condition $\pi^-(\nu)(\widehat{E}, \widehat{B}) = 0$ is a natural choice for the stretched equations with $\tau > 0$. If this choice is made for $\tau > 0$, then by analytic continuation $\pi^-(\nu)(\widehat{E}, \widehat{B}) = 0$ is satisfied on $\text{Re } \tau > M$. The conclusion is that for the rectangular domain \mathcal{Q} , a natural boundary condition for the stretched equations is $\pi^-(\nu)(\widehat{E}, \widehat{B}) = 0$ on $\partial\mathcal{Q} \setminus \mathcal{S}$. Theorem 1.11 solves the resulting mixed initial boundary value problem.

The stability result for the stretched equations is the following. The assertion involves boundary traces. If $W = (F, G) \in L^2(\mathcal{Q})$ satisfies the stretched Maxwell equations with source j, ρ vanishing on a neighborhood of $\partial\mathcal{Q}$ then on a neighborhood of the boundary, $W, \text{div } W, \text{curl } W$ are square integrable. Proposition 5.4 shows that W has a well defined trace in $H^{-1/2}(\partial\mathcal{Q})$. Therefore, the boundary traces in **ii** and **iv** of Theorem 1.11 make sense.

Definition 1.10 *If $\mathcal{O} \subset \mathbb{R}^3$ is open and $K \subset \mathcal{O}$ is compact, define $L_K^2(\mathcal{O}) := \{f \in L^2(\mathcal{O}) ; \text{supp } f \subset K\}$. The spaces $C_K^\infty(\mathcal{O})$ and $H_K^1(\mathcal{O})$ are defined similarly,*

Theorem 1.11 *Assume that Assumption 1.2 is satisfied. There is an $M_0 > 0$ so that if $M > M_0$, J and R satisfying $-\text{div } J = \tau R$ are holomorphic on $\text{Re } \tau > M$ with values in $L_{\mathcal{Q}_I}^2(\mathcal{Q})$, then there is a unique $W(\tau) = (F(\tau), G(\tau))$ that satisfies **i**, **ii**, **iii**, **iv**.*

i. W is holomorphic on $\{\text{Re } \tau > M\}$ with values in $L^2(\mathcal{Q})$.

ii. W_{tan} is holomorphic on $\{\text{Re } \tau > M\}$ with values in $L^2(\cup\Gamma_k)$.

iii. For all $\text{Re } \tau > M$, one has the following equations on \mathcal{Q} ,

$$\tau F - \widetilde{\text{curl}} G = -J, \quad \tau G + \widetilde{\text{curl}} F = 0, \quad \widetilde{\text{div}} G = 0, \quad \widetilde{\text{div}} F = R. \quad (1.17)$$

iv. $F_{tan} = G_{tan} \wedge \nu$ (equivalently $\pi^-(\nu)(F, G) = 0$) on $\cup\Gamma_k$.

In addition, with constant independent of R, J, τ, M ,

$$(\text{Re } \tau)^2 \|W(\tau)\|_{L^2(\mathcal{Q})}^2 + (\text{Re } \tau) \|W_{tan}(\tau)\|_{L^2(\cup\Gamma_k)}^2 \lesssim \|R(\tau), J(\tau)\|_{L^2(\mathcal{Q})}^2. \quad (1.18)$$

Remark 1.4 *In addition, $W|_{\partial\mathcal{Q}}$ is holomorphic with values in $L^2(\partial\mathcal{Q})$ with $\|W|_{\partial\mathcal{Q}}\|_{L^2(\partial\mathcal{Q})} \lesssim |\tau|^{1/2}(\operatorname{Re} \tau)^{-1}\|R, J\|_{L^2(\mathcal{Q})}$ (see (8.6)). For $\eta < 1$, $\nabla W \in L^2(\eta\mathcal{Q})$ with $\|\nabla W\|_{L^2(\eta\mathcal{Q})} \leq C_\eta|\tau|(\operatorname{Re} \tau)^{-1}\|R, J\|_{L^2(\mathcal{Q})}$ (see (8.7)).*

Example 1.1 Perfect matching. *Theorem 1.11 implies perfect matching when $\mathcal{Q} = \mathbb{R}^3$. Compare W with a second family of solutions, W_0 , corresponding to the unstretched case $\sigma = 0$. Perfect matching asserts that $W|_{\mathcal{Q}_I} = W_0|_{\mathcal{Q}_I}$.*

Proof of perfect matching from [15]. Equation (1.15) asserts that $\tilde{\partial}_j$ in the x coordinates is equal to $\partial/\partial X_j$ in the X coordinates. Therefore for positive real τ , the change of variable conjugates the stretched operator in x to the unstretched operator (equivalently the operator with $\sigma = 0$) in X . In addition for τ real both operators have easy estimates. Both of these operators are invertible for $\tau \in]M, \infty[$. The change of variable is equal to the identity on \mathcal{Q}_I . Therefore, when one solves these different equations with the same source supported in \mathcal{Q}_I the two solutions agree on \mathcal{Q}_I .

As both W and W_0 are holomorphic on $\operatorname{Re} \tau > M$, it follows by analytic continuation that $W|_{\mathcal{Q}_I} = W_0|_{\mathcal{Q}_I}$ for $\operatorname{Re} \tau > M$. \square

Remark 1.5 *For a bounded computational domain \mathcal{Q} the perfect matching is destroyed. Waves reaching the external boundary are partly reflected and pollute the computation in \mathcal{Q}_I . This effect is mitigated by the decay of wave in the layers. The imperfections decrease with increasing thickness of the layers. Proving that the PML for Maxwell's equations have this observed dissipative behavior is an outstanding open problem.*

1.5 Main Theorem for the split equations

Theorem 1.12 *Assume that assumptions 1.1 and 1.2 are satisfied. There is an $M_0 > 0$ so that if $\mu > M_0$ and $\rho, j \in e^{\mu t}L^2(\mathbb{R} \times \mathcal{Q}_I)$ then there is a unique solution $\tilde{U} = (U^1, U^2, U^3) \in L^2(\mathbb{R}; H^{-1}(\mathcal{Q}))$, satisfying $\tilde{U} = 0$ for $t \leq 0$ and the split boundary value problem (1.8, 1.9) on $\mathbb{R} \times \mathcal{Q}$ with,*

$$\begin{aligned} V &:= U^1 + U^2 + U^3 := (F, G) \in e^{\mu t}L^2(\mathbb{R} \times \mathcal{Q}), \\ V_{tan} &\in e^{\mu t}L^2(\mathbb{R} \times \Gamma_k), \quad 1 \leq k \leq 6, \\ F_{tan} &= G_{tan} \wedge \nu \quad (\text{equivalently } \pi^-(\nu)(F, G) = 0) \quad \text{on } \mathbb{R} \times \cup \Gamma_k. \end{aligned}$$

In addition, V and the components \tilde{U}^k satisfy

$$\begin{aligned} \mu \|e^{-\mu t} V\|_{L^2(\mathbb{R} \times \mathcal{Q})} + \mu^{1/2} \|e^{-\mu t} V_{tan}\|_{L^2(\mathbb{R} \times \cup \Gamma_k)} + \\ \mu \|e^{-\mu t} \{U^k, \partial_t U^k\}\|_{L^2(\mathbb{R}; H^{-1}(\mathcal{Q}))} \lesssim \|e^{-\mu t} \{\rho, j\}\|_{L^2(\mathbb{R} \times \mathcal{Q})}. \end{aligned} \quad (1.19)$$

Remark 1.6 i. *The output of Bérenger's algorithm is the restriction of V to \mathcal{Q}_I . With this in mind, estimate (1.19) resembles the estimate for the Maxwell's equations themselves.*

ii. *The restriction of U^k to $\eta\mathcal{Q}$ with $\eta < 1$ satisfies (use equation (8.7))*

$$\mu \|e^{-\mu t} U^k\|_{L^2(\mathbb{R} \times \eta\mathcal{Q})} \leq C_\eta \|e^{-\mu t} \{\partial_t \rho, \partial_t j\}\|_{L^2(\mathbb{R} \times \mathcal{Q})}. \quad (1.20)$$

Remark 1.7 Stability. *For PMLs for Maxwell's system with variable absorptions and absorbing boundary conditions, stability even for smooth boundaries, is not proved before this paper.*

Uniform stability. *Estimate (1.19) allows the possibility of exponential growth in time. Proving uniform boundedness, suggested by computational experience, is an outstanding open problem.*

For the special case of $\mathcal{Q} = \mathbb{R}^3$ with constant σ_k , Bécache-Joly [5] prove the boundedness of $\|E, B\|_{L^2(\mathbb{R}^3)}$ for $t \geq 0$ by exact solution formulas.

For $\partial_t^2 - \tilde{\Delta}$ with constant absorptions on \mathbb{R}^3 or on \mathcal{Q} with periodic or Dirichlet boundary conditions Bécache-Joly [5] and Baffet et. al. [2] bound $\sup_{t \geq 0} \|u_t\|_{L^2(\mathcal{Q})}$ for smooth solutions. For these boundary conditions, the boundary terms in their energy method vanish identically. This is very different from the outward flux of energy desired in numerical methods. Deceptively, it sounds as if the outward flux should make the problem simpler.

1.6 Outline of the proof

The key result is Theorem 1.11 asserting existence uniqueness and holomorphy in τ of solutions of the stretched Maxwell system. Just as solutions of Maxwell's system are solutions of D'Alembert's wave equation, the solution of the stretched Maxwell system satisfies a Helmholtz system, introduced in §3.

On $\partial\mathcal{Q}$ this Helmholtz system requires six boundary conditions. Two come from the boundary condition $\pi^-(\nu)(E, B) = 0$. An additional pair are the divergence relations $\operatorname{div} E = \operatorname{div} B = 0$. The final pair comes from the stretched equations themselves. The additional condition, introduced in

§3.1.2, is $\pi^+(\mathcal{M}\nu)(\tau - \widetilde{\text{curl}} E, \tau + \widetilde{\text{curl}} B) = 0$ with \mathcal{M} from Definition 3.1. The resulting boundary value problem is solved on a domain \mathcal{Q}^δ that is a smoothing of \mathcal{Q} on a δ -neighborhood of the singular set \mathcal{S} .

Though solutions of the stretched Maxwell boundary value problem automatically satisfy the resulting Helmholtz boundary value problem, the converse is proved only for real τ and only for smooth solutions. That smoothness requires the smoothed domain. That the stretched boundary value problem is satisfied for general τ follows by analytic continuation.

Section 4 shows that the Helmholtz boundary value problem satisfies Lopatin-ski's coercivity condition, and depends holomorphically on τ . The boundary data for this problem are sections of vector bundles that depend holomorphically on τ .

The hardest part of the analysis is the derivation of estimates for the solutions of the stretched Maxwell boundary value problem on \mathcal{Q}^δ . One set of estimates comes from the Helmholtz boundary value problem and depends on Green's identities established in §5. As for the analysis of the absorbing boundary conditions for Maxwell's equations, the associated quadratic form leading to the Laplace operator is the div – curl form and not the Dirichlet form [13]. Proposition 5.10 gives important relation between the boundary terms in Green's identity and the spectral projections of §2.

A system sharing the Lorenz invariance of Maxwell's but with stronger ellipticity is Pauli's equations analysed in [17]. That analysis uses the Dirichlet form. The Helmholtz boundary value problem is significantly simpler with slightly stronger estimates. The difficulty of proceeding from Pauli to Maxwell surprised us.

In §5.3 the Green's identities are applied to the stretched Maxwell's equations with $\tau \in]0, \infty[$ to prove invertibility. This complements the Lopatin-ski analysis of §4.

For complex τ there are two difficulties in applying the identities of §5. The first is the choice of appropriate multipliers. The second is the boundary terms that no longer have a sign.

Section 6 presents the technique for overcoming the second obstacle. We derive estimates for the stretched Maxwell system that are analogues of those of Jerison-Kenig [19] for harmonic functions and M. Mitrea [25] for Maxwell's equations. They assert that the boundary trace up to a small correction is of the same size as the $\pi^+(\mathcal{M}\nu)$ part of the trace. Green's identity has a term with $\pi^+(\mathcal{M}\nu)$ part appearing with favorable sign.

The natural multiplier in Green's identity is not the complex conjugate of

(\widetilde{E}, B) . The natural space is the set of fields with square integrable $\widetilde{\text{div}}$ and $\widetilde{\text{curl}}$. The complex conjugate of such a field need not satisfy this constraint. In §7, appropriate multipliers are found. The crucial estimates required to solve the stretched Maxwell boundary value problem on \mathcal{Q}^δ with bounds uniform in δ is proved in §7.

Section 8.1 shows, by analytic continuation from τ real, that the solution of the Helmholtz boundary value problem satisfies the stretched Maxwell equations on \mathcal{Q}^δ . Section 8.2 solves the stretched Maxwell equations on \mathcal{Q} with its trihedral corners by passage to the limit $\delta \rightarrow 0$. The holomorphy in τ makes it easier to prove convergence. The uniqueness of the solution is proved by showing that the Laplace transform vanishes for τ real. That follows from the uniqueness theorem in Part 1 of [16]. Section 8.3 solves the split equations.

Part 1 of [16], treats domains with trihedral corners for symmetric hyperbolic problems with strictly dissipative boundary conditions. The existence proofs were easy thanks to straightforward energy identities. For the present article, the existence step is difficult.

Maxwell's equations on smooth domains with the absorbing boundary conditions is a classical problem. It, and the stretched version with $\tau \in]0, \infty[$, is a strictly dissipative symmetric hyperbolic system with boundary characteristic of constant multiplicity [21, 14]. Similar results can also be obtained by variational methods [12, 13, 27], or layer potential methods [26]. For complex τ these problems lose their easy positivity and symmetry properties. We treat the Laplace transform with methods with the flavor of the variational methods. coupled with the method pioneered by Jerison and Kenig. This strategy bypasses deep harmonic analysis estimates, like those of [11]. Such estimates are likely valid in the present context.

2 Symbol spectral decomposition

The notation from Definition 1.4 is used in the next Lemma.

Lemma 2.1 *For $\xi \in \mathbb{R}^3 \setminus 0$, the real 6×6 symmetric matrix $\mathcal{A}(\xi)$ from (1.5) has eigenvalues $\pm|\xi|$ and 0. Each eigenvalue has multiplicity two. The eigenspaces, positive homogeneous of degree zero in ξ , are given by*

$$\begin{aligned} \mathcal{E}^0(\mathcal{A}(\xi)) &= \{(\mathbf{e}, \mathbf{b}) : \mathbf{e} \parallel \xi \text{ and } \mathbf{b} \parallel \xi\}, \\ \mathcal{E}^+(\mathcal{A}(\xi)) &= \{(\mathbf{e}, \mathbf{b}) : \mathbf{b} \perp \xi \text{ and } \mathbf{e} = \mathbf{b} \wedge \xi/|\xi|\}, \\ \mathcal{E}^-(\mathcal{A}(\xi)) &= \{(\mathbf{e}, \mathbf{b}) : \mathbf{b} \perp \xi \text{ and } \mathbf{e} = -\mathbf{b} \wedge \xi/|\xi|\}. \end{aligned} \tag{2.1}$$

Proof. By homogeneity, it suffices to consider $|\xi| = 1$. Each of the subspaces on the right is two dimensional. For the eigenvalues ± 1 , compute

$$\mathcal{A}(\xi) \begin{pmatrix} \mathbf{e} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & -\xi \wedge \\ \xi \wedge & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} -\xi \wedge \mathbf{b} \\ \xi \wedge \mathbf{e} \end{pmatrix}.$$

It follows that (\mathbf{e}, \mathbf{b}) in the formulas are eigenvectors with eigenvalue ± 1 . The eigenvalue 0 is easier. Since the spaces on the right span \mathbb{C}^6 it follows that they contain all the eigenvectors. \square

Example 2.1 *It follows that for $\tau \in \mathbb{C}$ and $\xi \in \mathbb{R}^3 \setminus 0$*

$$\det(\tau I - \mathcal{A}(\xi)) = \tau^2(\tau^2 - \xi \cdot \xi)^2. \quad (2.2)$$

Since the left hand side is a homogeneous degree 6 polynomial in (τ, ξ) , it follows (for example by Taylor's theorem) that this identity holds for all $(\tau, \xi) \in \mathbb{C}^{1+3}$.

Definition 2.2 *When dealing with the Maxwell system we use the shorthand $\mathcal{E}^\pm(\xi)$ and $\mathcal{E}^0(\xi)$ for $\mathcal{E}^\pm(\mathcal{A}(\xi))$ and $\mathcal{E}^0(\mathcal{A}(\xi))$. Similarly for the spectral projectors.*

Example 2.2 *For a real unit vector ξ the map $\mathbf{b} \mapsto (\pm \mathbf{b} \wedge \xi, \mathbf{b})/\sqrt{2}$ is an isometry from the set of vectors orthogonal to ξ to $\mathcal{E}^\pm(\xi)$.*

Definition 2.3 *Define*

$$\mathcal{Z} := \{\xi \in \mathbb{C}^3 \setminus 0 : \xi_1^2 + \xi_2^2 + \xi_3^2 = 0\}.$$

\mathcal{Z} is a closed conic subset of $\mathbb{C}^3 \setminus 0$. $(\mathbb{C}^3 \setminus 0) \setminus \mathcal{Z}$ is open, conic, connected, and, contains $\mathbb{R}^3 \setminus 0$.

Example 2.3 *For $\xi \in \mathbb{C}^3 \setminus 0$, define subspaces $W_j(\xi) \subset \mathbb{C}^3$ by*

$$W_1 := \mathbb{C}\xi, \quad \text{and,} \quad W_2 := \{v \in \mathbb{C}^3 : v \cdot \xi = 0\}.$$

Then $\dim W_1 = 1$ and $\dim W_2 = 2$. In addition, $W_1 \cap W_2 = \{0\}$ if and only if $\xi \notin \mathcal{Z}$. In that case, $\mathbb{C}^3 = W_1 \oplus W_2$.

Definition 2.4 *For $0 \neq \xi \in \mathbb{C}^3 \setminus \mathcal{Z}$, denote π_ξ and π_{ξ^\perp} the projections in $\text{Hom}(\mathbb{C}^3)$ given by*

$$\pi_\xi v := \frac{v \cdot \xi}{\xi \cdot \xi} \xi, \quad \text{and,} \quad \pi_{\xi^\perp} := I - \pi_\xi.$$

The decomposition $I = \pi_\xi + \pi_{\xi^\perp}$ corresponds to $W_1 \oplus W_2 = \mathbb{C}^3$.

Remark 2.1 i. For ξ in $\mathbb{R}^3 \setminus 0$, π_{ξ^\perp} is equal to the orthogonal projection onto the orthogonal complement of ξ . When ξ is not real, π_{ξ^\perp} is not equal to the orthogonal projection onto the orthogonal complement of ξ .

ii. The map $\{(\mathbb{C}^3 \setminus 0) \setminus \mathcal{Z}\} \ni \xi \mapsto \pi_{\xi^\perp}$ is holomorphic with values in $\text{Hom}(\mathbb{C}^3)$. For ξ in $\mathbb{R}^3 \setminus 0$, π_{ξ^\perp} is equal to the orthogonal projection onto the orthogonal complement of ξ . These properties characterize π_{ξ^\perp} .

iii. For $c \in \mathbb{C} \setminus 0$, $\pi_{c\xi} = \pi_\xi$ and $\pi_{(c\xi)^\perp} = \pi_{\xi^\perp}$.

iv. For $\xi \in \mathbb{R}^3 \setminus 0$,

$$\xi \wedge (\xi \wedge w) = -(\xi \cdot \xi) \left(w - \frac{(\xi \cdot w)\xi}{\xi \cdot \xi} \right) = -(\xi \cdot \xi) \pi_{\xi^\perp} w. \quad (2.3)$$

The terms of the identity are holomorphic functions of ξ on the connected set $\mathbb{C}^3 \setminus \mathcal{Z}$. Identity (2.3) extends by analytic continuation to $\xi \in \mathbb{C}^3 \setminus \mathcal{Z}$.

Lemma 2.5 Suppose that $\xi \in \mathbb{R}^3$ is a unit vector.

i. The span of the non zero eigenspaces of $\mathcal{A}(\xi)$ is the set of \mathbf{e}, \mathbf{b} with $\mathbf{e} \cdot \xi = 0$ and $\mathbf{b} \cdot \xi = 0$. In addition, for arbitrary (\mathbf{e}, \mathbf{b}) ,

$$\pi^0(\xi)(\mathbf{e}, \mathbf{b}) = \left((\mathbf{e} \cdot \xi)\xi, (\mathbf{b} \cdot \xi)\xi \right).$$

ii. For (\mathbf{e}, \mathbf{b}) satisfying $\mathbf{e} \cdot \xi = 0$, and $\mathbf{b} \cdot \xi = 0$,

$$\pi^+(\xi)(\mathbf{e}, \mathbf{b}) = \frac{1}{2} \left(\mathbf{b} \wedge \xi + \mathbf{e}, \mathbf{b} - \mathbf{e} \wedge \xi \right).$$

iii. For arbitrary (\mathbf{e}, \mathbf{b}) ,

$$\begin{aligned} \pi^+(\xi)(\mathbf{e}, \mathbf{b}) &= \pi^+(\xi)(\pi_{\xi^\perp} \mathbf{e}, \pi_{\xi^\perp} \mathbf{b}) \\ &= \frac{1}{2} \left((\pi_{\xi^\perp} \mathbf{b}) \wedge \xi + \pi_{\xi^\perp} \mathbf{e}, \pi_{\xi^\perp} \mathbf{b} - (\pi_{\xi^\perp} \mathbf{e}) \wedge \xi \right). \end{aligned}$$

iv. The projection on the negative eigenspace is given by $\pi^-(\xi) = \pi^+(-\xi)$.

Proof. i. The span of the eigenspaces with non zero eigenvalues is the orthogonal complement of the kernel. Part **i** follows from the description of the kernel in Lemma 2.1.

ii. Using Example 2.2, there are unique vectors \mathbf{b}_\pm perpendicular to ξ so that

$$(\mathbf{e}, \mathbf{b}) = (\mathbf{b}_+ \wedge \xi, \mathbf{b}_+)/\sqrt{2} + (-\mathbf{b}_- \wedge \xi, \mathbf{b}_-)/\sqrt{2}, \quad (2.4)$$

with the \mathbf{b}_\pm term being the projection on $\mathcal{E}^\pm(\xi)$. Equation (2.4) holds if and only if

$$\mathbf{b}_+ + \mathbf{b}_- = \sqrt{2}\mathbf{b}, \quad \mathbf{b}_+ \wedge \xi - \mathbf{b}_- \wedge \xi = \sqrt{2}\mathbf{e}.$$

The wedge product of the second with ξ shows that it is equivalent to

$$-\mathbf{b}_+ + \mathbf{b}_- = \sqrt{2}\mathbf{e} \wedge \xi.$$

Adding (resp. subtracting) from the first equation implies that $\mathbf{b}_- = (\mathbf{b} + \mathbf{e} \wedge \xi)/\sqrt{2}$ (resp. $\mathbf{b}_+ = (\mathbf{b} - \mathbf{e} \wedge \xi)/\sqrt{2}$). Therefore

$$\begin{aligned} \pi^+(\xi)(\mathbf{e}, \mathbf{b}) &= (\mathbf{b}_+ \wedge \xi, \mathbf{b}_+)/\sqrt{2} = ((\mathbf{b} - \mathbf{e} \wedge \xi) \wedge \xi, \mathbf{b} - \mathbf{e} \wedge \xi)/2 \\ &= (\mathbf{b} \wedge \xi + \mathbf{e}, \mathbf{b} - \mathbf{e} \wedge \xi)/2, \end{aligned}$$

proving **ii**.

iii. For general (\mathbf{e}, \mathbf{b}) , part **i** implies that the projection on the span of the nonzero eigenspaces is equal to $(\pi_{\xi^\perp}\mathbf{e}, \pi_{\xi^\perp}\mathbf{b})$. Then **ii** implies **iii**.

iv. Follows from $\mathcal{A}(-\xi) = -\mathcal{A}(\xi)$. \square

Example 2.4 For real unit vectors ξ , the span of the nonpositive eigenspaces $\mathcal{E}^-(\xi) \oplus \mathcal{E}^0(\xi) = \text{Ker } \pi^+(\xi)$ is equal to the set of vectors \mathbf{e}, \mathbf{b} satisfying $\pi_{\xi^\perp}\mathbf{b} - (\pi_{\xi^\perp}\mathbf{e}) \wedge \xi = 0$. Taking the wedge product with ξ shows that this is equivalent to $(\pi_{\xi^\perp}\mathbf{b}) \wedge \xi + \pi_{\xi^\perp}\mathbf{e} = 0$. In addition,

$$\frac{\|\pi^+(\xi)(\mathbf{e}, \mathbf{b})\|}{\sqrt{2}} \leq \|\pi_{\xi^\perp}\mathbf{b} - (\pi_{\xi^\perp}\mathbf{e}) \wedge \xi\| = \|(\pi_{\xi^\perp}\mathbf{b}) \wedge \xi + \pi_{\xi^\perp}\mathbf{e}\| \leq \sqrt{2} \|\pi^+(\xi)(\mathbf{e}, \mathbf{b})\|.$$

Example 2.5 Equation (2.2) implies that for $\xi \in \mathbb{C}^3 \setminus 0$ the eigenvalues of $\mathcal{A}(\xi)$ are equal to 0 and the two square roots of $\xi \cdot \xi$. Analytic continuation on paths that wind about \mathcal{Z} can pass from one root to the other. One cannot assign a square root that is holomorphic on $\mathbb{C}^3 \setminus \mathcal{Z}$.

Definition 2.6 Let

$$\Omega := \{\xi \in \mathbb{C}^3 : \text{Re}(\xi \cdot \xi) > 0\} = \{\xi \in \mathbb{C}^3 : |\text{Im } \xi| < |\text{Re } \xi|\}.$$

For $\xi \in \Omega$, define $(\xi \cdot \xi)^{1/2}$ to be the unique square root of $\xi \cdot \xi$ that has strictly positive real part. The function $\Omega \ni \xi \mapsto (\xi \cdot \xi)^{1/2}$ is holomorphic.

The eigenvalues are holomorphic on larger sets, for example $\{\xi \in \mathbb{C}^3 : \xi \cdot \xi \in \mathbb{C} \setminus]-\infty, 0]\}$. The extension to Ω suffices for our needs.

Lemma 2.7 For Ω from Definition 2.6 and $\xi \in \Omega$, $\mathcal{A}(\xi)$ has eigenvalues 0 and $\pm(\xi \cdot \xi)^{1/2}$. Each eigenvalue has an eigenspace of dimension two. The spectral projections are holomorphic on Ω and are given by,

$$\begin{aligned}\pi^+(\xi)(\mathbf{e}, \mathbf{b}) &= 2^{-1}((\xi \cdot \xi)^{-1/2} \mathbf{b} \wedge \xi + \pi_{\xi^\perp} \mathbf{e}, \pi_{\xi^\perp} \mathbf{b} - (\xi \cdot \xi)^{-1/2} \mathbf{e} \wedge \xi), \\ \pi^0(\xi)(\mathbf{e}, \mathbf{b}) &= (\pi_\xi \mathbf{e}, \pi_\xi \mathbf{b}), \\ \pi^-(\xi) &= -\pi^+(-\xi).\end{aligned}$$

Proof. The eigenvalues and their algebraic multiplicities follows from (2.2). The contour integral representations show that the spectral projections are holomorphic in ξ . For $\xi \in \mathbb{R}^3 \setminus 0$ one has the formulas for the projectors and,

$$\mathcal{A}(\xi)\pi^\pm(\xi) = \pm(\xi \cdot \xi)^{1/2}\pi^\pm(\xi), \quad \mathcal{A}(\xi)\pi^0(\xi) = 0. \quad (2.5)$$

By analytic continuation, the identities (2.5) and the formulas for the spectral projections extend to Ω . \square

Example 2.6 For $\xi \in \Omega$, $\pi^\pm(\xi)$, π_ξ and π_{ξ^\perp} are related by

$$\begin{pmatrix} \pi_{\xi^\perp} & 0 \\ 0 & \pi_{\xi^\perp} \end{pmatrix} = \pi^+(\xi) + \pi^-(\xi), \quad \text{and,} \quad \begin{pmatrix} \pi_\xi & 0 \\ 0 & \pi_\xi \end{pmatrix} = \pi^0(\xi). \quad (2.6)$$

3 Stretched Helmholtz boundary value problem

The stretched Maxwell equations are solved on smoothed domains \mathcal{Q}^δ . Section 3.1.2 shows that on the smoothed parts there is a boundary condition forced by the desired holomorphy in τ . The same section shows that solutions of the resulting boundary value problem satisfy a Helmholtz boundary value problem on \mathcal{Q}^δ . When τ is real, the converse statement that solutions of the Helmholtz boundary value problem satisfy the stretched Maxwell system is proved. The proof of the converse statement for τ real requires the smoothness of \mathcal{Q}^δ . Section 8.1 uses this converse for real τ to show that for the holomorphic family the stretched Maxwell system is satisfied for all τ .

3.1 Smoothed domains

The construction of a holomorphic family of solutions $E(\tau), B(\tau)$ to the stretched Maxwell equations on \mathcal{Q} is carried out by solving the same equations on an increasing sequence of smaller domains \mathcal{Q}^δ for $0 < \delta \leq \delta_0$. The \mathcal{Q}^δ are obtained by smoothing the corners and edges of \mathcal{Q} .

3.1.1 Choice of smoothed domains Q^δ

Take advantage of the fact that Q is strictly star shaped with center at the origin. Express Q in polar coordinates as a graph $r < \phi(\omega)$ with ϕ lipschitzian on S^2 . Choose sequence of smooth domains Q^δ given by $r < \phi^\delta(\omega)$ and so that Q and Q^δ are identical outside a δ -neighborhood of the singular set $\mathcal{S} \subset \partial Q$. As $\delta \searrow 0$, the Q^δ increase to Q . The ϕ^δ are chosen satisfying $\|\nabla_\omega \phi^\delta\|_{L^\infty(S^2)} \leq \|\nabla_\omega \phi\|_{L^\infty(S^2)}$.

Remark 3.1 *At most points of the boundary of Q^δ the normal vector ν is parallel to one of the coordinate axes.*

Assumption 3.1 *With Assumption 1.1 in mind, choose δ_0 so that Q^{δ_0} contains $\overline{Q_I}$ in its interior.*

3.1.2 Boundary conditions on ∂Q^δ

This subsection chooses a boundary condition to impose on the curved parts of Q^δ . On the flat parts of the boundary of Q^δ belonging to the faces Γ_k impose the condition $\pi^-(\nu)(E, B) = 0$. This convention implies that taking the limit as $\delta \rightarrow 0$ recovers the boundary conditions on ∂Q in Theorem 1.12. The choice of boundary conditions on the curved parts requires thought.

Definition 3.1 *Define*

$$\Pi(\tau, x) := \prod_{i=1}^3 \frac{\tau + \sigma_i(x_i)}{\tau},$$

and,

$$\mathcal{M}(\tau, x) := \begin{pmatrix} \mathcal{M}_{11} & 0 & 0 \\ 0 & \mathcal{M}_{22} & 0 \\ 0 & 0 & \mathcal{M}_{33} \end{pmatrix}, \quad \text{with } \mathcal{M}_{ii}(\tau, x) := \prod_{j \neq i} \frac{\tau + \sigma_j(x_j)}{\tau}.$$

Then

$$\mathcal{M}_{11} \mathcal{M}_{22} \mathcal{M}_{33} = \Pi^2, \quad \frac{\tau}{\tau + \sigma_i(x_i)} = \Pi^{-1} \mathcal{M}_{ii}, \quad \tilde{\partial} = \Pi^{-1} \mathcal{M} \partial. \quad (3.1)$$

The operator $\mathcal{A}(\tilde{\partial})$ is given by

$$\sum_i \mathcal{A}_i \tilde{\partial}_i = \sum_i \frac{\tau}{\tau + \sigma_i(x_i)} \mathcal{A}_i \partial_i = \mathcal{A}(\Pi^{-1} \mathcal{M} \partial) = \Pi^{-1} \mathcal{A}(\mathcal{M} \partial). \quad (3.2)$$

The normal matrix associated to $\mathcal{A}(\tilde{\partial})$ is equal to

$$\sum_i \frac{\tau}{\tau + \sigma_i(x_i)} \mathcal{A}_i \nu_i = \Pi^{-1} \mathcal{A}(\mathcal{M}\nu). \quad (3.3)$$

For τ real, it is natural to impose at all $x \in \partial\mathcal{Q}^\delta$ the boundary condition

$$\pi^-(\mathcal{M}\nu)(E, B) = 0. \quad (3.4)$$

This choice is characterized by the following three desirable features. First, it gives the desired answer when ν is parallel to one of the coordinate axes. Therefore it gives the right limit when $\delta \rightarrow 0$ since in the limit the normals are parallel to coordinate axes. Second, it depends holomorphically on τ . Third it maximizes the outward energy flux when τ is real. Indeed, $A(\mathcal{M}\nu)$ has eigenvalues $\pm\|\mathcal{M}\nu\|, 0$ each with multiplicity two. Therefore the outward flux is equal to one half of

$$(A(\mathcal{M}\nu)u) \cdot u = \|\mathcal{M}\nu\| \|\pi^+(\mathcal{M}\nu)u\|^2 - \|\mathcal{M}\nu\| \|\pi^-(\mathcal{M}\nu)u\|^2. \quad (3.5)$$

The boundary condition (3.4) enforces the vanishing of the nonpositive or incoming part. The resulting outward flux is as positive as possible.

Seek E, B holomorphic in τ . In that case the left hand side of (3.4) is holomorphic in τ . The preceding argument shows that it is natural to impose that it vanishes when τ is real. By analytic continuation it must vanish for all τ .

3.2 Near equivalence of Maxwell and Helmholtz

Proposition 3.2 *Suppose that K is a compact subset of \mathcal{Q}^δ , $\text{Re } \tau > 0$ and $\rho(x), j(x) \in L^2_K(\mathcal{Q}^\delta)$ satisfy $\tau\rho = -\widetilde{\text{div}} j$. Consider two boundary value problems for $(E, B) \in H^2(\mathcal{Q}^\delta)$ with first boundary condition $\pi^-(\mathcal{M}\nu)(E, B) = 0$ on $\partial\mathcal{Q}^\delta$.*

i. E, B satisfies the stretched Maxwell's equations,

$$\tau E - \widetilde{\text{curl}} B = -j, \quad \tau B + \widetilde{\text{curl}} E = 0, \quad \widetilde{\text{div}} E = \rho, \quad \widetilde{\text{div}} B = 0. \quad (3.6)$$

ii. E, B satisfies the boundary value problem for the stretched Helmholtz equation with additional boundary conditions,

$$\begin{aligned} \tau^2 E - \widetilde{\Delta} E &= -\tau j - \widetilde{\text{grad}} \rho && \text{on } \mathcal{Q}^\delta, \\ \tau^2 B - \widetilde{\Delta} B &= \widetilde{\text{curl}} j && \text{on } \mathcal{Q}^\delta, \\ \pi^+(\mathcal{M}\nu)(\tau E - \widetilde{\text{curl}} B, \tau B + \widetilde{\text{curl}} E) &= 0 && \text{on } \partial\mathcal{Q}^\delta, \\ \widetilde{\text{div}} E &= \widetilde{\text{div}} B = 0 && \text{on } \partial\mathcal{Q}^\delta. \end{aligned} \quad (3.7)$$

They are related as follows.

I. i \implies ii.

II. There is an $M > 0$ so that for real $\tau \in]M, \infty[$, **ii \implies i.**

Proof of Proposition 3.2. I. The identity $\mathcal{A}(\partial)^2 = -\text{curl curl } I$ implies that

$$\begin{aligned} (\tau I - \mathcal{A}(\partial)) (\tau I + \mathcal{A}(\partial)) &= (\tau^2 + \text{curl curl}) I_{6 \times 6} \\ &= (\tau^2 - \Delta + \text{grad div}) I_{6 \times 6}. \end{aligned}$$

The stretched version is

$$\begin{aligned} (\tau I - \mathcal{A}(\tilde{\partial})) (\tau I + \mathcal{A}(\tilde{\partial})) &= (\tau^2 + \widetilde{\text{curl curl}}) I_{6 \times 6} \\ &= (\tau^2 - \tilde{\Delta} + \widetilde{\text{grad div}}) I_{6 \times 6}. \end{aligned}$$

Applying to (E, B) using (3.6) yields

$$(\tau^2 + \widetilde{\text{curl curl}})(E, B) = (\tau I - \mathcal{A}(\partial))(-j, 0) - (\widetilde{\text{grad div}} E, 0).$$

This is exactly the first two lines in (3.7).

The third line of (3.7) is satisfied by assumption. The fourth line follows from Maxwell's equations. This completes the proof of **I.**

II. Suppose $\tau \in]0, \infty[$. Define

$$(F, G) := \left(\tau I + \sum_{i=1}^3 \mathcal{A}_i \tilde{\partial}_i \right) (E, B) + (j, 0).$$

The second line of (3.7) implies that $\widetilde{\text{curl}} j \in L^2(\mathcal{Q}^\delta)$. The continuity equation implies that $\widetilde{\text{div}} j \in L^2(\mathcal{Q}^\delta)$. Since j is supported in K the fact that $(\widetilde{\text{curl}}, \widetilde{\text{div}})$ is an elliptic system on \mathbb{R}^3 implies that $j \in H_K^1(\mathcal{Q}^\delta)$. Since $E \in H^2(\mathcal{Q}^\delta)$ it follows that $(F, G) \in H^1(\mathcal{Q}^\delta)$.

The dynamic stretched Maxwell's equations,

$$\tau E - \widetilde{\text{curl}} B = -j, \quad \tau B + \widetilde{\text{curl}} E = 0, \quad (3.8)$$

are satisfied if and only if $(F, G) = 0$.

The third line of equation (3.7) together with the fact that ρ and j vanish at the boundary imply that on $\partial\mathcal{Q}^\delta$,

$$\pi^+(\mathcal{M}\nu)(F, G) = \pi^+(\mathcal{M}\nu) \left(\tau E - \widetilde{\text{curl}} B, \tau B + \widetilde{\text{curl}} E \right) = 0. \quad (3.9)$$

Define $W := (F, G)$ so the stretched Helmholtz equation implies that $(\tau I - \mathcal{A}(\tilde{\partial}))W = 0$. Note the minus sign. Compute using the symmetry of \mathcal{A}_k in the second step to show that

$$\begin{aligned} \int_{\mathcal{Q}^\delta} W \cdot \mathcal{A}(\tilde{\partial})W \, dx &= \sum \int_{\mathcal{Q}^\delta} W \cdot \frac{\tau}{\tau + \sigma_k} \mathcal{A}_k \partial_k W \, dx \\ &= \sum \int_{\mathcal{Q}^\delta} \frac{\tau}{\tau + \sigma_k} \mathcal{A}_k W \cdot \partial_k W \, dx \end{aligned}$$

An integration by parts, with ν denoting the unit outward normal to \mathcal{Q}^δ , yields

$$\begin{aligned} \int_{\mathcal{Q}^\delta} W \cdot \mathcal{A}(\tilde{\partial})W \, dx &= \\ - \int_{\mathcal{Q}^\delta} \sum \mathcal{A}_k \partial_k \left(\frac{\tau}{\tau + \sigma_k} W \right) \cdot W \, dx &+ \int_{\partial \mathcal{Q}^\delta} \sum \frac{\tau \nu_k}{\tau + \sigma_k} \mathcal{A}_k W \cdot W \, d\Sigma. \end{aligned} \quad (3.10)$$

The first term on the right of (3.10) is equal to

$$- \int_{\mathcal{Q}^\delta} \mathcal{A}(\tilde{\partial})W \cdot W \, dx - \int_{\mathcal{Q}^\delta} \left(\sum \mathcal{A}_k \partial_k \frac{\tau}{\tau + \sigma_k} \right) W \cdot W \, dx. \quad (3.11)$$

The second term on the right of (3.10) is equal to

$$\int_{\partial \mathcal{Q}^\delta} \sum \frac{\tau \nu_k}{\tau + \sigma_k} \mathcal{A}_k W \cdot W \, d\Sigma = \int_{\partial \mathcal{Q}^\delta} \Pi^{-1} \mathcal{A}(\mathcal{M}\nu)W \cdot W \, d\Sigma. \quad (3.12)$$

In the last four identities use $\mathcal{A}(\tilde{\partial})W = \tau W$ to find

$$\int_{\mathcal{Q}^\delta} \left[2\tau - \sum \mathcal{A}_k \partial_k \left(\frac{\tau}{\tau + \sigma_k} \right) \right] W \cdot W \, dx = \int_{\partial \mathcal{Q}^\delta} \Pi^{-1} \mathcal{A}(\mathcal{M}\nu)W \cdot W \, d\Sigma. \quad (3.13)$$

Equation (3.9) yields $\pi^+(\mathcal{M}\nu)W = 0$ on $\partial \mathcal{Q}^\delta$. Therefore, the integrand in the boundary integral on the right of (3.13) is nonpositive thanks to (3.5). Thus, the integral is nonpositive.

Choose $M > 0$ so that for $\tau > M$ the real symmetric matrix in square brackets on the left is everywhere greater than the identity matrix. For such τ , the left hand side is ≥ 0 . The only way that (3.13) can hold is if both sides vanish. The vanishing of the left hand side yields $W = 0$. This completes the proof. \square

4 Lopatinski, Fredholm, and analyticity

This section shows that second boundary value problem in the Proposition 3.2 satisfies Lopatinski's ellipticity condition. The boundary value problem has unknowns E, B of dimension six followed by two boundary conditions of Dirichlet type and four involving first derivatives,

$$\begin{aligned}
 \tau^2 E - \widetilde{\Delta} E &= f_1 && \text{on } \mathcal{Q}^\delta, \\
 \tau^2 B - \widetilde{\Delta} B &= f_2 && \text{on } \mathcal{Q}^\delta, \\
 \pi^-(\mathcal{M}\nu)(E, B) &= g_1 && \text{on } \partial\mathcal{Q}^\delta, \\
 \pi^+(\mathcal{M}\nu)(\tau E - \widetilde{\text{curl}}B, \tau B + \widetilde{\text{curl}}E) &= g_2 && \text{on } \partial\mathcal{Q}^\delta, \\
 \widetilde{\text{div}} E &= g_3 && \text{on } \partial\mathcal{Q}^\delta, \\
 \widetilde{\text{div}} B &= g_4 && \text{on } \partial\mathcal{Q}^\delta.
 \end{aligned} \tag{4.1}$$

Three properties are proved with constant M independent of δ .

- For $\text{Re } \tau > M$, problem (4.1) satisfies Lopatinski's criterion describing coercive elliptic boundary value problems. The associated operator is therefore Fredholm.
- The operators on the left depend holomorphically on $\tau \in \mathbb{C}$ for $\text{Re } \tau > M$.
- The operator is invertible for $\tau \in]M, \infty[$. The last is proved in Section 5.3 after appropriate Green's identities are found.

The Analytic Fredholm Theorem implies that for each δ the operator is invertible except at a discrete subset $\mathbb{D}(\delta) \subset \{\text{Re } \tau > M\}$.

The hard work of deriving uniform estimates for solutions of the stretched Maxwell boundary value problem is carried out after Section 5.3.

4.1 Lopatinski condition and Fredholm property

The verification of Lopatinski's condition begins with the unstretched boundary value problem. The stretched case is then a corollary.

Proposition 4.1 *For each τ with $\text{Re } \tau > 0$ and $\delta \in]0, \delta_0]$, the unstretched*

boundary value problem

$$\begin{aligned}
\tau^2 B - \Delta B &= f_1 && \text{on } \mathcal{Q}^\delta, \\
\tau^2 E - \Delta E &= f_2 && \text{on } \mathcal{Q}^\delta, \\
\pi^-(\nu)(E, B) &= g_1 && \text{on } \partial\mathcal{Q}^\delta, \\
\pi^+(\nu)(\tau E - \text{curl } B, \tau B + \text{curl } E) &= g_2 && \text{on } \partial\mathcal{Q}^\delta, \\
\text{div } E &= g_3 && \text{on } \partial\mathcal{Q}^\delta, \\
\text{div } B &= g_4 && \text{on } \partial\mathcal{Q}^\delta,
\end{aligned} \tag{4.2}$$

satisfies Lopatinski's condition characterizing coercive elliptic problems.

Proof. The verification at a point of $\partial\mathcal{Q}^\delta$ demands that one replaces \mathcal{Q}^δ by the half-space with the same tangent plane and outward normal ν as \mathcal{Q}^δ . Next one drops lower order terms in the equations and the boundary conditions. The resulting problem is rotation invariant. Thus it suffices to verify for the half space $\{x_1 > 0\}$ with $\nu = (-1, 0, 0)$.

The Lopatinski condition concerns solutions of the resulting problem with all source terms equal to zero. It considers solutions of the form

$$e^{i(\xi_2 x_2 + \xi_3 x_3)} w(x_1), \quad (\xi_2, \xi_3) \in \mathbb{R}^2 \setminus 0, \quad w(x_1) \rightarrow 0 \text{ when } x_1 \rightarrow +\infty.$$

The condition is satisfied when zero is the only such solution.

That boundary value problem with source terms equal to zero is

$$\begin{aligned}
\Delta E = \Delta B &= 0 && \text{on } \{x_1 > 0\}, \\
\pi^-(\nu)(E, B) &= 0 && \text{on } \{x_1 = 0\}, \\
\pi^+(\nu)(-\text{curl } B, \text{curl } E) &= 0 && \text{on } \{x_1 = 0\}, \\
\text{div } E = \text{div } B &= 0 && \text{on } \{x_1 = 0\}.
\end{aligned}$$

The Laplace equations and decay at infinity hold if and only if there are constant vectors $\mathbf{e} \in \mathbb{C}^3$ and $\mathbf{b} \in \mathbb{C}^3$ so that

$$w = e^{i\xi_1 x_1}(\mathbf{e}, \mathbf{b}), \quad \xi_1^2 + \xi_2^2 + \xi_3^2 = 0, \quad \text{Im } \xi_1 > 0. \tag{4.3}$$

Lopatinski's condition requires that the only such solution that satisfies the boundary conditions is the trivial solution. For each ξ_2, ξ_3 there is a six dimensional space of solutions w to (4.3). This shows that six boundary conditions are needed.

By rotational invariance it suffices to verify Lopatinski's condition for $(\xi_2, \xi_3) = (0, 1)$ in which case, $\xi = (\xi_1, \xi_2, \xi_3) = (i, 0, 1)$. The boundary conditions hold if and only if

$$\pi^-(\nu)(\mathbf{e}, \mathbf{b}) = \mathbf{0}, \quad \pi^+(\nu)(-\xi \wedge \mathbf{b}, \xi \wedge \mathbf{e}) = \mathbf{0}, \quad \text{and} \quad \xi \cdot \mathbf{e} = \xi \cdot \mathbf{b} = \mathbf{0}.$$

First analyse the boundary condition $0 = \pi^-(\nu)(\mathbf{e}, \mathbf{b}) = \pi^-(\nu)(\mathbf{e}_{tan}, \mathbf{b}_{tan})$. Lemma 2.1 shows that this is equivalent to

$$(\mathbf{e}_{tan}, \mathbf{b}_{tan}) \in \mathcal{E}^+(\nu) \Leftrightarrow \mathbf{e}_{tan} = \mathbf{b}_{tan} \wedge \nu \Leftrightarrow (0, e_2, e_3) = (0, b_3, -b_2).$$

The divergence conditions $i\mathbf{e}_1 + \mathbf{e}_3 = 0$ and $i\mathbf{b}_1 + \mathbf{b}_3 = 0$ yield,

$$\mathbf{e} = (-ib_2, b_3, -b_2), \quad \text{and} \quad \mathbf{b} = (ib_3, b_2, b_3).$$

The curl boundary condition is

$$0 = \pi^+(\nu)(-\xi \wedge \mathbf{b}, \xi \wedge \mathbf{e}) = \pi^+(\nu)(-(\xi \wedge \mathbf{b})_{tan}, (\xi \wedge \mathbf{e})_{tan}).$$

Equivalently, $(-(\xi \wedge \mathbf{b})_{tan}, (\xi \wedge \mathbf{e})_{tan}) \in \mathcal{E}^-(\nu)$. Lemma 2.1 implies that this holds if and only if

$$-(\xi \wedge \mathbf{b})_{tan} = -(\xi \wedge \mathbf{e})_{tan} \wedge \nu. \quad (4.4)$$

For any vector \mathbf{v} , $(\xi \wedge \mathbf{v})_{tan} = (0, v_1 - iv_3, iv_2)$. Therefore

$$(\xi \wedge \mathbf{b})_{tan} = (0, 0, ib_2), \quad (\xi \wedge \mathbf{e})_{tan} = (0, 0, ib_3), \quad (\xi \wedge \mathbf{e})_{tan} \wedge \nu = (0, ib_3, 0).$$

Comparing the first and last shows that if (4.4) holds then $b_2 = b_3 = 0$. Therefore $\mathbf{e} = \mathbf{b} = \mathbf{0}$ verifying Lopatinski's condition. \square

Corollary 4.2 *There is an $M > 0$ so that for $|\tau| > M$ and $0 < \delta < \delta_0$, the stretched boundary value problem (4.1), satisfies Lopatinski's condition characterizing coercive elliptic problems.*

Proof. The problems obtained by dropping lower order terms converge to those of the preceding Proposition when $|\tau| \rightarrow \infty$. Since the Lopatinski condition is inherited by nearby operators, the result follows. \square

The source term g_1 must take values in the negative spectral subspace of $\mathcal{A}(\mathcal{M}\nu)$. Consider the action of the operators $\tau^2 - \tilde{\Delta}$ on $H^2(\mathcal{Q}^\delta)$. Since g_1 is a trace at the boundary, the natural space for g_1 is

$$\left\{ g \in H^{3/2}(\partial\mathcal{Q}^\delta) : \pi^-(\mathcal{M}\nu)g = g \right\}. \quad (4.5)$$

Denote by $\mathcal{E}^-(\tau, x)$ the negative spectral subspace of $\mathcal{A}(\mathcal{M}(\tau, x)\nu(\tau, x))$. Then an equivalent definition is that g_1 is an $H^{3/2}$ section of the vector bundle over $\partial\mathcal{Q}^\delta$ whose fiber at x is $\mathcal{E}^-(\tau, x)$. Similarly, the natural space for g_2 is

$$\left\{ g \in H^{1/2}(\partial\mathcal{Q}^\delta) : \pi^+(\mathcal{M}\nu)g = g \right\}. \quad (4.6)$$

Definition 4.3 For $\text{Re } \tau > 0$ the closed subspace of $H^{3/2}(\partial\mathcal{Q}^\delta)$ defined in (4.5) is denoted $H_{\mathcal{E}^-(\tau, x)}^{3/2}(\partial\mathcal{Q}^\delta)$. The space (4.6) is denoted $H_{\mathcal{E}^+(\tau, x)}^{1/2}(\partial\mathcal{Q}^\delta)$.

Analysis of this space of sections of a τ dependent vector bundle is reduced to the case of a τ independent Sobolev space as follows. Treat $H_{\mathcal{E}^+(\tau, x)}^{1/2}(\partial\mathcal{Q}^\delta)$. The space $H_{\mathcal{E}^-(\tau, x)}^{3/2}(\partial\mathcal{Q}^\delta)$ is analogous. For $g \in H^{1/2}(\partial\mathcal{Q}^\delta)$,

$$g = \pi^-(\mathcal{M}\nu)g \iff \pi^+(\mathcal{M}\nu)g = 0 \quad \text{and} \quad \pi^0(\mathcal{M}\nu)g = 0.$$

Lemma 2.7 shows that with $g = (\mathbf{e}, \mathbf{b})$ this is equivalent to

$$\mathbf{e} \cdot (\mathcal{M}\nu) = \mathbf{b} \cdot (\mathcal{M}\nu) = 0 \quad \text{and} \quad (\pi_{(\mathcal{M}\nu)^\perp} \mathbf{b}) \wedge (\mathcal{M}\nu) + \pi_{(\mathcal{M}\nu)^\perp} \mathbf{e} = 0.$$

For $|\tau|$ large, $\mathcal{M}\nu = \nu + O(1/|\tau|)$ so this two dimensional space is very close to $\mathcal{E}^-(\nu)$ that is given by the pair of equations independent of τ ,

$$\mathbf{e} \cdot \nu = \mathbf{b} \cdot \nu = 0 \quad \text{and} \quad (\pi_{\nu^\perp} \mathbf{b}) \wedge \nu + \pi_{\nu^\perp} \mathbf{e} = 0.$$

A vector \mathbf{e}, \mathbf{b} satisfies these equations if and only if $(\mathbf{e}, \mathbf{b}) = (\mathbf{e}_{tan}, \mathbf{b}_{tan})$ and $\mathbf{b}_{tan} \wedge \nu + \mathbf{e}_{tan} = 0$. Therefore, the linear map

$$\mathcal{E}^-(\nu(x)) \ni (\mathbf{e}(x), \mathbf{b}(x)) \mapsto \mathbf{e}(x)_{tan} \in T_x(\partial\mathcal{Q}^\delta)$$

is an invertible map from a two dimensional space to another. It depends smoothly on τ, x and analytically on τ . It follows that the nearby map

$$\mathcal{E}^-(\mathcal{M}\nu(x)) \ni (\mathbf{e}(x), \mathbf{b}(x)) \mapsto \mathbf{e}(x)_{tan}$$

is also invertible. Denote by $R^-(\tau, x)$ its inverse. The equation $\pi^-(\mathcal{M}\nu)g = g$ holds if and only if $(\mathbf{e}, \mathbf{b}) = R^-(\tau, x)\mathbf{e}_{tan}$. The space $H_{\mathcal{E}^-(\tau, x)}^{1/2}(\partial\mathcal{Q}^\delta)$ is exactly the set of functions of the form $R^-\mathbf{e}_{tan}$ where \mathbf{e}_{tan} is a vector field tangent to $\partial\mathcal{Q}^\delta$ with regularity $H^{1/2}(\partial\mathcal{Q}^\delta)$. The set of such fields is independent of τ . Analogously, $\mathcal{E}^+(\mathcal{M}\nu)$ has equation $(\mathbf{e}, \mathbf{b}) = R^+(\tau, x)\mathbf{e}_{tan}$.

Definition 4.4 *A continuous linear operator from one Banach space to another is Fredholm when its kernel is finite dimensional and its range is closed with finite dimensional cokernel. The index of a Fredholm operator S is defined as $\dim \ker S - \dim \operatorname{coker} S$.*

Recall that a continuous curve of Fredholm operators has constant index.

Corollary 4.5 *The map that sends $E, B \in H^2(\mathcal{Q}^\delta)$ to*

$$\left(\tau^2 E - \tilde{\Delta} E, \tau^2 B - \tilde{\Delta} B, \pi^-(\mathcal{M}\nu)(E, B)|_{\partial\mathcal{Q}^\delta}, \right. \\ \left. \pi^+(\mathcal{M}\nu)(\tau E - \widetilde{\operatorname{curl}} B, \tau B + \widetilde{\operatorname{curl}} E)|_{\partial\mathcal{Q}^\delta}, (\widetilde{\operatorname{div}} E)|_{\partial\mathcal{Q}^\delta}, (\widetilde{\operatorname{div}} B)|_{\partial\mathcal{Q}^\delta} \right)$$

is continuous with values in

$$L^2(\mathcal{Q}^\delta) \times L^2(\mathcal{Q}^\delta) \times H_{\mathcal{E}^-(\tau, x)}^{3/2}(\partial\mathcal{Q}^\delta) \times H_{\mathcal{E}^+(\tau, x)}^{1/2}(\partial\mathcal{Q}^\delta) \times H^{1/2}(\partial\mathcal{Q}^\delta) \times H^{1/2}(\partial\mathcal{Q}^\delta). \quad (4.7)$$

It is a Fredholm operator. Both the kernel and the annihilator of the range contain only smooth functions. The index of the operator is equal to zero.

Proof. The Fredholm and smoothness assertions are the basic results of the theory of elliptic boundary value problems satisfying Lopatinski's condition (see Agmon Douglis Nirenberg [1], Taylor [35, Vol II], Hörmander [18, Vol III]).

Consider the continuous family of Fredholm maps that is generated by sending the imaginary part of τ to zero, then sending the σ_j to zero. Want to invoke the invariance of index under continuous deformation. Multiplication by R^- is a linear isomorphism from $H_{\mathcal{E}^-(\tau, x)}^{3/2}(\partial\mathcal{Q}^\delta)$ to the fixed Hilbert space of $H^{3/2}(\partial\mathcal{Q}^\delta)$ tangential fields. Similarly for $H_{\mathcal{E}^+(\tau, x)}^{1/2}(\partial\mathcal{Q}^\delta)$. In this way the Fredholm maps are conjugated to maps with values in a fixed space. The index of the resulting family is constant. Therefore the index of the original family is constant. To prove that the index of the original family is zero it suffices to prove that the index is equal to zero when the absorptions vanish and $\tau \in]0, \infty[$.

To do that, define the operator L_0 on $L^2(\mathcal{Q}^\delta)$ whose domain is the set of functions in $H^2(\mathcal{Q}^\delta)$ that satisfy the homogeneous boundary conditions. The elliptic regularity theorem implies that L_0 is selfadjoint. In addition,

$$\operatorname{Re} (L_0 u, u) \geq \operatorname{Re} \tau \|u\|_{L^2(\mathcal{Q}^\delta)}^2.$$

This implies that L_0 is an invertible operator from its domain onto $L^2(\mathcal{Q}^\delta)$. It follows immediately that the operator of the Corollary is injective.

Since the operator of the Corollary has closed range, to prove surjectivity it suffices to show that the range includes the dense set of smooth elements of (4.7). To prove that, it is sufficient to subtract a smooth function that satisfies the inhomogeneous boundary conditions. The difference needs to satisfy a problem with smooth data and homogeneous boundary conditions. That is solvable thanks to the invertibility of L_0 . \square

4.2 Analyticity

The boundary value problem (4.1) takes the form,

$$\begin{aligned}
(\tau^2 - \tilde{\Delta})(E, B) &= (f_1, f_2) && \text{on } \mathcal{Q}^\delta, \\
\pi^-(\mathcal{M}\nu)(E, B) &= R^-(\tau, x)\underline{g}_1 && \text{on } \partial\mathcal{Q}^\delta, \\
\pi^+(\mathcal{M}\nu)(\tau E - \widetilde{\text{curl}} B, \tau B + \widetilde{\text{curl}} E) &= R^+(\tau, x)\underline{g}_2 && \text{on } \partial\mathcal{Q}^\delta, \\
(\widetilde{\text{div}} E, \widetilde{\text{div}} B) &= (g_3, g_4) && \text{on } \partial\mathcal{Q}^\delta.
\end{aligned} \tag{4.8}$$

The source term \underline{g}_1 takes values in the space of $H^{3/2}(\partial\mathcal{Q}^\delta)$ tangential fields. and \underline{g}_2 in the $H^{1/2}(\partial\mathcal{Q}^\delta)$ tangential fields. The coefficients of the operators depend differentiably on τ, x and analytically on τ .

Proposition 4.6 *Suppose that $\underline{\tau} \in \mathbb{C}$ and that the Fredholm mapping from Corollary 4.5 is invertible. Then the mapping is invertible on a neighborhood of $\underline{\tau}$. If the source terms f, \underline{g}_j depend analytically on τ , then the corresponding solution of (4.8) is an analytic function of τ with values in $H^2(\mathcal{Q}^\delta)$.*

Proof. The invertibility for $\tau = a + ib$ near $\underline{\tau}$ follows by Neumann's series. Standard elliptic theory shows that the map $a, b \mapsto (E, B)$ is infinitely differentiable with values in $H^2(\mathcal{Q}^\delta)$. The derivatives satisfy the system obtained by differentiating the system and boundary conditions with respect to a, b . To prove analyticity it suffices to show that $\partial(E, B)/\partial\bar{\tau} = 0$. All the coefficients and the f, g are analytic in τ . Therefore, differentiating the boundary value problem shows that $\partial(E, B)/\partial\bar{\tau}$ satisfies system (4.8) with all sources equal to zero. That $\partial(E, B)/\partial\bar{\tau} = 0$ follows from the invertibility. \square

5 Green's identities

A priori estimates require Green's identities for $\widetilde{\operatorname{div}}$ and $\widetilde{\operatorname{curl}}$. They lead to trace theorems for the spaces $H_{\widetilde{\operatorname{curl}}}$ and $H_{\widetilde{\operatorname{div}}}$. The unstretched versions are standard [8, 9]. Proposition 5.10 relates the boundary terms in Green's identities to the spectral projections from §2. The Green's identities have positivity properties when $\tau \in]0, \infty[$ that yields a proof of existence in Section 5.3.

5.1 Stretched Green's identities

5.1.1 Stretched div, curl and grad identities

Lemma 5.1 *The following differential identities hold for $v \in C^1$,*

$$\Pi(\tau, x) \widetilde{\operatorname{div}} v = \operatorname{div}(\mathcal{M}v), \quad (5.1)$$

$$\widetilde{\operatorname{div}}(\Phi \wedge \Psi) = (\widetilde{\operatorname{curl}} \Phi) \cdot \Psi - (\widetilde{\operatorname{curl}} \Psi) \cdot \Phi, \quad (5.2)$$

$$\Pi(\tau, x) \widetilde{\operatorname{grad}} \phi = (\partial_1(\mathcal{M}_{11}\phi), \partial_2(\mathcal{M}_{22}\phi), \partial_3(\mathcal{M}_{33}\phi)) = \mathcal{M} \operatorname{grad} \phi. \quad (5.3)$$

Proof. For (5.1) compute using (3.1),

$$\begin{aligned} \Pi(\tau, x) \widetilde{\operatorname{div}} v &= \left(\prod_{j=1}^3 \frac{\tau + \sigma_j(x_j)}{\tau} \right) \left(\sum_{i=1}^3 \frac{\tau}{\tau + \sigma_i(x_i)} \partial_i v_i \right) \\ &= \sum_{i=1}^3 \left(\prod_{j \neq i} \frac{\tau + \sigma_j(x_j)}{\tau} \right) \partial_i v_i = \sum_{i=1}^3 \mathcal{M}_{ii} \partial_i v_i \\ &= \sum_{i=1}^3 \partial_i (\mathcal{M}_{ii} v_i) \quad (\text{since } \partial_i \mathcal{M}_{ii} = 0) \\ &= \operatorname{div}(\mathcal{M}v). \end{aligned} \quad (5.4)$$

For (5.3) use $\partial_i \mathcal{M}_{ii} = 0$ again to compute,

$$\begin{aligned} \Pi(\tau, x) \widetilde{\operatorname{grad}} \phi &= \Pi(\tau, x) \left(\frac{\tau}{\tau + \sigma_1} \partial_1 \phi, \frac{\tau}{\tau + \sigma_2} \partial_2 \phi, \frac{\tau}{\tau + \sigma_3} \partial_3 \phi \right) \\ &= (\mathcal{M}_{11} \partial_1 \phi, \mathcal{M}_{22} \partial_2 \phi, \mathcal{M}_{33} \partial_3 \phi) \\ &= (\partial_1(\mathcal{M}_{11}\phi), \partial_2(\mathcal{M}_{22}\phi), \partial_3(\mathcal{M}_{33}\phi)) = \mathcal{M} \operatorname{grad} \phi. \end{aligned}$$

The third identity is similar. It is left to the reader. \square

Lemma 5.2 For ϕ, Φ, Ψ in $H^1(\mathcal{Q}^\delta)$ the following integral identities hold,

$$\begin{aligned} \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \widetilde{\text{curl}} \Phi \cdot \Psi \, dx - \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \widetilde{\text{curl}} \Psi \cdot \Phi \, dx \\ = \int_{\partial \mathcal{Q}^\delta} (\Psi \wedge (\mathcal{M} \nu)) \cdot \Phi \, d\Sigma, \end{aligned} \quad (5.5)$$

$$\begin{aligned} \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \Phi \cdot \widetilde{\text{grad}} \phi \, dx + \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \phi \, \widetilde{\text{div}} \Phi \, dx \\ = \int_{\partial \mathcal{Q}^\delta} \phi \Phi \cdot (\mathcal{M} \nu) \, d\Sigma. \end{aligned} \quad (5.6)$$

Proof. For (5.5), using (5.1) with Gauss' divergence theorem shows that for Φ, Ψ in $H^1(\mathcal{Q}^\delta)$,

$$\int_{\mathcal{Q}^\delta} \Pi(\tau, x) \widetilde{\text{div}}(\Phi \wedge \Psi) \, dx = \int_{\mathcal{Q}^\delta} \text{div}(\mathcal{M}(\Phi \wedge \Psi)) \, dx = \int_{\partial \mathcal{Q}^\delta} (\mathcal{M}(\Phi \wedge \Psi)) \cdot \nu \, dx.$$

Since \mathcal{M} is diagonal,

$$(\mathcal{M}(\Phi \wedge \Psi)) \cdot \nu = (\Phi \wedge \Psi) \cdot \mathcal{M} \nu = -(\Psi \wedge \Phi) \cdot \mathcal{M} \nu = (\Psi \wedge \mathcal{M} \nu) \cdot \Phi.$$

This proves (5.5) using (5.2).

Formula (5.6) comes from (5.3). Indeed,

$$\begin{aligned} \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \widetilde{\text{grad}} \phi \cdot \Phi \, dx &= \int_{\mathcal{Q}^\delta} \sum \Phi_j \partial_j (\mathcal{M}_{jj} \phi) \, dx \\ &= - \int_{\mathcal{Q}^\delta} \sum \mathcal{M}_{jj} \phi \partial_j \Phi_j \, dx + \int_{\partial \mathcal{Q}^\delta} \sum \phi \mathcal{M}_{jj} \Phi_j \nu_j \, d\Sigma \\ &= - \int_{\mathcal{Q}^\delta} \phi \sum \mathcal{M}_{jj} \partial_j \Phi_j \, dx + \int_{\partial \mathcal{Q}^\delta} \phi \sum \Phi_j \mathcal{M}_{jj} \nu_j \, d\Sigma \\ &= - \int_{\mathcal{Q}^\delta} \Pi \phi \sum \frac{\mathcal{M}_{jj}}{\Pi} \partial_j \Phi_j \, dx + \int_{\partial \mathcal{Q}^\delta} \phi \Phi \cdot (\mathcal{M} \nu) \, d\Sigma \\ &= - \int_{\mathcal{Q}^\delta} \Pi \phi \, \widetilde{\text{div}} \Phi \, dx + \int_{\partial \mathcal{Q}^\delta} \phi \Phi \cdot (\mathcal{M} \nu) \, d\Sigma. \quad \square \end{aligned}$$

Definition 5.3 Define Hilbert spaces and their analogues for \mathcal{Q} ,

$$\begin{aligned} H_{\widetilde{\text{curl}}}(\mathcal{Q}^\delta) &:= \left\{ \Phi \in L^2(\mathcal{Q}^\delta) : \widetilde{\text{curl}} \Phi \in L^2(\mathcal{Q}^\delta) \right\}, \\ H_{\widetilde{\text{div}}}(\mathcal{Q}^\delta) &:= \left\{ \Phi \in L^2(\mathcal{Q}^\delta) : \widetilde{\text{div}} \Phi \in L^2(\mathcal{Q}^\delta) \right\}. \end{aligned}$$

The set of restrictions of $C_0^\infty(\mathbb{R}^3)$ to \mathcal{Q}^δ (denoted $C_{(0)}^\infty(\mathcal{Q}^\delta)$) is dense in each of the three spaces $H_{\text{curl}}^\sim(\mathcal{Q}^\delta)$, $H_{\text{div}}^\sim(\mathcal{Q}^\delta)$, and $H_{\text{div}}^\sim(\mathcal{Q}^\delta) \cap H_{\text{curl}}^\sim(\mathcal{Q}^\delta)$. The density is proved by convolution with a kernel supported in cones towards the interior. Similarly for \mathcal{Q} .

Proposition 5.4 *The map*

$$\Phi, \Psi \mapsto \int_{\partial\mathcal{Q}^\delta} (\Phi \wedge \Psi) \cdot \mathcal{M}\nu \, d\Sigma \in \mathbb{C}$$

extends uniquely from $C_{(0)}^\infty(\mathcal{Q}^\delta) \times C_{(0)}^\infty(\mathcal{Q}^\delta)$ to a continuous bilinear form on $H_{\text{curl}}^\sim(\mathcal{Q}^\delta) \times H^{1/2}(\partial\mathcal{Q}^\delta)$. Therefore, for $\Phi \in H_{\text{curl}}^\sim(\mathcal{Q}^\delta)$, $\Phi|_{\partial\mathcal{Q}^\delta} \wedge \mathcal{M}\nu$ is a well defined element of $H^{-1/2}(\partial\mathcal{Q}^\delta)$. Similarly, the map

$$\phi, \Psi \mapsto \int_{\partial\mathcal{Q}^\delta} \phi \Psi \cdot \mathcal{M}\nu \, d\Sigma$$

is a continuous bilinear form on $H^{1/2}(\partial\mathcal{Q}^\delta) \times H_{\text{div}}^\sim(\mathcal{Q}^\delta)$. For $\Psi \in H_{\text{div}}^\sim(\mathcal{Q}^\delta)$, $\Psi|_{\partial\mathcal{Q}^\delta} \cdot \mathcal{M}\nu$ is a well defined element of $H^{-1/2}(\partial\mathcal{Q}^\delta)$. The same assertions are true for \mathcal{Q} .

Proof. For the trace, prove the $\widetilde{\text{curl}}$ assertions. Given $\Psi \in H^{1/2}(\partial\mathcal{Q}^\delta)$ choose $\underline{\Psi} \in H^1(\mathbb{R}^3)$ whose trace at $\partial\mathcal{Q}^\delta$ is equal to Ψ and $\|\underline{\Psi}\|_{H^1(\mathbb{R}^3)} \lesssim \|\Psi\|_{H^{1/2}(\partial\mathcal{Q}^\delta)}$. Using (5.5), shows that with constants independent of δ ,

$$\begin{aligned} \left| \int_{\partial\mathcal{Q}^\delta} (\Phi \wedge \mathcal{M}\nu) \cdot \Psi \, d\Sigma \right| &\lesssim \|\Phi, \widetilde{\text{curl}}\Phi\|_{L^2(\mathcal{Q}^\delta)} \|\underline{\Psi}, \widetilde{\text{curl}}\underline{\Psi}\|_{L^2(\mathcal{Q}^\delta)} \\ &\lesssim \|\Phi, \widetilde{\text{curl}}\Phi\|_{L^2(\mathcal{Q}^\delta)} \|\Psi\|_{H^{1/2}(\partial\mathcal{Q}^\delta)}. \end{aligned}$$

The second assertion follows from (5.6) in the same way. \square

Corollary 5.5 *The map $C_{(0)}^\infty(\mathcal{Q}) \ni \Phi \mapsto \Phi|_{\partial\mathcal{Q}}$ extends uniquely to a continuous map from $H_{\text{curl}}^\sim(\mathcal{Q}) \cap H_{\text{div}}^\sim(\mathcal{Q})$ to $H^{-1/2}(\partial\mathcal{Q})$.*

Proof. That $C_{(0)}^\infty(\mathcal{Q})$ is dense in $H_{\text{curl}}^\sim(\mathcal{Q}) \cap H_{\text{div}}^\sim(\mathcal{Q})$ is proved by convolution. To prove continuity, test Φ by dense elements of $H^{1/2}(\partial\mathcal{Q})$. The set of restrictions of element of $C_0^\infty(\mathbb{R}^3)$ so that Ψ vanishes in a neighborhood of \mathcal{S} is dense in $H^{1/2}(\partial\mathcal{Q})$ (see [16, Lemma 2.7.i] where the assertion has a typo. It should be $H^{-1/2}(\mathbb{R}^3)$). For such test functions

$$\int_{\partial\mathcal{Q}} \Phi \cdot \Psi \, d\Sigma = \sum_k \int_{\Gamma_k} \Phi \cdot \Psi \, d\sigma.$$

Use the identity valid for all $\mathbf{w} \in \mathbb{C}^3$ and $\xi \in \mathbb{C}^3 \setminus \mathcal{Z}$,

$$\mathbf{w} = \frac{(\xi \cdot \mathbf{w})\xi}{\xi \cdot \xi} - \frac{\xi \wedge (\xi \wedge \mathbf{w})}{\xi \cdot \xi}.$$

Use this on Γ_k with $\xi = \mathcal{M}\nu$, $\text{Re } \tau > 0$ and $\mathbf{w} = \Phi$. On each face Γ_k , $\mathcal{M}\nu$ is a constant vector that is not in \mathcal{Z} . Find that

$$\int_{\Gamma_k} \Phi \cdot \Psi \, d\Sigma = \int_{\Gamma_k} \left(\frac{((\mathcal{M}\nu) \cdot \Phi)(\mathcal{M} \cdot \nu)}{(\mathcal{M}\nu) \cdot (\mathcal{M}\nu)} - \frac{(\mathcal{M}\nu) \wedge ((\mathcal{M}\nu) \wedge \Phi)}{(\mathcal{M}\nu) \cdot (\mathcal{M}\nu)} \right) \cdot \Psi \, d\Sigma.$$

Since Ψ is compactly supported in Γ_k one has the upper bound

$$\lesssim \left(\|(\mathcal{M}\nu) \cdot \Phi\|_{H^{-1/2}(\partial\mathcal{Q})} + \|(\mathcal{M}\nu) \wedge \Phi\|_{H^{-1/2}(\partial\mathcal{Q})} \right) \|\Psi\|_{H^{1/2}(\partial\mathcal{Q})}.$$

The result follows from the Proposition 5.4. \square

5.1.2 Stretched curl curl and grad div identities

Lemma 5.6 For $w, \Phi \in H^2(\mathcal{Q}^\delta) \times H^1(\mathcal{Q}^\delta)$ one has the two identities,

$$\begin{aligned} \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \widetilde{\text{curl}} \widetilde{\text{curl}} w \cdot \Phi \, dx - \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \widetilde{\text{curl}} w \cdot \widetilde{\text{curl}} \Phi \, dx \\ = - \int_{\partial\mathcal{Q}^\delta} (\widetilde{\text{curl}} w \wedge \mathcal{M}\nu) \cdot \Phi \, d\Sigma, \end{aligned} \quad (5.7)$$

$$\begin{aligned} \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \widetilde{\text{grad}} \widetilde{\text{div}} w \cdot \Phi \, dx + \int_{\mathcal{Q}^\delta} \Pi \widetilde{\text{div}} w \widetilde{\text{div}} \Phi \, dx \\ = \int_{\partial\mathcal{Q}^\delta} \widetilde{\text{div}} w \Phi \cdot (\mathcal{M}\nu) \, d\Sigma. \end{aligned} \quad (5.8)$$

Remark 5.1 When τ is not real, $\mathcal{M}\nu$ is a complex three vector. It is a not a normal.

Proof of Lemma 5.6. Applying (5.5) to $\Psi := \widetilde{\text{curl}} w$ proves identity (5.7). Choosing $\phi = \widetilde{\text{div}} w$ in (5.6) yields (5.8). \square

Lemma 5.7 For $\Phi \in H^2(\mathcal{Q}^\delta)$ and $\Psi \in H^1(\mathcal{Q}^\delta)$,

$$\begin{aligned} \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \Psi \cdot (\tau^2 \Phi - \widetilde{\Delta} \Phi) \, dx = \\ \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \left(\tau^2 \Phi \cdot \Psi + \widetilde{\text{curl}} \Phi \cdot \widetilde{\text{curl}} \Psi + \widetilde{\text{div}} \Phi \widetilde{\text{div}} \Psi \right) \, dx \\ - \int_{\partial\mathcal{Q}^\delta} (\widetilde{\text{curl}} \Phi \wedge (\mathcal{M}\nu) + \widetilde{\text{div}} \Phi (\mathcal{M}\nu)) \cdot \Psi \, d\Sigma. \end{aligned} \quad (5.9)$$

Proof. Equation (1.16) shows that for any $\Phi \in H^1(Q^\delta)$,

$$- \int_{Q^\delta} \Pi(\tau, x) \Phi \cdot \tilde{\Delta} \Psi \, dx = - \int_{Q^\delta} \Pi(\tau, x) \Phi \cdot (\widetilde{\text{grad}} \widetilde{\text{div}} \Psi - \widetilde{\text{curl}} \widetilde{\text{curl}} \Psi) \, dx.$$

Using the two second order formulas from Lemma 5.6 yields

$$\begin{aligned} - \int_{Q^\delta} \Pi(\tau, x) \Psi \cdot \tilde{\Delta} \Phi \, dx &= \int_{Q^\delta} \Pi(\tau, x) \widetilde{\text{curl}} \Phi \cdot \widetilde{\text{curl}} \Psi \, dx + \int_{Q^\delta} \Pi(\tau, x) \widetilde{\text{div}} \Phi \widetilde{\text{div}} \Psi \, dx \\ &\quad - \int_{\partial Q^\delta} (\widetilde{\text{curl}} \Phi \wedge (\mathcal{M}\nu)) \cdot \Psi \, d\Sigma - \int_{\partial Q^\delta} \Psi \cdot (\mathcal{M}\nu) \widetilde{\text{div}} \Phi \, d\Sigma. \end{aligned}$$

Adding the τ^2 term to each side yields (5.9). \square

5.2 The bilinear form $a(E, B, E', B')$

For the next definition, Proposition 5.4 proves that $\pi_{(\mathcal{M}\nu)^\perp} \Phi$ is a well defined element of $H^{-1/2}(\partial Q^\delta)$.

Definition 5.8 *With the notations from Definition 2.4, define*

$$\mathbb{V} = \left\{ \Phi \in H_{\widetilde{\text{curl}}}(\mathcal{Q}^\delta) \cap H_{\widetilde{\text{div}}}(\mathcal{Q}^\delta) : \pi_{(\mathcal{M}\nu)^\perp} \Phi \in L^2(\partial \mathcal{Q}^\delta) \right\}.$$

Define a symmetric continuous bilinear form $b : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}$ by

$$\begin{aligned} b(\Phi, \Psi) &:= \tau^2 \int_{Q^\delta} \Pi(\tau, x) \Phi \cdot \Psi \, dx + \\ &\quad \int_{Q^\delta} \Pi(\tau, x) (\widetilde{\text{div}} \Phi \cdot \widetilde{\text{div}} \Psi + \widetilde{\text{curl}} \Phi \cdot \widetilde{\text{curl}} \Psi) \, dx + \tau \int_{\partial Q^\delta} \pi_{(\mathcal{M}\nu)^\perp} \Phi \cdot \pi_{(\mathcal{M}\nu)^\perp} \Psi \, d\Sigma. \end{aligned}$$

Define a symmetric continuous bilinear form $a : (\mathbb{V} \times \mathbb{V}) \times (\mathbb{V} \times \mathbb{V}) \rightarrow \mathbb{C}$ by

$$a((E, B), (E', B')) := b(E, E') + b(B, B').$$

For $\{E, B\} \subset H_{\widetilde{\text{curl}}}(\mathcal{Q}^\delta) \cap H_{\widetilde{\text{div}}}(\mathcal{Q}^\delta)$ define $\mathcal{D}(E, B)$ as

$$\begin{aligned} \mathcal{D}(E, B) &:= \left(\tau \pi_{(\mathcal{M}\nu)^\perp} E + \widetilde{\text{curl}} E \wedge (\mathcal{M}\nu) + \widetilde{\text{div}} E (\mathcal{M}\nu), \right. \\ &\quad \left. \tau \pi_{(\mathcal{M}\nu)^\perp} B + \widetilde{\text{curl}} B \wedge (\mathcal{M}\nu) + \widetilde{\text{div}} B (\mathcal{M}\nu) \right). \end{aligned} \tag{5.10}$$

Lemma 5.9 For $\{E, B\} \subset H^2(\mathcal{Q}^\delta)$, and $\{E', B'\} \subset H^1(\mathcal{Q}^\delta)$,

$$\begin{aligned} a((E, B), (E', B')) &= \int_{\mathcal{Q}^\delta} \Pi(\tau, x) ((\tau^2 - \tilde{\Delta})(E, B)) \cdot (E', B') dx \\ &\quad + \int_{\partial\mathcal{Q}^\delta} \mathcal{D}(E, B) \cdot (E', B') d\Sigma. \end{aligned} \quad (5.11)$$

Proof of Lemma 5.9. Rewriting identity (5.9) yields

$$\begin{aligned} &\int_{\mathcal{Q}^\delta} \Pi(\tau, x) \left(\tau^2 \Phi \cdot \Psi + \widetilde{\text{curl}} \Phi \cdot \widetilde{\text{curl}} \Psi + \widetilde{\text{div}} \Phi \widetilde{\text{div}} \Psi \right) dx = \\ &\int_{\mathcal{Q}^\delta} \Pi(\tau, x) \Psi \cdot (\tau^2 \Phi - \tilde{\Delta} \Phi) dx + \int_{\partial\mathcal{Q}^\delta} (\widetilde{\text{curl}} \Phi \wedge (\mathcal{M}\nu) + \widetilde{\text{div}} \Phi \mathcal{M}\nu) \cdot \Psi d\Sigma. \end{aligned}$$

This yields

$$\begin{aligned} b(\Phi, \Psi) &= \int_{\mathcal{Q}^\delta} \Pi(\tau, x) (\tau^2 \Phi - \tilde{\Delta} \Phi) \cdot \Psi dx + \\ &\int_{\partial\mathcal{Q}^\delta} \tau (\pi_{(\mathcal{M}\nu)^\perp} \Phi) \cdot (\pi_{(\mathcal{M}\nu)^\perp} \Psi) + (\widetilde{\text{curl}} \Phi \wedge (\mathcal{M}\nu) + \widetilde{\text{div}} \Phi \mathcal{M}\nu) \cdot \Psi d\Sigma. \end{aligned}$$

Equivalently,

$$\begin{aligned} b(\Phi, \Psi) &= \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \Psi \cdot (\tau^2 \Phi - \tilde{\Delta} \Phi) dx \\ &\quad + \int_{\partial\mathcal{Q}^\delta} \left(\tau \pi_{(\mathcal{M}\nu)^\perp} \Phi + \widetilde{\text{curl}} \Phi \wedge (\mathcal{M}\nu) + \widetilde{\text{div}} \Phi \mathcal{M}\nu \right) \cdot \Psi d\Sigma. \end{aligned} \quad (5.12)$$

Combine (5.12) for (E, E') and (B, B') to obtain

$$\begin{aligned} a((E, B), (E', B')) &= \int_{\mathcal{Q}^\delta} \Pi(\tau, x) (E', B') \cdot (\tau^2 (E, B) - \tilde{\Delta} (E, B)) dx \\ &\quad + \int_{\partial\mathcal{Q}^\delta} \left(\tau \pi_{(\mathcal{M}\nu)^\perp} E + \widetilde{\text{curl}} E \wedge (\mathcal{M}\nu) + \widetilde{\text{div}} E \mathcal{M}\nu \right) \cdot E' d\Sigma \\ &\quad + \int_{\partial\mathcal{Q}^\delta} \left(\tau \pi_{(\mathcal{M}\nu)^\perp} B + \widetilde{\text{curl}} B \wedge (\mathcal{M}\nu) + \widetilde{\text{div}} B \mathcal{M}\nu \right) \cdot B' d\Sigma. \end{aligned} \quad (5.13)$$

This completes the proof of Lemma 5.9. \square

The next proposition yields connections between the boundary terms from Green's identities and the spectral projections of the stretched operators. These identities show that the form a is extremely well adapted to the stretched Maxwell equations. When $\tau = 0$ and the absorptions are all equal to zero, this result is closely related to formula (1.8) in [12].

Proposition 5.10 For $\{E, B\} \subset H^2(Q^\delta)$,

$$\begin{aligned}\pi^0(\mathcal{M}\nu)\mathcal{D}(E, B) &= \left((\widetilde{\operatorname{div}} E)\mathcal{M}\nu, (\widetilde{\operatorname{div}} B)\mathcal{M}\nu \right), \\ \pi^+(\mathcal{M}\nu)\mathcal{D}(E, B) &= \pi^+(\mathcal{M}\nu)\left(\tau E - \widetilde{\operatorname{curl}} B, \tau B + \widetilde{\operatorname{curl}} E \right) (\mathcal{M}\nu \cdot \mathcal{M}\nu)^{1/2}, \\ \pi^-(\mathcal{M}\nu)\mathcal{D}(E, B) &= \pi^-(\mathcal{M}\nu)\left(\tau E - \widetilde{\operatorname{curl}} B, \tau B + \widetilde{\operatorname{curl}} E \right) (\mathcal{M}\nu \cdot \mathcal{M}\nu)^{1/2}.\end{aligned}$$

Proof. Use Lemma 2.7. The π^0 identity is an immediate consequence of the definition (5.10) of \mathcal{D} . Indeed when $\pi^0(\mathcal{M}\nu)$ from (2.6) is applied to the right hand side of (5.10) it annihilates all the terms except $(\operatorname{div} E)\mathcal{M}\nu$ and $(\operatorname{div} B)\mathcal{M}\nu$ yielding the desired conclusion.

For the $\pi^+(\mathcal{M}\nu)$ identity, compute

$$\begin{aligned}\pi^+(\mathcal{M}\nu)(E, B) &= \pi^+(\mathcal{M}\nu)\left((\pi_{\mathcal{M}\nu} + \pi_{(\mathcal{M}\nu)^\perp})E, (\pi_{\mathcal{M}\nu} + \pi_{(\mathcal{M}\nu)^\perp})B \right) \\ &= \pi^+(\mathcal{M}\nu)\left(\pi_{(\mathcal{M}\nu)^\perp}E, \pi_{(\mathcal{M}\nu)^\perp}B \right).\end{aligned}\tag{5.14}$$

Use the formula of part **ii** of Lemma 2.5. Compute for arbitrary \mathbf{v} and \mathbf{w} and $\xi \in \mathbb{R}^3 \setminus 0$. Since $\mathbf{v} \wedge \xi$ and $\mathbf{w} \wedge \xi$ are orthogonal to ξ ,

$$\begin{aligned}\pi^+(\xi)(\mathbf{v} \wedge \xi, \mathbf{w} \wedge \xi) &= \frac{1}{2}\left(\mathbf{v} \wedge \xi + (\mathbf{w} \wedge \xi) \wedge \xi/|\xi|, \mathbf{w} \wedge \xi + (-\mathbf{v} \wedge \xi) \wedge \xi/|\xi| \right) \\ &= \frac{1}{2}\left(-|\xi| \mathbf{w} + \mathbf{v} \wedge \xi, |\xi| \mathbf{v} + \mathbf{w} \wedge \xi \right) = |\xi| \pi^+(\xi)(-\mathbf{w}, \mathbf{v}).\end{aligned}$$

This implies that for $\tau \in]0, \infty[$

$$\pi^+(\mathcal{M}\nu)(\mathbf{v} \wedge \mathcal{M}\nu, \mathbf{w} \wedge \mathcal{M}\nu) = \pi^+(\mathcal{M}\nu)(-\mathbf{w}, \mathbf{v})(\mathcal{M}\nu \cdot \mathcal{M}\nu)^{1/2}.$$

The terms of the identity are holomorphic in $|\tau| > M$. By analytic continuation it follows that the identity holds for all such τ . Therefore,

$$\pi^+(\mathcal{M}\nu)(\widetilde{\operatorname{curl}} E \wedge \mathcal{M}\nu, \widetilde{\operatorname{curl}} B \wedge \mathcal{M}\nu) = \pi^+(\mathcal{M}\nu)(-\widetilde{\operatorname{curl}} B, \widetilde{\operatorname{curl}} E)(\mathcal{M}\nu \cdot \mathcal{M}\nu)^{1/2}.\tag{5.15}$$

Equations (5.14) and (5.15) yield the $\pi^+(\mathcal{M}\nu)$ identity.

For the $\pi^-(\mathcal{M}\nu)$ identity compute, using the $\pi^+(\mathcal{M}\nu)$ identity for the second equality,

$$\begin{aligned}\pi^-(\mathcal{M}\nu)\mathcal{D} &= \pi^+(-\mathcal{M}\nu)\mathcal{D} = \pi^+(-\mathcal{M}\nu)\left(\tau E - \widetilde{\operatorname{curl}} B, \tau B + \widetilde{\operatorname{curl}} E \right) \\ &= \pi^-(\mathcal{M}\nu)\left(\tau E - \widetilde{\operatorname{curl}} B, \tau B + \widetilde{\operatorname{curl}} E \right).\end{aligned}$$

This completes the proof of the Proposition 5.10. \square

5.3 Invertibility for $\tau \in]0, \infty[$

The first application of the preceding results is for τ real. In that case the operators are real and the identities are relatively simple as is the classical proof of the next result.

Proposition 5.11 *For $\delta \in]0, 1]$ and $\tau \in]0, \infty[$ the Fredholm map from Corollary 4.5 is invertible.*

Proof. Suppose that $E, B \in C^\infty(\mathcal{Q}^\delta)$ belongs to the kernel. Since the operator is real it is sufficient to prove that there are no real elements of the kernel. For such elements take $E', B' = E, B$. Then equation (5.11) and Proposition 5.10 imply that $a((E, B), (E, B)) = b(E, E) + b(B, B) = 0$. Each of $b(E, E)$ and $b(B, B)$ is a sum of three nonnegative terms. Therefore all six terms vanish. In particular $\int_{\mathcal{Q}^\delta} \tau \Pi(\tau, x) (E \cdot E + B \cdot B) dx = 0$. This proves that $E = B = 0$ establishing injectivity.

Corollary 4.5 proves that the operator has index zero. Surjectivity follows. \square

Proposition 5.12 *For $\delta \in]0, 1]$ there is a discrete set $\mathbb{D}(\delta) \subset \{\text{Re } \tau > M\}$ so that for τ in the complement of $\mathbb{D}(\delta)$, the Fredholm map from Corollary 4.5 is invertible. The inverse is meromorphic with values in the set of bounded operators from the space (4.7) to $H^2(\mathcal{Q}^\delta)$.*

Proof. Once invertibility is established for $\tau \in]0, \infty[$, Proposition 5.12 follows from the Analytic Fredholm Theorem [32, Theorem VI.14]. To apply this theorem, rewrite the boundary value problem in the form (4.8) with source terms $\underline{g}_1, \underline{g}_2$ in τ -independent Sobolev spaces. The map $E, B \mapsto (f, \underline{g}_1, \underline{g}_2, g_3, g_4)$ is then Fredholm. It is defined for $|\tau| > M$ and is analytic on that set. It is invertible for τ large and real. The Analytic Fredholm Theorem implies that it is invertible on the complement of a discrete subset of $|\tau| > M$ and the inverse is meromorphic. \square

For $\tau \in \{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$, denote by $(\tau^2 - \tilde{\Delta})^{-1} \in \text{Hom}(L^2(\mathcal{Q}^\delta); H^2(\mathcal{Q}^\delta))$ the inverse operator of the problem *with homogeneous boundary data*. In Section 7.3 we prove that for M sufficiently large, $(\tau^2 - \tilde{\Delta})^{-1}(\widetilde{\text{curl } j}, -\tau j - \widetilde{\text{grad } \rho})$ has no poles in $\text{Re } \tau > M$. This follows from uniform estimates on $\{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$. Their proofs occupy Sections 6 and 7.

6 Boundary estimate à la Jerison-Kenig-Mitrea

The identity and estimate of this section has a long history for harmonic functions dating to Rellich [33] and Payne-Weinberger [28]. It was promoted to a result of central importance in studying boundary value problems in Lipschitz domains by Jerison and Kenig [20, 19]. Mitrea [25] introduced a version appropriate to Maxwell's equations. The present section extends Mitrea's work to the stretched Maxwell system. The estimate for the stretched equations is only slightly weaker than in the unstretched case. Recall the case of harmonic functions on \mathcal{Q}^δ from [20, 19]. The Dirichlet condition $u = f$ on $\partial\mathcal{Q}^\delta$ immediately controls $\|\nabla_{tan} u\|_{L^2(\mathcal{Q}^\delta)}$. The Neumann condition $\nu \cdot \nabla u = g$ controls $\|\nabla_n u\|_{L^2(\partial\mathcal{Q}^\delta)}$. The Rellich-Payne-Weinberger-Jerison-Kenig estimate shows that for harmonic functions $\|\nabla_{tan} u\|_{L^2(\partial\mathcal{Q}^\delta)}$ and $\|\nabla_n u\|_{L^2(\partial\mathcal{Q}^\delta)}$ are of the same magnitude. This controls the complete gradient for both the Dirichlet and Neumann problems.

The next Proposition is the main result of the section. It shows that for solutions of the stretched Maxwell equations, an estimate for $\|\pi_{(\mathcal{M}\nu)^\perp}(E, B)\|_{L^2(\partial\mathcal{Q}^\delta)}$ yields a bound for the entire boundary trace $\|E, B\|_{L^2(\partial\mathcal{Q}^\delta)}$.

Proposition 6.1 *With a constant M independent of $\delta \in]0, \delta_0[$, $\tau \in \{\operatorname{Re} \tau > M\}$, and, $E \in H^1(\mathcal{Q}^\delta)$,*

$$\begin{aligned} \|E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 &\lesssim \|\pi_{(\mathcal{M}\nu)^\perp} E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \\ &+ \frac{1}{|\tau|} \|E\|_{L^2(\mathcal{Q}^\delta)}^2 + \|E\|_{L^2(\mathcal{Q}^\delta)} \|\widetilde{\operatorname{div}} E, \widetilde{\operatorname{curl}} E\|_{L^2(\mathcal{Q}^\delta)}. \end{aligned} \quad (6.1)$$

The remainder of this section is devoted to a proof of Proposition 6.1.

6.1 Unstretched estimate of Mitrea

Definition 6.2 *Define Θ to be the radial vector field (x_1, x_2, x_3) .*

Mitrea [25] proves the following identity for complex E ,

$$\begin{aligned} \operatorname{div} \left\{ \frac{1}{2} |E|^2 \Theta - \operatorname{Re}((\overline{E} \cdot \Theta) E) \right\} = \\ \operatorname{Re} \left\{ \frac{1}{2} |E|^2 - \overline{E} \cdot \Theta \operatorname{div} E + (\overline{E} \wedge \Theta) \cdot \operatorname{curl} E \right\}. \end{aligned} \quad (6.2)$$

Integrating this identity, the flux through the boundary has density

$$\frac{1}{2} |E|^2 \Theta \cdot \nu - \operatorname{Re}((\overline{E} \cdot \Theta) (E \cdot \nu)). \quad (6.3)$$

Decompose $\Theta = \Theta_{tan} + \Theta_n$ with $\Theta_n = (\Theta \cdot \nu)\nu$. Similarly $E_n = (E \cdot \nu)\nu$. There is a $c > 0$ so that for all δ , $\Theta \cdot \nu \geq c$ on $\partial\mathcal{Q}^\delta$.

The contribution of Θ_n to the second summand in (6.3) is equal to

$$\operatorname{Re}((\bar{E} \cdot \nu) (\Theta \cdot \nu) (E \cdot \nu)) = (\Theta \cdot \nu)|E_n|^2.$$

Since $|E|^2 = |E_{tan}|^2 + |E_n|^2$, the flux density is equal to

$$\frac{1}{2} \left(|E_{tan}|^2 - |E_n|^2 \right) \Theta \cdot \nu - \operatorname{Re}((\bar{E}_{tan} \cdot \Theta_{tan}) (E \cdot \nu)). \quad (6.4)$$

The second term is $\lesssim |E_{tan}||E_n|$. Integrating (6.2) over \mathcal{Q}^δ and using the strict positivity of $\Theta \cdot \nu$ yields Mitrea's estimate,

$$\begin{aligned} \left| \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2 - \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \right| &\lesssim \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} + \\ &+ \|E\|_{L^2(\mathcal{Q}^\delta)} \left(\|E\|_{L^2(\mathcal{Q}^\delta)} + \|\operatorname{div} E, \operatorname{curl} E\|_{L^2(\mathcal{Q}^\delta)} \right). \end{aligned}$$

6.2 Stretched Mitrea estimate

Proposition 6.3 *With constant independent of $\delta \in [0, \delta_0]$, $E \in H^1(\mathcal{Q}^\delta)$, and, τ with $\operatorname{Re} \tau > 1$,*

$$\begin{aligned} \left| \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2 - \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \right| &\lesssim \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} \\ &+ \frac{1}{|\tau|} \|E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \|E\|_{L^2(\mathcal{Q}^\delta)} \left(\|E\|_{L^2(\mathcal{Q}^\delta)} + \|\widetilde{\operatorname{div}} E, \widetilde{\operatorname{curl}} E\|_{L^2(\mathcal{Q}^\delta)} \right). \end{aligned} \quad (6.5)$$

Remark 6.1 *This estimate is weaker than the unstretched version because of the $1/|\tau|$ term on the right.*

The next lemma extends (6.2) to the stretched operators $\widetilde{\operatorname{div}}$ and $\widetilde{\operatorname{curl}}$.

Lemma 6.4 *If $E \in C^1(\mathcal{Q}^\delta)$ and $X(\tau, x)$ is from Definition 1.9, then for all $\tau \in \{\operatorname{Re} \tau > 0\}$,*

$$\begin{aligned} \widetilde{\operatorname{div}} \left\{ \frac{1}{2} |E|^2 X(\tau, x) - \operatorname{Re}((\bar{E} \cdot X(\tau, x))E) \right\} = \\ \operatorname{Re} \left\{ \frac{1}{2} |E|^2 - (\bar{E} \cdot X(\tau, x)) \widetilde{\operatorname{div}} E + (\bar{E} \wedge X(\tau, x)) \cdot \widetilde{\operatorname{curl}} E \right\}. \end{aligned} \quad (6.6)$$

Proof of Lemma 6.4. For $\tau \in]0, \infty[$, $X(\tau, x)$ maps \mathcal{Q}^δ to its image $\underline{\mathcal{Q}}^\delta(\tau)$. For a vector field $F_{\mathcal{Q}^\delta}$ defined on \mathcal{Q}^δ , associate the field $F_{\underline{\mathcal{Q}}^\delta}$ on $\underline{\mathcal{Q}}^\delta$ defined

by $F_{\underline{\mathcal{Q}}^\delta}(X(\tau, x)) = F_{\mathcal{Q}^\delta}(x)$. The value of $F_{\underline{\mathcal{Q}}^\delta}$ at $X(\tau, x)$ is equal to the value of $F_{\mathcal{Q}^\delta}$ at x . Formula (1.15) yields

$$\operatorname{div}_X F_{\underline{\mathcal{Q}}^\delta} \Big|_{X=X(\tau, x)} = \widetilde{\operatorname{div}} F_{\mathcal{Q}^\delta}(x), \quad \text{and} \quad \operatorname{curl}_X F_{\underline{\mathcal{Q}}^\delta} \Big|_{X=X(\tau, x)} = \widetilde{\operatorname{curl}} F_{\mathcal{Q}^\delta}(x).$$

For the radial field $\Psi(X) = (X_1, X_2, X_3)$, the corresponding field in x is $X(\tau, x)$. The identity (6.2) on $\underline{\mathcal{Q}}^\delta(\tau)$ is

$$\begin{aligned} \operatorname{div}_X \left\{ \frac{1}{2} |\underline{E}|^2 \Psi - \operatorname{Re}((\overline{\underline{E}} \cdot \Psi) \underline{E}) \right\} = \\ \operatorname{Re} \left\{ \frac{1}{2} |\underline{E}|^2 - (\overline{\underline{E}} \cdot \Psi) \operatorname{div}_X \underline{E} + (\overline{\underline{E}} \wedge \Psi) \cdot \operatorname{curl}_X \underline{E} \right\}. \end{aligned} \quad (6.7)$$

There is equality of the corresponding fields in x . That proves (6.6) for $\tau \in]0, \infty[$.

For all $x \in \mathcal{Q}^\delta$, each of the terms of (6.6) is holomorphic in $\{\operatorname{Re} \tau > 0\}$. Analytic continuation from $]0, \infty[$ implies that (6.6) holds on $\operatorname{Re} \tau > 0$. \square

Proof of Proposition 6.3. Define

$$\begin{aligned} \mathcal{N} := & \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} + \\ & \|E\|_{L^2(\mathcal{Q}^\delta)} \left(\|E\|_{L^2(\mathcal{Q}^\delta)} + \|\widetilde{\operatorname{div}} E, \widetilde{\operatorname{curl}} E\|_{L^2(\mathcal{Q}^\delta)} \right). \end{aligned} \quad (6.8)$$

Multiply (6.6) by $\Pi(\tau, x)$. Using (5.1) and integrating over \mathcal{Q}^δ yields the bound,

$$\left| \int_{\partial\mathcal{Q}^\delta} \mathcal{M} \nu \cdot \left(\frac{1}{2} |E|^2 X(\tau, x) - \operatorname{Re}((\overline{\underline{E}} \cdot X(\tau, x)) E) \right) d\Sigma \right| \lesssim \mathcal{N}. \quad (6.9)$$

From Definition 3.1, $\mathcal{M} = I + O(1/|\tau|)$. In addition $X(\tau, x) - \Theta(x) = O(1/|\tau|)$ uniformly on compact sets in x . Therefore

$$\begin{aligned} \left| \int_{\partial\mathcal{Q}^\delta} \mathcal{M} \nu \cdot \left(\frac{1}{2} |E|^2 X(\tau, x) - \operatorname{Re}((\overline{\underline{E}} \cdot X(\tau, x)) E) \right) d\Sigma - \right. \\ \left. \int_{\partial\mathcal{Q}^\delta} \nu \cdot \left(\frac{1}{2} |E|^2 \Theta - \operatorname{Re}((\overline{\underline{E}} \cdot \Theta) E) \right) d\Sigma \right| \lesssim \frac{1}{|\tau|} \|E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \end{aligned} \quad (6.10)$$

Equation (6.4) implies that

$$\begin{aligned} \left| \nu \cdot \left(\frac{1}{2} |E|^2 \Theta - \operatorname{Re}((\overline{\underline{E}} \cdot \Theta) E) \right) - \frac{1}{2} \left(|E_{tan}|^2 - |E_n|^2 \right) (\Theta, \nu) \right| \\ \lesssim \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} \end{aligned} \quad (6.11)$$

Therefore,

$$\begin{aligned} \left| \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2 - \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \right| &\lesssim \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} \\ &+ \left| \int_{\partial\mathcal{Q}^\delta} \nu \cdot \left(\frac{1}{2} |E|^2 \Theta - \operatorname{Re}((\bar{E} \cdot \Theta)E) \right) d\Sigma \right|. \end{aligned} \quad (6.12)$$

Combining (6.8), (6.9), (6.10), and, (6.12) completes the proof of (6.5). \square

Proof of Proposition 6.1. Denote

$$\mathcal{R} := \frac{1}{|\tau|} \|E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \|E\|_{L^2(\mathcal{Q}^\delta)} \left(\|E\|_{L^2(\mathcal{Q}^\delta)} + \|\widetilde{\operatorname{div}} E, \widetilde{\operatorname{curl}} E\|_{L^2(\mathcal{Q}^\delta)} \right).$$

Proposition 6.3 shows that

$$\left| \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2 - \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \right| \lesssim \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} + \mathcal{R},$$

so

$$\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \lesssim \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)} + \mathcal{R}.$$

Bound $\|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}$ by a small constant times $\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}^2$ plus a large constant times $\|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2$ and absorb the E_n term on the left to find

$$\|E_n\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \lesssim \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \mathcal{R}.$$

Adding $\|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2$ to both sides yields

$$\|E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \lesssim \|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \mathcal{R}. \quad (6.13)$$

Use $\mathcal{M} = I + O(1/|\tau|)$ to show that

$$\mathcal{M}\nu - \nu = O(1/|\tau|), \quad \text{so,} \quad \pi_{(\mathcal{M}\nu)^\perp} - \pi_{\nu^\perp} = O(1/|\tau|). \quad (6.14)$$

Since $\pi_{\nu^\perp} E = E_{tan}$, it follows that

$$\|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)} \lesssim \|\pi_{(\mathcal{M}\nu)^\perp} E\|_{L^2(\partial\mathcal{Q}^\delta)} + \frac{1}{|\tau|} \|E\|_{L^2(\partial\mathcal{Q}^\delta)},$$

so,

$$\|E_{tan}\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \lesssim \|\pi_{(\mathcal{M}\nu)^\perp} E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \frac{1}{|\tau|^2} \|E\|_{L^2(\partial\mathcal{Q}^\delta)}^2.$$

Insert this estimate in (6.13) to find

$$\|E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \lesssim \|\pi_{(\mathcal{M}\nu)^\perp} E\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \mathcal{R} + \frac{1}{|\tau|^2} \|E\|_{L^2(\partial\mathcal{Q}^\delta)}^2.$$

Choose M so that for $|\tau| > M$ the last term can be absorbed on the left. This completes the proof of Proposition 6.1. \square

7 Stretched Maxwell estimate on Q^δ

This section combines Green's identities with Proposition 6.1 to derive *a priori* estimates. A key aspect is the choice of test functions to use in Green's identity. That choice relies on the approximation identity in the next subsection. In §8 the main theorems are proved by combining the *a priori* estimates with the Fredholm theory of §4.

$$7.1 \quad \widetilde{\text{curl}} \Phi \approx \text{curl} (\mathcal{M}^{-1} \Phi)$$

Definition 7.1 i. $Q := \mathcal{M} \overline{\mathcal{M}}^{-1}$ is the diagonal matrix with entries $\mathcal{M}_{ii} (\overline{\mathcal{M}_{ii}})^{-1}$ of modulus equal to one.

ii.

$$\mathcal{N} := \begin{pmatrix} 0 & -\mathcal{M}_{33} \partial_3 \mathcal{M}_{22} & \mathcal{M}_{22} \partial_2 \mathcal{M}_{33} \\ \mathcal{M}_{33} \partial_3 \mathcal{M}_{11} & 0 & -\mathcal{M}_{11} \partial_1 \mathcal{M}_{33} \\ -\mathcal{M}_{22} \partial_2 \mathcal{M}_{11} & \mathcal{M}_{11} \partial_1 \mathcal{M}_{22} & 0 \end{pmatrix}.$$

Remark 7.1 i. The coordinate directions are eigenvectors of \mathcal{M} and Q .
ii. Q is unitary. **iii.** The functions $\Pi - I$, $\mathcal{M} - I$, $Q - I$, and \mathcal{N} are all $O(1/|\tau|)$.

Lemma 7.2 For all $\Phi \in H^1$,

$$\widetilde{\text{curl}} \Phi = \Pi \mathcal{M}^{-1} \text{curl} (\mathcal{M}^{-1} \Phi) + \frac{1}{\Pi} \mathcal{N} \mathcal{M}^{-1} \Phi. \quad (7.1)$$

Proof of Lemma 7.2. Writing vectors vertically, compute

$$\begin{aligned}
\Pi(\tau, x) \widetilde{\text{curl}} \Phi &= \Pi(\tau, x) \begin{pmatrix} \widetilde{\partial}_1 \\ \widetilde{\partial}_2 \\ \widetilde{\partial}_3 \end{pmatrix} \wedge \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{11} \partial_1 \\ \mathcal{M}_{22} \partial_2 \\ \mathcal{M}_{33} \partial_3 \end{pmatrix} \wedge \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{M}_{11} \partial_1 \\ \mathcal{M}_{22} \partial_2 \\ \mathcal{M}_{33} \partial_3 \end{pmatrix} \wedge \begin{pmatrix} \mathcal{M}_{11} \frac{\Phi_1}{\mathcal{M}_{11}} \\ \mathcal{M}_{22} \frac{\Phi_2}{\mathcal{M}_{22}} \\ \mathcal{M}_{33} \frac{\Phi_3}{\mathcal{M}_{33}} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{M}_{22} \partial_2 \left(\mathcal{M}_{33} \frac{\Phi_3}{\mathcal{M}_{33}} \right) - \mathcal{M}_{33} \partial_3 \left(\mathcal{M}_{22} \frac{\Phi_2}{\mathcal{M}_{22}} \right) \\ \mathcal{M}_{33} \partial_3 \left(\mathcal{M}_{11} \frac{\Phi_1}{\mathcal{M}_{11}} \right) - \mathcal{M}_{11} \partial_1 \left(\mathcal{M}_{33} \frac{\Phi_3}{\mathcal{M}_{33}} \right) \\ \mathcal{M}_{11} \partial_1 \left(\mathcal{M}_{22} \frac{\Phi_2}{\mathcal{M}_{22}} \right) - \mathcal{M}_{22} \partial_2 \left(\mathcal{M}_{11} \frac{\Phi_1}{\mathcal{M}_{11}} \right) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{M}_{22} \mathcal{M}_{33} \left(\partial_2 \frac{\Phi_3}{\mathcal{M}_{33}} - \partial_3 \frac{\Phi_2}{\mathcal{M}_{22}} \right) + (\mathcal{M}_{22} \partial_2 \mathcal{M}_{33}) \frac{\Phi_3}{\mathcal{M}_{33}} - (\mathcal{M}_{33} \partial_3 \mathcal{M}_{22}) \frac{\Phi_2}{\mathcal{M}_{22}} \\ \mathcal{M}_{33} \mathcal{M}_{11} \left(\partial_3 \frac{\Phi_1}{\mathcal{M}_{11}} - \partial_1 \frac{\Phi_3}{\mathcal{M}_{33}} \right) + (\mathcal{M}_{33} \partial_3 \mathcal{M}_{11}) \frac{\Phi_1}{\mathcal{M}_{11}} - (\mathcal{M}_{11} \partial_1 \mathcal{M}_{33}) \frac{\Phi_3}{\mathcal{M}_{33}} \\ \mathcal{M}_{11} \mathcal{M}_{22} \left(\partial_1 \frac{\Phi_2}{\mathcal{M}_{22}} - \partial_2 \frac{\Phi_1}{\mathcal{M}_{11}} \right) + (\mathcal{M}_{11} \partial_1 \mathcal{M}_{22}) \frac{\Phi_2}{\mathcal{M}_{22}} - (\mathcal{M}_{22} \partial_2 \mathcal{M}_{11}) \frac{\Phi_1}{\mathcal{M}_{11}} \end{pmatrix}.
\end{aligned}$$

Using the definition of \mathcal{N} this identity is equivalent to

$$\Pi(\tau, x) \widetilde{\text{curl}} \Phi = \begin{pmatrix} \mathcal{M}_{22} \mathcal{M}_{33} & 0 & 0 \\ 0 & \mathcal{M}_{11} \mathcal{M}_{33} & 0 \\ 0 & 0 & \mathcal{M}_{11} \mathcal{M}_{22} \end{pmatrix} \text{curl}(\mathcal{M}^{-1} \Phi) + \mathcal{N} \mathcal{M}^{-1} \Phi.$$

Dividing by Π and using Definition 3.1 yields (7.1). \square

7.2 Lower bound for $a((E, B), (E', B'))$ for well chosen E', B'

Seek E' so that $\widetilde{\text{curl}} E' \approx \overline{\widetilde{\text{curl}} E}$ so $\widetilde{\text{curl}} E \cdot \widetilde{\text{curl}} E' \approx |\widetilde{\text{curl}} E|^2$. Since $\widetilde{\text{curl}}$ is not real, it is not sufficient to take $E' = \overline{E}$. The result of the preceding section yields

$$\overline{\widetilde{\text{curl}} E'} \approx \overline{\text{curl}(\mathcal{M}^{-1} E')} = \text{curl}(\overline{\mathcal{M}^{-1} E'}), \quad \widetilde{\text{curl}} E \approx \text{curl}(\mathcal{M}^{-1} E).$$

Comparing suggests the important idea of choosing E', B' so that

$$\begin{aligned}
\mathcal{M}^{-1} E' &= \overline{\mathcal{M}^{-1} E}, \quad \text{equivalently,} \quad E' = \mathcal{M} \overline{\mathcal{M}^{-1} E} := Q \overline{E}, \\
\mathcal{M}^{-1} B' &= \overline{\mathcal{M}^{-1} B}, \quad \text{equivalently,} \quad B' := Q \overline{B}.
\end{aligned}$$

When τ is real the coercivity in the next proposition would suffice by itself to prove well posedness. When τ is complex this is not the case. The critical

juncture is at (7.7) where the full trace appears as an error term. To estimate this term requires Proposition 6.1.

Proposition 7.3 *There are positive constants C, M so that for all τ with $\operatorname{Re} \tau > M$, $\delta \in]0, \delta_0[$, and all $W = \{E, B\} \subset H^1(\mathcal{Q}^\delta)$ with $\operatorname{supp}\{\widetilde{\operatorname{div}} E, \widetilde{\operatorname{div}} B\} \subset \mathcal{Q}_I$, one has with $W' := (E', B') := (Q\bar{E}, Q\bar{B})$,*

$$\begin{aligned} (\operatorname{Re} \tau) |\tau| \|W\|_{L^2(\mathcal{Q}^\delta)}^2 + |\tau| \|\pi_{(\mathcal{M}\nu)^\perp}(W)\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \\ + \frac{\operatorname{Re} \tau}{|\tau|} \|\widetilde{\operatorname{curl}} W, \widetilde{\operatorname{div}} W\|_{L^2(\mathcal{Q}^\delta)}^2 \leq C |a(W, W')|. \end{aligned} \quad (7.2)$$

The proof begins with a sequence of Lemmas analysing the individual terms in $a((E, B), (E', B'))$.

Lemma 7.4 *With $E' := Q\bar{E}$, one has*

$$\widetilde{\operatorname{curl}} E' = (I + O(1/|\tau|)) \widetilde{\operatorname{curl}} \bar{E} + O(1/|\tau|)\bar{E}. \quad (7.3)$$

where $O(1/|\tau|)$ denotes a matrix valued function with $L^\infty(\mathbb{R}^3)$ -norm $\lesssim 1/|\tau|$ as $|\tau| \rightarrow \infty$. For the divergence one has

$$\widetilde{\operatorname{div}} E' = \widetilde{\operatorname{div}} \bar{E} + O(1/|\tau|)\bar{E} + O(1/|\tau|)\overline{\nabla} \bar{E}. \quad (7.4)$$

Remark 7.2 *The term $O(1/|\tau|)\overline{\nabla} \bar{E}$ involves all derivatives of E and not just $\widetilde{\operatorname{div}} E$ and $\widetilde{\operatorname{curl}} E$.*

Proof of Lemma 7.4. Apply (7.1) to E' . Since $Q = \mathcal{M}\bar{\mathcal{M}}^{-1}$,

$$\begin{aligned} \widetilde{\operatorname{curl}} E' &= \Pi \mathcal{M}^{-1} \operatorname{curl} (\mathcal{M}^{-1} E') + \frac{1}{\Pi} \mathcal{N} \mathcal{M}^{-1} E' \\ &= \Pi \mathcal{M}^{-1} \operatorname{curl} (\overline{\mathcal{M}^{-1} E}) + \frac{1}{\Pi} \mathcal{N} \overline{\mathcal{M}^{-1} E}. \end{aligned} \quad (7.5)$$

Apply (7.1) to E and conjugate to find

$$\widetilde{\operatorname{curl}} \bar{E} = \overline{\Pi \mathcal{M}^{-1}} \operatorname{curl} (\overline{\mathcal{M}^{-1} E}) + (\overline{\Pi})^{-1} \overline{\mathcal{N} \mathcal{M}^{-1} E}.$$

Solve this equation for $\operatorname{curl} (\overline{\mathcal{M}^{-1} E})$ and insert in (7.5) to find,

$$\widetilde{\operatorname{curl}} E' = (\Pi / \overline{\Pi}) \mathcal{M}^{-1} \overline{\mathcal{M}} \left(\widetilde{\operatorname{curl}} \bar{E} - (\overline{\Pi})^{-1} \overline{\mathcal{N} \mathcal{M}^{-1} E} \right) + \frac{1}{\Pi} \mathcal{N} \overline{\mathcal{M}^{-1} E}.$$

Using Remark 7.1, this gives (7.3).

For the divergence identity, compute using (5.1)

$$\overline{\Pi} \widetilde{\operatorname{div} E} = \overline{\operatorname{div} \mathcal{M} E} = \operatorname{div} \overline{\mathcal{M} E}. \quad (7.6)$$

In parallel, since Q and \mathcal{M} are diagonal so commute,

$$\Pi \widetilde{\operatorname{div} E'} = \Pi \widetilde{\operatorname{div} Q \overline{E}} = \operatorname{div} \mathcal{M} Q \overline{E} = \operatorname{div} Q \mathcal{M} \overline{E}.$$

The definition of Q yields

$$Q \mathcal{M} \overline{E} = Q \mathcal{M} \overline{\mathcal{M}^{-1} \mathcal{M} E} = Q^2 \overline{\mathcal{M} E}.$$

Therefore

$$\begin{aligned} \Pi \widetilde{\operatorname{div} E'} &= \operatorname{div} Q^2 \overline{\mathcal{M} E} = \operatorname{div} \overline{\mathcal{M} E} + \operatorname{div} ((I - Q^2) \overline{\mathcal{M} E}) \\ &= \overline{\Pi} \widetilde{\operatorname{div} E} + \operatorname{div} ((I - Q^2) \overline{\mathcal{M} E}). \end{aligned}$$

Therefore

$$\widetilde{\operatorname{div} E'} = \overline{\operatorname{div} E} + (I - \Pi^{-1} \overline{\Pi}) \overline{\operatorname{div} E} + \Pi^{-1} \operatorname{div} ((I - Q^2) \overline{\mathcal{M} E}).$$

When the derivatives in the divergence hit $(I - Q^2) \overline{\mathcal{M} E}$ they generate terms $O(1/|\tau|) \overline{E}$. When the derivatives hit the field \overline{E} , they generate $O(1/|\tau|) \nabla \overline{E}$. This completes the proof of the Lemma. \square

The next lemma shows that the terms in $a((E, B), (E', B'))$ are nearly positive.

Lemma 7.5 *Define $E' := Q \overline{E}$. There are constants independent of $\delta \in]0, \delta_0[$ so that for all $x \in \mathcal{Q}^\delta$, one has the pair of estimates*

$$\begin{aligned} \left| \Pi \widetilde{\operatorname{curl} E} \cdot \widetilde{\operatorname{curl} E'} - |\widetilde{\operatorname{curl} E}|^2 \right| &\lesssim \frac{1}{|\tau|} |\widetilde{\operatorname{curl} E}| (|\widetilde{\operatorname{curl} E}| + |E|), \\ \left| \Pi \widetilde{\operatorname{div} E} \cdot \widetilde{\operatorname{div} E'} - |\widetilde{\operatorname{div} E}|^2 \right| &\lesssim \frac{1}{|\tau|} |\widetilde{\operatorname{div} E}| (|\nabla E| + |E|), \\ \left| \pi_{(\mathcal{M}\nu)^\perp} E \cdot \pi_{(\mathcal{M}\nu)^\perp} E' - |\pi_{(\mathcal{M}\nu)^\perp} E|^2 \right| &\lesssim \frac{1}{|\tau|} |E|^2 \quad \text{on } \partial \mathcal{Q}^\delta. \end{aligned}$$

Proof of Lemma 7.5. The first two estimates follow from (7.3) and (7.4) together with Remark 7.1. For the third, compute

$$\pi_{(\mathcal{M}\nu)^\perp} E' = \pi_{(\mathcal{M}\nu)^\perp} Q \overline{E} = \pi_{\nu^\perp} \overline{E} + (\pi_{(\mathcal{M}\nu)^\perp} Q - \pi_{\nu^\perp}) \overline{E}.$$

To estimate the second summand on the right use,

$$\|\pi_{(\mathcal{M}\nu)^\perp} - \pi_{\nu^\perp}\|_{L^\infty(\partial\mathcal{Q}^\delta)} = O(1/\tau), \quad \text{and,} \quad \|Q - I\|_{L^\infty(\partial\mathcal{Q}^\delta)} = O(1/\tau).$$

Thanks to the reality of π_{ν^\perp} ,

$$\pi_{(\mathcal{M}\nu)^\perp} E' = \pi_{\nu^\perp} \bar{E} + O(1/|\tau|) \bar{E} = \overline{\pi_{\nu^\perp} E} + O(1/|\tau|) \bar{E}.$$

Continuing,

$$\pi_{\nu^\perp} E = \pi_{(\mathcal{M}\nu)^\perp} E + (\pi_{\nu^\perp} - \pi_{(\mathcal{M}\nu)^\perp}) E = O(1/|\tau|) E.$$

Therefore

$$\left| \pi_{(\mathcal{M}\nu)^\perp} E' - \overline{\pi_{(\mathcal{M}\nu)^\perp} E} \right| \lesssim \frac{1}{|\tau|} |E|.$$

The desired estimate follows. \square

Proof of Proposition 7.3. Define

$$\mathcal{G} := \tau^2 \|W\|_{L^2(\mathcal{Q}^\delta)}^2 + \tau \|\pi_{(\mathcal{M}\nu)^\perp} W\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \|\widetilde{\operatorname{div}} W, \widetilde{\operatorname{curl}} W\|_{L^2(\mathcal{Q}^\delta)}^2.$$

The summands agree with those of $a(W, W')$ to leading order in τ .

• Lemmas 7.4 and 7.5 imply that

$$\begin{aligned} \left| \mathcal{G} - a(W, W') \right| &\lesssim |\tau| \|W\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \frac{1}{|\tau|} \int_{\mathcal{Q}^\delta} |\widetilde{\operatorname{div}} E| |\nabla E| \, dx \\ &+ \frac{1}{|\tau|} \|\widetilde{\operatorname{div}} W, \widetilde{\operatorname{curl}} W\|_{L^2(\mathcal{Q}^\delta)} \left(\|\widetilde{\operatorname{div}} W, \widetilde{\operatorname{curl}} W\|_{L^2(\mathcal{Q}^\delta)} + \|W\|_{L^2(\mathcal{Q}^\delta)} \right). \end{aligned}$$

To estimate the integral on the right it is important that $\widetilde{\operatorname{div}} E$ has support in \mathcal{Q}_I so the integral is $\leq \|\widetilde{\operatorname{div}} W\|_{L^2(\mathcal{Q}_I)} \|\nabla E\|_{L^2(\mathcal{Q}_I)}$. The overdetermined system $\widetilde{\operatorname{div}}, \widetilde{\operatorname{curl}}$ is elliptic uniformly in $\operatorname{Re} \tau \geq 1$. It follows that with a constant independent of such τ and δ ,

$$\|\nabla E\|_{L^2(\mathcal{Q}_I)} \lesssim \|\widetilde{\operatorname{div}} E, \widetilde{\operatorname{curl}} E\|_{L^2(\mathcal{Q}^\delta)} + \|E\|_{L^2(\mathcal{Q}^\delta)}.$$

Define

$$N_1 := \|W\|_{L^2(\mathcal{Q}^\delta)}^2, \quad N_2 := \|\pi_{(\mathcal{M}\nu)^\perp} W\|_{L^2(\partial\mathcal{Q}^\delta)}^2, \quad N_3 := \|\widetilde{\operatorname{div}} W, \widetilde{\operatorname{curl}} W\|_{L^2(\mathcal{Q}^\delta)}^2.$$

Therefore,

$$\left| \mathcal{G} - a(W, W') \right| \lesssim \|W\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + |\tau| N_1 + \frac{1}{|\tau|} \left(N_3 + (N_3 N_1)^{1/2} \right). \quad (7.7)$$

Proposition 6.1 implies that

$$\|W\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \lesssim |\tau|^{-1}N_1 + N_2 + (N_1N_3)^{1/2}. \quad (7.8)$$

Combining (7.7) and (7.8) yields

$$\left| \mathcal{G} - a(W, W') \right| \lesssim |\tau| N_1 + N_2 + |\tau|^{-1}N_3 + (N_1N_3)^{1/2}. \quad (7.9)$$

• Derive a lower bound for \mathcal{G} by considering its real and imaginary parts,

$$\begin{aligned} \operatorname{Re} \mathcal{G} &= ((\operatorname{Re} \tau)^2 - (\operatorname{Im} \tau)^2)N_1 + \operatorname{Re} \tau N_2 + N_3, \\ \operatorname{Im} \mathcal{G} &= (2 \operatorname{Im} \tau \operatorname{Re} \tau)N_1 + \operatorname{Im} \tau N_2. \end{aligned} \quad (7.10)$$

Where $|\operatorname{Im} \tau| \leq \operatorname{Re} \tau/2$, the equation for the real part of \mathcal{G} implies that for $\operatorname{Re} \tau > M_1$,

$$|\tau|^2 N_1 + |\tau| N_2 + N_3 \lesssim |\operatorname{Re} \mathcal{G}|. \quad (7.11)$$

For the more difficult region $\{|\operatorname{Im} \tau| > \operatorname{Re} \tau/2\}$, the lower bound for $|\mathcal{G}|$ is weaker. When $\operatorname{Im} \tau \neq 0$, the second equation in (7.10) is equivalent to

$$\frac{\operatorname{Im} \mathcal{G}}{\operatorname{Im} \tau} = (2 \operatorname{Re} \tau)N_1 + N_2. \quad (7.12)$$

Where $\{|\operatorname{Im} \tau| > \operatorname{Re} \tau/2\}$, this implies

$$(\operatorname{Re} \tau)N_1 + N_2 \lesssim \frac{|\operatorname{Im} \mathcal{G}|}{|\tau|}. \quad (7.13)$$

Multiplying by $|\tau|^2/(\operatorname{Re} \tau)$ yields for $\operatorname{Re} \tau > M_2$,

$$|\tau|^2 N_1 + \frac{|\tau|^2}{\operatorname{Re} \tau} N_2 \lesssim \frac{|\tau|}{\operatorname{Re} \tau} |\operatorname{Im} \mathcal{G}|. \quad (7.14)$$

Estimate N_3 using the first equation in (7.10) together with (7.14) to find,

$$N_3 \lesssim |\tau|^2 N_1 + |\operatorname{Re} \mathcal{G}| \lesssim \frac{|\tau|}{\operatorname{Re} \tau} |\operatorname{Im} \mathcal{G}| + |\operatorname{Re} \mathcal{G}|$$

This together with (7.14) yields

$$|\tau|^2 N_1 + \frac{|\tau|^2}{\operatorname{Re} \tau} N_2 + N_3 \lesssim \frac{|\tau|}{\operatorname{Re} \tau} |\mathcal{G}|.$$

Multiplying by $(\operatorname{Re} \tau)/|\tau|$ yields

$$(\operatorname{Re} \tau)|\tau|N_1 + |\tau|N_2 + \frac{\operatorname{Re} \tau}{|\tau|}N_3 \lesssim |\mathcal{G}|. \quad (7.15)$$

One has the larger lower bound (7.11) on the complementary region. Therefore (7.15) holds for all $\operatorname{Re} \tau > M_2$. The left hand side of (7.15) and the left hand side of (7.2) are the same. Denote these left hand sides as

$$\mathcal{H} := (\operatorname{Re} \tau)|\tau|N_1 + |\tau|N_2 + \frac{\operatorname{Re} \tau}{|\tau|}N_3 \lesssim |\mathcal{G}|. \quad (7.16)$$

• Next show that

$$|\tau|N_1 + N_2 + |\tau|^{-1}N_3 + (N_1N_3)^{1/2} \lesssim \frac{1}{\operatorname{Re} \tau} \mathcal{H}. \quad (7.17)$$

The equality in (7.16) estimates the first three terms. For the last, estimate

$$(N_1N_3)^{1/2} = (|\tau|N_1)^{1/2}(|\tau|^{-1}N_3)^{1/2} \leq \frac{1}{2}(|\tau|N_1 + |\tau|^{-1}N_3) \leq \frac{1}{\operatorname{Re} \tau} \mathcal{H}.$$

With the shorthand $a = a(W, W')$ and $\operatorname{Re} \tau > M_3$ estimate (7.9) yields

$$\mathcal{H} \lesssim |\mathcal{G}| \lesssim |a| + |\mathcal{G} - a| \lesssim |a| + \frac{1}{\operatorname{Re} \tau} \mathcal{H}.$$

For $\operatorname{Re} \tau > M_3$, $\mathcal{H} \lesssim |a|$. This completes the proof of Proposition 7.3. \square

7.3 Stretched Maxwell equation estimate on \mathcal{Q}^δ

Theorem 1.11 concerning the stretched Maxwell equations on \mathcal{Q} follows from the next estimate.

Theorem 7.6 *There are positive constants C, M depending on \mathcal{Q}_I and independent of $\delta \in]0, \delta_0[$, $\tau \in \{\operatorname{Re} \tau > M\}$, so that for $(j, \rho) \in L^2_{\mathcal{Q}_I}(\mathcal{Q}^\delta)$, and $W = (E, B) \in H^2(\mathcal{Q}^\delta)$ that satisfies the stretched Maxwell boundary value problem,*

$$\begin{aligned} \tau E - \widetilde{\operatorname{curl}} B &= -j & \text{on } \mathcal{Q}^\delta, \\ \tau B + \widetilde{\operatorname{curl}} E &= 0 & \text{on } \mathcal{Q}^\delta, \\ \widetilde{\operatorname{div}} B &= 0 & \text{on } \mathcal{Q}^\delta, \\ \widetilde{\operatorname{div}} E &= \rho & \text{on } \mathcal{Q}^\delta, \\ \pi^-(\mathcal{M}\nu)W &= 0 & \text{on } \partial\mathcal{Q}^\delta, \end{aligned} \quad (7.18)$$

one has

$$(\operatorname{Re} \tau)^2 \|W\|_{L^2(\mathcal{Q}^\delta)}^2 + (\operatorname{Re} \tau) \|\pi_{(\mathcal{M}\nu)^\perp} W\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \leq C \|\rho, j\|_{L^2(\mathbb{R}^3)}^2. \quad (7.19)$$

Proof. The stretched Maxwell equations imply the stretched Helmholtz equation $(\tau^2 - \widetilde{\Delta})W = (-\tau j - \widetilde{\text{grad}} \rho, \widetilde{\text{curl}} j)$ on \mathcal{Q}^δ .

Use the notation \mathcal{H} from (7.16). The source terms in the Helmholtz equations are

$$(f_1, f_2) := (-\tau j - \widetilde{\text{grad}} \rho, \widetilde{\text{curl}} j). \quad (7.20)$$

Inequality (7.2) shows that $\mathcal{H} \lesssim |a((E, B), (E', B'))|$. Lemma 5.9 implies that

$$a((E, B), (E', B')) = \int_{\mathcal{Q}^\delta} \Pi(f_1 \cdot E' + f_2 \cdot B') dx + \int_{\partial \mathcal{Q}^\delta} \mathcal{D}(E, B) \cdot (E', B') d\Sigma.$$

The formulas for the spectral projections of $\mathcal{D}(E, B)$ in Proposition 5.10 show that $\mathcal{D}(E, B) = 0$ for the solutions of (7.18), so

$$a((E, B), (E', B')) = \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \left((-\tau j - \widetilde{\text{grad}} \rho) \cdot E' + \widetilde{\text{curl}} j \cdot B' \right) dx.$$

Since j and ρ vanish at the boundary, integration by parts using Lemma 5.2 yields

$$a((E, B), (E', B')) = \int_{\mathcal{Q}^\delta} \Pi(\tau, x) \left((-\tau j \cdot E' + \rho \cdot \widetilde{\text{div}} E' + j \cdot \widetilde{\text{curl}} B') \right) dx.$$

The Cauchy-Schwarz inequality together with the inequalities of Lemma 7.4 yield,

$$|a((E, B), (E', B'))| \lesssim \|\rho, j\|_{L^2(\mathcal{Q}^\delta)} \left(\|\tau W\|_{L^2(\mathcal{Q}^\delta)} + \|\widetilde{\text{div}} W, \widetilde{\text{curl}} W\|_{L^2(\mathcal{Q}^\delta)} \right).$$

The Maxwell equations imply that

$$\|\tau W\|_{L^2(\mathcal{Q}^\delta)} \lesssim \|\rho, j\|_{L^2(\mathcal{Q}^\delta)} + \|\widetilde{\text{div}} W, \widetilde{\text{curl}} W\|_{L^2(\mathcal{Q}^\delta)}.$$

Therefore,

$$|a((E, B), (E', B'))| \lesssim \|\rho, j\|_{L^2(\mathcal{Q}^\delta)}^2 + \|\rho, j\|_{L^2(\mathcal{Q}^\delta)} \|\widetilde{\text{div}} W, \widetilde{\text{curl}} W\|_{L^2(\mathcal{Q}^\delta)}.$$

Estimate the second summand on the right as

$$\begin{aligned} & \left(\frac{|\tau|}{\text{Re } \tau} \|\rho, j\|_{L^2(\mathcal{Q}^\delta)}^2 \right)^{1/2} \left(\frac{\text{Re } \tau}{|\tau|} \|\widetilde{\text{div}} W, \widetilde{\text{curl}} W\|_{L^2(\mathcal{Q}^\delta)}^2 \right)^{1/2} \\ & \leq \frac{1}{2} \left(\frac{1}{\epsilon} \frac{|\tau|}{\text{Re } \tau} \|\rho, j\|_{L^2(\mathcal{Q}^\delta)}^2 + \epsilon \frac{\text{Re } \tau}{|\tau|} \|\widetilde{\text{div}} W, \widetilde{\text{curl}} W\|_{L^2(\mathcal{Q}^\delta)}^2 \right). \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{H} \leq C|a| &\leq \frac{C}{2\epsilon} \frac{|\tau|}{\operatorname{Re} \tau} \|\rho, j\|_{L^2(\mathcal{Q}^\delta)}^2 + \frac{C\epsilon}{2} \frac{\operatorname{Re} \tau}{|\tau|} \|\widetilde{\operatorname{div}} W, \widetilde{\operatorname{curl}} W\|_{L^2(\mathcal{Q}^\delta)}^2 \\ &\leq \frac{C}{2\epsilon} \frac{|\tau|}{\operatorname{Re} \tau} \|\rho, j\|_{L^2(\mathcal{Q}^\delta)}^2 + \frac{C\epsilon}{2} \mathcal{H}. \end{aligned}$$

Choose $\epsilon = 1/C$. Absorb the second summand on the right in the left hand side to find $\mathcal{H} \lesssim (|\tau|/\operatorname{Re} \tau) \|\rho, j\|_{L^2(\mathcal{Q}^\delta)}^2$. Multiplying by $(\operatorname{Re} \tau)/|\tau|$ yields

$$\begin{aligned} (\operatorname{Re} \tau)^2 \|W\|_{L^2(\mathcal{Q}^\delta)}^2 + \operatorname{Re} \tau \|\pi_{(\mathcal{M}\nu)^\perp} W\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \\ + |\tau|^{-1} \|\widetilde{\operatorname{curl}} W, \widetilde{\operatorname{div}} W\|_{L^2(\mathcal{Q}^\delta)}^2 \lesssim \|\rho, j\|_{L^2(\mathbb{R}^3)}. \end{aligned} \quad (7.21)$$

This completes the proof of Theorem 7.6. \square

Remark 7.3 *In our earlier work on the internal corners for Bérenger's algorithm in Part II of [16], and also for the external corners for the Pauli system [17], the estimates analogous to (7.21) were stronger. The variational forms of the Helmholtz systems used Dirichlet's integral rather than a div-curl form. The divergence and curl in (7.21) were replaced by the full gradient.*

8 Proofs of the three main theorems

The order in these proofs is crucial. The derivation of the stretched equations from the Helmholtz equations requires the smooth boundary. First solve Helmholtz on \mathcal{Q}^δ . Derive the stretched Maxwell equations on \mathcal{Q}^δ . Use the estimates for the stretched equations on \mathcal{Q}^δ to pass to the limit $\delta = 0$ solving the stretched equations on \mathcal{Q} . Use [16] for uniqueness of the stretched equations on \mathcal{Q} when τ is real. Uniqueness for all τ follows by an analytic continuation argument. The solution of the stretched equations yields a solution of the split equations via the Laplace transform. Uniqueness for the split equations follows from uniqueness for the stretched equations.

8.1 The stretched Maxwell equations on \mathcal{Q}^δ

Theorem 8.1 *Suppose that $\mu > M$ from Theorem 7.6. Suppose that $\delta > 0$. Suppose that $\rho, j \in e^{\mu t} L^2(\mathbb{R}_t \times \mathcal{Q})$ satisfy Assumption 1.1. Then there is a unique $W(\tau) = (F(\tau), G(\tau))$ holomorphic from $\{\operatorname{Re} \tau > M\}$ to $H^2(\mathcal{Q}^\delta)$*

that satisfies the stretched Maxwell boundary value problem with absorbing boundary condition

$$\begin{aligned}
\tau F - \widetilde{\text{curl}} G &= -\widehat{j} && \text{on } \mathcal{Q}^\delta, \\
\tau G + \widetilde{\text{curl}} F &= 0 && \text{on } \mathcal{Q}^\delta, \\
(\widetilde{\text{div}} F, \widetilde{\text{div}} G) &= (\widehat{\rho}, 0) && \text{on } \mathcal{Q}^\delta, \\
\pi^-(\mathcal{M}\nu)(F, G) &= 0 && \text{on } \partial\mathcal{Q}^\delta.
\end{aligned} \tag{8.1}$$

The solution satisfies with constant independent of δ and τ ,

$$\begin{aligned}
(\text{Re } \tau)^2 \|W(\tau)\|_{L^2(\mathcal{Q}^\delta)}^2 + (\text{Re } \tau) \|\pi_{(\mathcal{M}\nu)^\perp} W(\tau)\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \\
\lesssim \|\widehat{\rho}(\tau), \widehat{j}(\tau)\|_{L^2(\mathcal{Q}^\delta)}^2.
\end{aligned} \tag{8.2}$$

Proof. Existence. For $\tau \in \{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$ denote by $(\tau^2 - \widetilde{\Delta})^{-1}$ the inverse of the Helmholtz boundary value problem with homogeneous boundary conditions from Proposition 5.12. Define

$$(F, G) := W = (\tau^2 - \widetilde{\Delta})^{-1}(-\tau\widehat{j} - \widetilde{\text{grad}} \widehat{\rho}, \widetilde{\text{curl}} \widehat{j}).$$

Then W is holomorphic on $\{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$ and satisfies the Helmholtz boundary value problem with homogeneous boundary conditions.

For τ real and large, Part II of Proposition 3.2 implies that W satisfies the stretched Maxwell boundary value problem (8.1).

The equations in each line of the stretched Maxwell boundary value problem are analytic on $\{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$ and are satisfied for τ large and real. Since $\{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$ is connected, they vanish for all $\tau \in \{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$ by analytic continuation. Therefore, (8.1) is satisfied for $\tau \in \{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$.

Estimate (7.19) therefore applies for all $\tau \in \{\text{Re } \tau > M\} \setminus \mathbb{D}(\delta)$. Since $\mathbb{D}(\delta)$ is discrete, the singularities of the function $W(\tau)$ at the points of $\mathbb{D}(\delta)$ are isolated. The estimate shows that W is bounded on punctured neighborhood of each singularity so they are all removable. Therefore W is holomorphic throughout $\{\text{Re } \tau > M\}$ and satisfies the boundary value problem (7.18) and the estimate (7.19) for all such τ . This proves (8.1) and (8.2).

Uniqueness. It suffices to show that the only solution with sources $\rho = 0$ and $j = 0$ is identically zero. Since it is holomorphic on $\text{Re } \tau > M$ it suffices to show that it vanishes on $]c, \infty[$ for c sufficiently large. For those c the operator $\tau I + \sum_k \mathcal{A}_k \widetilde{\partial}_k$ is symmetric and strictly positive in the sense of

Friedrichs. In addition the boundary condition is dissipative and the solution belongs to $H^2(\mathcal{Q}^\delta)$. Uniqueness follows immediately from the identity

$$\operatorname{Re} \left((\tau I + \sum_k \mathcal{A}_k \tilde{\partial}_k) W, W \right)_{L^2(\mathcal{Q}^\delta)} = 0,$$

after an integration by parts as in the proof of Proposition 3.2. \square

8.2 The stretched Maxwell equations on \mathcal{Q} , Theorem 1.11

Proof of Theorem 1.11. Existence. Define $W^\delta = (F^\delta, G^\delta)$ on \mathcal{Q}^δ to be the solution from Theorem 8.1. With constant independent of δ, τ ,

$$(\operatorname{Re} \tau)^2 \|W^\delta\|_{L^2(\mathcal{Q}^\delta)}^2 + (\operatorname{Re} \tau) \|\pi_{(\mathcal{M}\nu)^\perp} W^\delta\|_{L^2(\partial\mathcal{Q}^\delta)}^2 \lesssim \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q}^\delta)}^2 \leq \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}^2. \quad (8.3)$$

Proposition 6.1 yields the following estimate for $(\operatorname{Re} \tau) \|W^\delta\|_{L^2(\partial\mathcal{Q}^\delta)}^2$,

$$\begin{aligned} (\operatorname{Re} \tau) \|W^\delta\|_{L^2(\partial\mathcal{Q}^\delta)}^2 &\lesssim (\operatorname{Re} \tau) \|\pi_{(\mathcal{M}\nu)^\perp} W^\delta\|_{L^2(\partial\mathcal{Q}^\delta)}^2 + \\ &\frac{\operatorname{Re} \tau}{|\tau|} \|W^\delta\|_{L^2(\mathcal{Q}^\delta)}^2 + (\operatorname{Re} \tau) \|W^\delta\|_{L^2(\mathcal{Q}^\delta)} \|\widetilde{\operatorname{div}} W^\delta, \widetilde{\operatorname{curl}} W^\delta\|_{L^2(\mathcal{Q}^\delta)}. \end{aligned} \quad (8.4)$$

The first two terms on the right are estimated by (8.3). The first two lines of (8.1) together with (8.3) imply

$$\|\widetilde{\operatorname{div}} W^\delta, \widetilde{\operatorname{curl}} W^\delta\|_{L^2(\mathcal{Q}^\delta)} \lesssim \|\tau W^\delta\|_{L^2(\mathcal{Q}^\delta)} + \|\widehat{j}\|_{L^2(\mathcal{Q}^\delta)} \lesssim \frac{|\tau|}{\operatorname{Re} \tau} \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}.$$

Therefore,

$$(\operatorname{Re} \tau) \|W^\delta\|_{L^2(\mathcal{Q}^\delta)} \|\widetilde{\operatorname{div}} W^\delta, \widetilde{\operatorname{curl}} W^\delta\|_{L^2(\mathcal{Q}^\delta)} \lesssim \frac{|\tau|}{(\operatorname{Re} \tau)} \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}^2. \quad (8.5)$$

Using (8.5) in (8.4) yields

$$\|W^\delta\|_{L^2(\partial\mathcal{Q}^\delta)} \lesssim \frac{|\tau|^{1/2}}{\operatorname{Re} \tau} \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}. \quad (8.6)$$

The coefficients of the operators $\widetilde{\operatorname{div}}$ and $\widetilde{\operatorname{curl}}$ differ from those of div and curl by $O(1/|\tau|)$. Therefore $\widetilde{\operatorname{div}}, \widetilde{\operatorname{curl}}$ is an overdetermined elliptic system uniformly in $|\tau| > M$. Thus $W^\delta := (F, G)$ satisfies

$$\|\nabla W^\delta\|_{L^2(\eta\mathcal{Q})} \leq C_\eta \|W^\delta, \widetilde{\operatorname{div}} W^\delta, \widetilde{\operatorname{curl}} W^\delta\|_{L^2(\mathcal{Q}^\delta)} \leq \frac{C_\eta |\tau|}{\operatorname{Re} \tau} \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}. \quad (8.7)$$

Choose a sequence $\delta(n)$ decreasing to zero. For $m \leq n$, $\mathcal{Q}^{\delta(n)} \supset \mathcal{Q}^{\delta(m)}$ and $\partial\mathcal{Q}^{\delta(n)} \cap \Gamma_k \supset \partial\mathcal{Q}^{\delta(m)} \cap \Gamma_k$. Therefore by the Cantor diagonal process there exists a subsequence, still denoted $\delta(n)$ so that for each m ,

- $W^{\delta(n)}$ converges weakly in $L^2(\mathcal{Q}^{\delta(m)})$ to a limit W_m ,
- $W^{\delta(n)}|_{\Gamma_k \cap \partial\mathcal{Q}^{\delta(m)}}$ converges weakly in $L^2(\Gamma_k \cap \partial\mathcal{Q}^{\delta(m)})$ to a limit $\Lambda_{m,k}$.

Estimate (8.3) implies that with constants independent of m ,

$$(\operatorname{Re} \tau)^2 \|W_m\|_{L^2(\mathcal{Q}^{\delta(m)})}^2 + (\operatorname{Re} \tau) \sum_k \|\pi_{(\mathcal{M}\nu)^\perp} \Lambda_{m,k}\|_{L^2(\Gamma_k \cap \mathcal{Q}^{\delta(m)})}^2 \lesssim \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}^2. \quad (8.8)$$

Since the $\mathcal{Q}^{\delta(n)}$ are increasing, for each $m > 1$, one has for $n \geq m$, $W_n = W_m$ on $\mathcal{Q}^{\delta(m)}$. Define $W \in L^2(\mathcal{Q})$ by $W = W_m$ on $\mathcal{Q}^{\delta(m)}$.

Similarly for $n \geq m$, $\Lambda_{n,k} = \Lambda_{m,k}$ on $\Gamma_k \cap \partial\mathcal{Q}^{\delta(m)}$. Define $\Lambda_k \in L^2(\Gamma_k \cap \partial\mathcal{Q})$ by $\Lambda_k = \Lambda_{m,k}$ on $\Gamma_k \cap \partial\mathcal{Q}^{\delta(m)}$. Estimate (8.8) implies,

$$(\operatorname{Re} \tau)^2 \|W\|_{L^2(\mathcal{Q})}^2 + (\operatorname{Re} \tau) \sum_k \|\pi_{(\mathcal{M}\nu)^\perp} \Lambda_k\|_{L^2(\Gamma_k)}^2 \lesssim \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}^2.$$

The stretched Maxwell equations on \mathcal{Q} follow from the equations on $\mathcal{Q}^{\delta(n)}$ on passing to the limit $n \rightarrow \infty$. In particular, $W \in H_{div}(\mathcal{Q}) \cap H_{curl}(\mathcal{Q})$.

Define $\Lambda \in L^2(\partial\mathcal{Q})$ to be the unique element so that $\Lambda|_{\Gamma_k} = \Lambda_k$ for all k . Corollary 5.5 shows that W has a well defined trace in $H^{-1/2}(\partial\mathcal{Q})$. By continuity of the trace, the restriction of the trace of W to Γ_k is equal to Λ_k . Therefore

$$W|_{\partial\mathcal{Q}} - \Lambda \in H^{-1/2}(\partial\mathcal{Q}), \quad \text{and} \quad \operatorname{supp}(W|_{\partial\mathcal{Q}} - \Lambda) \subset \mathcal{S}.$$

The only element of $H^{-1/2}(\partial\mathcal{Q})$ with such small support is 0 (see [16, Lemma 2.7.i]). It follows that $W|_{\partial\mathcal{Q}} = \Lambda \in L^2(\partial\mathcal{Q})$. In addition,

$$(\operatorname{Re} \tau)^2 \|W\|_{L^2(\mathcal{Q})}^2 + (\operatorname{Re} \tau) \|\pi_{(\mathcal{M}\nu)^\perp} W\|_{L^2(\partial\mathcal{Q})}^2 \lesssim \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}^2. \quad (8.9)$$

Since ν is a unit vector parallel to the k^{th} axis one has

$$\|\pi_{(\mathcal{M}\nu)^\perp} W\|_{L^2(\Gamma_k)}^2 = \|(\mathcal{M}_{kk})^2 W_{tan}\|_{L^2(\Gamma_k)}^2 \gtrsim \|W_{tan}\|_{L^2(\Gamma_k)}^2,$$

yielding

$$(\operatorname{Re} \tau)^2 \|W\|_{L^2(\mathcal{Q})}^2 + (\operatorname{Re} \tau) \sum_k \|W_{tan}\|_{L^2(\Gamma_k)}^2 \lesssim \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}^2. \quad (8.10)$$

For any $\underline{\delta} > 0$ the holomorphy of $W : \{\operatorname{Re} \tau > M\} \rightarrow L^2(\mathcal{Q}^{\underline{\delta}})$ follows from the fact that it is the weak limit of a bounded family of holomorphic functions $W^{\delta(n)}(\tau)$. Therefore, for any $\underline{\delta}$, $W : \{\operatorname{Re} \tau > M\} \rightarrow L^2(\mathcal{Q}^{\underline{\delta}})$ is holomorphic.

To show that W is holomorphic with values in $L^2(\mathcal{Q})$ it is sufficient to show that $\tau \mapsto \ell(W(\tau))$ is holomorphic for each ℓ in the dual of $L^2(\mathcal{Q})$. Since $W \in L^\infty(\{\operatorname{Re} \tau > M\}; L^2(\mathcal{Q}))$, it suffices to show that $\ell(v(\tau))$ is holomorphic for ℓ in a dense subset. Indeed if ℓ is the limit of ℓ_j for which the result is true, estimate

$$|\ell(W(\tau)) - \ell_j(W(\tau))| \leq \|\ell - \ell_j\| \sup_{\operatorname{Re} \tau > M} \|W(\tau)\|_{L^2(\mathcal{Q})}, \quad \text{on } \operatorname{Re} \tau > M.$$

This proves that $\ell(W(\tau))$ is the uniform limit of the holomorphic functions $\ell_j(W(\tau))$.

Take the dense set to be the functionals $W \mapsto \int W \cdot \Phi dx$ with $\Phi \in C_0^\infty(\mathcal{Q})$. For each such Φ , $\Phi \in C_0^\infty(\mathcal{Q}^{\delta(n)})$ for $n \geq m$. Then for $n \geq m$, $\ell(W(\tau)) = \lim \ell(W^{\delta(n)}(\tau))$ with $\ell(W^{\delta(n)}(\tau))$ holomorphic from Theorem 8.1.

An entirely analogous argument proves the analyticity of $W(\tau)|_{\partial\mathcal{Q}}$ with values in $L^2(\partial\mathcal{Q})$. In this case the functionals $\int_{\partial\mathcal{Q}} W \cdot \Psi d\Sigma$ with Ψ smooth on the boundary and vanishing on a neighborhood of \mathcal{S} are dense. This completes the proof of existence.

Uniqueness. Need to show that if the source terms ρ, j are identically equal to zero, then the solution must vanish. By analytic continuation, it suffices to show that the solution vanishes for τ large and real.

For $\tau > 0$ real, the change of variable $X(\tau, x)$ from Definition 1.9 maps \mathcal{Q} to the rectangle $\underline{\mathcal{Q}}$. The stretched Maxwell equations are conjugated to the unstretched Maxwell equations in the variable X . For τ real and ν parallel to one of the coordinate axes $A(\mathcal{M}\nu)$ is a positive multiple of $A(\nu)$ so $\pi^-(\mathcal{M}\nu) = \pi^-(\nu)$.

Therefore $V(X)$ on $\underline{\mathcal{Q}}$ defined by $V(X(\tau, x)) = W(x)$ belongs to $L^2(\underline{\mathcal{Q}})$ and satisfies

$$\left(\tau I + \sum_k \mathcal{A}_k \frac{\partial}{\partial X_k} \right) V = 0 \quad \text{on } \underline{\mathcal{Q}}, \quad \pi^-(\nu)V = 0, \quad \text{on } \partial\underline{\mathcal{Q}} \setminus \underline{\mathcal{S}}.$$

The trace $V|_{\partial\underline{\mathcal{Q}} \setminus \underline{\mathcal{S}}} \in L^2(\partial\underline{\mathcal{Q}} \setminus \underline{\mathcal{S}})$. Denote by Λ the unique element of $L^2(\partial\underline{\mathcal{Q}})$ whose restriction to $\partial\underline{\mathcal{Q}} \setminus \underline{\mathcal{S}}$ is equal to $V|_{\partial\underline{\mathcal{Q}} \setminus \underline{\mathcal{S}}}$. Then $V|_{\partial\underline{\mathcal{Q}}} - \Lambda$ is an element of $H^{-1/2}(\partial\underline{\mathcal{Q}})$ supported in $\underline{\mathcal{S}}$. The difference therefore vanishes so $V|_{\partial\underline{\mathcal{Q}}} = \underline{\Lambda} \in L^2(\partial\underline{\mathcal{Q}})$.

This shows that V satisfies the hypotheses of the uniqueness Theorem 2.13 of [16] that implies $V = 0$. Pulling back to the x variables shows that $W = 0$. It follows that F and G vanish for $\tau \in]M, \infty[$. By analytic continuation they vanish on $\{\operatorname{Re} \tau > M\}$. This completes the proof of uniqueness and therefore Theorem 1.11. \square

8.3 Bérénger's split equations on \mathcal{Q} , Theorem 1.12

The solution is constructed from its Laplace transform. The Paley-Wiener Theorem 8.2 yields properties of the solution from those of its Laplace transform. The Laplace transform of a distribution F supported in $t \geq 0$ and so that $e^{-Mt}F \in L^2(\mathbb{R})$, is defined for all $\operatorname{Re} \tau > M$ by

$$\widehat{F}(\tau) := \int e^{-\tau t} F(t) dt.$$

Our functions F take values in a Hilbert space H . The Laplace transform takes values in H . It is defined and holomorphic in a half space $\operatorname{Re} \tau > M$.

Theorem 8.2 *The Laplace transforms of functions F supported in $t \geq 0$ and so that $e^{-Mt}F \in L^2(\mathbb{R}; H)$ are exactly the functions $G(\tau)$ holomorphic in $\operatorname{Re} \tau > M$ with values in H and so that*

$$\sup_{\lambda > M} \int_{\operatorname{Re} \tau = \lambda} \|\widehat{F}(\tau)\|_H^2 |d\tau| < \infty.$$

In this case $\widehat{F}(\tau)$ has trace at $\operatorname{Re} \tau = M$ that is square integrable and

$$\int e^{-2Mt} \|F(t)\|_H^2 dt = \sup_{\lambda > M} \int_{\operatorname{Re} \tau = \lambda} \|\widehat{F}(\tau)\|_H^2 |d\tau| = \int_{\operatorname{Re} \tau = M} \|\widehat{F}(\tau)\|_H^2 |d\tau|.$$

Proof of Theorem 1.12. Define M from Theorem 1.11.

Uniqueness. Suppose that $\widetilde{U} = (U^1, U^2, U^3)$ is a solution.

The Laplace transform of the condition $V := \sum U^j \in e^{\mu t} L^2(\mathbb{R} \times \mathcal{Q})$ is that $W := \widehat{U}^1 + \widehat{U}^2 + \widehat{U}^3$ satisfies **i** of Theorem 1.11.

The Laplace transform of the condition $V_{tan} \in e^{\mu t} L^2(\Gamma_k)$ implies that $W_{tan} : \{\operatorname{Re} \tau > \mu\} \rightarrow L^2(\Gamma_k)$ is holomorphic.

The Laplace transform of the split equations (1.8) imply the split equations

$$(\tau + \sigma_k(x_k)) \widehat{U}^k + \mathcal{A}_k \partial_k W = \widehat{f}_k, \quad k = 1, 2, 3. \quad (8.11)$$

Summing implies that W satisfies **iii** of Theorem 1.11.

The Laplace transform of the boundary condition $\pi^-(A(\nu))V = 0$ shows that W satisfies **iv** of Theorem 1.11.

Therefore W must be equal to the unique solution of the stretched Maxwell system from Theorem 1.11. This uniquely determines W .

The split equations (8.11) then determine \widehat{U}^k .

This completes the proof that \widehat{U}^k is uniquely determined. Since the Laplace transform is injective, this proves the uniqueness of \widetilde{U} .

Existence. • Construction of a candidate solution. Define $W(\tau)$ to be the solution from Theorem 1.11. Consider five holomorphic functions on $\text{Re } \tau > M$,

$$\begin{aligned} W(\tau) & \text{ with values in } L^2(\mathcal{Q}), \\ (\tau + \sigma_k(x_k))^{-1} A_k \partial_k W(\tau) & \text{ with values in } H^{-1}(\mathcal{Q}), \quad k = 1, 2, 3, \\ W_{tan}(\tau) & \text{ with values in } L^2(\cup \Gamma_k). \end{aligned}$$

The estimate of Theorem 1.11 implies that

$$\begin{aligned} (\text{Re } \tau)^2 \|W(\tau)\|_{L^2(\mathcal{Q})}^2 + (\text{Re } \tau)^2 |\tau| \|(\tau + \sigma_k(x_k))^{-1} A_k \partial_k W(\tau)\|_{H^{-1}(\mathcal{Q})}^2 \\ + (\text{Re } \tau) \sum_k \|W_{tan}(\tau)\|_{L^2(\Gamma_k)}^2 \lesssim \|\widehat{\rho}(\tau), \widehat{j}(\tau)\|_{L^2(\mathcal{Q})}. \end{aligned}$$

Theorem 8.2 implies that there are uniquely determined functions

$$\begin{aligned} V(t) & \in e^{\mu t} L^2(\mathbb{R}; L^2(\mathcal{Q})), \\ U^k(t) & \in e^{\mu t} L^2(\mathbb{R}; H^{-1}(\mathcal{Q})), \quad k = 1, 2, 3, \\ \Psi(t) & \in e^{\mu t} L^2(\mathbb{R}; L^2(\partial \mathcal{Q})), \end{aligned}$$

supported in $t \geq 0$ so that,

$$\widehat{V} = W, \quad \widehat{U}^k = -(\tau + \sigma_k(x_k))^{-1} A_k \partial_k W, \quad \widehat{\Psi}|_{\cup \Gamma_k} = W_{tan}(\tau). \quad (8.12)$$

Not only is \widehat{U}^k estimated in terms of W but so is $|\tau| \widehat{U}^k$. This yields,

$$\begin{aligned} \mu^2 \|e^{-\mu t} V\|_{L^2(\mathbb{R}; L^2(\mathcal{Q}))}^2 + \mu \sum_k \|e^{-\mu t} \Psi\|_{L^2(\mathbb{R}; L^2(\Gamma_k))}^2 \\ \mu^2 \|e^{-\mu t} \{U^i, \partial_t U^i\}\|_{L^2(\mathbb{R}; H^{-1}(\mathcal{Q}))}^2 \lesssim \|e^{-\mu t} \{\rho, j\}\|_{L^2(\mathbb{R}; L^2(\mathcal{Q}))}^2. \end{aligned} \quad (8.13)$$

• Verification of Bérenger's split equations for V .

The stretched equation satisfied by W is equivalent to the fact that the Laplace transform of $V - \sum_k U^k$ vanishes identically. Conclude that $V = \sum_i U^i$. The middle equation in (8.12) shows that the Laplace transform of $(\partial_t + \sigma_i(x_i))U^i + \mathcal{A}_i \partial_i (\sum_i U^i)$ vanishes identically. Uniqueness of the Laplace transform implies that the U^i satisfy the Bérenger split equations on $\mathbb{R} \times \mathcal{Q}$.

• **Verification of the boundary condition $\pi^-(\nu)V = 0$ on Γ_k .**

Write $V = (E, B)$. Part **iii** of Lemma 2.5 shows that on $\cup_k \Gamma_k$,

$$\pi^-(\nu)V = \left(E_{tan} - B_{tan} \wedge \nu, B_{tan} + E_{tan} \wedge \nu \right).$$

By construction the right hand side belongs to $e^{\mu t} L^2(\mathbb{R} \times \cup \Gamma_k)$ and is supported in $t \geq 0$. To prove that the right hand side vanishes it suffices to show that its Laplace transform vanishes. The transform is holomorphic on $\text{Re } \tau > M$ with values in $L^2(\cup \Gamma_k)$. To show that it vanishes it suffices to show that it vanishes for $\tau \in]M, \infty[$.

By construction, $\pi^-(\mathcal{M}\nu)\widehat{V}(\tau) = 0$ on $\cup \Gamma_k$ for $\text{Re } \tau > M$. When τ is real $\mathcal{M}(\tau, x)$ is a positive diagonal matrix. When $x \in \Gamma_k$, the normal is parallel to one of the coordinate axes. Therefore $\mathcal{M}\nu$ is a positive multiple of ν . Thus for those τ, x , $\pi^-(\mathcal{M}\nu) = \pi^-(\nu)$. Therefore, for $\tau \in]M, \infty[$, $\pi^-(\nu)\widehat{V}(\tau) = 0$ on Γ_k . By analytic continuation in τ it follows that $\pi^-(\nu)\widehat{V}(\tau) = 0$ on Γ_k holds on $\text{Re } \tau > M$. This is the Laplace transform of $\pi^-(\nu)V(t)|_{\Gamma_k}$. Therefore $\pi^-(\nu)V(t)|_{\Gamma_k} = 0$. This completes the proof of Theorem 1.12. \square

Proof of part ii of Remark 1.6. Equation (8.11) implies that

$$(\text{Re } \tau) \|\widehat{U}^k\|_{L^2(\eta\mathcal{Q})} \leq \frac{\text{Re } \tau}{|\tau|} (\|\nabla W\|_{L^2(\eta\mathcal{Q})} + \|\widehat{j}\|_{L^2(\mathcal{Q})}) \leq C_\eta \|\widehat{\rho}, \widehat{j}\|_{L^2(\mathcal{Q})}.$$

The Paley-Wiener Theorem 8.2 then implies (1.20). \square

9 Reducing other PML to the stretched equations

The analysis of Bérenger's perfectly matched layers is a straightforward consequence of the analysis of the stretched Maxwell system in Theorem 1.11. In our experience, all perfectly matched layers for Maxwell's equations are built around the stretched Maxwell system. Sometimes, this structure is not easily found. This section analyses a second perfectly matched layer to illustrate the point. It covers a handful of other methods.

Bérenger's model had several drawbacks. First, the Cauchy problem for the split equations even with constant coefficients is only weakly well-posed.

Secondly the split equations do not resemble familiar problem from physics. The anisotropic medium of Sacks et al [34] addresses both. It leads to the same stretched system of Chew and Weedon see [10]. It was used by Ziolkowski [39] to produce potentially realizable absorbers. It was then modified by Abarbanel and Gottlieb and extended by Turkel-Yefet to $x \in \mathbb{R}^3$, [36].

Our second perfectly matched layer is the Turkel-Yefet system, equation (5.4) in [36]. We add a non zero current j . The unknowns are modified electric and magnetic fields $(\tilde{E}, \tilde{B}) \in \mathbb{R}^3 \times \mathbb{R}^3$, augmented by 6 auxiliary functions $(P, Q) \in \mathbb{R}^3 \times \mathbb{R}^3$ to yield a twelve dimensional system. Define diagonal constant coefficient matrices,

$$\begin{aligned} S &:= \text{diag}(\sigma_1, \sigma_2, \sigma_3), \\ M &:= \text{diag}(\sigma_2 + \sigma_3 - \sigma_1, \sigma_3 + \sigma_1 - \sigma_2, \sigma_1 + \sigma_2 - \sigma_3), \\ N &:= \text{diag}((\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3), (\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1), (\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)). \end{aligned}$$

Assumptions 1.1 and 1.2 imply that S, M, N vanish on \mathcal{Q}_I .

The unknowns $\tilde{E}, \tilde{B}, P, Q$ are required to satisfy the symmetric hyperbolic system with nonvanishing lower order terms

$$\begin{aligned} \partial_t \tilde{E} - \text{curl} \tilde{B} + M\tilde{E} + NP &= -j, \\ \partial_t \tilde{B} + \text{curl} \tilde{E} + M\tilde{B} + NQ &= 0, \\ \partial_t P + SP &= \tilde{E}, \\ \partial_t Q + SQ &= \tilde{B}. \end{aligned} \tag{9.1}$$

The auxiliary variable P resembles the polarization in electromagnetism and the variable Q is a magnetic analogue. The electric and magnetic fields of interest are linear combinations of the auxiliary variables given by

$$E := \tilde{E} - SP, \quad \text{and} \quad B := \tilde{B} - SQ.$$

One has $E = \tilde{E}$ and $B = \tilde{B}$ on \mathcal{Q}_I .

The Laplace transformed system, with hats omitted for easy of reading, implies

$$(\tau I + S)P = \tilde{E}, \quad \text{and} \quad (\tau I + S)Q = \tilde{B}.$$

Inserting this into the first equations yields

$$\begin{aligned} (\tau I + M + N(\tau I + S)^{-1})\tilde{E} - \text{curl} \tilde{B} &= -j, \\ (\tau I + M + N(\tau I + S)^{-1})\tilde{B} + \text{curl} \tilde{E} &= 0. \end{aligned}$$

Factor $(\tau I + S)^{-1}$ to find,

$$\begin{aligned} ((\tau I + M)(\tau I + S) + N)(\tau I + S)^{-1}\tilde{E} - \operatorname{curl}\tilde{B} &= -j, \\ ((\tau I + M)(\tau I + S) + N)(\tau I + S)^{-1}\tilde{B} + \operatorname{curl}E &= 0. \end{aligned}$$

Multiply by τ and use $(\tau I + S)E = \tau\tilde{E}$ and $(\tau I + S)B = \tau\tilde{B}$, to find

$$\begin{aligned} ((\tau I + M)(\tau I + S) + N)E - \operatorname{curl}(\tau I + S)B &= -\tau j, \\ ((\tau I + M)(\tau I + S) + N)B + \operatorname{curl}(\tau I + S)E &= 0. \end{aligned} \tag{9.2}$$

All the matrices involved are diagonal, so commute. The first coefficient, $(\tau I + M)(\tau I + S) + N$, is equal to

$$\begin{aligned} (\tau + \sigma_2 + \sigma_3 - \sigma_1)(\tau + \sigma_1) + (\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) \\ = \tau^2 + \tau(\sigma_2 + \sigma_3) + \sigma_2\sigma_3 = (\tau + \sigma_2)(\tau + \sigma_3). \end{aligned}$$

The first component of $\operatorname{curl}(\tau I + S)B$ is equal to $\partial_2((\tau + \sigma_3)B_3) - \partial_3((\tau + \sigma_2)B_2)$. Since σ_j depends only on x_j , this is equal to

$$(\tau + \sigma_3)\partial_2 B_3 - (\tau + \sigma_2)\partial_3 B_2.$$

The first line of (9.2) is equal to

$$(\tau + \sigma_2)(\tau + \sigma_3)E_1 - ((\tau + \sigma_3)\partial_2 B_3 - (\tau + \sigma_2)\partial_3 B_2) = -\tau j_1.$$

Multiply by τ and divide by $(\tau + \sigma_2)(\tau + \sigma_3)$ to obtain

$$\tau E_1 - \left(\frac{\tau}{\tau + \sigma_2}\partial_2 B_3 - \frac{\tau}{\tau + \sigma_3}\partial_3 B_2 \right) = -\frac{\tau^2}{(\tau + \sigma_2)(\tau + \sigma_3)}j_1 = -j_1.$$

A similar computation for τE_j , shows that E, B satisfy the stretched Maxwell system

$$\tau E - \widetilde{\operatorname{curl}} B = -j, \quad \tau B + \widetilde{\operatorname{curl}} E = 0.$$

Conversely, suppose that E, B is a solution of the stretched equations. The equation for P is

$$\partial_t P + SP = \tilde{E} = E + SP, \quad \text{so,} \quad \partial_t P = E.$$

Similarly $\partial_t Q = B$. Together with $(\tilde{E}, \tilde{B}) = (E + SP, B + SQ)$ they generate a solution of the Turkel-Yefet system. Perfect matching when the computational domain is \mathbb{R}^3 follows.

The Turkel-Yefet system (9.1) on \mathcal{Q} with boundary condition $\pi^-(\nu)(E, B) = 0$ on $\cup\Gamma_k$, implies that their E, B satisfy exactly the same stretched boundary value problem as the E, B of Bérenger. Wellposedness of the Turkel-Yefet system follows the analysis of the stretched equations.

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