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Hocus Pocus. Wittgenstein's critical philosophy of mathematical practice

Frank Scheppers

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Hocus Pocus

Wittgenstein's critical philosophy of mathematical practice

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Abstract

In this study I interpret Wittgenstein’s work on mathematics as an integral part of his philosophical endeavor at large, in terms of the stated aims and methods of this endeavor, and the (critical, ethical and aesthetical) agendas underlying it. My focus is on Wittgenstein’s later work, but strong continuities in some of the lines of thought I am interested in, lead to many naturally occurring references to earlier material.

The study is based on a close reading of extended passages of the manuscripts (as published in the online Bergen edition of the Nachlass), with special attention to the critical remarks that are mostly neglected if not shunned by Wittgenstein-scholarship.

I show that Wittgenstein foreshadows many of the themes that are prevalent in 21st century Philosophy of Mathematical Practice,¹ which however lacks the critical bias that is proper to Wittgenstein’s work and argue that the Wittgensteinian themes focused on in the present

¹ ‘Philosophy of Mathematical Practice’ (PhilMathPract) here refers to the research tradition represented by, for instance, (Van Kerkhove and van Bendegem 2007), (Mancosu 2008), (Van Kerkhove 2009), (Ferreirós 2016). For a historical overview, see (van Bendegem 2014). See also section 3.2 below.

study still can offer worthwhile contributions to the Philosophy of Mathematics at large, and to Philosophy of Mathematical Practice in particular.

Preface

The aim of this study is to bundle some of the main points of my work in Philosophy of Mathematics, mostly dating from the period 2008-2017.² As my professional and personal circumstances did not appear to allow me to prepare a publishable text in the short term, I decided in the spring of 2022 to make available what I had in the form of -what became- the present rather rough draft.³ As it turns out, this exercise did yield a publishable text focused mainly on Wittgenstein's critical remarks (see Part 2 of the present document). Still, as the process of distilling a small book out of this material may take *some* time and will involve trimming down at least *some* of the bulk, it seemed a good idea to consolidate the present draft version as it is, which would also allow me to solicit feedback from an expert readership.

As the present text is intended to function as a stand-alone document but does build on other work of mine, I was forced to sometimes repeat myself. As it did not make sense in the present circumstances to rewrite passages of unpublished ⁴ work for the purposes of this study, equally not intended for publication in its present form, I allowed myself to simply cannibalize some of my previous work.

I thank Joep Hoekstra, Michel Schorokoff and Sorin Bangu, who have -each in their own way- given me the impetus to do the work necessary to produce this draft. I also thank Jip Van

² Versions of this material have been presented on numerous informal and formal occasions, among which I would like to mention the following more formal ones and thank those who offered me feedback: (1) lecture "Hocus Pocus 101. Wittgenstein's critical remarks on (meta-)mathematics: meaning, everydayness and epistemic authenticity", as part of the Masterclass on Mathematical Practices with Karine Chemla, Centrum voor Logica en Wetenschapsfilosofie (V.U.B.), 2018-05-18; (2) lecture "Hocus Pocus 2.0. Wittgenstein's philosophy of mathematics, epistemic authenticity, and the pragmatics of formalism", Centrum voor Logica en Wetenschapsfilosofie (V.U.B.), 2018-06-18; (3) lecture "Meaningfulness / meaninglessness and epistemic authenticity / fakeness in Wittgenstein's philosophy of mathematical practice", as part of the Conference on Virtue Epistemology of Mathematical Practice (V.U.B., July 13-14 2018), 2018-07-13; lecture "Wittgenstein's philosophy of mathematical practice and the ethics of formalism", Centre for Logic and Philosophy of Science, V.U.B., 2021-02-03.

³ By rough draft I mean that some sections are underdeveloped and other sections are overdeveloped for the purpose they serve in the context of this text, that references to the literature and cross-references are incomplete and unequally distributed over the text, that some sections have been taken from other projects and have not been sufficiently integrated into their new context, that the references to, and quotations from, Wittgenstein's text have not been harmonized: although I refer to the on-line Bergen edition of the Nachlass in principle, I sometimes still refer to the standard editions; I have not yet decided on the way I should integrate LW's original text into mine (German original, English translation; in the body of the text, in footnote); this decision will depend on the venue the final version will eventually be published at.

⁴ I use the terms 'published' and 'unpublished' in their currently prevailing commercial sense, as much of this 'unpublished' work has been readily available via academia.edu and other informal channels.

Besouw, Bart Van Kerkhove and Koen Vermeir for their comments, which helped improve the present draft considerably, but will have even more impact on future versions.

This draft version (especially sections 1.1.3(C), 1.3 and part 2) is still a 'happy' text, in a way that eventual later versions will probably not be: this draft is the first time these contents are written out after my initial notes and the writing has not yet gone stale by successive rewrites and strategic considerations. If I were my readers, I would prefer to read this version, rather than whatever versions will follow, even if those versions will surely contain less mistakes and a more thorough interaction with the literature.

My work on Wittgenstein's PhilMath is part of a larger, more long-term effort centered around the concept of 'practice',⁵ which in its turn emerged from previous work in linguistics.⁶ In the somewhat longer run, this effort should give rise to a book-length study, of which my work on Wittgenstein, and my work on mathematics are only a small part.

Abbreviations

Throughout this draft I use the following abbreviations of my own:

- PhilMath = Philosophy of Mathematics
- PhilMathPract = Philosophy of Mathematical Practice
- LW = Ludwig Wittgenstein

As for references to LW's work on mathematics that is central to this study, I have tried to refer directly to the manuscripts as published in the online Bergen University edition of the Nachlass (<http://wab.uib.no/transform/wab.php?modus=opsjoner>). I use the numbering of the manuscripts and typescripts used there (but ultimately based on (von Wright 1997).

However, for practical reasons (mostly ease of reference, in those cases in which there is no added value in citing the manuscript, or when the reference to the standard edition is actually more meaningful than a reference to a manuscript), I still sometimes refer to the standard editions.

I use the following abbreviated references to LW's published work:

- *PhU* = *Philosophische Untersuchungen / Philosophical Investigations*;
- *PhPF* = *Philosophie der Psychologie - Ein Fragment / Philosophy of Psychology - A Fragment* (as of the fourth edition of *PhU* (Wittgenstein 2009), *PhPF* is the title for what had -controversially- been published as Part II of *PhU*; I will also follow the new numbering into paragraphs);
- *ÜG* = *Über Gewißheit / On Certainty*;
- *BGM* = *Bemerkungen über die Grundlagen der Mathematik / Remarks on the Foundations of Mathematics*;
- *BPhP* = *Bemerkungen über die Philosophie der Psychologie / Remarks on the Philosophy of Psychology*.
- *LSPhP* = *Letzte Schriften über die Philosophie der Psychologie / Last Writings on the Philosophy of Psychology*
- *TLP* = *Logisch-philosophische Abhandlung / Tractatus Logico-Philosophicus*

Except if mentioned otherwise I quote these standard texts from the editions prepared by his literary heirs, as published in the Suhrkamp *Werkausgabe*: Wittgenstein 1989a; Wittgenstein 1989b; Wittgenstein 1989c; Wittgenstein 1989d; Wittgenstein 1989e. For *PhU* and *PhPF* I consulted the Blackwell 4th edition (Wittgenstein 2009). One last abbreviation that I will use, is the following:

- *LFM* = *Lectures on the Foundations of Mathematics, Cambridge, 1939* ((Wittgenstein 1976))

⁵ (Scheppers 2009); (Scheppers 2017). The material in these texts gave rise to a few half-hearted attempts at publication, but personal circumstances prevented me from following through with the peer review process. It may be worth mentioning that this research activity also gave rise to a research project proposal "The ontology of the 'practice turn' in the philosophy of scientific and mathematical practice: towards a radically pragmatic framework", submitted to the FWO in 2018, not selected for funding.

⁶ (Scheppers 1993); (Scheppers 1997); (Scheppers 2003); (Scheppers 2004); (Scheppers 2011); (Scheppers 2018)

0. Introduction

LW's PhilMath has not been well received (see section 0.1 below). There are plenty of reasons why the reception of LW's writings on this matter (and others) may have gone wrong: the aims and the methods of LW's philosophy, as well as the way his work is presented, appear quite different from what is expected by both his contemporaries and many present-day scholars, including even those who claim to be inspired by LW's work (see section 0.2 below). As far as LW's aims and methods go, the scholarship focusing on LW's writings on mathematics has been particularly bad at taking these into account, apparently preferring to focus on what LW can contribute to existing issues in PhilMath at large, rather than reading the texts on their own terms. The question as to how LW's work on math fits in with the agenda underlying his work at large will therefore be a central concern in this study.

Sections 0.1, 0.2 and 0.3 of the present introduction aims at briefly sketching a few aspects of the broader context of LW's PhilMath that are relevant to its interpretation. In section 0.4, I briefly explain how the present study is organized.

0.1 Wittgenstein's bizarre (?) philosophy of mathematics

In 1944, Wittgenstein -supposedly-⁷ asked John Wisdom to include a final sentence to a short biographical paragraph that Wisdom had written about him for a biographical dictionary: "Wittgenstein's chief contribution has been in the philosophy of mathematics". Whatever the value of this testimony, it does appear that Wittgenstein first got interested in philosophy through mathematics-related problems ⁸ and that this was why he first contacted Frege and Russell. It is also true that after his return to philosophy he wrote and taught prolifically about mathematics, especially between 1929 and 1934 and again in the period September 1937-April 1944.

Still, despite his own commitment to PhilMath and despite his reputation as one of the major philosophers in 20th century philosophy at large, Wittgenstein has -generally speaking- a bad reputation in PhilMath circles (and this includes -perhaps surprisingly- PhilMathPract circles). In some cases, even mentioning LW's name is almost religiously (though not always successfully) avoided. In any case, LW's work seems to have had remarkably little direct

⁷ This story, told by Rush Rhees, was not corroborated by John Wisdom (Monk 1990, p. 466 and p. 628, note ad p. 466); Monk also mitigates the importance of this anecdote by pointing out that LW started to shift his focus to other topics only a few months after the moment of its supposed occurrence.

⁸ Cf. e.g. (McGuinness 1988) pp. 73-77.

influence on mainstream PhilMath and his work is rarely quoted in the field,⁹ with the exception perhaps of authors interested in aspects of mathematics that are closely related to logic, for instance, authors interested in Gödel, Tarski, Turing, or Russell.

Thus, scholarship taking LW's work on mathematics (more or less) seriously is restricted to a productive but rather small niche, more or less isolated from what may be called the 'mainstream' of PhilMath (although some of the big names do have mainstream credibility through other work of theirs).¹⁰

⁹ Thus, Bangu (Bangu n.d.) (s.d., §1) speaks of, as well as illustrates, "Wittgenstein-phobia" in *Philosophy of Mathematics*. Similarly, see also Mühlhölzer 2010, p. 10 (Mühlhölzer 2010) on the bad reputation of LW's work on mathematics within the field of PhilMath at large.

As a case in point, handbooks or collections with a wide scope in PhilMath seem to often ignore Wittgenstein's contributions to the field:

- Linnebo's *Philosophy of Mathematics* (Linnebo 2017) mentions Wittgenstein a few times, but -bizarrely- not his work on math;
- Incurvati's 2020 *Conceptions of Set and the Foundations of Mathematics* (Incurvati 2020) appears to not mention LW at all;
- the volume *Philosophy of mathematics* within the series *Handbook of the Philosophy of Science* (Irvine 2009) mentions LW in 5 separate passages;
- in Heaton's *A Brief History of Mathematical Thought* (Heaton 2017), it is claimed on p. 5 that "This book is related to the work of various philosophers (particularly Ludwig Wittgenstein), [...]", and LW is mentioned a few times, but the book does not discuss LW's work on math in any detail;
- Friend's 2007 *Introducing Philosophy of Mathematics* (Friend 2014) explicitly states "There are several glaring omissions in this book, noticeably Wittgenstein's philosophy of mathematics. By way of excuse I can say that this is not meant as an encyclopaedia of the philosophy of mathematics, but only an introduction, so it is not intended to cover all philosophies. Nevertheless, the omission of Wittgenstein's philosophy of mathematics bears further justification. I am no expert on Wittgenstein, and I am not sure I would trust second-hand sources, since many disagree with each other profoundly. I do not have the expertise to favour one interpretation over others, so I leave this to my more able colleagues".
- Notable exceptions are (Shapiro 2005), (Shanker 1996).

This also goes for contributions to the emerging field of PhilmathPract. References to LW in the seminal collection *Philosophy of Mathematical Practice* (Mancosu 2008) are limited to one contribution; in (Van Kerkhove and van Bendegem 2007), LW's *PhU* comes up in the bibliography of two contributions; in (Van Kerkhove 2009), one contribution (Desmet 2009) deals at some length with math-related lines of thought in LW's work (although other aspects of LW's work are exploited in a few other contributions). It is symptomatic for LW's reception in *Philosophy of Mathematics* that Ferreirós, in his 2016 primer on *Philosophy of Mathematical Practice* (Ferreirós 2016), mentions LW only for a 'colorful' quotation, according to Ferreirós "expressing views akin to logical positivism and strict formalism" (pp. 89-90). However, the context of the quote is an exposition of views very much akin to Ferreirós' own account, though LW is more radical in his turn towards practice. [[By the way, Ferreirós history of set theory (Ferreirós 2007) briefly mentions LW and his *TLP* on a few occasions (among which I like "The strange features of the famous *Tractatus* by his student Wittgenstein [1921] are thus more a symptom than a deviation." (footnote 1 on p. 332), which elucidates the following sentence in the body text: "'Russell's peculiar conflation of syntax and semantics has the effect that his work is dealing with philosophical logic, and even metaphysics, throughout"), but does not mention LW's remarks on set theory at all.]] A notable exception is Ravn & Skovsmose's *Connecting Humans to Equations: A Reinterpretation of the Philosophy of Mathematics* (Ravn and Skovsmose 2019).

¹⁰ The following authors come to mind (list in alphabetical order, not necessarily the order in which they come to mind): Sorin Bangu; Juliet Floyd; Pasquale Frascolla; Jaakko Hintikka; Georg Kreisel; Timm Lampert; Penelope Maddy; Felix Mühlhölzer; Victor Rodych; S.G. Shanker; Mark Steiner;

(A) weird claims

In part, this lack of appreciation for LW in PhilMath circles may be the result of the fact that LW's PhilMath was/is mainly known in the form of a few dogmatic claims that sounded, and to many still sound, distinctively weird:

- “we don't know the meaning of a theorem unless we know the way to prove it (e.g. Fermat, '777 in π ')”;
- “continuing the decimal expansion of an irrational number is an expansion of math”;¹¹
- “math is a matter of grammar: mathematical statements are not propositions but instructions on how to use certain words”;
- “mathematical advances are inventions, not discoveries” (which got LW lumped in with various forms of social-constructivism that bear very little resemblance to his own work, qua conceptual framework, but especially qua methods and aims);
- “math is defined by its applications”.

All of these claims will be addressed in later sections within the present study.

(B) wild criticism

One of the reasons why LW's work has met with such resistance within PhilMath, is that he appears to strongly reject contributions to mathematics that are universally (or almost universally) accepted as canonical parts of mainstream mathematics, sometimes using very strong language in the process:

- he calls Cantor's diagonal argument 'hocus pocus';
- he calls Gödel's famous paper 'unphilosophical' and his (and most mathematicians') concepts 'slimy';
- he calls the set-theoretical construction that leads to Russell's paradox a cancerous tumor and considers set theory in general “pernicious”, a symptom of the “illness of our time”, and the mathematics of the previous hundred years “instinctless”.

Even among those commentators that are generally speaking sympathetic towards LW's PhilMath, nobody appears to be willing to defend LW's critical outbursts against such august and canonized parts of the mathematical mainstream as Cantor and Dedekind's diagonal methods, certain conceptions of infinity, set theory as a foundational theory, well-established interpretations of Gödel's results, etc. Most Wittgenstein apologists with respect to his PhilMath either (1) avoid, ignore and/or explain away or (2) explicitly disavow LW's more

¹¹ To be fair: LW is aware of the bizarreness of this claim: “So seltsam es klingt: Die Weiterentwicklung einer irrationalen Zahl ist eine Weiterentwicklung der Mathematik” (Ms-126,133, d.d. 19421214).

overtly critical rants and focus on a selection of what appear to be more technical lines of thought.¹²

I will take the opposite route and try to show that *LW's philosophical activity, including his PhilMath, is profoundly and essentially critical* (in several senses of that word) and that *these overtly critical remarks show a fundamental aspect of LW's work*.

(C) continuing exegetical controversies

It is remarkable that many central aspects in the exegesis of LW's work on mathematics are still controversial, despite LW's high profile in the history of 20th century philosophy, despite the many years that have passed since his death, and even since the posthumous publication of his works, and despite the enormous exegetical efforts spent on his oeuvre. Thus, scholars are still debating such questions as "is LW a finitist?", "is LW a constructivist?", "is LW a formalist or an anti-formalist?", etc.¹³

Interestingly, scholars are also still disagreeing on the issue as to whether LW is a revisionist or not, i.e. as to whether we have to take his claim that "philosophy leaves everything as it is" serious, or the fact that LW does seem to criticize a number of things (see section 0.2 here below). Similarly, scholars are still debating on what exactly LW was objecting against in his apparent critique of Gödel, or in his overt criticism of Dedekind and Cantor.

Many times, I've heard people blame LW's 'obscurity' for this lack of consensus (and use this as a justification for not having to engage with this body of work), but I don't think that this is fair: one of the main points of this study is that most of the misunderstandings are based on

¹² For instance, Mühlhölzer refuses to defend LW's more critical rants, and calls them "engherzig" (narrow-hearted) ((Mühlhölzer 2010), p 15). Maddy speaks of LW's non-revisionism and the tension with the critical strands of his work in terms of 'false modesty': "Surely, one cannot deny the law of the excluded middle or rule out non-constructive existence proofs and at the same time leave "mathematics as it is". But what is the motivation for this prohibition? If philosophy provides compelling reasons to abandon the Platonistic picture, if current mathematical practice is based on that picture, why shouldn't the result of philosophical analysis be allowed to reform that practice? Mightn't Wittgenstein's reluctance be a form of false modesty? This reading of Wittgenstein's late views uncovers a tension between the upshot of his philosophical views and his insistence that philosophy alters nothing.(5) It tempts us to downplay the non-interference remarks in favor of the presumed payoffs of his contentful philosophical conclusions. A directly opposed approach - my focus in this paper - would give pride of place to the non-interference claims and adjust the reading of the rest to match." ((Maddy 1993), p. 55).

For a fair-minded but sympathetic account of the historical background of LW's work on math and the way it was written and published, see Floyd 2015, pp. 9-12 (Floyd 2015); still, even Floyd does not seem to have any sympathy for LW's critical rants: "By 1939 Wittgenstein's knowledge of the foundations of mathematics as an ongoing mathematical pursuit was minimal, even by his own contemporaries' standards, as he himself emphasized with his Cambridge students [1989, pp. 13-14]. Yet he frequently dares to vituperate in his notebooks, especially on bad days when he is driving himself hard: he compares set theory (to choose only one among other famous examples) to a cancerous growth on mathematics (BGM VII, §7), as it were, sucking out its healthy marrow. The manuscript writings of this ambivalent philosopher are laced, far more even than the published versions, with continual expressions of ire, aspersion, hesitation, rejection, criticism, and revision." (p. 11).

¹³ To make things worse, one could even argue that some aspects on which there is little controversy, most notably LW's anti-Platonism, are less clear-cut than the literature seems to suggest.

a refusal to read LW's texts on their own terms, i.e. in terms of the aims, problems, methods, etc. these texts claim for themselves.

0.2 Wittgenstein's work on mathematics in the context of his oeuvre as a whole

Most scholarship on LW's PhilMath has approached it from the point of view of PhilMath in general, focusing on the question as to what LW's work may or may not contribute to this or that issue in PhilMath at large. Even in the works of scholars that are obviously aware of the problem, the problem remains pervasive, perhaps not in the least because of external constraints (peer review *oblige*).¹⁴

In the present study, I will radically adhere to the basic idea that Wittgenstein's PhilMath should be read as an integral part of his philosophy as a whole and that all of its aspects should be interpreted in terms of the same underlying agenda, biases, methods, and themes. Trying to read isolated remarks taken from LW's manuscripts that appear to deal with a topic one happens to be interested in, in the hope that these remarks will shed light on that topic, is the standard hermeneutic in this field, but need not be the most fruitful approach, neither to obtain a proper understanding of LW's text for its own sake (obviously), nor to learn something from LW's work with respect to the topic one is interested in.

(A) style and presentation

The format and style in which LW wrote and was published is far removed from what is usual in contemporary academic publications:

- the writings are sometimes presented as collections (LW called (what became) *PhU* an 'album') of separate paragraphs, as collections of fragments rather than as continuous lines of argument;
- LW often asks his reader to consider some alternative way of looking at a familiar issue, sometimes a quite plausible way, sometimes a wildly counterfactual way; for instance, to imagine mathematical calculations as applied in accounting or engineering practices, or as part of a ceremony, or as a way to produce wallpaper, to communicate with ghosts, etc. (cf. section 1.3 below);

¹⁴ The problem is also that LW adopts an outsider's perspective that is directly at odds with the mathematical exceptionalism that is prevalent in PhilMath and that inspires the idea that only mathematicians are qualified to say something about mathematics (cf. section 0.2(C) above and section 3.2.2(B) and 3.2.3(A) below).

- similarly, LW often switches from one opinion on a particular topic to one radically opposed to it in quick succession and without really resolving the tensions or explicitly coming to a conclusion.¹⁵

On top of this, the editorial history of his works was a catastrophe, and the appearance of fragmentation and misunderstandings of what is and what isn't at stake in individual texts was enhanced by the butchering that the manuscripts received at the hands of the editors;¹⁶ the original manuscripts, while often still not taking the shape of clearly stated theses which then are argued for, often do feel a lot more cohesive than the text presented in the standard editions. It may be worth mentioning that starting to read LW's work on mathematics directly from the Bergen edition of the Nachlass did make a big difference in the way I understood LW's work.¹⁷ In what follows, I will present the results of a close reading of extended passages from MS-117, MS-121, MS-122, MS-122, MS-126 and MS-161. Furthermore, in section 2 of the

¹⁵ Cf. (Sass 2001), pp. 104-105: "Any attempt to correlate the life and thought of a major philosopher is likely to be difficult, of course, but several features of Wittgenstein's thinking and writing make such an attempt particularly hazardous. First there is the difficulty of identifying or recognizing precisely who is speaking in his texts. Understanding Wittgenstein requires that one read with an ear cocked to the dramatic ironies and other complexities that make up the ebb and flow of his argument. This means recognizing that we are confronted not with a textbook or treatise so much as with a series of conversational dialogues in which it can be difficult to discern the location, or even the existence, of a settled point of view attributable to Wittgenstein. What at first may seem the assertion of a philosophical view often turns out to be the provision of a target or a stalking horse for his criticism. A remark from 1951 is apropos: "But see, I write one sentence, and then I write another – just the opposite. And which shall stand?". For the context of this last -very funny- remark, see (Bouwsma 1999), p. 122

Mühlhölzer (2010, pp. 12-14) discusses this aspect of LW's style under the heading "Aber-Dialektik" ("the dialectic of but"), (disapprovingly) quoting (p.13) Hintikka's interpretation of this feature in terms of a "defensive" attitude, "insecurity" and even "paranoia" on the part of LW. I believe this kind of interpretation is based on a misunderstanding of LW's aims, his method and his inherently polyphonic style. The purpose, and the value, of this demarche resides in the experience of going through this process, internalizing the different points of view and reactions that emerge from working through the material one is working on. It is not necessarily fruitful to try and determine which voice is LW's "own voice". In the context of the original manuscript notebooks, all of these 'voices' (?) have to be taken seriously, qua participants in the debate at hand.

The difficulties resulting from this quasi-dialogical / polyphonic style for the interpretation of these texts are very clear in Kripke's attempt (Kripke 1982) and the very rich ensuing debate about the correctness of Kripke's interpretation: Kripke goes as far as interpreting the *Phil* as a dialogue between LW and a character he calls "the sceptic"; a substantial part of the interpretative debate is about determining "who said what" and what statements are made by the real "LW". For an overview of the debates, see (Miller and Wright 2002); (Kusch 2006).

I find it hard to comprehend why professional philosophers have difficulties recognizing what LW is doing in his writings: they may disapprove of the format for a publishable text, but I would have thought that the process of internalizing and working through opposing (or perpendicular) points of view would be instantly recognizable. In other words: to me, LW's manuscripts look like an adequate representation of the way philosophers normally (?) work.

¹⁶ For the history of the publication of LW's writings, see (Erbacher 2015), (Erbacher 2016); (Toynton 1997); (Venturinha 2010); (McDougall n.d.). Perhaps the time will come in which it becomes possible to write an appropriately critical account without generating unfruitfully emotional polemics.

¹⁷ LW's manuscripts, as opposed to the album-like standard editions, do give the impression of a philosopher thinking in a coherent way, going over various aspects of a limited set of topics for prolonged periods of time. Apparently, the editors decided that this kind of writing was not publishable, and perhaps that was the case back when they were supposed to deliver published materials, but with hindsight, it would perhaps have been better, if they had selected a few manuscripts that do present a sustained development of coherent lines of thought (even if they do not take the shape of 'thesis and arguments') and published them as they were. One day, someone should write a masterpiece about the real-life circumstances that lead the trustees to not trust in the quality of the legacy they were curating.

present study, we will encounter a few spectacular cases of how the editorial practices that gave rise to the standard editions can lead scholars astray.

(B) the aims and methods of philosophy: therapy and critique

As for **aims**, LW makes a very clear distinction between the aims of (his) philosophy and the aims of scientific endeavors. Philosophy does not consist in constructing true propositional knowledge at all. Philosophical problems are not to be considered as questions one should answer, but rather as an undesirable or even pathological state of confusion, which should disappear completely (*PhU* §133; cf. already *TLP* 6.521). In *PhU*, LW uses a number of different metaphors expressing this basic idea:

- the philosophical problem as an illness and his philosophical method as a therapy (*PhU* §133;¹⁸ §255: “Die Philosophie behandelt eine Frage; wie eine Krankheit”; cf. also *PhU* §593, where ‘philosophische Krankheite’ are said to be caused by an ‘einseitiges Diät’ and see section 2.2(C) below for the “Krankheit einer Zeit” remark in *BGM* 2, 23);¹⁹
- the philosophical problem as a case of being lost or trapped, and philosophy as pointing the way out. Cf. e.g. “Was ist dein Ziel in der Philosophie? – Der Fliege den Ausweg aus dem Fliegenglas zeigen.” (*PhU* §309). Cf. also the philosophical problem as ‘Glatteis’ (*PhU* §107); language as a “labyrinth” (*PhU* §203); the dead-end street of doing philosophy (*PhU* §436: “Sackgasse des Philosophierens”), ... ;
- the philosophical problem as a case of enchantment (“Verhexung”) of our minds (*PhU* §109), of ‘superstition’ due to ‘grammatical illusions’ that give rise to ‘philosophical pathos’ (*PhU* §110),²⁰ of being captured by a picture (*PhU* §115).²¹

These aims are obviously at odds with the overtly scientific objectives of mainstream (post-)analytic English-language philosophy in general and PhilMath in particular.

¹⁸ Wir wollen nicht das Regelsystem für die Verwendung unserer Worte in unerhörter Weise verfeinern oder vervollständigen. // Denn die Klarheit, die wir anstreben, ist allerdings eine *vollkommene*. Aber das heißt nur, daß die philosophischen Probleme *vollkommen* verschwinden sollen. // Die eigentliche Entdeckung ist die, die mich fähig macht, das Philosophieren abzubrechen, wann ich will. - Die die Philosophie zur Ruhe bringt, so daß sie nicht mehr von Fragen gepeitscht wird, die *sie selbst* in Frage stellen. - Sondern es wird nun an Beispielen eine Methode gezeigt, und die Reihe dieser Beispiele kann man abbrechen. - Es werden Probleme gelöst (Schwierigkeiten beseitigt), nicht *ein* Problem. // Es gibt nicht *eine* Methode der Philosophie, wohl aber gibt es Methoden, gleichsam verschiedene Therapien.

¹⁹ To these we could add the notion of ‘mental cramp’, which occurs in the first paragraph of (and on several other occasions in) the *Blue Book* (1933-1934).

²⁰ »Die Sprache (oder das Denken) ist etwas Einzigartiges« - das erweist sich als ein Aberglaube (nicht Irrtum!), hervorgerufen selbst durch grammatische Täuschungen. // Und auf diese Täuschungen, auf die Probleme, fällt nun das Pathos zurück. (Cf. section 4.3 below).

²¹ Ein *Bild* hielt uns gefangen. Und heraus konnten wir nicht, denn es lag in unsrer Sprache, und sie schien es uns nur unerbittlich zu wiederholen.

Interestingly, philosophy itself is sometimes seen as an undesirable obsession: we have to learn to stop philosophizing when we want to (*PhU* §133);²² philosophical behavior can easily be confused for madness and should perhaps not be performed in public (*ÜG* §467), etc. This leads to an interesting paradox that I dealt with elsewhere (Scheppers 2017), Chapter 3, §1, and that is closely related to the problem of LW's overt anti-revisionism, but apparent criticism (cf. sections 0.2(D), 1.2.3(C) and 3.1.1(C8) below).

LW's **methods** are closely related to his therapeutic aims. In a number of passages, LW speaks out more or less clearly against theorizing or even explanation (not even, or perhaps even especially not, scientific explanation) as a proper and adequate method of philosophy. LW's philosophy consists entirely in clearing up misunderstandings (or so he claims): time and time again, LW wants to show that this or that 'philosophical' proposition is based on an erroneous understanding of the meaning of the words used in that utterance (cf. e.g. *PhU* §§90-92).²³ In other words: philosophical problems originate when words are taken out of the everyday contexts in which they belong; putting these words back into their contexts is sufficient to 'dissolve' the problem; philosophical analysis consists in merely presenting the everyday use of certain words in an easily overseenable way (cf. e.g. *PhU* §122). Thus, the method consists in 'staying at the surface' (*PhU* §92) and avoiding to yield to the temptation of looking for 'depth' where there is no depth.²⁴ This stance is especially alien to the one taken in most PhilMath, which appears to cultivate exactly what LW is combating (cf. sections 2.0.3, 2.4.3(C) and Appendix 4.3 below).

It follows from these considerations that LW did not have any ambitions to construct a systematic **terminology** either: although LW did post hoc contribute a number of seminal terms to the standard philosophical jargon (Language Game, Form of Life, Grammar, Hurly-Burly, ...), only the term Language Game (and perhaps Grammar) could be argued to fulfill the function of a terminus technicus within LW's own work.²⁵

²² The issue appears to have been a real-life one for LW. For instance, Rush Rhees reports that LW in conversation confessed: "In my book I say that I am able to leave off with a problem in philosophy when I want to. But that's a lie; I can't." (Wittgenstein and Rhees 2015, 54); I am not so sure that Baker and Hacker's deflationary comment "but this was transforming a metaphor into a literalism" (Baker and Hacker 2005, 252) is actually to the point. The concept 'to stop talking' / 'to be silent' is mentioned elsewhere as well (e.g. *BPP2* §402, and of course *TLP* §7). Cf. also LW, Ms-127,82: Friede in den Gedanken. || Das ist das ersehnte Ziel dessen, der philosophiert.

²³ Already in *TLP* (6.53) it was stated that the only method in philosophy consisted in pointing out that no meaning had been attributed to certain words in a 'metaphysical' claim.

²⁴ Cf. Baker and Hacker's formula "The flatness of philosophical grammar" (Baker and Hacker 2009, 19–21). Cf. also *ÜG* §471: "Es ist so schwer, den *Anfang* zu finden. Oder besser: schwer, am Anfang anzufangen. Und nicht zu versuchen, weiter zurückzugehen", and *BPP1* §509: "Das psychologische Phänomen *nicht* erklären, sondern *hinnehmen*, ist das schwere".

²⁵ For an important remark on the terminology used in the present study, I refer to section 1.1.1(C).

(C) LW's anthropological approach: "observing math from the outside"

It has been said that LW's evolution between his early work, before he left philosophy for a number of years, and the classical later work, is mostly a shift away from a purely **logical** point of view and toward an **anthropological** point of view,²⁶ perhaps under the influence of Sraffa ((Mühlhölzer 2010), p. 392, quoting (Monk 1990), 260 ff.).

In the manuscripts studied here, LW explicitly states that he views math as an "anthropological phenomenon" (LW, Ms-124,116; cf. also Ms-162b,26v) and repeatedly says that he approaches mathematics "from the outside".²⁷

This has for an effect that certain aspects of math which may be trivial (and therefore invisible) from within, become highlighted and clarify in what respects and to what extent the mathematical phenomena are similar or dissimilar to other anthropological phenomena. LW is very much aware of the fact that this means that he is not actually speaking about the things that mathematicians and philosophers of math like to talk about.²⁸ This basic attitude runs against the mathematical exceptionalism (i.e. the idea that math is unlike other human endeavors) that is prevalent within PhilMath (cf. sections 2.4.3(D) and 3.2.3(A) below), but fits in with LW's basic critical stance. For an example that nicely illustrates all this, see my analysis of LW, Ms-124,115-119, in which LW attacks Gödel's 'slimy' concepts in terms of his own vision of math as an "anthropological phenomenon" (cf. section 2.3(E) below).

(D) philosophy as criticism and critique and the issue of LW's (anti-)revisionism

LW claims that his philosophy does not aim at criticizing language games and changing what people say ('non-revisionism', or 'anti-revisionism'); "*philosophy leaves everything as it is*" (PhU §124). Interestingly, LW refers in the same paragraph to the idea that philosophy cannot offer a foundation for everyday language either, which highlights the extent to which his

²⁶ From an approach "sub specie aeternitatis" to an approach "sub specie humanitatis" ((Gakis 2015) p. 928.). This coincides with the difference between logic and grammar: a grammar implies the details of how people actually speak and highlights the historical and cultural contingency and variation, whereas logic has the connotation of universality and a-temporality. Cf. also my account in terms of the shift from a semantic approach to a pragmatic approach to meaning.

²⁷ Cf. for instance:

- Wer das Wesen der Mathematik verstehen will, muß nicht aus ihrem Fenster heraus, sondern von außen hinein schauen. (LW, Ms-123,17v-18r, 19401116)
- Meine Aufgabe ist es nicht, Russells Logik von innen anzugreifen, sondern von außen. D.h.: nicht, sie mathematisch anzugreifen – sonst triebe ich Mathematik – sondern ihre Stellung, ihr Amt. (LW, Ms-124, 82)
- Die philosophische Betrachtung der Mathematik hat eine andere Pointe als die mathematische von math. Sätzen & Beweisen. (LW, Ms-161,63r)

²⁸ Meine Aufgabe ist es nicht über den Gödelschen Beweis (z.B.) | | , z.B., zu reden; sondern an ihm vorbei zu reden. (LW, Ms-124, 84)

engagement with the ‘Grundlagen-debate’ (cf. section 0.3 here below and section 3.1.3(B)) is intertwined with the very core of his later philosophy.²⁹

At the same time, a critical strand was present in LW’s philosophy from the beginning (e.g. *TLP* §6.53) and LW is often interpreted in such a way that he apparently does criticize certain ways of speaking, most notably certain typically philosophical uses of language. For instance, in *PhU* §116, LW appears to be critical of philosophical/metaphysical language, and explicitly contrasts the way in which “the philosophers” use words such as ‘knowing’, ‘being’, ‘object’, etc. and his own philosophical practice, which consists in bringing those words back to their *everyday* use.³⁰

Both strands in LW’s thought taken together immediately give rise to a paradox: if it is true that it is Wittgenstein’s aim in philosophy to ‘leave everything as it is’ and merely describe existing language games as they are, why doesn’t he seem to ‘leave alone’ a number of ‘non-ordinary’ (‘philosophical’ / ‘metaphysical’ / ‘theoretical’ ...) ways of using language? In other words: why does LW claim to want to ‘leave language games as they are’ and at the same time condemn a number of types of language use? Elsewhere ((Scheppers 2017), Ch. 3), I deal with the philosophical significance of this paradox for its own sake; for the purposes of the present study, I am mostly concerned with its impact on the interpretation of LW’s overtly critical remarks about certain types of mathematical discourse.

In the literature, we encounter two ways of dealing with this tension (if it is dealt with at all):³¹

- deflating the non-revisionist claim and accepting that LW actually claims that -say- set theory is not proper math (Steiner 2009; Maddy 1993);
- accepting non-revisionism as central to LW’s purpose and trying to interpret the apparent criticism in that light (Dawson 2015).

It is one of the main claims of this study that LW’s philosophy is primarily and pervasively critical, and that the same thing goes for his PhilMath (in actual fact, LW’s PhilMath is a very good example of this aspect of LW’s philosophy).

²⁹ *PhU* §124: “Die Philosophie darf den tatsächlichen Gebrauch der Sprache in keiner Weise antasten, sie kann ihn am Ende also nur beschreiben. Denn sie kann ihn auch nicht begründen. Sie läßt alles, wie es ist. [...]”

³⁰ *PhU* §116: “Wenn die Philosophen ein Wort gebrauchen – ‘Wissen’, ‘Sein’, ‘Gegenstand’, ‘Ich’, ‘Satz’, ‘Name’ – und das Wesen des Dings zu erfassen trachten, muss man sich immer fragen: Wird denn dieses Wort in der Sprache, in der es seine Heimat hat, je tatsächlich so gebraucht? – *Wir* führen die Wörter von ihrer metaphysischen, wieder auf ihre alltägliche Verwendung zurück.”

³¹ For the fact that most Wittgenstein-apologists with respect to his PhilMath either (1) avoid, ignore and/or explain away or (2) explicitly disavow LW’s more overtly critical rants and focus on a selection of what appear to be more technical lines of thought, see section 0.2(B) above.

(E) LW's conceptual framework (LW's pragmatism, LW's holism, LW's structuralism, LW's everydayism)

Besides the fact that LW's conception of the aims and methods of philosophy are quite different from most of the mainstream of PhilMath from his day up until now (2022), LW also operates with a conceptual framework that may not be immediately palatable to the reader.

Throughout Part 1, I will highlight a number of aspects of the conceptual framework that LW develops in his later work, and show how these aspects should inform our way of understanding LW's PhilMath.

- pragmatism: meaningfulness is ultimately a matter of embedding in practices, and practice is ontologically irreducible;
- holism and structuralism: practices are holistic structures, within which practical, linguistic, epistemic, biological, etc. dimensions cannot be reduced to each other;
- everydayism: the difference between everyday and non-everyday practices is a pervasive and fundamental one in LW's outlook.

(F) LW and his readers/commentators (conclusion)

The fact that LW's aims, methods and conceptual framework are very far removed from those of most people who work in the field of PhilMath (especially in the so-called 'analytic' Anglo-Saxon traditions) does not bide well for an easy reception of LW's work on mathematics. Furthermore, the fact that LW offers an 'anthropological' perspective "from the outside looking in" is directly at odds with the overtly exceptionalist attitude prevalent in PhilMath. It is my contention that many of LW's commentators have looked at LW's texts (often in a more or less edited format, that already aimed at making the texts more 'palatable') without necessarily taking the distance between LW's concerns and those of mainstream PhilMath into account and have tried to extract answers from the text to questions that were not necessarily relevant from the point of view of LW's own concerns. The omnipresent focus on -isms (finitism, constructivism, Platonism, normativism, ...) is a case in point: these stances are only relevant if one is interested in the questions that these -isms are supposed to be an answer to (cf. section 3.1.1 below).

0.3 Wittgenstein's philosophy of mathematics in the context of the Grundlagen-debates

The topic in PhilMath that attracted LW to philosophy in the first place ((McGuinness 1988) pp. 73-77) and that continued to be the backdrop for all of his work on mathematics is the issue of the foundations of math, heavily debated in the first part of the 20th century. All the

topics that LW dealt with (set theory, formalism, Gödel, applications, ...) are relevant to him insofar as they are related to this topic.

We should not forget that LW was not only present but was actively contributing when Bertrand Russell was still one of the major players in the field (though his most seminal work may have already been produced). By all accounts, LW's early contributions were taken very seriously by Russell, who appeared to have seen a worthy successor in LW, who would be able to continue the more technical work in logic and PhilMath that he felt himself less capable and/or inclined to pursue (cf. e.g. (McGuinness 1988), pp. 104-15 et passim). In this respect, it is interesting to observe that Kurt Gödel appears to have blamed LW's influence for Russell's leaving behind his earlier "epistemological" (i.e. correspondence-based) views on mathematics in favor of his 'classical' logic-based ones (Floyd and Kanamori 2016).

In the later works that we mostly focus on in this study, LW's main contribution was pointing out that the things that were being proposed as foundations, actually were not that in any real sense: according to LW, math in actual fact is not rooted in axiomatic systems, on the contrary. The roots (but this may not even be the right word in this context) of math are in the end not even propositional; historically/genetically as well as synchronically/structurally, math is rooted in (what was then called) 'applications',³² a heterogeneous, contingent and unstable set of practical activities, which existed long before math as a coherent body of knowledge. Another aspect is the fact that Grundlagen are typically presented as more simple, universal and somehow a priori necessary, whereas LW argues that complexity and contingency are irreducible.

Thus, in this study, I point out that LW's critique operates at a much more general level than the more technical issues that mainstream PhilMath (incl. most studies concerned with the exegesis of LW's PhilMath) focuses on, and that LW's stance antagonizes a number of different strands within mainstream PhilMath in a quite radical manner (radically anti-foundational; radically anti-unitarian/anti-monist; radically anti-teleological/'anti-naturalizing'³³).

³² Note that the term 'applications' presupposes that there is something pre-existing to be applied, which seems to be a bizarre choice if one is arguing that the applications are prior. This terminological choice shows to what extent LW's contribution is rooted in contemporary debates about the foundations of math: applications is simply the default word that was available in this historical context to express the idea of practical math-like practices.

³³ I have not been able to come up with a good term for the position that opposes claims to the effect that one's own ideological preferences are 'facts of nature'.

0.4 Structure

After the present introductory section “0”, this study is organized as follows:

1. *LW's philosophy of (mathematical) practice*, in which I show how LW's PhilMath, including some of its more controversial aspects, fits in with his philosophical approach at large and a number of ways in which it still can contribute to present-day PhilMath and PhilMathPract;
2. *LW's philosophy (of mathematics) as critique*, in which I show how the infamous critical remarks on set theory, various diagonal methods, Gödel, etc. actually fit in with some of the core concerns and agendas underlying LW's world view at large;
3. *Conclusions*, in which I summarize the different lines of thought that came up throughout the study and try to show how they are relevant to present-day Wittgenstein-scholarship as well as to PhilMath at large;
4. *Appendices*, in which I present materials that are directly related to the subject matter(s) dealt with in this study but go beyond the interpretation of LW's text.

Part 1. Wittgenstein's philosophy of (mathematical) practice

In the first part of this study, I show how LW's PhilMath is best understood as an integral part of his philosophical approach to meaning and practice at large:

- Section 1.1 gives an overview of some of the main features of LW's philosophical approach in general (his pragmatic account of meaning as embeddedness in practice, and his holistic and structuralist conception of practice) and how this informs a number of aspects of his PhilMath: his focus on 'applications' (incl. exotic ones), his opinions on formalism, the issue of the 'freedom' of pure math, etc. I also show how many of these lines of thought foreshadow much later developments in PhilMathPract.
- Section 1.2 discusses the role of the notion of the "everyday" in LW's work (incl. his PhilMath).
- Section 1.3 consists of a close reading of a longer passage that offers a good illustration of some of the lines of thought focused on in this study.
- In section 1.4 I briefly summarize the main lines of thought that I developed in Part 1.

The focus in this part of the study is not so much on the fine-grained exegesis of the Wittgensteinian corpus as on showing how some of the main features of LW's philosophy have not (not yet?) been picked up in present-day PhilMath (incl. PhilMathPract) and still could offer a powerful contribution to this field.

1.1 Wittgenstein's pragmatic approach to meaning: meaning as embedding in a practice

There is something to be said for the idea that 'meaning' is the central issue throughout LW's work (Rodych 1997; Floyd 2001):³⁴ the opposition sense vs. nonsense is a pervasive one throughout LW's works, and in Part 2 of this study, I will show how it is connected with the ethical, aesthetical and existential biases that underlie and motivate LW's philosophy as a whole.³⁵

³⁴ Cf. also the title *Understanding and Meaning* that Baker and Hacker gave to the first volume of their classic commentary of the PhU (Baker and Hacker 2005).

³⁵ However, there is also a lot to be said for the view held by Paul Horwich: "I have been arguing that, early and late, it is Wittgenstein's view of philosophy, rather than his view of meaning, that plays the pivotal role in his thought" (Horwich 2004: 105). Nothing hinges on it for the present purposes.

More precisely, LW's intervention regarding meaning coincides with the shift from semantics, as a correspondence-based approach to meaning, to a pragmatic approach to meaning, in terms of embedding in practices. It is not my aim to give a full account of LW's contribution to -say- the theory of meaning or even of LW's own conception on the issues in a technical way. But in order to be able to formulate what I have to say about LW's work on mathematics, and make this understandable to the reader, I will have to say something about these broader issues, though my account here will have to be very succinct and I can only briefly summarize a few lines of thought.³⁶

1.1.1 Wittgenstein's pragmatism

Here (as elsewhere), I use the adjective 'pragmatic' in the very general sense of "based in (the study of) practice" or "focusing on practice" and 'pragmatism' for any pragmatic approach. No reference to the historical American pragmat(ic)ism of Peirce, James, etc. is intended (though their choice for the term was of course not gratuitous either).

(A) LW's shift from a semantic approach to a pragmatic perspective on meaning

Semantic approaches to meaning -and most (if not all) traditional approaches fit in this category-³⁷ construe meaning as a matter of correspondence between several strata of structure. For instance, in linguistics, phonological structure (sound) is related to conceptual structure (depending on the theory one adheres to, passing through various layers of syntactic structure).³⁸ Relating linguistic structure to the real (or a virtual) world ("reference") is again a matter of correspondence.

³⁶ I also believe it should not be controversial to state that the critical bias that is omnipresent throughout LW's work and life (see here above) is often expressed in terms of meaninglessness (nonsense) or loss of meaning (see also section 1.2 below).

³⁷ Of course, pragmatics has its own traditional precursors in (for example) rhetoric, which also may have had an influence on argumentation theory, but these are not usually construed as theories of meaning and lack the direct link to core issues of logic and mathematics.

³⁸ For instance, in linguistic approaches to natural language, a **grammar** would be construed as follows: a phonological structure is linked with a semantic/conceptual structure, perhaps mediated by a morphosyntactic level; each of these levels is characterized by its own system of categorial distinction and its own combinatorial rules (its own syntax). For the sake of clarity, consider, for instance, the following differences/similarities:

- at the phonological level: cat / mat / sat / flat / begat / ... vs. flow / though / mow / doe ...;
- at the semantic level: cat / dog / bird / cow / ... vs. crayon / pen / pencil / marker / ...;
- at the morphosyntactic level: I beat my horse / The train reaches Antwerp / This paper concerns entomology / ... vs. the man on the moon / the best of the best / the road to the mountains /

For an account in the tradition of generative grammar that highlights the 'correspondence' aspect in a particularly clear manner, see for instance (Jackendoff 1985), (Jackendoff 2002))

What LW tries to introduce, is what would nowadays be called the **pragmatics** of natural language (cf. Mey (Mey 2001), (Mey 1998); for the links and differences between semantics and pragmatics, see e.g. Scheppers 2011 (Scheppers 2011), §13.1.4 and §13.2(1)). From a pragmatic point of view the following may be functionally similar: "Close that window!"; "It's cold in here"; "Do you want to kill me?"; [Speaker shuts the window himself].

LW started out with a particular type of correspondence-theory about meaning, which has been described as the ‘**picture-theory**’ about truth (Johnston 2017). For the LW of *TLP*, there was no meaning outside the proposition (which is defined as something that has to be either true or false).

Interestingly, the overall purpose of the *TLP* appears to be to push the correspondence approach to meaning (the “picture theory”) to its limits: to show that this approach gives a coherent account of the full subject matter of logic, but that this does not amount to anything of real importance, as LW explicitly states in Preface to the *TLP*:

Dagegen scheint mir die *Wahrheit* der hier mitgeteilten Gedanken unantastbar und definitiv. Ich bin also der Meinung, die Probleme im Wesentlichen endgültig gelöst zu haben. Und wenn ich mich hierin nicht irre, so besteht nun der Wert dieser Arbeit zweitens darin, daß sie zeigt, wie wenig damit getan ist, daß diese Probleme gelöst sind.³⁹

(Ts-202,IIr = *TLP*, ‘Vorwort’)

A similar idea, emphasizing the ultimately nonsensical nature of the approach taken in the *TLP*, reappears in the famous ‘ladder’ image at the very end of the *TLP*:

Meine Sätze erläutern dadurch, dass sie der, welcher mich versteht, am Ende als unsinnig erkennt, wenn er durch sie—auf ihnen—über sie hinausgestiegen ist. (Er muss sozusagen die Leiter wegwerfen, nachdem er auf ihr hinaufgestiegen ist.)⁴⁰

The main insight behind LW’s later philosophy appears to be that he let go of the picture-theory as his main account of meaning. Already in 1931, LW had let go of the notion of elementary proposition and started experimenting with the notion of ‘**grammar**’, in the sense of ‘a set of rules for the use of words’, as an account of meaning (Manninen 2011). The notion ‘Sprachspiel’/‘Language Game’ occurs first in the *Blue Book* (1933-1934) to refer to simplified games that LW introduces to make a specific point, but the denotation of the term soon enough expanded to cover real-life patterns, as well as imaginary, exotic, and/or counterfactual games. This is one of the few terms coined by LW that actually took on the function of a true terminus technicus within LW’s own work. In the literature, this view is

³⁹ On the other hand the *truth* of the thoughts communicated here seems to me unassailable and definitive. I am, therefore, of the opinion that the problems have in essentials been finally solved. And if I am not mistaken in this, then the value of this work secondly consists in the fact that it shows how little has been done when these problems have been solved.

⁴⁰ My propositions are elucidatory in this way: he who understands me finally recognizes them as senseless, when he has climbed out through them, on them, over them. (He must so to speak throw away the ladder, after he has climbed up on it.)

often summarized under the slogan “meaning is use”,⁴¹ but this is too reductive a summary to cover LW’s understanding of pragmatic meaning and a less narrow conceptualization of meaning -for instance, as ‘embedding in, or function within, an encompassing practice’, would be more appropriate.⁴²

(B) meaning as embedding in practice: Language Games, Forms of Life

In LW’s later work, the shift from semantics to pragmatics is marked by the introduction of a number new concepts/terms in his vocabulary. The most obvious and most systematically used of these terminological innovations is perhaps ‘**Language Game**’.

The **list of examples** in the famous paragraph *PhU* §23 shows what kind of things LW has in mind when he speaks of Language Games.⁴³ This list on its own already allows us to see that most of these behavioral patterns cannot be understood as strictly verbal/linguistic:

- some of the Language Games may be more or less purely linguistic: ‘reporting an event’, ‘telling a joke’, ‘translating’;
- verbal and non-verbal aspects are equally essential to ‘giving and obeying orders’;
- a number of these patterns do not necessarily imply language use at all: ‘construct an object by means of a picture’, ‘solve an applied math problem’, ‘play-acting’.

This observation already shows why the notion of Language Game can be viewed as a precursor of the notion of **practice** in general.⁴⁴

This list occurs in the paragraph in which the notion of Language Game takes its proper shape for the first time in *PhU*, and the way LW introduces this concept is very explicit about the point he intends to make. Let us try to reconstruct LW’s line of thought.

Starting point is the statement that an account of meaning in terms of a fixed number of sentence types (say: statements, questions, orders) should be abandoned in favor of an account in terms of large array of Language Games, that is furthermore subject to contingency,

⁴¹ Admittedly based on remarks by LW himself, such as *PhU* §43 (“Man kann für eine *große* Klasse von Fällen der Benützung des Wortes “Bedeutung” a wenn auch nicht für *alle* Fälle seiner Benützung a dieses Wort so erklären: Die Bedeutung eines Wortes ist sein Gebrauch in der Sprache.”).

⁴² The main problem with the concept of ‘use’ is that it suggests that the locus of the meaning is ultimately the ‘user’, whereas this is not compatible with what LW says elsewhere, perhaps most famously from the ‘private language argument’ (see section 1.1.2 below, on LW’s structuralism).

⁴³ Befehlen, und nach Befehlen handeln – Beschreiben eines Gegenstands nach dem Ansehen, oder nach Messungen – Herstellen eines Gegenstands nach einer Beschreibung (Zeichnung) – Berichten eines Hergangs – Über den Hergang Vermutungen anstellen – Eine Hypothese aufstellen und prüfen – Darstellen der Ergebnisse eines Experiments durch Tabellen und Diagramme – Eine Geschichte erfinden; und lesen – Theater spielen – Reigen singen – Rätsel raten – Einen Witz machen; erzählen – Ein angewandtes Rechenexempel lösen – Aus einer Sprache in die andere übersetzen – Bitten, Danken, Fluchen, Grüßen, Beten.

⁴⁴ It may be useful to remind ourselves at this point that LW had no ambition to develop a systematic terminological apparatus.

variation and change.⁴⁵ In other words, LW no longer tries to account for the difference between meaningful discourse and nonsense in terms of the relation between a proposition and reality, but in terms of the relation between an utterance and the practice in which it occurs. Of course, once one has taken this step, one has to give up the hope to be able to reduce the possible ways for an utterance to be meaningful to a few universal types of sentences.

It is remarkable that LW chooses **mathematical practice as a good example** of the historical contingency of practices. One wonders whether this is a deliberately provocative move: while historically and ‘anthropologically’ correct, it goes against the grain of what mainstream mathematicians and philosophers of mathematics seem to think of math.

Interestingly, LW explicitly states that the very purpose of the word Language Game is to highlight the **link with Form of Life** (cf. also Whiting 2017).⁴⁶ It is suggested that language is a mere part of something more encompassing, that LW first calls ‘an activity’ and, in immediate apposition to ‘activity’, a Form of Life. It is as if he uses the term Form of Life to correct the use of the word ‘activity’, as if ‘activity’ was still not general enough a term. The point is that meaning cannot be reduced to a language-internal, local phenomenon, but should be approached holistically (see section 1.1.2 below).⁴⁷ The extensive list of examples of Language Games helps highlighting both the practical aspect and the heterogeneity. The paragraph closes by contrasting the ‘new’ approach with traditional logic (including the *TLP*), in which only propositional (true or false) sentences were considered relevant.

⁴⁵ Wieviele Arten der Sätze gibt es aber? Etwa Behauptung, Frage und Befehl? – Es gibt *unzählige* solcher Arten: unzählige verschiedene Arten der Verwendung alles dessen, was wir »Zeichen«, »Worte«, »Sätze«, nennen. Und diese Mannigfaltigkeit ist nichts Festes, ein für allemal Gegebenes; sondern neue Typen der Sprache, neue Sprachspiele, wie wir sagen können, entstehen und andre veralten und werden vergessen. (Ein *ungefährtes* Bild davon können uns die Wandlungen der Mathematik geben.)

⁴⁶ Das Wort »Sprachspiel« soll hier hervorheben, daß das Sprechen der Sprache ein Teil ist einer Tätigkeit, oder einer Lebensform. Führe dir die Mannigfaltigkeit der Sprachspiele an diesen Beispielen, und anderen, vor Augen:
[hereafter follows the list of Language Games already quoted above].

– Es ist interessant, die Mannigfaltigkeit der Werkzeuge der Sprache und ihrer Verwendungsweisen, die Mannigfaltigkeit der Wort- und Satzarten, mit dem zu vergleichen, was Logiker über den Bau der Sprache gesagt haben. (Und auch der Verfasser der Logisch-Philosophischen Abhandlung.)

⁴⁷ Baker and Hacker’s commentary on *PhU* §23 includes the following observation: “It is unclear what principle of classification (if any) is employed. It is not obvious, e.g., that requesting and thanking, which are speech-acts, are on the same level as forming and testing a hypothesis or as acting on-stage, which are not.” ((Baker and Hacker 2005), p. 87). Searle’s notion of ‘**speech act**’, which despite its currency in philosophical circles has never been a very fruitful way of analyzing actual discourse ((Mey 2001), pp. 212–217), evidently cannot be projected onto LW’s thought and is not only subject to a criticism very similar to the one LW applies to ‘sentence types’, but also is unable to account for the hierarchical nature of intentionality and for the similarity -or rather identity- of the role of intentionality in the case of verbal behavior and non-verbal behavior. For an approach to discourse coherence and pragmatic ‘sense’ that is more in tune with the Wittgensteinian lines of thought analyzed here, see (Scheppers 2003).

(C) terminology

Some of LW's coinages have come to have a life on their own in the secondary literature, though they do not necessarily fulfill a singular terminological function with LW's work. Perhaps the most obvious quasi-technical term in LW's later work is Language Game / Sprachspiel (Language Game). The term that most famously covers the notion of structural patterns beyond the simple Language Game and in the literature has been treated as a quasi-technical term and a key concept in LW's approach, despite its very low frequency in LW's oeuvre, is 'Lebensform'/'Form of Life' ((Baker and Hacker 2009, 218–23); (Hacker 2015); (Moyal-Sharrock 2015). The word 'Lebensform' as such occurs only 7 times in LW's writings as represented in the standard editions (*PhU* §19, §23 and §241; *PhPF* i §1 and xi §345; *BPhP1* §630; *ÜG* §358).⁴⁸

We might be tempted to try and demarcate/define LW's key concepts with respect to each other by saying things like: Form of Life designates larger scale and/or more general patterns than Language Game, or: Language Game denotes formal linguistic patterns, and Form of Life more holistic entities. This also appears to be the way in which these terms live on in the literature. But in fact, LW's ways of relating Language Games and Form of Life fluctuate a lot. A first aspect concerns the relative complexity of the patterns referred to: some passages suggest that a Form of Life is something 'larger' or 'more encompassing' than a Language Game (cf. *PhU* §19, *PhU* §23, *PhU* §§240–242); in other passages, more small-scale or limited patterns are also called Forms of Life: being able to speak or not (*BPhP1* §630), the fact that we assume that certain physical aspects of the world remain stable (*PhPF* xi §345), a certain kind of certainty, expressed through the 'I know' game (*ÜG* §358).

The term Form of Life accordingly refers to patterns that coincide with a language as a whole, or the way of life of an entire community/culture, as well as to patterns that are not 'larger' than a single Language Game. In the above we have seen that LW uses the term Language Game to denote a number of practices that do not even have to involve language.

For the purposes of this study, I have therefore chosen to deviate from LW's own terminological practice and use the term "**practice**" as an **umbrella term** for many different terms within LW's text (including the very common "Language Game").⁴⁹ When there is no

⁴⁸ Baker and Hacker add two cases from the manuscripts that do not appear in the standard editions, plus a reference to *Philosophical Occasions*, but do not mention *BPP1* §630 (Baker and Hacker 2005:74–75; see also more recent contributions, e.g. by P.M.S. Hacker (Hacker 2015) and Anna Boncompagni (Boncompagni 2015), for a few more relevant excerpts from the Nachlaß.

⁴⁹ Let me point out that this is not that different from using the Wittgensteinian term Form of Life as a common terminus technicus when discussing LW's work: Form of Life /Lebensform was not used systematically, as a terminus technicus, by LW either.

emphasis on agency or activity, I will sometimes also use terms such as ‘Forms of Life’ or ‘our lives’, loosely, not unlike LW’s own use of these terms.

(D) LW’s philosophy of mathematical practice: mathematical practices embedded in applications / Forms of Life

So, within the context of the Grundlagen-debate, it is one of LW’s most significant moves to attract the attention away from axiomatic ‘foundations’ to actual practices involving operations such as counting, measuring, drawing, proving, etc. Many times, LW also insists on applications at a very practical level, in the context of everyday buying and selling things, building things, engineering, accounting, graphic design, music. Any reader of the published later works (say: *PhU*) recognizes this as one of the most typical Wittgensteinian moves, and many of the standardly quoted examples involve math-related activities. Thus, *PhU* §1 already involves the example of buying five red apples and the use of number words in various practical contexts continues to be a recurrent example. Various types of measuring and calculating prices as part of buying and selling also are recurrent examples, as are examples related to engineering applications (building houses, bridges, machines, etc.).

LW often goes to (relatively) great lengths to evoke the details of the practice, thus e.g. in the case of the sometimes rather exotic ⁵⁰ measurement procedures in *BGM* 1, §§143-151. In this passage, LW makes us consider various ways in which **woodsellers** could determine the price of timber:

- by piling up the wood, measuring height, width and length of the pile, multiplying the outcomes of the three measurements and calling the outcome of the multiplication the price in pennies;
- by piling up the wood and measuring the surface it covers;
- LW offers some alternative scenarios: pricing by weight, by labor time or by effort (take the age and skill of the woodsman into account!), or simply giving the wood away.⁵¹

In this case, LW is quite explicit about some of the points he wants to make:

- he insists on the fact that what these people do is part of a practical context (the buying/selling of wood for the purpose of building a house): the measurements and

⁵⁰ For LW’s practice of conjuring up imaginary practices, see section 1.3 below.

⁵¹ Comparing the standard edition of this passage with Ms-118,33v-36r would -by the way- be a good way to illustrate the impact of the editorial practices of LW’s literary executioners: the published text does look a lot more continuous and smooth (the editors took out LW’s many self-interruptions), but we also lose the connection with the more abstract issues that LW tried to come to grips with by means of this example, as well as the sense of intense involvement on the part of the author. The addition of “151. (A society acting in this way would perhaps remind us of the Wise Men of Gotham.)” in the standard edition, was taken from a previous version of this text (MS-117) and it remains to be seen whether it was a good idea to compile the two versions the way the editors did: referring to legendary fools in this context does make a difference for how one reads the text.

calculations (if that is what they are) are a way to determine what the builder needs to pay to woodsalesman for his timber;

- he points out that the propositional nature of equations (which some of us might be tempted to consider essential to calculation) need not play a role in such practices at all;
- it is suggested that what counts as a correct calculation of a price (which criteria are considered relevant, e.g. the surface covered by the pile, the weight of the wood, the age of the woodsman, ...) depends entirely on how the calculation fits in with its wider practical context and that there is no a priori way to know what would count as correct/acceptable outside an actual practical context.

NB: none of the scenarios that LW imagines is so outlandish that it could not be found in the historical or ethnographic record.

(E) pragmatics first vs. the primacy of truth and propositional knowledge

A pragmatic account of meaning implies that *meaning in the case of linguistic behavior* (Language Games in the more literal sense of the word) is conceived of as entirely *parallel to meaning in non-communicative activities*.⁵² This appears to be one of the recurring themes in LW's later work and is already evident if you look at the list of examples at the beginning of *PhU* quoted above.

Note that this is contrary to the assessment that LW was mainly a philosopher of language. Of course, LW was dealing with philosophy and logic as the main topic of his work, and philosophy is essentially a verbal endeavor, which means that he is mainly focusing on things that are essentially of a linguistic nature. But his main contribution to the study of language is clearly that he showed that meaning is not primarily a matter of language: *what is meaningful about language is not primarily linguistic*.⁵³

A well-documented aspect of LW's intervention in the history of philosophy is that the 'pragmatic first' idea displaces the primary status of **truth**. LW's later work documents the realization that conveying information in such a way that the truth of it is crucial to its meaning, is only one among many ways in which language can function; in other words, truth

⁵² In a completely different context, I have been arguing for the importance and correctness of the Pragmatics First claim (including the parallelism between communicative/linguistic meaning and non-linguistic/non-communicative meaning), which I here attribute to LW, since the 1990s (most important publications (Scheppers 2003) and (Scheppers 2011), but see already (Scheppers 1997)).

⁵³ LW's later philosophy was *wrongly* understood as a 'philosophy of language' from the beginning, as becomes clear from the following report of LW's lectures in the years 1930-1933 by G.E. Moore: "[...] he held that though the "new subject" must say a great deal about language, it was only necessary for it to deal with those points about language which have led, or are likely to lead, to definite philosophical puzzles or errors. I think he certainly thought that some philosophers now-a-days have been misled into dealing with linguistic points which have no such bearing, and the discussion of which therefore, in his view, forms no part of the proper business of a philosopher." (G. E. Moore 1955, 27)

is now seen as only one of the possible effects (and in some cases only a side-effect) of the way language works.

In the prolongation of that line of thought, lies the notion that at the bottom of propositional (i.e. truth-based) constructions, there is **non-propositional**, even not propositionalizable, stuff. Hence LW's persistent focus on phenomena like understanding gestures and music, physiognomy (recognizing faces and facial expressions) as a paradigm for meaning (see also paragraph (F) here below and section 2.0.2 below). This is one of the key notions that we have to keep in mind when we try to come to grips with LW's PhilMath.

It seems that LW's emerging practice-based account of meaning, or at least the way this is expressed in his writings, is often mistaken for other "-isms". To make my point as simply as I can: *LW's agenda is to show that meaning need not be a matter of correspondence, the other lines of thought are subordinate to this main move.* Even in those cases in which language does refer to things out there, this still is always and irreducibly mediated by the practice which serves as the immediate context for this reference, as well as the encompassing structures (Form of Life, ...) of which this practice is a part.

(F) LW's shift from a semantic to a pragmatic account of meaning and the interpretation of his remarks on mathematics

I believe an adequate understanding of how exactly LW's view on meaning in general shifted from a semantic one to a pragmatic one helps account for a few apparently problematic issues concerning his (also evolving) views on mathematics.

LW's apparent **normativism in PhilMath** is a case in point: at a certain stage of his development, LW experimented with the idea that mathematical utterances, like the sentences of logic (which already in the *TLP* were shown to be tautologies and therefore not really meaningful) are grammatical sentences, parts of a grammar, a set of rules for the use of words ((Frascolla 1994); (Rodych 2011)). This intermittent but long-lasting experiment with this **concept of grammar** as an account for meaning should be understood within his even longer-lasting and more fundamental concern with non-propositionality, starting with the realization in the *TLP* that logical (as well as ethical and aesthetical) sentences are tautological and therefore by definition transcendental.⁵⁴

⁵⁴ If you define the world as the set of all facts and facts are the referents of propositions (defined in terms of binary truth values), then it follows that tautologies do not refer to the world, i.e. are by definition transcendental. Cf. also LW's use of the term 'mystical' to refer to the non-propositional (Breitenbach 2008 (Breitenbach 2008)); while this term does make sense in the context of *TLP*, in which propositional language is considered the only type of meaningful language, his later work is characterized by the realization that our everyday use of language makes sense without being rooted in propositional truth, and from this point of view, it would be strange to call all of these non-propositional aspects of our everyday lives 'mystical'.

One interesting instance of **the non-propositional underlying the propositional** is the way we should view **axiomatic systems** from this perspective, which also impacts our view on how natural language works and how mathematics works: the function of axiomatic systems within mathematics is completely displaced, in that it can no longer be seen as a foundation, on the contrary, formal math is now seen as grounded in everyday practices such as counting, measuring, selling and buying, building, etc. rather than the other way round (see also section 2.3 below, in which we analyze passages in which LW explicitly argues against the foundational function of axiomatic systems).

(G) LW's pragmatic account of mathematical meaning vs. model theory, proof theory, etc.

The move towards a pragmatic account of meaning also opposes LW to the prevailing views of meaning in math, which seemed to have moved in the opposite direction in the course of the 20th century:⁵⁵ instead of emphasizing those aspects that math has in common with other meaningful human behavior, there has been an ongoing tendency for math to try and incorporate more and more aspects of its own functioning within its own formalism: model theory, category theory, proof theory are supposed to somehow express the self-understanding of math, by turning key aspects of its own functioning into mathematical objects.

Thus, formal **proof theory** (as part of mathematical logic) views mathematical proofs as mathematical objects, the features of which can be studied by mathematical means. However, from a pragmatic and/or anthropological point of view, this approach has the obvious disadvantage that mathematical proofs are no longer studied qua proofs, in the normal 'human' sense of the word: as human actions that intend to convince one of the truth of some claim. Thus, LW's work on proofs asks basically the following question: how can something that does not convince a human mathematician of the truth of a claim be considered a proof? Other aspects of LW's work on proofs (e.g. his insistence of surveyability etc.) can readily be understood from this point of view. This part of LW's PhilMath has been thoroughly covered in the literature (cf. for instance Felix Mühlhölzer's monumental book (Mühlhölzer 2010); see also (Mühlhölzer 2006), (Floyd 2001)); suffice it here to point out that LW's opinions on the matter of proof are clearly related to his basic 'pragmatic' stance: the meaningfulness of something equals the way it functions within the practice(s) in which it actually functions.

Similarly, **model theory** is an attempt to capture the meaning of formal theories in mathematics in terms of their relation with the mathematical structures for which their

⁵⁵ Interestingly, the 'pragmatics first' aspect of LW's approach also is completely different from Carnapian pragmatics, and from the way pragmatics is still construed in most grammatical approaches within linguistics.

statements are true. Again, this approach moves in the opposite direction from LW's anthropological/pragmatic approach: whereas model theory tries to incorporate the semantics of mathematical formalism into mathematical formalism itself, LW emphasizes the fact that the meaningfulness of any mathematical entity ultimately depends on how it is embedded in 'our lives'. Whatever model theory may achieve otherwise, it will not be able to capture the fact that the meaningfulness of mathematics (or of items within mathematics) is fundamentally similar to meaningfulness in general, i.e. as it applies to any other human endeavor.

(H) LW on "dead signs" and "mindless calculation"

A very similar idea is expressed in terms of the difference between live and dead signs.⁵⁶ One of the early occurrences of this idea in LW's work is in the seminal passage towards the beginning of the so-called *Blue Book*, in which we also find an instance of the idea that meaning is use:

Ts-309 [*The Blue Book*, 1933–1934],6-7:

Frege ridiculed the formalist conception of mathematics by saying that the formalists confused the unimportant thing, the sign, with the important, the meaning. Surely, one wishes to say, mathematics does not treat of dashes on a bit of paper. Frege's idea could be expressed thus: the propositions of mathematics, if they were just complexes of dashes, would be dead and utterly uninteresting, whereas they obviously have a kind of life. And the same, of course, could be said of any proposition: Without a sense, or without the thought, a proposition would be an utterly dead and trivial thing. And further it seems clear that no adding of inorganic signs can make the proposition live. And the conclusion which one draws from this is that what must be added to the dead signs in order to make a live proposition is something immaterial with properties different from all mere signs.

But if we had to name anything which is the life of the sign, we should have to say that it was its *use*.

The idea is that mathematics requires a non-mechanical interpretation/use, in exactly the same way that the mere manipulation of forms (i.e. without any verbal semantics) is not logic,

⁵⁶ The image is not limited to LW and scholars influenced by LW (e.g. Mühlhölzer's "On Live and Dead Signs in Mathematics" (Mühlhölzer 2014)), but is pervasive in all corners of the literature, see e.g. (Ferreirós 2016): p.42: "formal systems come to life"; p.45: "maths are not cold and bloodless"; (Livingston 2015) p. 204: "When professional provers read mathematical argumentation, they seem to always, unavoidably, seek to find and maintain the association of the text with proving's lived work. If they are not doing this, they are not fully engaged in the professional practice of doing mathematics."

A very similar idea is expressed in terms of "emptiness", see e.g. Dieudonné's expression "mathématiques vides" in the title of his article "Mathématiques vides et mathématiques significatives" (Dieudonné 1982), and "les mathématiques non motivées ou le délayage" in the body of the text; in this article, the expression does (of course!) not refer to formalism in math as such, but to the fact that a large part of what is being published by professional mathematicians is not motivated by any genuine interest in anything genuinely mathematical. Timothy Lampert, admittedly acquainted with LW's work and admittedly in a preprint, dares to use the astounding formula "sound but empty proofs" (Lampert 2017).

qua model of human reasoning. By stating that “no adding of inorganic signs can make the proposition live”, LW appears to directly attack the model-theoretic approach, which does exactly that.

A similar, if not the same, idea is developed in MS-126,30-32, d.d. 19421028, a.k.a. *BGM V*, §2, in which LW asks whether the purely formal, mindless, manipulation of signs, whether by a machine or by drilled humans, counts as calculating.⁵⁷ And his answer to the question is straightforward and completely consistent with the rest of his account of meaning (in math and in general), as analyzed above:

[...] it is essential to mathematics that its signs are also employed in civilian clothing. It is the use outside mathematics, and so the meaning of the signs, that makes the sign-game into mathematics.

In the next few pages, LW explores in great detail the idea that people might use calculating machines for a wide variety of purposes, even if they have no understanding of math whatsoever. It is important to understand that LW does not suggest that the difference between meaningful and meaningless depends on the involvement of a psychological subject, which would go directly against the holism expressed in e.g. his private language argument (see below). What secures meaningfulness is embedding in a practical, everyday situation. A normal human form of life -of course- does imply an agent with a number of psychological attributes, but the ‘subject’ is not the proper locus for meaning. This is one of the points that I want to cover in the following section.

1.1.2 Wittgenstein’s holism and structuralism about practices and Forms of Life

Anti-reductionism is a well-recognized feature of LW’s approach. Practices (incl. Language Games etc.) and Forms of Life are the embodiment of this anti-reductionism, in that they are multi-dimensional and resist reduction to any of the dimensions contained within them. I distinguish two separate aspects:

⁵⁷ Rechnet die Rechenmaschine?

Denk Dir, eine Rechenmaschine wäre durch Zufall entstanden; & nun drückt Einer durch Zufall auf ihre Knöpfe (oder ein Tier läuft über sie) & sie rechnet das Produkt 25×20 . –

Ich will sagen: Es ist der Mathematik wesentlich, daß ihre Zeichen auch im Zivil gebraucht werden.

Es ist der Gebrauch außerhalb der Mathematik, also die Bedeutung der Zeichen, was das Zeichenspiel zur Mathematik macht.

So wie es ja auch kein logischer Schluß ist, wenn ich ein Gebilde in ein anderes transformiere (eine Anordnung von Stühlen etwa in eine andere) wenn diese Anordnungen nicht außerhalb dieser Transformation einen sprachlichen Gebrauch haben.

Aber ist nicht das wahr, daß Einer, der keine Ahnung von der Bedeutung der Russellschen Zeichen hätte, Russells Beweise nachrechnen könnte? | | der nichts von der Bedeutung der Russellschen Zeichen wüßte, die Russellschen Beweise nachrechnen könnte? Undalso in einem wichtigen Sinne prüfen könnte ob sie richtig seien oder falsch?

- holism: various dimensions that are relevant to a domain are considered parts of a single system, rather than viewing them as external to each other;
- structuralism: the identity of various items within a system is considered in terms of their place within the system, rather than as prior to the system.

In the context of the present study, I cannot show that -or rather to what extent- the 'ontological' picture that I sketch below accurately represents LW's own views, but I do want to claim that the picture emerges naturally from LW's text and for the purposes of this study, the following, admittedly somewhat blunt and dogmatic presentation will have to do.

Beyond the matter of Wittgenstein-exegesis, I will also insist on the ways in which LW's holistic and structuralistic conceptualization of practice, which have not been picked up in present-day PhilMathPract, could still contribute to current issues.

(A) holism: practices as multidimensional structures

The way LW speaks of Language Games and Forms of Life already shows that he conceives of them as multidimensional. Elsewhere ((Scheppers 2017) §4), I distinguished the following dimensions:

- a linguistic dimension: obviously, LW insists a lot on the way words are used in the context of various practices, if only because his subject matter is philosophy, a mostly verbal kind of practice;
- a pragmatic dimension: what an agent does within the framework of a practice, is an obviously relevant aspect of it;
- a social/cultural/historical dimension: LW often makes his readers picture various exotic populations or primitive cultures so as to highlight the variability and contingency that underlies our forms of life;
- a mental/cognitive/psychological dimension: what agents perceive, think, feel, etc. are major aspects of how they experience practices;⁵⁸
- a biological dimension: our biological constitution (e.g. the fact that we have hands, eyes, ... etc.), different from the constitution of other species, is a major factor in how our form(s) of life evolved (cf. LW's occasional comparisons with dogs and lions or the evocation of societies in which everyone is color blind, ...);

⁵⁸ LW's main points about psychology (not only in the works that have been edited under such titles as *Bemerkungen über die Philosophie der Psychologie* or *Letzte Schriften über die Philosophie der Psychologie*, but also in the famous 'private language argument' in *Phil*) intend to show how psychological aspects cannot be understood outside the context of the practices within which their expression occurs, but the structural relation (see below) goes both ways: psychological aspects are an integral part of the practices in which they occur as well.

- a physical/material dimension: the physical properties of our world are obviously constitutive of our forms of life (e.g. counting would not have been as important if the objects we would encounter in daily life had had less stable identities, like clouds; measuring would not be as viable if the materials at our disposal changed shape more readily; weighing cheese would not make sense if cheese expanded and shrunk; without apparent cause (*PhU* §142); etc.);
- an epistemic dimension: although the net intended result of LW's contribution may have been to give epistemic aspects (knowledge, truth, certainty, ...) a much more peripheral role than they have in the philosophical tradition, especially in fields like PhilMath, epistemic considerations continue to play a role in LW's analysis of certain types of language use.

Thus, LW's approach foreshadows a number of developments that occurred much later in the development of some branches of PhilMath or initiated outside PhilMath and turned out to be philosophically relevant, most notably the realization that mathematics shows other philosophically relevant aspects besides the epistemic ones (cf. section 3.2.1(C) below).

(A1) holism at work (1): practice-based vs. agent-based vs. community based;

In a large part of the philosophical tradition, the default locus for meaning has been the subject. In many avatars of the practice-turn, the subject is merely revamped as the agent and continues to function in exactly the same way as the subject did. One of the advantages of this version of the practice turn is that it remains compatible with various more or less common-sense versions of -what remains basically- reductionism/physicalism/naturalism: because the subject/agent coincides with a biological organism, we can continue to entertain the idea that science will at least in principle be able to deal with whatever we want to describe in terms of practice, in a unified manner, i.e. in terms of biology, chemistry, physics.

As far as practice-based accounts of mathematics go, this is the point of view adopted, for instance in José **Ferreirós'** primer *Mathematical Knowledge and the Interplay of Practices* (Ferreirós 2016), in which practice-based means mostly agent-based.

However, for both practical/empirical reasons (complex practices can obviously not be reduced to the agency of a single human agent) and more theoretical reasons (cf. the 'private language argument'), most authors quickly come to the conclusion that practices cannot directly be reduced to individual agents. In that case, '**the community**' is often what is invoked as an ersatz-subject. Kripke (Kripke 1982) -and in his defense of Kripke, Martin Kusch (Kusch 2006)- attribute this vision to LW. However, this use of the notion of 'community'

retains the disadvantages of reductionism (because it *is* a form of reductionism) and is not viable as an interpretation of LW's work:⁵⁹ the foundationalism and reductionism of Kripke's way of invoking 'community consensus' as the ultimate ground for meaning is incompatible with LW's holism, as embodied by -amongst others- the concept of Form of Life, as well as the anti-foundationalism he demonstrates throughout his work on math.⁶⁰

Furthermore, there are **methodological objections** to simply positing communities as prior to practices,⁶¹ as well as empirical ones. If a relatively fixed group of individuals share many different aspects of life, as in nuclear families, or a remote village, perhaps an office in which colleagues work closely on a daily basis; then it seems completely natural to speak of a community, but it seems arbitrary to speak of the "community of mathematicians", let alone even larger and/or more heterogeneous collections of people that have very few real contacts with each other and/or very little in common.

Thus, we should ask at least the following simple questions. In what sense is mathematics actually based in an actual 'community'? Is this always the case? In what sense are the interactions between mathematicians that we observe stable and intensive enough to be qualified as 'communities'? Are there perhaps historical cases in which we can see actual communities at work and cases in which we can't? Etc.

A case in point example would be Netz's work on **the social aspects of Ancient Greek math** (Netz 1999) (Netz 2009). If Netz is right about Ancient Greek math, Ancient Greek elite mathematics was not based in communities in any real sense at all (whatever your interpretation of 'community' may be), in that most mathematicians worked (or is it 'played'?)

⁵⁹ Severin Schroeder comes to the same conclusion ((Schroeder 2021), §7.1 'Rule-following and community').

⁶⁰ See also Floyd (Floyd 2021) p. 56, for basically the same argument.

⁶¹ As a matter of **method**, if communities are *a priori* considered prior to practices, this *parti pris* will inform the way we will construe the basic phenomena, in that it will be impossible to even perceive human interactions that cannot easily be construed under this label. Of course, if it can be shown that certain practices in certain cases coincide with, or depend on, or give rise to, etc. the existence of a community (under some definition of that term), then this would be a noteworthy empirical fact. For instance, if you do find a community somewhere, for instance a group of monks living together in an abbey and practicing math together, then that would be an interesting result, especially if you can -for example- show that their math is different from the math done by other contemporary networks or groups. The notion of 'community' would be especially instructive if you can show that what makes their math distinctive is related to other distinctive features of other communal practices of theirs. That would be an interesting result.

But that is not at all the same thing as positing a priori that the social aspect of human practices is always supported by a community, let alone that meaning ultimately is a matter of consensus in a community, as is often presupposed by both Wittgenstein-scholars (Kripke (Kripke 1982) and many others) and philosophers of mathematical practice ((Ferreirós 2016)). Therefore, it is not correct to equate practice with community from the outset: this would tend to make both terms ('practice' and 'community') almost void, and would preclude a proper analysis of the actual role of communities (if any) in concrete cases.

The concept of 'community' as the somehow natural locus of human interaction is not politically/ideologically neutral either: the suggestion that communities are the 'natural' (with the connotation of 'desirable') way for people to experience their social nature, has strong right-wing, anti-humanist implications.

Alternative conceptualisations, without the undesirable connotations of 'community', are available. See, for instance, the notion of 'nexus of practice' in (Scollon 2001) and (Scollon and Scollon 2003).

in complete isolation, as far as real-life interaction goes (in some periods there simply may have been no mathematician active in the whole Greek world).⁶²

For the present Wittgenstein-related purposes, it suffices to point out that projecting the idea that communities could be construed as the ultimate locus of meaning does not make much sense if we consider LW's anti-reductionist (holist & structuralist) implementation of the notion of practice.

(A2) holism at work (2): practice-based vs. knowledge-centered

In a similar fashion, holism about practice also displaces **the status of propositional knowledge** somewhat to the periphery of practices, as compared to the center-stage role knowledge always has played in the philosophical tradition, especially in PhilMath.

Mainstream approaches to PhilMath are all basically epistemological, i.e. in these approaches, math is viewed as essentially a body of propositional contents, and mathematical practice is viewed as of only marginal interest to the understanding of the subject matter, at best. This epistemological emphasis takes several avatars, which do not all have the same consequences, but have in common that the propositional contents, whether they are construed as human knowledge or as objects independent of human cognition, are considered as essentially independent from human practice.

LW's holism about practice, as applied to mathematics, opposes this epistemological bias on a very fundamental level. The idea that mathematical practice may have some philosophical relevance has become acceptable in at least some parts of the academic landscape, but almost invariably the underlying research agenda, the main issues about mathematics that one is interested in, continue to be formulated in entirely epistemological terms.⁶³

In the context of LW's pragmatic view of meaning and his holistic conception of practice, propositional knowledge loses its self-contained status, in that its meaning is now construed as a matter of how it functions within practices and how it is embedded in our forms of life, in the most general sense of this term (i.e. including the physical, biological, and cognitive aspects, but also the highly variable, contingent, historical, cultural and social aspects).

In the texts that we focus on for the purposes of this study, we read several passages in which LW goes as far as pointing out that one can easily imagine instances in which activities that

⁶² Of course, Netz may be fundamentally underestimating the permeability of mathematics, i.e. overestimating the impermeability of the boundaries between mathematics as a 'ludic', 'elite', intellectual endeavor on the one hand, and applied geometrical and arithmetic (accounting) practices on the other: the elites that could afford to engage in ludic (i.e. non-professional, non-applied) math were typically also landowners and must have had at least a passive acquaintance with professional geometry (in the etymological sense of the word) and accounting.

⁶³ The title of Ferreirós' *Mathematical Knowledge and the Interplay of Practices* (Ferreirós 2016) is emblematic in this respect.

are basically indistinguishable from normal calculations, can perfectly do without any propositional knowledge being involved at all (see *BGM* 1, §§143-151, quoted in section 1.1.1(D) above; cf. also section 1.3 below for a number of potential applications of math-like techniques in which propositional truth need not play a role at all). For instance, in the context of cutting wood to size for carpentry, calculation-like techniques need not at all involve strings of signs that are evaluated for truth, as long as one distinguishes between doing it right and doing it wrong. It follows that the status of knowledge within mathematical practice should be an empirical issue, especially in the present ‘naturalist’ era. The question should be: when and how does knowledge and truth play a role in mathematical practice?

In any case, LW’s work does not take the propositional/epistemic nature of mathematics for granted and in all seriousness asks the question as to what the exact role of mathematical propositions is amongst the other aspects that make up mathematical practice.⁶⁴ My claim is that within a coherently practice-based account, the role of knowledge will ipso facto be displaced as compared to an epistemological account: one can’t coherently think in terms of ‘practice’ and then construe knowledge as external to the practice (nor vice versa, by the way), one can’t coherently attribute an autonomous ontological status to knowledge, independent of the practice in which it occurs. So my Wittgensteinian criticism of the way ‘practice’ is construed in most of *PhilMathPract*, is one of lack of internal coherence, not one of lack of Wittgensteinian orthodoxy.

(B) structuralism: identity as irreducibly relational

I show elsewhere ((Scheppers 2017), Ch. 1) that the only charitable reading of LW’s multidimensional account of practice implies that the relation between these dimensions is structural, internal to the structure of practices (otherwise the practice-based account would be vulnerable to the same objections as any other reductionist approach), which in its turn implies that none of these dimensions can be primary with respect to the other dimensions. This aspect is not clearly understood in mainstream, especially ‘naturalist’, versions of the practice turn, or has at least not (not yet?) been picked up. Again, this topic, in its most generic ‘ontological’ form, is as such not directly relevant to the subject matter of the present study (which is why I will not argue for it here at any length), but it has consequences for the interpretation of a few notable aspects of LW’s *PhilMath* (which is why I have to briefly mention it).

⁶⁴ Questioning the primacy of purely epistemic issues has been proven extremely fruitful in Philosophy of science and there is no reason why it should not be equally fruitful in *PhilMath*.

structuralism at work: "we don't know the meaning of a theorem unless we know the way to prove it"

One of the more controversial among LW's claims has always been that the meaning of a sentence (say: a conjecture, e.g. Fermat's) changes once it is proven, i.e. that a theorem has not the same meaning as the conjecture the same words/symbols used to express. I will here look at the way in which this idea is articulated in LW, Ms-126,59-61, d.d. 19421108-19421110.⁶⁵

Some of the ways in which LW tries to articulate his point in this excerpt should not be too hard to understand (or to swallow) at all: "The question [does '770' occur in the decimal development of pi, or not?] changes its status as soon as it becomes decidable. For a connection is made where there used to be none" should make sense to everyone, perhaps slightly trivially so: in actual practice, one can obviously do *other* things with the question and the concepts that are used as soon as one knows how to go about answering it, than was *previously* the case.

LW then compares this with an author who has not yet decided whether one of the characters in his upcoming book has a sister. This comparison and the point it is supposed to make should be clear and understandable, even if some/many of us may not want to follow LW where this point leads us.

LW's point is not that hard to understand from the point of view of his account of meaning in general: if there is literally no other way to establish the meaning of an utterance than by looking at how it is used within the practice in which it is actually used, then it follows that the meaning of the sentence is different when it is used in one context as compared to another

⁶⁵ 8.11.

Wie seltsam die Frage ist ob in der unendlichen Entwicklung von π die Figur φ (eine gewisse Anordnung von Ziffern, z.B. '770') vorkommen wird, sieht man erst wenn man die Frage in einer ganz hausbackenen Weise zu stellen versucht: Menschen sind darauf abgerichtet worden nach gewissen Regeln Zeichen zu setzen. Sie verfahren nun dieser Abrichtung gemäß & wir sagen es sei ein Problem, ob sie der gegebenen Regel folgend jemals die Figur φ anschreiben werden.

Was aber sagt der, der || welcher, wie Weyl, sagt, eines sei klar: man werde oder werde nicht, in der endlosen Entwicklung auf φ kommen?

Mir scheint, wer dies sagt, stellt schon selbst eine Regel, oder ein Postulat auf.

Wie, wenn man auf eine Frage hin erwiderte: 'Auf diese Frage gibt es bis jetzt noch keine Antwort'?

So könnte etwa der Dichter antworten der gefragt wird ob der Held seiner Dichtung eine Schwester hat oder nicht - wenn er nämlich noch nichts darüber entschieden hat.

Die Frage - will ich sagen - verändert ihren Status, wenn sie entscheidbar wird. Denn ein Zusammenhang wird dann gemacht, der früher nicht da war.

Man kann von dem Abgerichteten fragen: 'wie wird er die Regel für diesen Fall deuten?', oder auch 'wie soll er die Regeln für diesen Fall deuten'. Wie aber, wenn über diese Frage keine Entscheidung getroffen wurde? - Nun, dann ist die Antwort nicht: 'er soll sie so deuten, daß φ in der Entwicklung vorkommt' oder: 'er soll sie so deuten daß es nicht vorkommt', sondern: 'darüber ist noch nichts entschieden'.

Wir mathematisieren mit den Begriffen. - Und mit gewissen Begriffen mehr als mit andern.

10.11.

Ich will sagen: Es scheint, als ob ein Entscheidungsgrund bereits vorläge; & er muß erst erfunden werden. Käme das darauf hinaus, zu sagen: Man benutzt beim Reden || Denken über die gelernte Technik des Entwickelns das falsche Bild einer vollendeten Entwicklung (dessen, was man für gewöhnlich 'Reihe' nennt) & wird dadurch gezwungen unbeantwortbare Fragen zu stellen.

context.⁶⁶ In the case of a conjecture, before one knows how to go about proving it, it is literally not clear yet how it will fit in with other things one says or does; so, one is literally not clear about its meaning yet.

One does not necessarily need to agree with LW's assessment, but one should be able to see how it fits in with, and makes sense within the context of, LW's structuralism and pragmatism, i.e. it should be clear that this is not an outrageous, gratuitous statement, but a logical part of a wide-reaching and coherent account.

It may at this point be interesting to anticipate Part 2 of this study and point out that what LW is targeting here is the **naturalness** of the idea that "in the infinite decimal development of pi, we must either find or not find <770>" (which he attributes here to Weyl⁶⁷): from the point of view of LW's holism and structuralism about meaning, there simply is no way to isolate these contents from the way they function within our actual practices, and as it stands, the contents are freewheeling, have no real function within any actual mathematical practice, and LW blames Weyl for **pretending** they do have a definite meaning. Again, one may or may not want to follow LW in this line of thought, but one cannot deny its coherence and consistency.⁶⁸

(C) holism and structuralism at work: objecthood and objectivity

This section is not really based on an analysis of any aspects of LW's work at all. The reason why I chose to integrate it at this point in the present study, is that it shows a Wittgenstein-like holism-cum-structuralism at work, as was the case in the paragraphs here above, and that it shows how such a more radically pragmatic framework could contribute to present-day PhilMath, as was also the case here above.

The ontological issues concerning mathematical objects are interesting in this context because it is perhaps the last domain in which ontological problems are still hotly debated, and because it allows for a very concrete demonstration of the potential of a pragmatic approach. The prototypical example of an object is the mesoscopic physical object: it is physical, it can be readily perceived/cognized and manipulated by humans, it is relatively permanent and apparently independent of our perception of them... Mathematical objects can be considered 'objects' in that they are perceived/cognized as relatively permanent and manipulated within mathematical practices, in the same ways that other objects are manipulated and perceived in other practices, but mathematical objects are traditionally considered "special", perhaps because (1) they are "not physical", but (2) at the same time they appear to be relatively

⁶⁶ ("The door is open" may be an invitation to come in or a request to close the door)

⁶⁷ For LW's references to Weyl (and other authors), see (Biesenbach 2008c); (Biesenbach 2008a); (Biesenbach 2008b).

⁶⁸ It would be interesting to ask the question as to how conjectures actually function in real-life mathematical practices, e.g. as part of an ethnomethodological research project in the style of (Livingston 1986) (Livingston 2015).

independent of our perception and/or our imagination. But from a pragmatic (or even a more general phenomenological) point of view, this is not that special.⁶⁹

In the phenomenology of objecthood, i.e. the way in which objects occur in actual practice,⁷⁰ the physicality of physical objects does not really play a role. Not even in the case of eminently physical objects. When we use a table, it's the tablehood of the table that constitutes its objecthood, not its physicality, its molecular structure, its physical properties, etc. (unless these features become a part of what is wrong with the table, of course).⁷¹ This point is made quite clearly by cases in which we perceive tablehood, without underlying physicality, as in representations (say: pictures, films, etc.) of tables, including even completely fictitious tables, as in cartoons (or even dreams): what makes these tables understandable as tables is their function, the way they are used as a table, despite their not being physical.

A rich ontological taxonomy is needed, which shows how objects actually function within practices. We adopt a 'structural' approach to objecthood, i.e. objecthood and its different sub-categories are defined in terms of their function within a practice: something is a certain kind of thing by virtue of the way it is manipulated. From a pragmatic point of view, is an object what functions (is manipulated, transformed, ...) as an object within a practice, for instance: as a tool, as an ingredient, as a product, as infrastructure.

The details of such a typology should be an empirical matter (including not only pragmatic, ethnographic or otherwise phenomenological but perhaps also cognitive approaches) and is not the job of a philosopher, though of course the results of such empirical inquiries may turn out to be philosophically relevant. I am ready to believe that there are very specific features to mathematical objects, but I want to see that demonstrated by means of actual analysis of actual mathematical practices, not posited a priori.

Interestingly, this line of argument puts us (again) on the opposite side of the argument from José Ferreirós, this time with respect to the 'objectivity without objects' claim, which is central to Ferreirós *Mathematical Knowledge and the Interplay of Practice* (Ferreirós 2016); Chapter 9, carrying the title 'Objectivity in Mathematical Knowledge' (pp. 247-290), is the penultimate and climactic chapter of the book. Ferreirós states his aims as follows:

⁶⁹ A case in point would be 'fictional' stuff. NB I'm not a fictionalist about mathematics (but then again, I'm not particularly a fictionalist about Mickey Mouse either): the reality/fictionality of stuff is not necessarily a particularly relevant aspect.

⁷⁰ This is more or less exactly what Martin Heidegger shows in §§15-10 of his seminal *Sein und Zeit*, by pointing out that 'Zuhandenheit' (i.e. being available in the context of an everyday practice), is the default way for things to 'be'. Cf. Scheppers 2017, Chapter 2, §2.

⁷¹ Cf. Heidegger's notions of *Auffälllichkeit*, *Aufdringlichkeit*, and *Aufsässigkeit*, in §16 of *Sein und Zeit* ((Heidegger 1967)).

The celebrated objectivity of mathematical results led many authors to believe that the theorems of mathematics are apodictic, necessary, a priori. From the beginning, we have remained uncommitted to this aprioristic view of mathematics, moreover we have defended the position that advanced mathematics is marked by the presence of hypotheses at its roots. How can the idea of objectivity be rescued in this setting? Precisely by considering the interplay of knowledge and practices that takes place in mathematics.

From the point of view developed in the present study, there is no reason to want to ‘rescue objectivity’ (on the contrary: we consistently emphasize the variability and contingency of mathematics in general), but there is also no problem with the objecthood of mathematical objects, which do function like objects within mathematical practices. In other words: we could adopt the slogan ‘**objects without objectivity**’, if necessary.

The purpose of the above paragraphs was to show (1) how LW’s holism and structuralism about practice gives rise to a much more radically pragmatic approach to practice than is currently prevalent in PhilMathPract, and -in preliminary way- (2) how this approach can contribute (relatively) novel ways to deal with more or less current issues in PhilMath and PhilMathPract.

1.1.3 Integration vs. fragmentation: the local-global dimension of embedding

In his account of meaning, LW insists time and time again on both of the following opposite poles of the same dimension:

- on the one hand, meaning depends on embedding in the very local, small-scale context of prototypical practices / Language Games such as buying something, measuring a piece of wood, etc.;
- on the other hand, the meaningfulness of these very local practices depends on their integration within much larger structures (ultimately “our lives” as a whole, in all their multidimensional glory).⁷²

So, on the one hand, certain small-scale / local aspects of practices can only be understood in terms of that particular small-scale practice itself. It is at the level of these small-scale patterns that the enormous variability of human practice is most visible (cf. the seminal expression

⁷² NB: it is not true that LW used ‘Form of Life’ for larger-scale patterns and ‘Language Game’ for small-scale practices. Cf. Scheppers 2017 (Scheppers 2017), §3.2.

“das ganze Gewimmel” (standard translation: “the whole hurly-burly”, as introduced in *BPhP2* §629 (= *Zettel* §567 = Ms-137,54b = Ts-232,754 = Ts-233b,38)⁷³).⁷⁴

On the other hand, LW also repeatedly ⁷⁵ points out that the meaningfulness of practices depends on how they fit in with much larger patterns, whether cultural (cultures, tribes, historical eras, ...) or biological (species, ...) or even physical (the existence or not of entities that are stable enough to be reliably counted), etc.

Think of smoking. Grab a matchbox, take out a match, light the match. In order to understand the why’s of this pattern, it is important to know what you need the light for. If you light that match to smoke, this instance of smoking a cigarette perhaps coheres with many other times you smoked a cigarette, and it coheres more remotely with various aspects of the tobacco industry and the history of the tobacco trade (or whatever is relevant to your understanding), but it will not be cohesive with encompassing patterns including other adjacent activities of yours, in the way it would be if you needed the light to heat water to do the dishes. Smoking remains a lot more ‘local’ than doing the dishes: the chain (or rather the tree) of explanations climbs up in a lot more hierarchically articulated way in the case of doing the dishes.⁷⁶

(A) the fragmentation of math - the integration of math

And the same goes for math: it can be fruitfully argued that certain aspects of advanced 20th and 21st century math (say: theoretical set theory) are not directly linking up with elementary practices like counting in the same way that basic arithmetic is, that they can be seen as auto-sufficient and self-supporting and don’t need their historical links to more basic practices to be meaningful. And in a way, this is correct: the meaning of practices often is quite local, not all features derive from its embedding in encompassing practices.⁷⁷ In sections 1.1.3(C) and

⁷³ Cf. also Ms-171,4: “Unsere Begriffe, Urteile, Reaktionen erscheinen nie bloß in Verbindung mit einer einzelnen Handlung, sondern mit dem ganzen Gewimmel der menschlichen Handlungen”.

⁷⁴ “Wie könnte man die menschliche Handlungsweise beschreiben? Doch nur, indem man die Handlungen der verschiedenen Menschen, wie sie durcheinanderwimmeln, zeigt. Nicht, was Einer jetzt tut, sondern das ganze Gewimmel ist der Hintergrund, worauf wir eine Handlung sehen, und bestimmt unser Urteil, unsere Begriffe und Reaktionen.”

“Das ganze Gewimmel” is one of those memorable formulas that LW’s work is full of, which also in English translation (“the whole hurly-burly”) made history (see e.g. the title of chapter 3 of Lee Braver’s (2012) *Groundless Grounds: A Study of Wittgenstein and Heidegger*, where it is supposed to summarily refer to both LW’s and Heidegger’s holisms).

⁷⁵ Even the very sentence in which he uses the expression “das ganze Gewimmel” the focus is on the hurly-burly as whole, as opposed to single actions by single agents.

⁷⁶ For the representation of intentionality by means of a tree structure, see e.g. (Scheppers 2003) and (Scheppers 2011). (By the way: in this context, one could try and experiment with pseudo-quantitative approaches in terms of concepts such as ‘depth of embedding’ or ‘degrees of locality’, but in my experience (i.e. I have tried...), that line of thought doesn’t lead very far).

⁷⁷ I clearly remember this reluctance was part of some mathematicians-philosophers’ first reaction to Ferreirós’ *Mathematical Knowledge and the Interplay of Practices* (Ferreirós 2016) in the reading group at the Vrije Universiteit Brussel’s Centre for Logic and Philosophy of Science dedicated to that book in 2015-2016, before it came out.

1.3 below, we will see how LW insists at great length on the great variety of (what he calls) applications that can give meaning to mathematical or math-like techniques.

On the other hand, in actual practice, contemporary mathematics is not that independent from its roots: there is a structural relationship between them, not only genetically, but also synchronically, through the fact that every mathematician is the product of an education that starts from more elementary numeracy training before embarking on advanced stuff.

One could also argue that there is a more intricate structural relationship, as well, which would account for its ‘magical’ applicability:⁷⁸ all human endeavors have developed in a specific context, in which so much is already ‘given’: a physical environment with specific features (relatively stable mesoscopic objects are a prerequisite for counting), cultural practices (argumentation as part of judicial and political procedures are a prerequisite to proof),

Interestingly, Floyd (Floyd 2021) insists a number of times on the fragmentation of mathematical practice. For instance, on p. 53: “This allowance for plasticity in projecting concepts shows, not that our procedures are not rule-governed, but rather that that notion itself requires parochial elements.”

⁷⁸ **Excursus: applicability**

Apart from the idea that math as an academic discipline is deeply rooted in, or rather: is interconnected with, a complex web of heterogeneous everyday practices, there is also the niche issue of the ‘miraculous’ applicability of mathematics-internal developments to natural science. This niche, which -as a niche within present-day PhilMath- was famously started by Wigner (Wigner 1960), has had a lot of success and gave rise to remarkable pathos throughout its course.

The applicability of basic geometry and arithmetic is -of course- not even an issue: the application existed before the math. But even in the case of more advanced mathematical developments, it can be pointed out that the math and its application (the engineering, the astronomy, ...) grew up together, are both aspects of the same Form of Life: it is no wonder at all that math is reflected in nature, if math and physics and engineering are deeply intertwined. All this has been explored by the late great Mark Steiner (cf. e.g. (Steiner 1989); (Steiner 2009); (Bangu 2006)). However, there also appear to be a number of cases in which the application occurred after the development of the math, and in unpredictable ways that “should not have worked”.

In reaction to the latter cases, I would like to point out the following:

- First, it would be good to look at the details of how exactly the scientists that stumbled on the miraculous applicability went about: what methods were already in place, what exactly was new, etc. This usually mitigates any sense of miracle from the get-go.
- Second, the sense of miraculousness may also result from the fact that one may underestimate the depth of the embedding, that one is blind to iceberg of givenness underlying any phenomenon: so much is already in common between our world and our mathematical practice that it is very hard to not overestimate the autonomy of our math. Below, in section 3.3(C), I will try and argue that this account of givenness is at the core of LW’s critical philosophy, in the deeply Kantian sense of the word ‘critical’.
- Third, Whiggishness may also play a role here: there is a bias towards focusing on success stories, and this may result in a skewed view of what is “normal” and what is “miraculous”. For every interesting item, there may occur billions of uninteresting items. We have an inherent tendency to look at phenomena that display regularity and to ignore the chaos that we cannot readily describe around those islands of regularity.

If one’s perception of nature is deeply influenced by math, i.e. if your science is mostly interested in these aspects of reality that can readily be quantified or otherwise represented in mathematical terms, and vice versa, if one’s math is deeply intertwined with techniques that deal with our relationship to nature (engineering, physics, astronomy), it is no wonder that math and nature reflect each other.

I have nothing more to say about the applicability of math as such. But I will have something to say about the verbiage (“awe”, “miraculous”, ...) employed by some of the authors dealing with the phenomenon (see section 3.2.3(D) and Appendix 4.3(A)).

(B) the freedom/autonomy of pure math vs. its essential embeddedness in ‘applications’

Another interesting issue that may be captured under the integration vs. fragmentation heading is the inherent tension between the self-proclaimed freedom of pure math and its essential embeddedness in the ‘applications’ that make mathematics meaningful. In his work on math, LW tackles this issue more or less directly in his analysis of the idea that math is a ‘game’ (see section 1.3(B) below) and his insistence on the importance of applications is in direct opposition to this idea. Similarly, LW’s critical remarks, analyzed in Part 2 of this study, including its Spenglerian overtones (cf. section (D) here below), mostly exploit the idea that meaningfulness depends on the integrated nature of a healthy culture (as opposed to the fragmentation of a culture in decline).

Even if LW’s polemical position in the context of the Grundlagen-debates made him emphasize the ‘global’ aspect of the embeddedness of math, the conceptual apparatus that emerges from his work is capable of describing how certain mathematical practices can function more or less autonomously at the local level (whether LW likes it or not).

(C) Ein buntes Gemisch: the heterogeneity of math (LW, Ms-122, 68r-88r (19391231-19400108))

The heterogeneity of mathematics is one of the most recurrent points in LW’s PhilMath: almost all of the texts that we read and analyze in the present study are characterized by a continuous insistence on the variety of techniques that make up math, the variability of the applications in which they are rooted and the precariousness of what is supposed to keep them together.

To illustrate the topic, I have chosen to analyze the passage in which LW uses the colorful expression “**ein buntes Gemisch**”. This passage consists mainly of a long struggle with the idea that mathematical proofs should prove something, that we should be able to use them as an example for correct applications, and that they therefore should be surveyable, etc. Felix Mühlhölzer wrote the book on this, literally: *Braucht die Mathematik eine Grundlegung?: ein Kommentar des Teils III von Wittgensteins Bemerkungen über die Grundlagen der Mathematik* (Mühlhölzer 2012).

For our purposes, I would like to first briefly focus on the passage Ms-122, 69v-71v, in which LW first explores the idea that there is no fundamental reason why mathematics should operate with propositional axioms at all: it is easy to imagine that certain mathematical techniques only exist in the form of rules for building houses, without any theoretical underpinnings (?)⁷⁹ at all: truth need not play a role at all, but one can *do it wrong*, which is not

⁷⁹ If one can even say that theory ever underpins anything...

at all the same thing as conforming to some *truth*. LW then asks the somewhat rhetorical question as to whether this isn't a case of applied math without pure math.⁸⁰

LW then comes back to the specific topic of 'proving', struggling with what exactly happens in a "pure" mathematical proof (there are many more interesting things to unpack in LW's manuscripts than what we can afford to do here).⁸¹

In his entry for 19400108, LW then articulates the idea that the techniques that make up math (in this context, he's focusing on proof techniques, specifically) do not form a unity, introducing the seminal formula "buntes Gemisch".⁸²

Ich will | | möchte sagen: Die Mathematik ist ein buntes Gemisch .?. von Beweistechniken. --Und darauf beruht ihre mannigfache Anwendbarkeit & ihre Wichtigkeit.

Here, he focuses specifically on the case of formal systems (the one he is most familiar with is Russell's PM, but the same would go for ZFC etc.), and points out that, when someone codes a mathematical system (say: differential calculus) into this formalism, he actually established a new piece of math.⁸³ So, LW's point is not that he is for or against formalisms, nor that he is for or against the way these formalisms are used as a technique in contemporary mathematics or PhilMath. His point is that these formalisms are expanding math, that they cannot be construed as simply unifying pre-existing math, that they are add-ons to pre-existing math, not a new expression of pre-existing math.⁸⁴

In this connection, LW makes another interesting point (which may -to some of us- seem somewhat at odds with the previous point): he compares the case of someone inventing a

⁸⁰ Man könnte fragen: "Warum verwendet die Mathematik überhaupt satzförmige Axiome?"

Die Frage ist: Ist es wahr, daß, wie ich behauptet habe, die Mathematik wesentlich die Rolle der Grammatik ihrer Zeichen spielt? - Kann man denn das in dem Beispiel sagen (das ich gab), worin Leute eine Rechnung als Teil einer Technik des Hausbaus verwenden??

Ich sagte: bei dieser Rechnung gäbe es ein (sozusagen arithmetisches) Richtig oder Falsch, nämlich: der Regel gemäß oder der Regel zuwider.

Haben wir hier nicht, sozusagen, angewandte Mathematik, ohne reine Mathematik?

⁸¹ Ich wollte doch sagen: Wo die reine Mathematik von Satz zu Satz fortschreitet, da wird von einer Ausdrucksform zur andern fortgeschritten.

Immer bin ich hier zum Dogmatismus geneigt!

Ist denn das Charakteristische am Beweis nicht, daß das Bewiesene am Ende ohne den Beweis feststeht? (Obwohl der Beweis immer zur Grammatik des Bewiesenen gehört.)

⁸² Although the expression occurs only a few times, and in passing, in the context of MS122, it has been quoted quite often, as if it was an important terminus technicus (cf. also 'Forms of Life'). The expression reoccurs a little further on at Ms-122,96r-96v, 19400113: "Ich will die Buntheit der Mathematik erklären".

⁸³ Und das kommt doch auf das Gleiche hinaus, wie zu sagen: Wer ein System, wie das R.sche, besäße & aus diesem 'durch entsprechende Definitionen' Systeme, wie den Differentialkalkül, erzeugte, der erfände | | erzeugte ein neues Stück Mathematik. (Wie ich schon früher gesagt habe.)

⁸⁴ Cf. Ms-122, 3v-4r: "Ich will der Formulierung entgegen: "ich weiß jetzt mehr über den Kalkül", & statt ihrer die setzen: "ich habe jetzt einen andern Kalkül". Der Sinn hiervon ist, die Kluft zwischen einem mathematischen Wissen & nicht-mathematischem Wissen immer in ihrer vollen Größe vor Augen zu behalten."

formalism (like Russell did), with someone inventing a **notation**.⁸⁵ The point is: when you invent a notation, this is also an expansion of your math.⁸⁶ So, once again, LW confronts the idea that practical things like notations should be viewed as somehow external to math and thus displaces the idea that 'foundational' systems are somehow more at the core of mathematics than the techniques that make up actual mathematical practice.

LW also addresses the idea that an axiomatic system (a "proving system" / "Beweissystem") can coordinate several pre-existing systems by translating them into a common code (which is basically the approach adopted by Russell and his successors; i.e. almost all mathematicians involved in foundational issues). LW asks whether such a coordinated system, consisting of many more or less independent sub-systems constitutes one system or several systems and suggests that the translation into a single code in the end does not change much about the heterogeneity of the sub-systems: even if we started doing trigonometry using ZFC formalism (which, by the way, we obviously don't!), it would -presumably, supposedly- still be -in a way- the same trigonometry we always did; and the same goes for all the other different subsystems.⁸⁷

So, what is suggested in this extended passage as a whole is:

- (1) that what is presented as a foundation of mathematics, is actually an **add-on** (that does not change anything about the **fundamental heterogeneity** of mathematical techniques), not something central to the actual mathematical practices it is supposed to unify, and
- (2) vice-versa, that certain things that we might want to consider extrinsic to pure math, such as **notation** systems, may have a much more pervasive importance with respect to our actual mathematical techniques.

⁸⁵ Nun, man könnte doch einfach sagen: Wenn ein Mensch das Rechnen im Dezimalsystem erfunden hätte - der hätte doch eine mathematische Erfindung gemacht! - Auch wenn ihm Russell's Principia Mathematica bereits vorgelegen wären. -

⁸⁶ This notion that notation is not a peripheral, external aspect of mathematics, but an integral, intrinsic aspect of what mathematics is, anticipates Kenneth Manders seminal and rightly celebrated paper (Manders 2008) by 55 years.

⁸⁷ Wie ist es, wenn man ein Beweissystem einem anderen koordiniert? Es gibt dann eine Übersetzungsregel mittels derer man die in S1 bewiesenen Sätze in die in S2 || im einen bewiesenen Sätze in die im andern bewiesenen übersetzen kann.

Man kann sich doch aber denken, daß einige- oder alle - Beweissysteme der heutigen, Mathematik auf solche Weise einem System, etwa dem R.schen zugeordnet wären. So daß alle Beweise, wenn auch umständlich, in diesem System ausgeführt werden könnten. So gäbe es dann nur das eine System - & nicht mehr die vielen Systeme? - Aber es muß sich doch also von dem einen || einen System zeigen lassen, daß es sich in den vielen darstellen läßt. || , daß es sich in die vielen auflösen läßt. - Ein Teil des Systems wird die Eigentümlichkeiten der Trigonometrie besitzen, ein anderer die der Algebra, u.s.w. Man kann also sagen, daß in diesen Teilen verschiedene Techniken verwendet werden.

LW's intervention in the Grundlagen-debates is not at the level of various positions *within* these debates (finitism vs. infinitesimal approaches; formalism vs. anti-formalism; etc.),⁸⁸ but attacks presuppositions at a much more general level. LW is attacking:

- the idea that math is a coherent/unitary system of propositions;
- the idea that these propositional systems somehow *underlie* the heterogeneous collection of techniques that make up actual mathematical practice.

(D) The heterogeneity of mathematical practice vs. historical grand narratives

The concept of a "buntes Gemisch" and the lines of thought that imply it, also antagonizes encompassing grand narratives about the history of mathematics, e.g. Ferreirós' (Ferreirós 2016) proposal to conceptualize the relation between 'less advanced' to 'more advanced' mathematical (inc. the evolution from pre-mathematical techniques to proper math) in terms of the mechanism of 'abstraction'/'idealization', and in terms of a **stratification of practices**, i.e. the idea that advanced math is built on underlying 'layers' of less abstract math, which in their turn are grounded in technical practices, which are ultimately rooted in the most basic or elementary practices of counting, measuring, etc. The problem with this idea is that (1) basic counting is irreducibly complex too, and that (2) more 'advanced' practices acquire their own grounds for meaningfulness, that may separate them from -say- basic counting. A good illustration of the latter point is the fact that within the Frege & Russell-style logistic approaches to the Grundlagen debate, in which LW started out, defining the sequence of natural numbers becomes a complicated issue. So: it is obviously wrong that counting and other basic pre-, proto- or quasi-mathematical operations are rooted in axiomatic systems, but the alternative picture in which practices are neatly layered is equally wrong: complex networks of relations between various mathematical and non-mathematical practices exist at all levels, to such an extent that the idea of a level or layer becomes misleading.⁸⁹

⁸⁸ As for the question as to whether LW was a formalist, an anti-formalist or something else, the issue may simply not be relevant in this context: LW does not advocate the use of formal systems in mathematics, but neither does he object to inventing formalisms: he merely objects to pretending formal systems are not invented, that they are somehow a fact of nature, that they reveal an underlying unity, rather than superimposing unity on an underlying heterogeneity.

⁸⁹ Again (as was the case in my argument against positing 'communities' as the default locus for meaning; see section 1.1.2(A1)), I would like to argue that the idea of 'layers' should not be posited a priori but shown as an empirical result. If one starts from the idea that math as a ludic, autonomous endeavor is built upon more elementary, 'technical' practices, which in turn are based on even more elementary activities, then one prevents oneself from discovering how things are actually relate to each other. I would not object to the idea that something is 'built upon' something else in any particular case, if it is shown to me that this is the case in that case, but I am not ready to accept that this the default way in which math evolves and has evolved as a discipline. Stratification is a very specific concept, not something you can simply posit to be the default, because that would be a sure way to obscure the heterogeneity, the mess, one can expect in any historical process.

(E) Sass on LW on fragmentation

While working on this study, I stumbled on a few strands in LW's thinking that complicate the picture sketched here above, in that LW's emphasis on the need for -what I called- 'local' patterns to be embedded in 'global patterns' in order to be meaningful, was much more prominent (and much more problematic) in LW than I previously thought.

Suffice it here to follow what Louis Sass's 2001 article 'Deep Disquietudes: Reflections on Wittgenstein as Antiphilosopher' (also quoted throughout section 2.0) says about this topic ((Sass 2001) Sass 2001, pp. 119-120).

First of all, Sass points out "the antipathy Wittgenstein always felt toward the modern condition of cultural fragmentation and self-consciousness, in which basic cultural presuppositions come under scrutiny and can no longer serve as the taken-for-granted foundation of spontaneous thought and action", and refers to the following excerpt from a draft preface to the *Philosophische Bemerkungen*:

Ms-109,204-207, d.d. 19301101:

Zu einem Vorwort:

Dieses Buch ist für diejenigen | | die geschrieben, die dem Geist | | seinem Geist in dem es geschrieben ist freundlich gegenüberstehn. Dieser Geist ist, glaube ich, ein anderer als der der | | des Stromes der großen europäischen & amerikanischen Zivilisation. Der Geist dieser Zivilisation dessen Ausdruck die Industrie, Architektur, Musik, der Faschismus & Sozialismus der Jetztzeit | | unserer Zeit ist, ist ein dem Verfasser fremder & unsympathischer Geist | | dem Verfasser fremd & unsympathisch. Dies ist kein Werturteil. Nicht als ob ich nicht wüßte daß was sich heute als Architektur ausgibt nicht Architektur ist & nicht | | er glaubte daß ... Architektur wäre & nicht als ob er dem was moderne Musik heißt nicht das größte Mißtrauen entgegenbrächte ohne ihre Sprache zu verstehen, aber das Verschwinden der Künste rechtfertigt kein absprechendes Urteil über eine Menschheit. Denn echte & starke Naturen wenden sich eben in dieser Zeit von dem Gebiet der Künste ab & anderen Dingen zu & der Wert des Einzelnen kommt irgendwie zum Ausdruck. Freilich nicht wie zur Zeit einer großen Kultur. Die Kultur ist gleichsam eine große Organisation die jedem der zu ihr gehört seinen Platz anweist an dem er im Geist des Ganzen arbeiten kann und seine Kraft kann mit gewissem Recht an seinem Erfolg im Sinne des Ganzen gemessen werden.

Zur Zeit der Unkultur aber zersplittern sich die Kräfte und die Kraft des Einzelnen wird durch entgegengesetzte Kräfte & Reibungswiderstände verbraucht & kommt nicht in der Länge des durchlaufenen Weges zum Ausdruck sondern vielleicht nur in der Wärme die er beim Überwinden der Reibungswiderstände erzeugt hat. Aber Energie bleibt Energie & wenn so das Schauspiel das dieses Zeitalter bietet auch nicht das des Werdens eines großen Kulturwerkes ist in dem die Besten dem gleichen großen Ziele zuarbeiten sondern das wenig imposante Schauspiel einer Menge deren Beste nur privaten Zielen nachstreben so dürfen wir nicht vergessen, daß es auf das Schauspiel nicht ankommt.

Ist es mir so klar daß das Verschwinden einer Kultur nicht das Verschwinden menschlichen Wertes bedeutet sondern bloß gewisser Ausdrucksmittel dieses Werts so bleibt dennoch die Tatsache bestehen

daß ich dem Strom der europäischen Zivilisation ohne Sympathie zusehe, ohne Verständnis für die Ziele wenn sie welche hat. Ich schreibe also eigentlich für Freunde welche in Winkeln der Welt verstreut sind.

Ob ich von dem typischen westlichen Wissenschaftler verstanden oder geschätzt werde ist mir gleichgültig weil er den Geist in dem ich schreibe doch nicht versteht.

Unsere Zivilisation ist durch das Wort Fortschritt charakterisiert. Der Fortschritt ist ihre Form nicht eine ihrer Eigenschaften daß sie fortschreitet. Sie ist typisch aufbauend. Ihre Tätigkeit ist es ein immer komplizierteres Gebilde zu konstruieren. Und auch die Klarheit dient doch nur wieder diesem Zweck & ist nicht Selbstzweck.

Mir dagegen ist die Klarheit die Durchsichtigkeit Selbstzweck.

Es interessiert mich nicht ein Gebäude aufzuführen sondern die Grundlagen der möglichen Gebäude durchsichtig vor mir zu haben.

Mein Ziel ist also ein anderes als das der Wissenschaftler & meine Denkbewegung von der ihrigen verschieden.⁹⁰

Sass then makes the obvious link with Spengler (see also section 2.0.0 below):

The distinction between culture and civilization is the central theme of *The Decline of the West* by Oswald Spengler, one of the handful of authors whom Wittgenstein repeatedly cited as a major influence and a writer whose thinking Wittgenstein described as “completely in touch with what I have often thought myself” (D 17, 6.5.30). Spengler contrasts Kultur and Zivilisation as “the living body of a soul” versus “the mummy of it.” In his view, a crucial caesura in European history occurred around 1800. On one side of this frontier, Spengler sees “life in fullness and sureness of itself, formed by growth from within [i.e., culture] . . . on the other, the autumnal, artificial, rootless life of our great cities under forms fashioned by the intellect [civilization].” For what Spengler calls “the Gothic and Doric men, Ionic and Baroque men” of the earlier era, “the whole vast form world of art, religion, custom, state, knowledge, social life was easy. They could carry it and actualize it without ‘knowing’ it.” For culture, writes Spengler, is “the self-evident.” He remarks on the typical modern feelings of “strangeness” with regard to these cultural forms,

⁹⁰ MS 109 200: 5.11.1930

Sketch for a Foreword†4

This book is written for those†e who are in sympathy with the spirit in which it is written.†f This spirit is, I believe, different from that of the†g prevailing European and American civilization. The spirit of this civilization the expression of which is the industry, architecture, music, of present day†h fascism & socialism, is a spirit that is alien & uncongenial†i to the author. This is not a value judgement. It is not as though I did not know that†j what today represents itself as architecture is not architecture & not†k as though he did not approach what is called modern music with the greatest mistrust (without understanding its language), but the disappearance of the arts does not justify a disparaging judgement on a whole segment of humanity. For in these times genuine & strong characters simply turn away from the field of the arts & towards other things & somehow the value of the individual finds expression. Not, to be sure, in the way it would at a time of Great Culture. Culture is like a great organization which assigns to each of its members his place, at which he can work in the spirit of the whole, and his strength can with a certain justice be measured by his success as understood within that whole. In a time without culture, however, forces are fragmented and the strength of the individual is wasted through the overcoming of opposing forces & frictional resistances; it is not manifest in the distance travelled but rather perhaps in the heat generated through the overcoming of frictional resistances. But energy is still energy & even if the spectacle afforded by this age is not the coming into being of a great work of culture in which the best contribute to the same great end, so much as the unimposing spectacle of a crowd whose best members pursue purely private ends, still we must not forget that the spectacle is not what matters.

Even if it is clear to me then that the disappearance of a culture does not signify the disappearance of human value but simply of certain means of expressing this value, still the fact remains that I contemplate the current of European civilization without sympathy, without understanding its aims if any. So I am really writing for friends

“the idea that they are a burden from which creative freedom requires to be relieved,” and the “fatal imposition of thought upon the inscrutable quality of creativeness.” All these, he says, are “symptoms of a soul that is beginning to tire. Only the sick man feels his limbs.” In such a condition, we might say (paraphrasing Wittgenstein), life does not fit into a mold and hence what is problematic cannot disappear (CV 27/31).

Sass then also refers to a conversation between O.K. Bouwsma and LW, as remembered by the former:⁹¹

In a conversation on these issues recollected by O. K. Bouwsma, Wittgenstein spoke of changes in the kind of human beings we are in the modern world: “There was a time when our lives were furnished rather simply, a house, a place, tools so many, a beast, and a circle of people. In this simplicity and this stability one grew attached to a limited environment. This gave a life a certain quality – roots.”⁶³

This passage, as well as other passages out of LW’s writings, sheds light on LW’s discomfort with the fragmentation of modern culture and society. Sass points out the sharp contrast between simpler and more stable earlier times and the present era in LW’s thought. A similar sentiment with respect to modernity is expressed in the following excerpts from one LW’s notebooks:

27.8.

Etwas besser geschlafen. Lebendige Träume. Etwas niedergedrückt; Wetter & Befinden.

Die Lösung des Problems, das Du im Leben siehst, ist eine Art zu leben, die das Problemhafte zum Verschwinden bringt.

Daß das Leben problematisch ist, heißt, daß Dein Leben nicht in die Form des Lebens paßt. Du mußt dann Dein Leben verändern, & paßt es in die Form, dann verschwindet das Problematische.

Aber haben wir nicht das Gefühl, daß der, welcher nicht darin ein Problem sieht für etwas Wichtiges, ja das Wichtigste, blind ist? Möchte ich nicht sagen, der lebe so dahin – eben blind, gleichsam wie ein Maulwurf, & wenn er bloß sehen | | aufschauen könnte, so sähe er das Problem?

Oder soll ich nicht sagen: daß wer richtig lebt, das Problem nicht als Traurigkeit, also doch nicht problematisch, empfindet, sondern vielmehr als eine Freude; also gleichsam als einen lichten Äther um sein Leben, nicht als einen fraglichen Hintergrund.

[...]

(Ms-118,17r-17v, d.d. 19370827)

Beinahe ähnlich, wie man sagt, daß die alten Physiker plötzlich gefunden haben, daß sie zu wenig Mathematik verstehen, um die Physik bewältigen zu können, kann man sagen, daß die jungen Menschen heutzutage plötzlich in der Lage sind, daß der normale, gute Verstand für die seltsamen Ansprüche des Lebens nicht mehr ausreicht. Es ist alles so verzwickelt geworden, daß zu seiner Bewältigung | | , es zu bewältigen, ein ausnahmsweiser Verstand gehörte. Denn es genügt nicht mehr, das Spiel gut spielen zu

⁹¹ I quote Sass’s text, but can’t find the passage in (Bouwsma 1999).

können, sondern die Frage ist immer wieder: was für ein Spiel ist jetzt überhaupt zu spielen? || sondern immer wieder ist die Frage: ist dieses Spiel jetzt überhaupt zu spielen & welches ist das rechte Spiel? (Ms-118,20r-20v, d.d. 19370827)⁹²

Note that for LW, -here as elsewhere (cf. section 2.0 below)- culture-critical strands are intertwined with existential-biographical strands. In the same vein, Sass also refers to the following diary passage (Ms-183,45-47, d.d. 19301008), in which LW -again- expresses his alienation with respect to modern society:

8.10.

In der neuen Wohnung, sie paßt mir noch nicht, wie ein neuer Anzug. Ich fühle mich kalt & ungemütlich. Schreibe das nur um etwas zu schreiben & mit mir selbst zu reden. Ich könnte sagen: jetzt bin ich endlich mit mir allein & muß nach & nach mit mir ins Gespräch kommen.

In der Großstadt-Zivilisation || großstädtischen Zivilisation kann sich der Geist nur in einen Winkel drücken. Dabei ist er aber nicht etwa atavistisch & überflüssig sondern er schwebt über der Asche der Kultur als (ewiger) Zeuge – – quasi als Rächer Gottes. || der Gottheit.

Als erwarte er seine || eine neue Verkörperung (in einer neuen Kultur).

Wie müßte der große Satiriker dieser Zeit ausschauen? [...]⁹³

In subsequent versions of this study, I will have to attempt to process this material a lot further, so as to come to an integrated account of LW's outlook on contemporary culture and society (see also section 2.0 below), as sketched in the present section, and his account of meaning as embedding in practice, as sketched in previous sections. What counts most in the present context, is that there is a direct link between the notion of 'embedding in practice', the

⁹² CV, p. 27:

Earlier physicists are said to have found suddenly that they had too little mathematical understanding to cope with physics; and in almost the same way young people today can be said to be in a situation where ordinary common sense no longer suffices to meet the strange demands life makes. Everything has become so intricate that mastering it would require an exceptional intellect. Because skill at playing the game is no longer enough; the question that keeps coming up is: can this game be played at all now and what would be the right game to play?

The way to solve the problem you see in life is to live in a way that will make what is problematic disappear. The fact that life is problematic shows that the shape of your life does not fit into life's mould. So you must change the way you live and, once your life does fit into the mould, what is problematic will disappear.

But don't we have the feeling that someone who sees no problem in life is blind to something important, even to the most important thing of all? Don't I feel like saying that a man like that is just living aimlessly - blindly, like a mole, and that if only he could see, he would see the problem?

Or shouldn't I say rather: a man who lives rightly won't experience the problem as sorrow, so for him it will not be a problem, but a joy rather; in other words for him it will be a bright halo round his life, not a dubious background.

NB: the standard edition with the title *Culture and Value* from which I quote the English translation, prints the two paragraphs, admittedly from the same day, but separated by several pages, in the opposite order from the one in which they appeared in the manuscript.

⁹³ Translation Sass: "In the metropolitan civilization the spirit can only huddle in some corner. And yet it is for instance not atavist and superfluous but hovers above the ashes of culture as an (eternal) witness - as if an avenger of the deity. As if it were awaiting a new incarnation (in a new culture)"

idea of 'local vs. global embedding', and a number of culture-critical and existential issues, which we will explore later on.

(F) Integration vs. fragmentation: loose ends

Our understanding of human practice (incl. discourse) in general depends on its local embedding in easily recognizable practices of the types that LW usually has in mind when he speaks of Language Games, but also on the fact that these small-scale structures in turn are embedded in much more encompassing structures (whether conceived of as 'our lives' or as a particular culture, or whatever). This idea is important because it allows us to conceptualize *both* (1) the fragmentation of actual practices that we observe as an essential aspect of our everyday realities, *and* (2) the fact that for these 'local' phenomena to be meaningful, they need to be somehow integrated in our lives at large.

However, as a matter of Wittgenstein-exegesis, I am at present not yet able to accurately articulate the following tensions in LW's thought:

- between (1) LW's insistence on the heterogeneity of everyday practice (cf. the positive valuation of the 'hurly-burly') and (2) LW's negative attitude towards fragmentation: how is the hurly-burly different from fragmentation?
- between (1) LW's understanding of the contingency and heterogeneity of the hurly-burly of everyday practice that serves as the "bedrock" for meaning and (2) LW's attachment to singular and rigid cultural affiliation as essential to meaning: on the one hand, LW is fully aware of the fact that there is no simple, unique or universal bedrock that gives meaning to all human behavior, but on the other hand, LW appears to believe in the detrimental consequences of the fragmentation of modern civilization.

I must say that all this is not necessarily a real problem for my own thoughts on these matters: my personal sensibilities are at the opposite side of the spectrum from LW's with regard to the importance of cultural affiliation, the idea that there is something like 'the non-everyday', the importance of authenticity, etc. However, as a matter of Wittgenstein-scholarship, there are some loose ends here. In section 2.0, I will come back to some of these issues and some headway will be made.

1.2 Sense = embedding in the everyday; lack of embedding in the everyday = nonsense

Once we acquire the notion of meaningfulness as embedding, we immediately also gain access to the idea that nonsense can be defined as a lack of embedding, or rather that lack of embedding gives rise to nonsense. In this section, we explore the role that the notion of ‘everydayness’ plays in LW’s work in general and his PhilMath in particular, which will lay the groundwork for our analysis of LW’s critical remarks in Part 2.

1.2.1 Wittgenstein’s criticism of ‘nonsense’ in terms of ‘lack of embedding’

There is some immediate appeal to the notion that nonsense is a matter of lack of embedding, it seems to make sense at first sight. For instance, it is tempting to view metaphysical discourse (think of “What is time?” or other prototypical philosophical stuff) as divorced from actual language games, actual practices, etc., and LW has been interpreted as saying exactly that, not in the least by the first generation of “positivist” admirers of his in the Vienna Circle.

(A) nonsense and senselessness in the *Tractatus Logico-Philosophicus* (TLP)

In the *TLP* (and associated writings), ‘nonsense’ appears to be used almost as a technical term, at least according to some (standard/mainstream) interpretations.⁹⁴ Thus, for a proposition to be meaningful, it has to be ‘bivalent’, i.e. either true or false; otherwise, it is not a real proposition at all. Tautologies and contradictions do not refer to the world and are not strictly ‘neither true nor false’, but rather ‘always true’ resp. ‘always false’. In some sense, tautologies have therefore no sense either. The *TLP* does not use the term ‘unsinnig’ (nonsensical) for this case, but ‘sinnlos’ (senseless).⁹⁵ This aligns well with the so-called ‘picture theory’ of truth (and also meaning) adopted in the *TLP*: propositions mean something by being a picture of reality (Johnston 2017). The following formula summarizes this conception:

- meaningful = bivalent (true or false) = refers to reality;
- meaningless = not bivalent (always true, never true, neither true nor false) = does not refer to reality.

As gibberish, which does not even look or sound like normal language (“tweedly deedly”), is not a philosophical problem to LW (or anybody else, really) and was –accordingly– not discussed by him in *TLP*, ‘meaninglessness’ de facto only concerns **pseudo-language**, i.e.

⁹⁴ For overviews of the debates, see Bronzo 2012, Conant and Bronzo 2017, Cheung 2017.

⁹⁵ *TLP* 4.461; *TLP* 4.4611; *TLP* 4.462.

language that appears to refer to the world but actually doesn't.⁹⁶ For the present purposes, the main point about LW's talk about nonsense is that it concerns 'fake' utterances that *appear* to be propositions but in fact *are not*. This notion of 'fakeness' will become important in Part 2 below.

(B) sense as embedding / nonsense as lack of embedding

After his return to philosophy in the late 1920s, LW's vision of what language is and does had considerably expanded: he no longer focused only on truth and propositionality, but recognized a large array of types of language uses.

The language-critical strand in his thought persisted, though, but in a different guise: whether an utterance makes sense or not now depends on whether or not it has a proper function within an (everyday) context, and words and sentences can have a wide array of possible functions (cf. e.g. the metaphor of the toolbox or the cabin of a locomotive in *PhU* §§11-12, and the explicit discussion of the issue in §23). This is what LW's use of the concept 'use' ("Gebrauch") comes down to.

LW's mature formulation of this idea typically involved the notion of Language Game: a pattern in which linguistic and non-linguistic behavior, agent intentions and representations, and objects form a structural whole. These usually small-scale patterns are in turn part of more encompassing, holistic notions such as – famously but very infrequently – Form of Life, and similar holistic notions such as 'our lives'. Building on most notably Baker and Hacker's notion of 'internal relation' ((Baker and Hacker 2009), p. 75), I have argued elsewhere ((Scheppers 2017); see also section 1.1.2 above) that the relations between the various variables within Language Games and within Forms of Life should be understood as 'internal' or 'structural' relations, i.e. that they should not be viewed as relations between pre-existing entities, but as relations that define the very identity of these entities. In this vein, we can formulate the notion of 'sense' in the following holistic and structuralistic way:

sense = function within a context = **embedding** within a Language Game (and ultimately in a Form of Life (or: 'our lives'))

A corresponding implementation of 'nonsense'/'senselessness'⁹⁷ immediately follows:

⁹⁶ Still, I believe it would be correct to say that, if pressed, LW (at least at the time of *TLP*) would not have admitted that pseudo-language was any better than gibberish and that- in fact- logically speaking, there is only one type of nonsense, thus agreeing with the 'resolute readings'.

⁹⁷ From here onwards I will simply use the word nonsense as the umbrella term, covering all forms of defective language use, and no longer bother with the *TLP* distinction 'senseless' vs. 'nonsense'.

nonsense = lack of function within a context = **lack of embedding** in a Language Game / Form of Life

This reading of the concept of nonsense is corroborated by a number of formulations in LW's later work: "language on holiday" (*PhU* §38), "a wheel that is not part of the engine" (*PhU* §271), an engine "idling" (*PhU* §132: "wenn die Sprache leerläuft"; cf. (Guetti 1993)), a sham corbel that supports nothing (*PhU* §217: "Unsere Forderung ist eine architektonische; die Erklärung eine Art Scheingesims, das nichts trägt"), a sham knob that turns out to be a mere ornament not connected with the mechanism (*PhU* §270).

As we have seen above, LW famously and controversially claimed that conjectures have no meaning until we know how to prove them (e.g. Fermat's conjecture or the proposition that '770' will occur in the decimal development of π). Even if common sense at first may make it hard for us to come to grips with the idea that these words literally mean nothing, this claim does make sense from the 'meaning is embedding' point of view. The problem with an unproven conjecture is the same as with a 'square circle' or 'North of the pole': the words sound like they mean something, but you can't do anything with them, there is no straightforward function for them within a bona fide practice or Language Game. Of course, they may acquire or be given such a role (as poetry, or as a mantra, or as a motivational slogan), but as it is that role will be not even remotely similar to the one of a theorem. Was LW right to call conjectures actually meaningless? I'm not sure that the issue is worth fighting over, but it is clear how the claim fits in with his general outlook on meaning, and it is clear that the role of the expression changes completely as soon as it acquires a specific place within a network of proofs.

(C) paradoxes

An interesting case in point is LW's view on paradoxes. In MS-118, 111v-, in the context of his 'notorious' account of contradictions in formal systems (see section 2.3 below), LW briefly but incisively talks about logical paradoxes, and the liar's paradox in particular:

Schadet der Widerspruch der entsteht, wenn Einer sagt: "Ich lüge. – Also lüge ich nicht. – Also lüge ich etc." Ich meine: ist unsere Sprache dadurch weniger brauchbar, daß man in diesem Fall aus einem nach den gewöhnlichen Regeln sein Gegenteil & daraus wieder ihn folgern kann? – Der Satz (selbst) ist unbrauchbar, & ebenso dieses Schlüsseziehen; aber im übrigen kann man es tun, wenn man will. || warum soll man es nicht tun? Es ist (nur) eine brotlose Kunst. || ! – Es ist ein Sprachspiel das Ähnlichkeit mit dem Spiel des Daumenfangens hat. (Dies || Dieses wird so gespielt: Man hält den Daumen der

rechten Hand mit der linken, so daß seine Spitze noch oben aus der linken hervorschaut. Nun entzieht man die rechte Hand rasch dem Griff der linken Hand & trachtet die rechte Daumenspitze noch mit der rechten Hand zu fangen, ehe sie sich zurückzieht.)⁹⁸

LW clearly emphasizes the fact that this kind of language use is actually -in real life- (1) useless, and (2) harmless. At best, it is a childish game in which no result is ever reached and no point is made. Paradoxes are an intuitively very clear example of what is meant by 'lack of embedding in practice': the liar's game is never really part of any practical context, and never has any real-life use or any real-life consequences (except perhaps annoyance, or fun, or both). A similar approach is displayed in one of the lectures published as *LFM*, in which LW and Alan Turing discuss the nonsensicality of paradoxes (also quoted in (Fogelin 2009), p. 161):

Think of the case of the Liar. It is very queer in a way that this should have puzzled anyone – much more extraordinary than you might think: that this should be the thing to worry human beings. Because the thing works like this: if a man says "I am lying" we say that it follows that he is not lying, from which it follows that he is lying and so on. Well, so what? . . . It doesn't matter. . . .

Now suppose a man says "I am lying" and I say "Therefore you are not, therefore you are, therefore you are not . . ." – What is wrong? Nothing. Except that it is of no use; it is just a useless language-game, and why should anybody be excited?

....

Turing: What puzzles one is that one usually uses a contradiction as a criterion for having done something wrong. But in this case one cannot find anything done wrong.

Wittgenstein: Yes – and more: nothing has been done wrong. One may say, "This can only be explained by a theory of types." But what is there which needs to be explained? (*LFM*, lecture 20, pp. 206–7)

Again, the point is that from a pragmatic point of view, paradoxes like the liar do not pose any real problem ever and should not get anybody excited (for this undeserved excitement, cf. sections 2.0.3, 2.4.3(C) and 3.2.3(B)), on LW's criticism of pathos and sensationalism).

If the criterion for making sense is embeddedness in practice, then paradoxes of this type are plain and obvious nonsense, a somewhat childish joke at best. Remains the question as to what to do with the enormously vast literature in logic that does appear to take paradoxes very seriously... (see section 2.3 below).

⁹⁸ 12. Is there harm in the contradiction that arises when someone says: "I am lying.--So I am not lying.--So I am lying.--etc."? I mean: does it make our language less usable if in this case, according to the ordinary rules, a proposition yields its contradictory, and vice versa?--the proposition *itself* is unusable, and these inferences equally; but why should they not be made?--It is a profitless performance!--It is a language-game with some similarity to the game of thumb-catching.

(D) formalism as desemantization

The idea that nonsense can be defined in terms of a lack of embedding in practice also leads us to interesting implications regarding formalism, in the general sense of ‘the use of formal systems and theories’, within mathematics. Formalism can at least in part be defined in terms of ‘desemantization’ ((Dutilh Novaes 2012)) and perhaps in a certain sense even in terms of de pragmatization (in the sense of de-embedding), which makes it an interesting topic in the present context. If we take the definition of desemantization literally *and* we take the ‘nonsense = lack of embedding’ idea literally, formalism is strictly speaking nonsense. However, in actual practice, we can see that formal systems are embedded in rich and well-supported practices/discourses, with deep historical roots and wide expansions, and they seem to make sense to the agents that operate with them. I have nothing else to say about the concept of desemantization in the present context of LW’s PhilMath, but will take up the topic in Appendix 4.1 below. The reason why I mention it here, is that it is a good way to summarily introduce the issue of the meaning of formal systems from the point of view of LW’s pragmatic approach to meaning.

1.2.2 Embedded in what? Everydayness

Despite its prima facie intuitive appeal, the idea of ‘lack of embedding’ as what constitutes nonsense soon enough hits its limits: the idea that nonsense corresponds to lack of embedding in practices is open to an obvious and direct objection: in actual fact, nothing actually occurs without being somehow embedded. Thus, it’s actually and obviously not true that philosophical discourse, not even the most esoteric metaphysical verbiage, lacks embedding: philosophical discourses are embedded in rich and wide networks of practices, within and outside philosophy, they have considerable cultural and historical depth, they are supported by relatively large networks of agents, ... Merely saying that this does not count as embedding would be disingenuous and would undermine the coherence and applicability of the notion of embedding.

(A) embedding in everydayness

And this takes us to the next step, which has a lot of textual support in Wittgenstein’s work and its *Nachleben*: the key concept is not so much lack of embedding in general, but lack of embedding in ‘everyday’/‘ordinary’ practices and forms of life.⁹⁹ So: the relevant notion here

⁹⁹ This aspect of LW’s work was recognized early on and supposedly participated in the rise of the branch of analytical philosophy called ‘ordinary language philosophy’.

is **everydayness**. Throughout his work, LW routinely distinguished between ‘normal’ / ‘ordinary’ / ‘everyday’ (‘normal’ / ‘gewöhnlich’ / ‘alltäglich’) contexts and whatever their opposite may be.¹⁰⁰

This idea of everydayness has a certain intuitive appeal and we can easily grasp what kind of thing LW means by looking at his many examples: buying apples; building houses; measuring, buying and selling timber; engineering and operating machines, etc. are straightforward everyday activities, and language use in the context of these activities is straightforwardly unproblematic, in a way that e.g. metaphysical talk, logical paradoxes and Gödel’s code are not.

I have quoted some of the more seminal expressions of this idea in the above (section 0.2(D)): LW’s contrast between the ‘metaphysical’ use of words and the everyday language in which words have their home in *PhU* §116 and the idea that philosophy should leave everything as it is in *PhU* §124: philosophy can neither change actual language use, nor can it offer a foundation for it, so ultimately, it can only describe it.¹⁰¹ This is perhaps the right place to also briefly address the following lines in §124:

It also leaves mathematics as it is, and no mathematical discovery can advance it. A “leading problem of mathematical logic” is for us a problem of mathematics like any other.

The inclusion of this remark in the context of the programmatic part of the *PhU* highlights the fact that for LW, there is no such thing as philosophically relevant mathematical problems: a problem may be a mathematical problem, but from a philosophical point of view, they are all the same. This goes directly against the grain of such luminaries as Cantor (for whom work on the transfinites was “a mission from God” with -to him- clear theological implications) and Gödel (whose most famous contributions were intended to defend mathematical Platonism).¹⁰² Anticipating our analyses in sections 1.3, 2.1, 2.2 and 2.3 below, we can already conclude from this programmatic paragraph that (a) a critical stance towards some of the

¹⁰⁰ The importance of this concept and its counterdistinction from the ‘higher’, the ‘abstract’, the ‘sublime’, etc. within LW’s philosophy is rooted in the anti-rationalist, anti-positivist and generally anti-theoretical tendencies in the various brands of ‘**Lebensphilosophie**’ adopted by such philosophers as Arthur Schopenhauer (for the relations between LW and Schopenhauer, see e.g. (Jacquette 2017), Søren Kierkegaard, and Friedrich Nietzsche, a background also shared (and processed in his own distinctive way) by LW’s contemporary Heidegger. Cf. also section 2.0.0 below.

¹⁰¹ 124. Die Philosophie darf den tatsächlichen Gebrauch der Sprache in keiner Weise antasten, sie kann ihn am Ende also nur beschreiben. / Denn sie kann ihn auch nicht begründen. / Sie läßt alles wie es ist. / Sie läßt auch die Mathematik wie sie ist, und keine mathematische Entdeckung kann sie weiterbringen. Ein “führendes Problem der mathematischen Logik” ist für uns ein Problem der Mathematik, wie jedes andere.

¹⁰² See also (Rittberg 2016), in which it is argued that “that mathematics can actively influence metaphysics, i.e. that mathematicians can set up mathematics in such a way that by doing mathematics they can actively influence metaphysical debates.” (p. 287), which goes even further in the direction that LW objects against.

major voices in the Grundlagen-debate are -in LW's own mind- at the core of his philosophy, and (b) the everyday vs. non-everyday distinction is deeply intertwined with this agenda.

(B) the mathematical everyday

In the case of mathematics, LW makes abundantly clear what he means by everyday practices, by means of very extensive exemplification. LW's insistence on the importance of embedding in practices takes the shape of reflections on a very large number of examples in which calculations are put to use as an integral part of a wide variety of practical applications, which give meaning to these calculations (see section 1.3 below).

LW also makes his point in more abstract terms. Thus, for instance, he famously states that mathematical concepts, in order to be meaningful, have to also be used in 'civilian clothes':

BGM5 §2:

Es ist der Mathematik wesentlich, daß ihre Zeichen auch im Zivil gebraucht werden. Es ist der Gebrauch außerhalb der Mathematik, also die Bedeutung der Zeichen, was das Zeichenspiel zur Mathematik macht.¹⁰³

For instance, in section 1.3 below, we will encounter the example of calculating the surface of a sphere (MS-126, 37-38), about which LW remarks that whatever it means 'to acquire a new understanding of the surface of sphere', this new conceptualization, for it to be a conceptualization of the surface of a sphere, should still be applicable to actual spheres (section 1.3(B) below).¹⁰⁴ In exactly the same way, whatever number theory you may want to affiliate with, for it to be a number theory, it will have to deal with what we do when we count apples.

NB that the remark about the 'civilian clothes' occurs in a context in which our own 'lived' experience is contrasted with what a machine does (section 1.1.1(H)). Again, we can observe that the lines of thought concerning formalism in terms of 'live signs vs. dead signs', the lines of thought concerning the importance of applications, and the lines of thought in terms of embeddedness in the everyday are intertwined at a very fundamental level.

¹⁰³ "I want to say: it is essential to mathematics that its signs are also employed in mufti. It is the use outside mathematics, and so the meaning of the signs, that makes the sign-game into mathematics. Just as it is not logical inference either, for me to make a change from one formation to another (say from one arrangement of chairs to another) if these arrangements have not a linguistic function apart from this transformation." (*BGM/RFM*, V, §2) The choice for the slightly slangy translation "in mufti" when there is perfectly neutral and transparent English equivalent of LW's "im Zivil" available ("in civilian clothes") is yet another example of an amateurish choice on the part of the editors/translators.

¹⁰⁴ Was heißt es, einen neuen Begriff von der Oberfläche einer Kugel gewinnen? In wiefern ist das dann ein Begriff von der Oberfläche einer Kugel? Doch nur insofern er sich auf wirkliche Kugeln anwenden läßt.

1.2.3 What is the non-everyday? (On everydayness as a moralistic concept)

The distinction between the everyday and the non-everyday is at first intuitively attractive and there is no doubt that some version of it was supported by LW (and many others). However, there is problem with the contrast-class of the everyday:¹⁰⁵ it is hard to articulate what constitutes the non-everyday. In LW's text, the non-everyday is always exemplified by metaphysics, mathematical logic, set theoretical discourse, or even philosophy in general (cf. sections 0.2(D) and 2.4.2(A)).

(A) everyday calculations vs. parasitic prose

The literature has paid a lot of attention to LW's distinction between mathematical operations (calculating, proving, constructing geometrical diagrams, etc.) on the one hand, and the "prose" that surrounds these operations on the other: LW supposedly only criticized the prose and left alone the operations. Juliet Floyd summarizes the topic as follows:¹⁰⁶

Within mathematics – as Wittgenstein is the first to insist – there is often an important distinction to be drawn between an intuitive notion and a rigorous mathematical notion, between, for example, an intuitive mathematical argument and a formalized proof. The 'prose' surrounding a proof may be perfectly unobjectionable, even indispensable. But some prose has the tendency to mislead, and is mathematically inessential. 'There are true but unprovable propositions in mathematics' is misleading prose for the philosopher, according to Wittgenstein. It fools people into thinking that they understand Gödel's theorem simply in virtue of their grasp of the notions of *mathematical proof* and *mathematical truth*. And it fools them into thinking that Gödel's theorem supports or requires a particular metaphysical view. ((Floyd 2001), p. 299)

I have not much to add to the literature about this prose vs. calculation distinction, except perhaps that, independently of the question of how important the distinction is for LW's work on math, it participates in the obvious weaknesses surrounding the concept everydayness in general (see below) and I believe that the issues that LW is pointing at when using this distinction actually become more interesting (not less interesting), if we don't use the prose-calculation distinction: to my mind, one can view the talk about the practice as an integral part of the practice and study the relation between the talk and the other aspects of the practice (or the lack of such a relation) as a matter of the internal structure of the practice. But this disagreement would lead us beyond the topics at hand.

¹⁰⁵ For the problematic nature of the contrast-class of the everyday in LW, see (Baker 2002); (Read 2010); (Scheppers 2017), §3.1.

¹⁰⁶ See also e.g. (A. W. Moore 2017) p.322; (Kienzler and Grève 2016) p. 79, p. 81); (Stenlund 2015).

(B) healthy working mathematicians' (or engineers' or accountants') math vs. degenerate philosophers' math

The notion of everydayness in mathematics is not LW's invention: it occurs before and after LW, often opposing "working mathematicians" to logicians and other esotericists. For example, a number of publications since "Foundations of Mathematics for the Working Mathematician" ((Bourbaki 1949)) contain the expression "for the Working Mathematician". Dieudonné, a prominent member of the Bourbaki-collective, spends almost 5 pages of his 1982 article "Mathématiques vides et mathématiques significatives" on the subject matter of "Logique et mathématiques" and starts this section as follows:

Les philosophes et les logiciens ont une tendance, parfaitement naturelle et excusable, à croire que les mathématiciens s'intéressent beaucoup à ce qu'ils font. Détrompez-les, ce n'est pas vrai : 95% des mathématiciens se moquent éperdument de ce que peuvent faire tous les logiciens et tous les philosophes. Cela ne les intéresse absolument pas.

((Dieudonné 1982), p. 16)

And on the next page, Dieudonné says:

Alors, quand on vient nous parler de la logique du premier et du deuxième ordre, de fonctions récursives et de modèles, théories très gentilles et très belles qui ont obtenu des résultats remarquables, nous mathématiciens, nous ne voyons aucune objection à ce qu'on s'en occupe, mais cela nous laisse entièrement froids.¹⁰⁷

((Dieudonné 1982), p. 17)

LW thus participates in a trope that is quite common, but for LW, the distinction is an obvious avatar of the distinction between the everyday and non-everyday and takes on a much more central place in his philosophical work than is usual.

In this context, it may also be useful to point out that non-everyday mathematical discourse is invariably valuated in negative terms (sometimes hyperbolically negative terms) by LW: set-theoretical parlance is called "a tumor", "pernicious", "the illness of our time", etc. (see section 2.2 below) and in Ms-127,184-187 (= BGM V §46), he talks about "the curse of the invasion of mathematics by mathematical logic".¹⁰⁸ Again, we can point out that LW was not

¹⁰⁷ For a similarly disdainful attitude towards sociological accounts of mathematics, see (Dieudonné 1982), pp. 22-23; see also section 3.2.3(A) below.

¹⁰⁸ "Das ist der Fluch des Einbruchs der math. Logik in die Mathematik, daß nun jeder Satz sich in mathematischer Schreibung darstellen läßt & wir uns daher verpflichtet fühlen ihn zu verstehen. Obwohl ja diese Schreibweise nur die Übersetzung der vagen gewöhnlichen Prosa ist." (cf. Ms-126,108: "28.11. "Der unheilvolle Einbruch" der Logik in die Mathematik.").

the only one to use that kind of language in similar mathematical contexts. Thus, Cantor-biographer Joseph W. Dauben says in a 1982 lecture:

So shocking and counter-intuitive were Cantor's ideas at first that the eminent French mathematician Henri Poincaré condemned the theory of transfinite numbers as a "disease" from which he was certain mathematics would someday be cured. Leopold Kronecker, one of Cantor's teachers and among the most prominent members of the German mathematics establishment, even attacked Cantor personally, calling him a "scientific charlatan," a "renegade" and a "corrupter of youth."

(Dauben 1989)

So, whereas in the meantime set theoretical discourse has become part of the mainstream and -at the same time- the ways in which people normally voice their criticism in public may have changed, LW's opinions and the ways in which he expressed them, may appear more eccentric now than they used to be at the time.

(C) the inherent weakness of the concept of "everydayness": everydayness as an agenda not a result
Although the intuitive notion of 'everydayness' -as introduced here above- is the one we need to make sense of LW's critical remarks that we are going to look at in what follows, everydayness is also problematic: why should talk about "nothingness" not be part of the philosopher's everyday, and infinite sets part of the mathematician's everyday, in the same way that talk about eggplants is part of the greengrocer's and the cook's everyday?

Why make the distinction between the everyday and the non-everyday and why make the cut-off at the level of the 'theorist'? Why are 'working mathematicians' (whatever that is) or engineers 'normal', and philosophers and people who practice mathematical logic not? Isn't one person's exotic, another person's everyday? Isn't it also true that once we get acquainted with a certain Form of Life, it will start to make sense (and look more 'normal') to us? Isn't this intuitive sort of relativism what LW's explorations instill in us? In other words: *everydayness is inadequate as a neutral descriptive/empirical concept.*

I will not pursue this line of thought here, as I will not normally go beyond the limits of LW's own contributions in this part of my study.¹⁰⁹ But the observation that everydayness is not an

¹⁰⁹ In (Scheppers 2009) and (Scheppers 2017), I make the case for a very similar analysis of Heidegger's use of the concept of everydayness. Heidegger was LW's contemporary (though Heidegger lived longer) and "everydayness", along with the emphasis on 'practice' and a preoccupation with 'authenticity', were obviously part of the cultural common ground shared by many German-speaking intellectuals of that era. For similarities between LW and his contemporary and fellow candidate for the title of 'greatest philosopher of the 20th century", see e.g. (Braver 2012); (Egan, Reynolds, and Wendland 2013); (Egan 2019).

I will argue elsewhere that one of the main differences between both authors/thinkers can be illustrated beautifully by pointing out their different valuation of 'everydayness', as the locus of authenticity in LW's work, and as the source of inauthenticity in MFH.

empirical notion that emerged from the analysis of actual practices is important for our understanding of what is at stake in LW's philosophy at large, and his PhilMath in particular: it suggests that the distinction between the everyday and the non-everyday is part of the agenda underlying LW's work, part of the premises, not part of the results.

1.3 Wittgenstein's emphasis on fringe-applications (MS-126, 37-81, 19421030-19421115)

In this section, I present a reading of an extended passage that illustrates a few lines of thought that are at the core of the subject matter of this study.

LW seems to insist a lot on atypical, even fringe, applications, often even including stuff that he made up himself. In some of his manuscripts, he goes on for dozens of pages on end, investigating one example after another of atypical applications of math-like calculations, or bizarre circumstances in which math-like activities could or could not have existed. In this section, we will focus on an extended passage from MS-126 (38-81), partially published in an edited form as *BGM V*, §§5-8.

(A) a list of examples

First, I would like to present a simple list of examples taken from this passage. Among the examples, there are a few that refer to 'normal' applications of normal mathematical techniques:

- calculating the surface of a sphere (in theory and of an actual sphere) (MS-126, 37-38);
- the construction a force polygon (MS-126, 56-57).¹¹⁰

But the majority of the examples LW presents in this passage are more exotic, in many cases fictional:¹¹¹

- what if arithmetic was only used as cipher (MS-126, 39-41; = Suhrkamp, p. 260);¹¹²

¹¹⁰ 6.11.

Nimm die Konstruktion des Kräftepolygons: ist das nicht ein Stück angewandte Mathematik? & wo ist der Satz der reinen Mathematik der bei dieser graphischen Berechnung zu Hilfe genommen wird?

¹¹¹ Cf. also the wonderful case of equations used for designing wallpaper (*BGM VII*, §41; *LFM* (Wittgenstein 1976), Lecture III, pp. 36-37). These passages in LW are -by the way- not the only interesting link between math and wallpaper: apparently, Sofia Kovalevskaya "taught herself calculus from wallpaper made from a calculus book" (Martin & Roitman 2014, p. 67). As for poetic anecdotes: LW apparently shared a pencil with Otto Neugebauer, the historian of ancient mathematics, while both were prisoners of war in Monte Cassino ((Floyd 2016), p. 57; (Swerdlow 1993), p. 139; (Høyrup 2017), p. 4).

¹¹² 2.11.

Wenn die arithmetischen Operationen lediglich zur Konstruktion einer Chiffre dienen wäre ihre Verwendung natürlich grundlegend von der unsern verschieden. Wären diese Operationen dann aber überhaupt mathematische Operationen?

- 4D geometry as a means to study the living conditions of ghosts (MS-126, 41 = Suhrkamp, p. 260);¹¹³
- calculating with numbers above 1000 only used for studying ghosts (MS-126, 42);
- fictional math: infinite numbers as part of a fairy tale (MS-126, 54-55);¹¹⁴
- set theory as a parody of math (MS-126, 55-56);¹¹⁵
- oracular math and ceremonial math (MS-126, 57-58 = Suhrkamp, p. 265);¹¹⁶
- competitive math (MS126, 80 = Suhrkamp, p. 273);¹¹⁷
- calculating in rhyme;¹¹⁸
- math for studying ghosts (again).¹¹⁹

The question is, every time: is this still math, or is it not? (Seriously: is it or is it not?) We will see later on that LW's insistence on the question of demarcation in this context is not aimed at promoting a clear-cut demarcation at all, but that asking the questions helps him and his readers to come to a more adequate attitude towards the issue.

Kann man von Dem, der eine Regel des Entzifferns anwendet, sagen, er vollziehe mathem. Operationen? Und doch lassen sich seine Transformationen || Umformungen so auffassen. Denn er könnte doch sagen, er berechne, was bei der Entzifferung des Zeichens ... nach der und der Regel herauskommen müsse. || des Zeichens ... gemäß dem & dem Schlüssel herauskommen müsse. Und der Satz: daß die Zeichen ... dieser Regel gemäß entziffert ... ergeben ist ein mathematischer. Sowie auch der Satz: daß man beim Schachspiel von dieser Stellung zu jener kommen kann.

¹¹³ Denke Dir die Geometrie des vierdimensionalen Raums zu dem Zweck betrieben, die Lebensbedingungen der Geister kennen zu lernen. Ist sie darum nicht Mathematik? Und kann ich nun sagen sie bestimme Begriffe?

¹¹⁴ Denke Dir unendliche Zahlen in: einem Märchen gebraucht. Die Zwerge haben sovielen Goldstücke aufeinander gelegt || getürmt, als es Kardinalzahlen gibt, || - etc. Was in einem Märchen vorkommen kann, muß doch Sinn haben. -

¹¹⁵ Denke Dir die Mengenlehre wäre als eine Art Parodie der || auf die Mathematik von einem Satiriker erfunden worden. - Später hätte man dann einen Nutzen || einen vernünftigen Sinn in ihr gesehen & sie in die Mathematik einbezogen. (Denn wenn der eine sie als das Paradies der Mathematiker ansehen kann, warum nicht ein anderer als einen Scherz || Witz?)

Die Frage ist: ist sie nun als Scherz nicht auch offenbar Mathematik? -

¹¹⁶ Ist dies nicht ein Fall wie der des Stammes, welcher eine rechnerische Technik zum Zweck gewisser Vorhersagungen hat, aber keine Sätze der reinen Mathematik?

Die Rechnung die zur Ausführung einer Zeremonie dient. Es werde z.B. nach einer bestimmten Technik aus dem Alter des Vaters & der Mutter & der Anzahl ihrer Kinder die Anzahl der Worte einer Segensformel abgeleitet die auf das Haus der Familie anzuwenden ist. In einem Gesetz wie dem Mosaischen könnte man sich Rechenvorgänge beschrieben denken || solche Rechenvorschriften niedergelegt denken. Und könnte man sich nicht denken, daß das Volk das diese zeremoniellen Rechenvorschriften besitzt im praktischen Leben nie rechnet?

Dies wäre zwar ein angewandtes Rechnen, aber es würde nicht dem Zweck einer || der Vorhersage dienen.

¹¹⁷ Die Menschen könnten z.B. Rechnungen zum Zweck einer Art von Wettrennen gebrauchen. Wie Kinder ja wirklich manchmal um die Wette rechnen; nur daß diese Verwendung bei uns keine große || eine ganz untergeordnete Rolle spielt.

¹¹⁸ LW makes us imagine that multiplication would be a lot harder than it actually is for us, e.g. because one only calculates orally, and has to construe a rhymed poem for each calculation.

Oder das Multiplizieren könnte uns viel schwerer fallen, als es tut - wenn wir z.B. nur mündlich rechneten, & um uns eine Multiplikation zu merken, sie also zu erfassen, wäre es nötig sie in die Form eines gereimten Gedichts zu bringen. Wäre dies dann einem Menschen gelungen, so hätte er das Gefühl, eine große, wunderbare Wahrheit gefunden zu haben.

Es wäre sozusagen für jede neue Multiplikation eine neue individuelle Arbeit nötig.

¹¹⁹ Wenn diese Leute nun glaubten, die Zahlen wären Geister & durch ihre Rechnungen erforschten sie das Geisterreich, oder zwängen die Geister, sich zu offenbaren - wäre dies nun Arithmetik? Oder - wäre es auch dann Arithmetik, wenn diese Menschen die Rechnungen zu nichts anderm gebrauchten?

These explorations are good example of LW's 'polyphonic' style (see section 0.2(A) above): he explores the subject matter with which he struggles by internalizing various potential opinions and reactions one may want to express in response to the examples he conjures up; he does not appear to use the examples to illustrate a pre-established point. This does not mean that he starts with a blank slate: of course, he has biases, agendas and opinions, which guide his process. The sheer quantity of the examples is part of the process that is enacted in the text: it resets the reader's sense of what is a 'normal' case, perhaps in the same way it did for LW. Working through the entire passage is a worthwhile exercise that I recommend, but for the purposes of this study, I can only highlight a few excerpts.

(B) LW, Ms-126,37-39, d.d. 19421030-19421101: math as a game

LW starts by exploring the idea that mathematics would be a formal game,¹²⁰ in which one never appeals to an extra-mathematical application, and asks what that would even mean: does it mean (1) that one exits and then reenters math proper, or (2) that one transits from one type of mathematical inference to another type of mathematical inference?¹²¹ I think it is important to understand that it is not really important what answer to these questions (yes or no) one would ultimately want to give. The point of these questions is that they make us realize that the relation between 'pure' math and its application is by no means self-evident. In any case, this line of questioning already suggests that there is a tension between (1) the autonomy of an axiomatic system (its game-like nature) and (2) the more 'applied' math that it is supposed to formalize, and that (2) can do without (1), but (1) can't do without (2).

This idea is illustrated by the first example in our excerpt, which is 'calculating the surface of a sphere'. LW asks what it would mean to 'gain a new conception of the calculation of the surface of a sphere' and points out that the very identity of what we would call "calculating the surface of a sphere" still depends on whether we can use the technique to actually calculate the actual surface of an actual sphere.¹²²

LW then transitions to the following question: "To what extent does one need to have a concept of 'proposition' in order to understand Russell's mathematical logic?".¹²³ Though somewhat abrupt, this transition need not be cryptic at all: again, LW points out that the

¹²⁰ The theme of axiomatic systems had been introduced on p. 20 of the notebook, d.d. 19421025.

¹²¹ Zu sagen, die Math. sei ein Spiel, soll heißen: wir brauchen beim Beweisen nirgends an die Bedeutung der Zeichen appellieren, also an ihre außermathematische Anwendung. Aber was heißt es denn überhaupt, || : an diese appellieren? Wie kann so ein Appell etwas fruchten?

Heißt das, aus der Mathematik heraustreten & wieder in sie zurückkehren, oder heißt es aus einer math. Schlußweise in eine andre treten?

¹²² Was heißt es, einen neuen Begriff von der Oberfläche einer Kugel gewinnen? In wiefern ist das dann ein Begriff von der Oberfläche einer Kugel? Doch nur insofern er sich auf wirkliche Kugeln anwenden läßt.

¹²³ Wieweit muß man einen Begriff vom 'Satz' haben, um die Russellsche mathem. Logik zu verstehen?

formalized system probably cannot function without a link to a pre-formal notion of 'proposition': the pre-formal notion makes Russell's formalization meaningful. Again, the point is the primacy of the everyday application and the fact that the identity of a formalized mathematical technique still depends on its link with the applied technique it is intended to formalize.

The next day, LW continues to think about the topic of the importance of an intended application for math and considers the borderline case of an application that is 'fantastic' (in the sense of 'pure fantasy'), i.e. that is not understood adequately by the agents themselves. LW cannot help himself and gives away his game by suggesting in passing that this is actually the case in set theory.¹²⁴ Those of us that would be offended by this suggestion may want to let it slide and look at the many other examples that follow instead: math as cipher, oracular math, math for the study of ghosts, ... (see list above).

(C) LW, Ms-126,47-50, d.d. 19421104: "isn't math, accompanied by bullshit, still math?"

Let's pick up the thread a few days later. LW presents a series of scenarios, asking in each case whether we would call what is being done, mathematics:

- someone calculates competently with complex numbers and is able to apply these calculations in physics, while holding strange beliefs about the nature of $\sqrt{-1}$ and how it was discovered;¹²⁵
- someone expands math with new definitions and theorems in an apparently competent fashion, but seems to conceive of this expansion as the discovery of a new space (which he apparently thinks of as some kind of room) and talks a lot of nonsense when asked to explain;¹²⁶

¹²⁴ 1.11.42.

Wenn die intendierte Anwendung der Math. wesentlich ist, wie steht es da mit Teilen der Mathematik, deren Anwendung - wenigstens || oder doch das, was Mathematiker für eine || die Anwendung hielten || halten, - gänzlich phantastisch ist. So daß man, wie in der Mengenlehre, einen Zweig der Math. treibt, von dessen Anwendung man sich einen ganz falschen Begriff macht. Treibt man nun nicht doch Mathematik?

¹²⁵ Wer glaubt, die Mathematiker haben ein seltsames Wesen, die $\sqrt{-1}$, entdeckt, die || das quadriert nun doch - 1 ergebe || ergäbe, kann der nicht doch ganz gut mit komplexen Zahlen rechnen & solche Rechnungen in der Physik anwenden? Und sind's darum weniger Rechnungen?

In einer Beziehung steht freilich sein Verständnis auf schwachen Füßen; aber er wird mit Sicherheit seine Schlüsse ziehen, & sein, Kalkül wird auf festen Füßen stehen.

Wäre es nun nicht lächerlich, zu sagen, dieser triebe nicht Mathematik?

¹²⁶ Es erweitert Einer die Math., gibt neue Definitionen & findet neue Lehrsätze - & in gewisser Beziehung kann man sagen, er wisse nicht, was er tut. - Er hat eine vage Vorstellung, etwas entdeckt zu haben wie einen Raum (wobei er an ein || sein Zimmer denkt), ein Reich erschlossen zu haben, & würde, darüber gefragt, viel Unsinn reden.

- someone executes enormously large multiplications in order to “conquer gigantic new provinces of the land of numbers”;¹²⁷
- a jester invented calculating with $\sqrt{-1}$ as an absurdist joke, thinking that he is writing down and operating with impossible things.¹²⁸

LW then asks the following:

Mit andern Worten: Wer an die mathematischen Gegenstände glaubt & ihre seltsamen Eigenschaften, - kann der nicht doch Mathematik betreiben? Oder: - treibt der nicht auch Mathematik?¹²⁹

What does LW exactly mean here? What does he refer to by “those who believe in mathematical objects”? More precisely: what does the word “die” in “die mathematischen Gegenstände” imply? Two potential interpretations:

- does he mean the hypothetical people who believe in the bizarre objects mentioned above (“die” as a pronoun carrying emphasis, referring back to previous sentence)?
- or does he mean any actual person who believes in any mathematical object at all (“die” as the article carrying no emphasis and referring to objects in general)?

The second reading seems more natural, especially without added emphasis in the manuscript (I checked), and this is also the reading reflected in the standard English translation (without an article). But does this description not apply to most mathematicians (both in the 1940s and now)? In that case, what LW says is not only provocative but also heavily ironic, or is LW now simply (innocently?) speaking from his own point of view according to which it is an established fact that believing in mathematical objects is as crazy as the people in his previous examples?¹³⁰

In any case, the above fictional examples are obviously intended to be compared to what contemporary practitioners of PhilMath were saying and are an integral part of LW’s critical endeavors, which will make up the subject matter of Part 2 below.

¹²⁷ Denken wir uns den primitiven Fall, daß Einer ungeheure Multiplikationen ausführte um wie er sagt: dadurch neue riesige Provinzen des Zahlenreichs zu gewinnen.

¹²⁸ Denk Dir das Rechnen mit der $\sqrt{-1}$ wäre von einem Narren erfunden worden, der bloß vom Paradoxen der Idee angezogen die Rechnung als eine Art Gottesdienst || Gottes- oder Tempeldienst des Absurden treibt. Er bildet sich ein das Unmögliche || schlechthin Unmögliche aufzuschreiben & mit ihm zu operieren.

¹²⁹ In other words: if someone believes in mathematical *objects* and their queer properties--can't he nevertheless do mathematics? Or--isn't he also doing mathematics?

¹³⁰ Let us not forget that these are notebooks, not immediately intended for publication, and that it is not a priori clear who -besides himself- were the audience LW imagined writing for (if any). It should also be clear by now that working through different points of view is an integral part of LW’s working method and writing style.

(D) LW, Ms-126,55-58, d.d. 19421105-19421107: “a kind of parody”

Before turning to set theory, LW first asks us to imagine a scenario in which infinite numbers are used in a fairy tale: the dwarfs have accumulated as many gold coins as there are cardinal numbers. LW then remarks (in all seriousness?) that “what can occur in a fairy tale, has to make sense, doesn’t it?”¹³¹ Apart from the question as to whether this is funny or not, and (different question) as to whether it was intended to be funny or not, it is true *and relevant* that stuff that occurs within fairy tales has to make sense - at least, in a certain sense.

LW then turns to one of his favorite topics/targets, set theory, and asks us to imagine that set theory was invented by a satirist, and that only later on, one had found something useful or reasonable to it and included it within normal math. And -with an undoubtedly snide reference to Hilbert’s famous remark on Cantor- he adds in parentheses: “If one person can view set theory as the “**mathematicians’ paradise**”, why can’t someone else not see it as a joke?”¹³² The implication is that neither explanation says anything worthwhile about the math qua math.

LW then says: “The question is: is set theory as a joke not also evidently math?”¹³³ So, LW’s point is not (at least not in this context) that set theory is a joke, a parody of math.¹³⁴ What he is saying is that even if it is a joke (or part of a fairy tale), it could still be viewed as math. LW then explores several options as to why set theory is evidently math, referring to various opinions that were current within the context of the Grundlagen-debates:

- (1) He first suggests that perhaps it is because it is a symbolic game following a set of rules (this option obviously refers to formalism).
- (2) Apparently as an objection to the formalist option (1), he then suggests that even in set-theory-qua-joke certain concepts are being constructed (reference to the anti-formalist / ‘conceptualist’ party in the Grundlagen-debates), even if one is confused about the application of the concepts.

¹³¹ Denke Dir unendliche Zahlen in: einem Märchen gebraucht. Die Zwerge haben sovielen Goldstücke aufeinander gelegt || getürmt, als es Kardinalzahlen gibt, || - etc. Was in einem Märchen vorkommen kann, muß doch Sinn haben. -

¹³² Denke Dir die Mengenlehre wäre als eine Art Parodie der || auf die Mathematik von einem Satiriker erfunden worden. - Später hätte man dann einen Nutzen || einen vernünftigen Sinn in ihr gesehen & sie in die Mathematik einbezogen. (Denn wenn der eine sie als das Paradies der Mathematiker ansehen kann, warum nicht ein anderer als einen Scherz || Witz?)

¹³³ Die Frage ist: ist sie nun als Scherz nicht auch offenbar Mathematik? -

¹³⁴ Of course, LW may actually have thought that set theory is a joke, coincidentally. But even if he was inspired by his actual opinion “set theory is a joke” to use the example “set theory is a joke” to make his point here, he is still not actually saying that “set theory is a joke” in the present context.

(3) Apparently as an objection to (2), he then asks how it is possible to have a concept and not be clear about its application (which corresponds to one of the main points of his own work as a whole).¹³⁵

It appears that LW steers us away from both formalism and conceptualism as viable explanations of why set-theory-as-a-joke would still be math. Still, I don't think that LW tries to force the point that set-theory-as-a-joke is not -in some sense- math: let's not forget that this example occurs somewhere in the middle of a long list of examples that have in common (a) math-like operations and (b) a wide range of different practical contexts that make them somehow meaningful to practitioners. Within this context, the point seems to be that whatever you believe you're doing while you're calculating doesn't matter that much: you're still calculating and your calculations can still play a meaningful role in whatever practical context they occur in, even if you entertain completely nonsensical beliefs about the nature of your calculations. The snide remark in which he equals Hilbert's idea of set theory as "the mathematician's paradise" with his own fictional idea of set theory as a joke, suggests that a lot of what is said within the context of PhilMath is talk of exactly this type: unfortunate nonsense that -fortunately- is external to (1) what makes mathematical technique mathematical technique and (2) what makes a mathematical application that mathematical application.

(E) LW, Ms-126, 57-58, d.d. 19421106-07; 77-81, d.d. 19421115 "a family of activities, with a family of applications"

The next day (November 6), LW goes on with his list of examples: the construction of a force polygon (an application for which no propositions of pure mathematic are needed), a tribe that uses calculations for divinatory purposes, a people that uses ceremonial calculations but otherwise doesn't calculate.

The day after that (November 7), LW only writes the following: "7.11. Would it be any wonder if the technique of calculating had a family of applications?".¹³⁶ This sounds like a conclusion of some sorts: *calculating is not one thing, but a family of many different things.*

And LW was not ready with this idea: a week later (Ms-126,77-81, d.d. 19421115), after writing about 20 notebook pages worth of remarks on infinite decimal expansions and similar topics,

¹³⁵ Und warum ist sie offenbar Mathematik? – Weil sie ein Zeichenspiel nach Regeln ist?

Werden hier nicht doch offenbar Begriffe gebildet – auch wenn man sich über deren Anwendung nicht im Klaren ist?

Aber wie kann man einen Begriff haben & sich über seine Anwendung nicht im Klaren sein? | | nicht klar sein?
¹³⁶ 7.11.

Wäre es ein Wunder wenn die Technik des Rechnens eine Familie von Anwendungen hätte?!

he writes on November 15: "But where is the problem here? Why should I not say that what we call mathematics is a family of activities with a family of purposes?"¹³⁷ He illustrates what he means with another batch of imaginary examples:

- calculating in rhyme: LW makes us imagine that multiplication would be a lot harder than it actually is for us, e.g. because one only calculates orally, and has to construe a rhymed poem for each calculation;¹³⁸
- studying ghosts (again): LW asks whether it would be considered arithmetic if people thought that numbers were ghosts, and that calculations served the purpose of studying the spiritual plane, etc.¹³⁹

It is true that the problem of what still counts as mathematics stops to be a real problem once one accepts that mathematics is not one single unitary thing, but a family of quite heterogeneous activities, with a quite heterogeneous set of applications.¹⁴⁰

(F) summary

The text we just read does not contain many (if any) doctrinary or dogmatic statements, but consists of a lengthy back and forth, working through the material, struggling with a large number of atypical (sometimes made up¹⁴¹) applications, and a number of opposing potential

¹³⁷ 15.11.

Aber wo ist hier das Problem? Warum soll ich nicht sagen, was wir Mathematik nennen sei eine Familie von Tätigkeiten zu einer Familie von Zwecken.

¹³⁸ Oder das Multiplizieren könnte uns viel schwerer fallen, als es tut – wenn wir z.B. nur mündlich rechneten, & um uns eine Multiplikation zu merken, sie also zu erfassen, wäre es nötig sie in die Form eines gereimten Gedichts zu bringen. Wäre dies dann einem Menschen gelungen, so hätte er das Gefühl, eine große, wunderbare Wahrheit gefunden zu haben.

Es wäre sozusagen für jede neue Multiplikation eine neue individuelle Arbeit nötig.

¹³⁹ Wenn diese Leute nun glaubten, die Zahlen wären Geister & durch ihre Rechnungen erforschten sie das Geisterreich, oder zwängen die Geister, sich zu offenbaren – wäre dies nun Arithmetik? Oder – wäre es auch dann Arithmetik, wenn diese Menschen die Rechnungen zu nichts anderm gebrauchten?

¹⁴⁰ After the remarks illustrating the idea that mathematics is a family of activities with a family of applications, LW writes somewhat cryptically, as a separate paragraph, and between parentheses: "(Ich suche einen Abstieg.)" ("I am looking for a way to get off?"), after which he embarks on a longish development of the idea of "mathematical alchemy", including coded remarks on his not so rosey mental state.

¹⁴¹ I would like to point out that LW could have chosen real historical or ethnographical materials to do the same work (there is a lot of freaky historical and ethnographic material out there!); see section 3.2.1(C). Of course, LW did not have access to this material and, more importantly, for the properly philosophical process he takes us through, it doesn't matter if the cases are real or not.

Numerology may be a real-life case in point. Is numerology mathematics? It apparently has many things in common with mainstream math, some of the main objects (natural numbers) and operations involved in it appear to be identical, and in some historical contexts they were definitely intertwined, and historically, not all practitioners made a clear distinction between both. But then again, numerology involves a number of (religious or at least ritual) aspects that do not normally play a role in (what we would call) proper math and sometimes this yields results that would be unacceptable in normal math... Cf. Burkert's ((Burkert 1972), p. 398) comment "number symbolism smothers mathematics" regarding the case of Pythagorean mathematician Philolaus (some Pythagoreans were mathematicians by that time) was precluded from finding the 'right' solution to a musical problem by his religious and numerological commitments. Pythagorean tenet "the whole tone / octave cannot be dissected" is nonsense from a mathematical point of view, but not so from the point of view of the numerical symbolism that underlies Pythagoreanism (number as such is important, not proportions).

opinions about, or reactions to, this material. The text illustrates a few different of aspects of LW's PhilMath that I would like to briefly focus on and make explicit.

(a) pragmatism

The above thought experiments presuppose that the calculations are always recognizable as calculations of some kind, whatever the prose that comes with them, and whatever the applications they are part of. So, LW operates basically with three different aspects of the various practices he is evoking:

- (1) some sort of calculation, qua operational technique;
- (2) a practical application that (1) is part of / embedded in;
- (3) optionally, a discourse or a set of beliefs that comes with (1) and/or (2).

As opposed to mainstream approaches to PhilMath, LW emphasizes what people do (i.e. in all these cases, some kind of calculation) and how this activity fits in with an encompassing practical context ("applications"), which in its turn highlights the intertwining of the apparently mathematical and the clearly non-mathematical within each application. All of this fits in with the features of LW's approach to meaning and practice that we discussed in section 1.1 above.

(b) comparative / anthropological approach

Methodologically speaking, it is obvious that LW's approach is a **comparative** one: he wants to shed light on our normal mathematical practices by comparing them to a wide range of other activities that are in some respects like it.

Demarcation is a recurrent theme in the passage we read: time and time again, LW asks: "Is this still mathematics or is it not?" It does not really matter if we answer the questions with yes or no in any particular case. The point is that there is never a clear and **natural** ¹⁴² line separating math from non-math. This idea is expressed (here as elsewhere) in terms of a 'family of techniques' and a 'family of applications'.

The effect of working through the long series of examples (not unlike studying real ethnographic material) is that it forces us to make a comparison between the 'exotic' practices and our own 'normal' practices and to realize (1) that our 'normal' practice is only one option

By the way, in Ms-116,247, LW ridicules philosophers who collect empirical facts 'as if the factuality of these things was important to us' (cf. section 3.2.1(C) below).

¹⁴² Most of us would probably agree that there is a cut-off *at some point*, but -as LW points out- it is not self-evident at all that there is a single 'natural' cut-off. As a matter of fact, LW appears to specifically target this notion of 'naturalness' as the object of his critique. And that takes us closer to the critical aspects of LW's philosophy, that we will discuss in Part 2 below.

between many (and none of the options is more 'natural' than the next), and (2) there are also many structural analogies between the different options.

Thus, the relation between the prose about math that is characteristic of PhilMath and actual math is not fundamentally different from the relation between the beliefs any other practitioner may entertain about his own techniques: ultimately, what counts is the way in which the technique is actually applied in actual everyday practice (as a way to build houses, to calculate prices, to predict the future, to study the lives of the ghosts, ...).

(c) anti-foundationalism

The most recurrent explicit point that LW makes throughout his long series thought experiments is that mathematical or math-like calculations (as we have discussed above, there is no natural way to operate a clear demarcation between both), become meaningful by being embedded in wide variety of different practical applications, not by what practitioners say or believe about them.

Many (if not all) of the examples de-emphasize the link between calculation and 'foundational' talk by presenting practical applications that work perfectly fine without any foundational talk, or -interestingly- even function when accompanied by completely nonsensical, even moronic explanations.

That last point about obviously nonsensical or stupid accompanying talk is interesting in that it pre-empts a possible objection to LW's criticism of mainstream PhilMath. If the question is: "Is it even imaginable that mathematicians talk bullshit about their own technique?", LW's reply would be: "Yes, very much so". What is the difference between believing in mathematical objects and believing that numbers are spirits, or between Cantor's set theoretical discourse and the discourse of a lunatic that believes that he has discovered a new room in the building of math? Well, if you think through the details of it, not that much, seems to be LW's point. LW's suggestion is clearly that this is exactly what is the case in the contemporary debates on the foundations of math: for LW, that kind of talk has nothing to do with either the mathematical techniques themselves or the applications that make them meaningful, and constitutes some kind of folkloristic practice on its own.

(d) anti-unitarianism / anti-monism

One of the most colorful aspects of the above passage is the variety of potential practical contexts for calculations that LW evokes throughout it. LW's constant insistence on the question of demarcation did not give rise to the articulation of criteria for distinguishing

proper math from other, similar activities, on the contrary, it had for a result that fool-proof criteria looked less and less plausible.

The purpose, or at least the de facto result, of this exercise was to get us (and perhaps LW himself) ready for the idea that math is not a single and unique system of propositions, but rather “a family of techniques, with a family of applications”, which puts radical heterogeneity at the core of mathematics. We will see in section 2.3 below, that LW sometimes openly targets the idea of math as a coherent whole and what we’ve seen in the above passage should be understood in connection to this agenda.

1.4 Conclusions to Part 1

By way of conclusion to the first part of this study, let’s recapitulate a few of the above lines of thought.

First, in section 1, I showed that for LW meaning is a matter of being embedded in practices and Forms of Life:

- this ‘pragmatism’, when applied to math, takes the shape of LW’s insistence on the primacy of a wide variety of heterogeneous practices that give meaning to the techniques used within them;
- LW’s holism and structuralism about practice yields a vision of math in which propositional knowledge is no longer at the core, and in which the meaningfulness of math can no longer be reduced to, or located in, the epistemic dimension (whether conceived of as reference to a mathematical universe, or as a part of human cognition), nor to the agent, nor to a community of agents;
- this vision also emphasizes the variability and heterogeneity of math, in direct opposition to those participants in the Grundlagen debates who view axiomatic systems as the foundations of a unified mathematics: according to LW, application is what gives meaning to mathematics is the underlying hurly-burly of heterogeneous practices (‘applications’), whereas axiomatic systems are add-ons, additional mathematical techniques, alongside the old ones, which cannot serve as foundations for these techniques and cannot even unify them in any real sense of the word.

Then, in section 1.2, we explored the idea that if sense equals embedding, nonsense could be construed as a ‘lack of embedding’:

- this idea, in its naïve form, has a certain appeal, in that we intuitively understand that activities like buying apples or building bridges are somehow embedded in our lives in a way that metaphysics is perhaps not;
- however, everything that occurs always is embedded somehow and it is therefore impossible to coherently characterize certain types of language use as being ‘not embedded’, in any real sense of these terms;
- LW’s notion of ‘lack of embedding’ turned out to heavily depend on the notion of ‘everydayness’: lack of embedding appeared to boil down to lack of embedding in everydayness;
- everydayness, again, may have a lot of intuitive appeal, but there appears to be no empirical or rational reason to distinguish between ‘everyday’ activities and ‘non-everyday’ (?) activities.

It is not part of my aim in this study to articulate a fundamental critique of LW’s (and other philosophers’) concept of “everydayness”, but the simple observation that everydayness is not an adequate conceptual tool for the empirical analysis of practices, is important in that it gives rise to the question as to why does LW (and others) use it. And the answer has to be that everydayness is not a conclusion but a premise. In other words: the distinction between the everyday and the non-everyday is part of an agenda underlying LW’s philosophical work as a whole, not the result of this work. And this immediately leads us to the critical agendas underlying LW’s philosophy, addressed in Part 2 of this study.

In sections 1.1.3(C) and 1.3, we read extended passages in which we observed a number of the above-mentioned aspects at work, which also allowed us to highlight how even the details of LW’s work on mathematics fit in with his stance within (or rather: towards) the *Grundlagen*-debates that got him into philosophy in the first place, dissociating himself from the epistemic bias in *PhilMath* (the idea that math is primarily a body of propositional knowledge), from the very idea that mathematics would need or even could have foundations, and from *PhilMath*’s deep-rooted monism (i.e. the idea that mathematics forms single coherent (unitary and unique) system).

Part 2. Wittgenstein's critical philosophy (of mathematics)

The second part of this study is based on a close reading of extended passages taken from a number of LW's manuscripts dealing with mathematics: MS106, MS113, MS117, MS118, MS121, MS124, MS125, MS126, MS161, MS163. I focus on topics where LW appears to attack more or less universally well-accepted aspects of the (philosophy of) mathematics of his time. With respect to the critical remarks that are the main focus of this part of my study, scholars who have a vested interest in LW's status as a great philosopher, seem to shy away from even looking at these passages - perhaps for fear of what they may find-, whereas for scholars who already dislike LW, the critical remarks serve as a readily available argument to simply dismiss LW's contribution as a whole: someone who objects to some of Cantor's, Dedekind's, Gödel's most revered contributions obviously doesn't know what he's talking about and must be a crank. Unfortunately, it may therefore be necessary to repeat the following platitude at the beginning of this section: there is a difference between (1) establishing what LW actually said or thought or meant and (2) determining whether he was right or not (or perhaps: acceptable, according to whichever criteria one chooses to apply). I will focus on (1), trying to show how the remarks in question are internally coherent and fit in with LW's work as a whole, and most of the time, I will not even bother with (2).

In an introductory section (2.0), I very summarily sketch the general cultural and biographical background against which LW's philosophy developed, which allows me to identify a few modes of thought and expression that will reoccur more or less systematically in LW's life and work, and -as I will show below- are crucial for our understanding of LW's PhilMath.

Then, I will present a running commentary on a series of excerpts from LW's manuscripts. As LW deals with the same issues and topics over and over again, approaching them from different angles and exploiting them for apparently different purposes, it is impossible to (1) remain close to the dynamics of a longer stretch of text, and (2) deal with a single topic at the same time. This is why I chose to pick a number of excerpts that display similar lines of thought and present a close reading of those, as follows:

- in section 2.1, I comment on 3 passages in which LW discusses various diagonal techniques and the ways in which they are exploited in contemporary philosophical or quasi-philosophical discourse about mathematics;

- in section 2.2, I comment on 3 passages in which LW discusses set theory in broad culture-critical terms as “a sign of the times”, which allows me to illustrate the link between the broader cultural tendencies discussed in section 2.0.0 and the details of LW’s PhilMath;
- in section 2.3, I present a number of excerpts that illustrate LW’s account of contradictions in formal axiomatic systems (incl. some of the ‘notorious’ ones that are often interpreted as reactions to Gödel’s work).

Section 2.4 consists of a summary of some of the main results emerging from Part 2 of this study.

2.0 Background: Wittgenstein’s philosophy as critique: nonsense, fakeness, bad faith, and bad taste

LW’s philosophy contains a strong **ethical bias**, as well as a deep-rooted **aesthetical bias**, which also shows in his biography. In this section, I gather a number of heterogeneous elements that together shed light on some central aspects of LW’s outlook on the world. In this context, it is useful to point out that LW participates in a number of **broad cultural tendencies**, through his early readings and the general culture of the milieu in which he was raised.

2.0.0 Political and cultural context

There exists a body of work dealing with the influence on LW of classical German philosophers such as Kant and Schopenhauer, but also authors that were more contemporary to LW such as Fritz Mauthner, Karl Kraus, Oswald Spengler, Otto Weininger, etc. (see below, as well as other aspects of the cultural and general historical background from which LW and his philosophical work emerged.¹⁴³

A number of the more salient features of LW’s outlook, such as his emphasis on **everydayness** (as opposed to sublimity, etc.)¹⁴⁴ and on **practice** (as opposed to thought), take part in -what

¹⁴³ For instance: (Sass 2001); (Stern and Szabados 2004); (Jacquette 2017); (DeAngelis 2007); (Hanna 2017); (Nyíri 1982); (Nyíri 1992); (McGuinness 2002a) A lot of useful material can be found in the two major LW biographies (Monk 1990) and (McGuinness 1988). See also (Janik 1992); (Nyíri 1982); (Steinvorth 1979).

For an analysis, see Biletzki 2003, ‘Chapter 6. The Fifth Station: Over the Deep End, Or the Ethical Reading’ (pp. 95-105) and ‘Cultural and political readings’ (pp. 181-186) (Biletzki 2003).

¹⁴⁴ Cf. for instance, *PhU* §80:

Wir stehen mit diesen Überlegungen an dem Ort, wo das Problem steht: Inwiefern ist die Logik etwas Sublimes?

Denn es schien, daß ihr eine besondere Tiefe a allgemeine Bedeutung a zukomme. Sie liege, so schien es, am Grunde aller Wissenschaften.

has been called- '**Lebensphilosophie**'. The 1999 edition of the Cambridge Dictionary of Philosophy introduces the term as follows (Surber 1999):

Such philosophers as Dilthey and Eucken (1846 – 1926) frequently applied it to a general philosophical approach or attitude that distinguished itself, on the one hand, from the construction of comprehensive systems by Hegel and his followers and, on the other, from the tendency of empiricism and early positivism to reduce human experience to epistemological questions about sensations or impressions. Rather, a Lebensphilosophie should begin from a recognition of the variety and complexity of concrete and already meaningful human experience as it is "lived"; it should acknowledge that all human beings, including the philosopher, are always immersed in historical processes and forms of organization; and it should seek to understand, describe, and sometimes even alter these and their various patterns of interrelation without abstraction or reduction. Such "philosophies of life" as those of Dilthey and Eucken provided much of the philosophical background for the conception of the social sciences as interpretive rather than explanatory disciplines. They also anticipated some central ideas of phenomenology, in particular the notion of the Life-World in Husserl, and certain closely related themes in Heidegger's version of existentialism.

Note that many of the features mentioned in this quotation are applicable to LW's work, especially (or perhaps: more overtly) to his later work: the explicitly non-systematic character of the investigation, the emphasis on our "immersion" in a historical context, rather than on atemporal epistemological issues, etc.

One of the most spectacular contributions regarding the cultural context of LW and his work is **Allan Janik and Stephen Toulmin's 1973 classic *Wittgenstein's Vienna*** (Janik and Toulmin 1973).¹⁴⁵ Even if perhaps a bit overenthusiastic in its broad strokes and perhaps sometimes misguided in its technical-philosophical interpretation of LW's work, it is still an impressive account and especially an impressive collection of relevant materials, which would be unfair and counter-productive to simply dismiss. For our purposes, one of the main points (if not the main point) of the book is immediately relevant. Throughout the book, Janik and Toulmin show that society during the last decades of the Imperial & Royal ("kaiserlich und königlich") regime in Austria (a.k.a. "Kakania") was characterized by an increasing malaise due to the **discrepancy between social and political realities and public discourse** about these realities:

It was the consistent attempt to evade the social and political problems of Austria by the debasement of language -by the invention of "bogus language games," based on the pretense that the existing forms of life were other than they really were- that created the underlying occasion for men's universal confusions about the problems of expression and communication. This confusion found an outlet, both in the particular aesthetic critiques characteristic of all the different arts in late Habsburg Vienna, and also in the

¹⁴⁵ See also (Steinvorth 1979); (Seldes 1996); (Molnar 1975).

general philosophical critique of language as initiated by Mauthner and subsequently taken up by Wittgenstein himself.

(Janik and Toulmin 1973), pp. 273-274

In what follows, I show that this notion of ‘**fakeness**’, applied to means of expression, as well as the **negative assessment of the era** he lived in, are recurring themes in LW’s PhilMath.

LW famously quoted the following authors as the ones that influenced him most:

So haben mich Boltzmann, Hertz, Schopenhauer, Frege, Russell, Kraus, Loos, Weininger, Spengler, Sraffa beeinflusst.

(LW, Ms-154,16r)

The influence of Boltzman and Hertz, who appear to have steered LW from engineering to philosophy of science, and of Frege and Russell, as his main teachers in logic, of Loos, in the context of LW’s own activity as an architect, Sraffa, as a colleague with whom he had repeated conversations during a prolonged period of time, and perhaps even Schopenhauer, as the leading philosopher for any German-speaking intellectual of LW’s generation are easy to fit in with the mainstream image of LW’s work as a philosopher. However, the ones that are perhaps the most relevant for the purposes of this study are Kraus, Weininger and Spengler:

- **Karl Kraus** was a Viennese journalist, mostly known for his critical attitude towards the way in which the contemporary mainstream press used language in such a way that it obscured the social and political realities of Austria under the waning Habsburgian regime;
- **Otto Weininger** is mainly known for publishing *Geschlecht und Charakter* (Sex and Character), shortly before his suicide at age 23 on October 3 1903 in the house where Beethoven died; the book consists of a wide-ranging broad-strokes psychological theory (?) centered around such concepts as ‘character’, ‘sex’ (as in ‘the sexes’), and ‘race’, which impressed many a contemporary intellectual, including some prominent Nazis;¹⁴⁶
- **Oswald Spengler** was an amateur (I mean: non-academic) historian and philosopher of history, mostly known for his 1918 *Der Untergang des Abendlandes* (The Decline of the West), which offers a negative assessment of the state of Western civilization in the context of a grandiose theoretical framework.

¹⁴⁶ The popularity of such public intellectuals as Jordan Peterson shows that Weininger’s work may make a comeback soon. LW is said to have been critical of the contents of Weininger’s opinions (LW wrote in a letter to Moore d.d. 19310831: “It isn’t necessary or rather not possible to agree with him but the greatness lies in that with which we disagree. It is his enormous mistake which is great. I.e. roughly speaking if you just add a “~” to the whole book it says an important truth” (Wittgenstein 2008)- which -by the way- does not make sense as a way to distance oneself from an ideology, at all), but was mainly impressed by the way in which Weininger addressed ‘real problems’ head-on.

What these authors have in common (among other things) is their attempt to understand the era and the culture in which they lived as a whole and in terms of very general concepts, as well as their very **negative assessment of the state of this culture and era**. These features also characterize LW's way of thinking, as we will see below.

Perhaps the single most important source for this idea is **Spengler**, whose main idea was that cultures are organic units having a quasi-biological lifespan; when cultures start to die, they turn into -still according to Spengler- civilizations, in which the cultural and social patterns of the old culture survive, but as empty forms,¹⁴⁷ without whatever made them vital and meaningful previously. As Guter points out (cf. (Guter 2015), also quoted in section 2.0.2 below), LW shares with Spengler the idea that in an era of flourishing culture, there is an organic unity between the different aspects of such a culture (its literature, its music, its science, its politics, its patterns of everyday life and everyday discourse, ...), which makes all these aspects deeply meaningful. According to Spengler, Western culture entered the civilization phase of its lifespan in the 19th century. The loss of unity between the various manifestations of the culture in its turn coincides with the loss of their intelligibility. For instance, whereas (according to Spengler, and according to LW) classical music was transparently linked to contemporary literature etc., modern music (in the classical tradition) had become increasingly unintelligible.¹⁴⁸

There is a lot more to be said about the conceptual links between the Spenglerian ideas of 'organic unity' and 'decline', the idea of meaningfulness by embedding in the everyday, the concept of 'authenticity' and the subtle or less subtle differences in the way variations on these ideas manifest themselves in authors such as LW and Martin Heidegger. Obviously, these aspects can't be developed in the context of the present study, but my analyses of LW's PhilMath below can be read as a case study in which LW's deep affiliation with Spenglerian themes shows up everywhere: *much of LW's criticism of mathematical developments since the 19th century boils down to the Spenglerian-sounding idea that mathematics has lost its organic connection to the everyday applications that make it meaningful.*

Not much of the mainstream English-language literature concerned with more technical aspects of LW's philosophical work seems to take on board the ethical and aesthetic and other more general aspects of LW's outlook on the world and on philosophy. Thus, for instance, in mainstream handbooks about Wittgenstein ((Glock and Hyman 2017) ; (Sluga and Stern 2017); (Kuusela and McGinn 2011)), these aspects are typically relegated to separate, biographical or otherwise 'contextual', non-technical chapters, but not taken into account in

¹⁴⁷ This notion of an 'empty form' deserves a closer look, but not in the present context.

¹⁴⁸ For notes on Spengler's views on math and how they are similar to and different from LW's, see Appendix 4.2.

the chapters that deal with properly philosophical topics, a fortiori not in the chapters dealing with PhilMath. In the present Part 2, I will try to do exactly this: show *how LW's PhilMath fully participates in the culture-critical aspects of the worldview he grew up with, including his preoccupation with various avatars of the issue of fakeness.*¹⁴⁹

2.0.1 Ethical and aesthetic biases underlying Wittgenstein's philosophical agenda: biographical and 'existential' aspects

(A) aesthetics

There is an obvious biographical or 'existential' side to the importance of aesthetics for LW. Starting with his upbringing in a high-society family committed to sponsoring, entertaining, and savoring the *crème de la crème* of the Austrian art world, LW remained actively interested in aesthetic and artistic matters (music, literature, architecture, sculpture, etc.) throughout his life, as is –for instance– witnessed by the selection of notes published as *Vermischte Bemerkungen / Culture and Value (VB)*, but also elsewhere in his oeuvre (for a quick overview of the data, see e.g. (Hagberg 2014); for an account of a few of LW's remarks on music, see section 2.0.2 below).

However, there are also indications that LW's aesthetical bias ran deeper ((?) or is it 'wider'?) than his preoccupation with art, and in a way that is relevant to our purposes in the present study. In the notorious sequence of paragraphs following *TLP* §6.4, after having pointed out that ethics is necessarily non-propositional (i.e. ethics cannot refer to things that are within the world), LW also says (in §6.421, as quoted above) that ethics and aesthetics are one, and transcendental (just like logic, for that matter).¹⁵⁰ See section 2.0.3 below.

It has also been suggested that LW's approach to life was as much aesthetical as it was ethical: his moral objections often concerned the *how* rather than the *what* of people's behavior (cf. e.g. what Brian McGuinness observes about LW's use of the dictum "*Le style c'est l'homme même*" ((McGuinness 2002b), pp. 21-22) :

For him style, the way something was put, was of enormous importance, and that not only in the artistic sphere. He said once, it wouldn't matter what a friend had done but rather how he talked about it.

¹⁴⁹ In the present context, I can only briefly mention the fact that there are many philosophically relevant commonalities between LW and his contemporary (and fellow often-quoted candidate for 'greatest philosopher of the 20th century') Martin Heidegger. Cf. (Scheppers 2009), (Scheppers 2017), (Braver 2012), (Egan, Reynolds, and Wendland 2013), (Egan 2019).

¹⁵⁰ Of course, these theses (?) should be interpreted within the framework of the *TLP*, according to which the only valid type of speech was propositional, i.e. the kind that is either true or false; everything else was considered to be meaningless.

Similarly he used to insist on a careful reading of the dictum, *Le style c'est l'homme même*. One should note the word '*même*': the thought is that the real man reveals himself in his style. The meaning of the words, the content, is something secondary, and so likewise is the brute action performed. Of course, it is an important philosophical observation that actions cannot be separated from the way in which they are judged by him who performs them.

These remarks deserve our attention, as they highlight aspects that are immediately relevant to the interpretation of LW's philosophy: LW's problematization of the relation between what someone does and what that person says about it, and LW's focus on the how rather than on the what. These aspects will come back in our analysis of LW's PhilMath below.

(B) ethics

A number of biographical anecdotes suggest that LW had very strong ethical (moralistic?) reflexes throughout his life and was always ready to disapprove of other people's or his own behavior in the strongest possible terms,¹⁵¹ and his biographers mention his obsession with honesty and sincerity, and his lack of patience with lack of these qualities, with hypocrisy, with half-heartedness, with vanity, etc.¹⁵²

These characteristics occasionally show up in his manuscripts, often written in code.¹⁵³ See, for instance, his entry for November 25 1939 in Ms-122,36v-38r, in which we read a long parenthetical remark written in code (here below printed in italic),¹⁵⁴ literally in the middle of a sentence dealing with a more 'technical' philosophical topic.

Aber ich verwende nun das [I'm much *too slick* & all I produce is pretty *slick*. Es hat nicht genug Falten im Gesicht sondern ist *oberflächlich* & von *glatter* Stirn. Zugleich macht es *fälschlich den Eindruck der Tiefe*, denn es

¹⁵¹ A good example is the letter LW wrote to his friend and fellow-ex-POW Ludwig Hänsel in 1937, in which he comments on papers written by the latter, by calling them amongst other things "vomit" (Schulte 2001, 183); Schulte makes the following comment: "[...] what arouses Wittgenstein's interest is more the way one thinks or talks about a subject than the content of these thoughts or statements", which fits in nicely with some of the main points of the present study.

¹⁵² See e.g. (Monk 1990) pp. 44-45, but similar examples can be found throughout any biographical account. See also (Sass 2001), p. 110: "At times he felt profound disgust for average people ("I suffer much from the human, or rather inhuman, beings with whom I live"); their pettiness, greed, affectation, and general lack of honor was so overwhelming as to make them seem virtually subhuman - like "loathsome worms" or "one-quarter animal" (M 228, 212, also 89). Even Wittgenstein's best friends were likely to feel the force of his severity and ruthless judgments, which could turn suddenly upon them if they said or did something that Wittgenstein considered inauthentic, fatuous, or weak. Yet it was with himself that Wittgenstein was at his most severe."

¹⁵³ For LW's habit of using an easily decipherable code for writing some of his non-technical remarks in his notebooks and diaries, see e.g. Schulte (Schulte 2001), p. 178. Cf.. Gorlée 2020. (Gorlée 2020)

¹⁵⁴ The transcriptions of Wittgenstein's Nachlass available on-line at the website of the Wittgenstein Archives at the University of Bergen (<http://wab.uib.no/transform/wab.php?modus=opsjoner>, last consulted on January 6 2021) prints the whole parenthesis before the sentence "Aber ich verwende nun das Aufsagen oder Anschreiben einer Wortfolge || Zeichenfolge aus dem Gedächtnis als Kriterium der Zahlengleichheit, Mengengleichheit.", probably for the sake of readability, but at the same time obscuring a most remarkable (textual? literary? cognitive? psychological?) phenomenon.

ist von Einem geschrieben der sich so gern tief wüßte. Das Gesicht ist zu faltenlos; aber Falten kommen vom Kummer, nicht von der Bequemlichkeit. Wer auf dem Kummer schwimmen will, um ja nie unterzutauchen, wie sollte der Tiefe kennen. Mein ganzes Leben (inneres & äußeres) ist darauf angelegt, auf sicherem || im sicheren Boot auf dem Meere, auf der Oberfläche, zu schwimmen. Ich will doch gar nicht zahlen; wie sollte ich erhalten?] Aufsagen oder Anschreiben einer Wortfolge || Zeichenfolge aus dem Gedächtnis als Kriterium der Zahlengleichheit, Mengengleichheit.

In this case, as in many others,¹⁵⁵ LW directs his wrath onto himself and develops the notion of superficiality vs. depth to an almost allegorical degree of detail. For our present purposes, I would like to attract attention to the fact that LW's objections target not so much a lack of depth, but rather **fake depth**: pretending that there is depth, where in fact there is none. Interestingly, this concept of fake depth is also applied in more technical contexts, for instance in *PhU* §89, §97, §111, and in the context of LW's *PhilMath* (Ms-126,133-138, discussed in section 2.1(C) below).

(C) authenticity as an existential problem

LW's biography also shows that LW suffered 'existentially' from his logico-philosophical problems, in the same way other –perhaps more 'normal' –¹⁵⁶ people may suffer from problems of a moral order. For LW, philosophical problems are not a fun game to play, nor a 9 to 5 job, but deadly serious, existentially, as can also be seen in the following oft quoted passage from Bertrand Russell's *Autobiography*:

He used to come to see me every evening at midnight, and pace up and down my room like a wild beast for three hours in agitated silence. Once I said to him: 'Are you thinking about logic or about your sins?' 'Both', he replied, and continued his pacing. I did not like to suggest that it was time for bed, as it seemed probable both to him and me that on leaving me he would commit suicide.

(Russell 2009) p. 313

¹⁵⁵ For a similar example of LW's obsession with sincerity, again directed at himself (and Mahler), and again written in code, see MS 120, 72v 1937:

Sich über sich selbst belügen, sich über die eigene Unechtheit belügen, muß einen schlimmen Einfluß auf den Stil haben; denn die Folge wird sein, daß man in ihm nicht Echtes von Falschem unterscheiden kann. So mag die Unechtheit des Stils Mahlers zu erklären sein & in der gleichen Gefahr bin ich.

Wenn man vor sich selber schauspielert, so muß der Stil davon der Ausdruck sein. Er kann dann nicht der Eigene sein. Wer sich selbst nicht kennen will, der schreibt eine Art Betrug.

Wer in sich selbst nicht hinuntersteigen will, weil es zu schmerzhaft ist, bleibt natürlich auch mit dem Schreiben an der Oberfläche. (Wer nur das Nächstbeste will, kann doch nur das Surrogat des Guten erreichen.)

¹⁵⁶ Throughout the best-selling *Wittgenstein's poker* (Edmonds and Eidinow 2001), LW is presented as "not quite human" (thus, at the beginning of chapter 21, interestingly opposed to the "all too human" Popper); cf. also the beginning of chapter 16: "While Popper remains recognizably human despite his aggressive approach to debate and disagreement, there is an unearthly, even alien, quality to Wittgenstein's dealings with others".

The relevance of this anecdotal evidence, as well as the evidence I will quote below, for the present purpose is that it sheds light not only on the way he conceives of the aims of philosophy in terms of therapy (etc.), but also on the moral indignation that often accompanies his philosophical criticism.

As far as the actual contents of LW's ethical and aesthetical interventions go, I ((Scheppers 2009); (Scheppers 2017)) am not the only one to point out the central role of **authenticity**, i.e. the value that consists in avoiding fakeness of all kinds, pretense, illusion and delusion, etc..¹⁵⁷ For an extended account, focusing on the role of such topics as sincerity, authenticity, fatuity, theatricality, vanity and confession, one can refer to Louis Sass' article 'Deep Disquietudes: Reflections on Wittgenstein as Antiphilosopher' (Sass 2001). Sass links a number of features of LW's philosophy with LW's psychological profile (that he identifies as "schizoid"). Thus, e.g., Sass links LW's **ambivalence towards the everyday** (both the complete absence of the everyday in LW's early work, and its central status in his later work), as well as his generally speaking **negative attitude** and the **critical nature** of his "antiphilosophy"¹⁵⁸ and his preoccupation with issues of self-reflexivity, to his psychological make-up. I do not endorse Sass' psychological approach, but the collection of materials he uses is relevant here, and I

¹⁵⁷ The concept is absent from the main current Anglo-Saxon manuals (Kuusela and McGinn 2011) and (Glock and Hyman 2017). See however (Cahill 2004). As pointed out in section 2.0.0 above, "bogus" and similar notions are central to the classic (Janik and Toulmin 1973).

More recently, David Egan published extensive work on the significance of the concept of authenticity for the interpretation of LW's work (Egan 2019; 2013). As also suggested by Egan's work and others ((Scheppers 2009) (Scheppers 2017) (Braver 2012) (Egan, Reynolds, and Wendland 2013) (Egan 2019)), I would like to insist that a comparison between LW and Martin **Heidegger** is enlightening. I can't go into this aspect here, but I would like to point out that the important differences between both thinkers can be pinpointed by looking at the way in which they value everydayness with respect to their shared concern with **authenticity**: whereas LW evaluates everydayness positively as the source of all meaningfulness, MH sees everydayness as the oppressive rule of "das Man" ("They"). On another occasion, I will articulate how this difference correlates with their very different cultural and ideological profiles: LW the conservative high-society snob, and MH the revolutionary middle-class Nazi.

¹⁵⁸ Sass 2001 (Sass 2001), p. 122: "Yet Wittgenstein's strongest impulse, his own genius, if indeed that is the appropriate term, was for philosophizing, and philosophizing of a peculiarly negative sort. If genius is to be defined as creation and absorption - absorption in the service of creation - then one has to recognize the problematic status of Wittgenstein's own work, which both early and later in his career has a distinctively negative flavor. After all, it derives in large measure from a distantiated contemplation and critique of the philosophical discourse of others. The main goal of Wittgenstein's thinking may be the discouragement of philosophizing itself ("Philosophy is a tool which is useful only against philosophers and against the philosopher in us"), but, in some respects at least, it traffics in further alienation, merely recapitulating the condition of philosophy in a higher degree.⁷⁰

Wittgenstein's own antiphilosophizing is, after all, grounded not in absorption but in a kind of alienated critical self-consciousness - or perhaps we should speak of absorption in a kind of alienated critical self-consciousness. The purpose, in any case, is deconstruction, discouragement, perhaps therapy, certainly not the construction of an alternative philosophical edifice. Wittgenstein writes, "The philosopher is not a citizen of any community of ideas. That is what makes him into a philosopher."⁷¹ This would, however, seem to be doubly true of the Wittgensteinian philosopher: alienated not only from the language of household, workshop, and marketplace, but from normal philosophical conversation as well. Perhaps this is part of what Wittgenstein had in mind when he wrote: "It's only by thinking even more crazily than philosophers do that you can solve their problems" (CV 75/86)."

want to acknowledge that I got acquainted with some of the material that has turned out crucial for my understanding of LW's critical remarks via Sass' article.

Sass adduces a lot of material showing LW's preoccupation with **lack of sincerity, vanity and pretense**. A case in point is the fact that in the mid-1930s, LW went through a period in which he felt a strong urge to confess some of his 'sins' to various friends and acquaintances; these sins turned out to be rather petty cases of insincerity, which however bothered LW greatly.¹⁵⁹ Of course, the very fact of desiring to confess in its turn can easily be interpreted as a kind of vanity and theatricality, which in its turn LW did not fail to blame himself for ((Sass 2001), pp. 133-134). The same basic patterns, which Sass conveniently summarizes under the heading of '**inauthenticity**', reoccur when it comes to LW's condemnation of other peoples' behavior:

Similar attitudes and intuitions pervade Wittgenstein's more explicitly ethical or moral concerns. Perhaps the main object of his ethical condemnation was what he termed "vanity," a quality he associated largely with tendencies toward theatrical self-display - that is, with what he saw as the inauthenticity and lack of courage inherent in being overly concerned about the impression one makes on other people, and with the detached self-consciousness inherent in imagining oneself as a potential object of admiration for others (M 278).¹⁰⁸ In Wittgenstein's diaries of the 1930s, "vanity" is a central theme; he despises "vanity" yet is constantly discovering it in himself:

[...]

(Sass 2001)p. 132

Interestingly, Sass also points out a link with LW's tastes in artistic expression, and more specifically his dislike for **theatricality** in this respect as well:

Wittgenstein was steeped in these traditions, in romanticism, certainly, but also in the early modernist movements of fin de siècle and early 20th-century- Vienna.¹⁰³ His own proclivities are apparent in his dislike of any kind of explicit moralizing and didacticism in literature and in his preference for works of art that refuse to betray the purity, authenticity, or integrity of their being through theatrical self-consciousness or by attempting to say what can only be shown. Literary works that Wittgenstein appreciated for such qualities include Tolstoy's *Hadji Murat* (McG 33) and the detective stories of Norbert Davis and other American writers of the "hard-boiled" school. "A typical American film, naïve and silly, can - for all its silliness and even *by means of it* - be instructive," he wrote. "A fatuous, self-conscious English film can teach one nothing" (CV 57/65).¹⁰⁴ Wittgenstein accepted an aesthetics (and an ethics) of authenticity - a view that would equate detached or theatrical self-consciousness with a diminishment of

¹⁵⁹ This obsession with his own vanity and the social awkwardness that comes with it, perdured into the last years of LW's life. Cf. O.K.Bouwsma's account of a conversation between LW and himself d.d. 19490805 about a conversation they had had on 19490731: "He [sc. LW] hardly knew how to tell me [sc. OKB]. It was absurd, etc. 'I am a very vain person'. 'The talk wasn't good. Intellectually, it may have been, but that isn't the point'. 'My vanity, my vanity'" ((Bouwsma 1999), p. 102). This quotation is also interesting, because LW, here as elsewhere, appears to condemn behavior (in this case a conversation between philosophers) for its style (if that is the word), while conceding that 'intellectually' it was OK. For this trope and its importance for LW's philosophy, see section 2.0.2 below.

both the reality of one's existence and the distinctiveness of one's identity. "If I perform to myself, then it's this that the style expresses," wrote Wittgenstein. "And then the style cannot be my own."¹⁰⁵ ((Sass 2001), pp. 131-132)

We will see in our analyses below that the same aversion for pretense and theatricality comes back as a key component in LW's philosophical work.

Whatever the validity and relevance of Sass' psychological assessments, Sass' material does show that, in LW's case, it is very hard to separate the man's existential worries from his philosophical worries, especially in the light of what he says on the topic himself. So, this material does strengthen our claim that LW's work on PhilMath (or any other topic) should not be artificially separated from the overall aims (and -dare I say-: 'spirit') of his philosophy. In what follows, I will be able to point out a recurrent concern with fakeness in LW's PhilMath, in the sense of a claim that things are presented as different from what they are.¹⁶⁰

2.0.2 Wittgenstein on art in general and Mahler's music in particular

We can also take a closer look at exactly what LW's aesthetical judgments in an art-related context amounted to. The following excerpt from LW's biography by **Monk** is worth quoting here:

In discussing aesthetics, Wittgenstein was not attempting to contribute to the philosophical discipline that goes by that name. The very idea that there could be such a discipline was a consequence, or perhaps a symptom, of the 'other'. He was, instead, trying to rescue questions of artistic appreciation from that discipline, particularly from the idea that there could be a kind of science of aesthetics:

You might think that Aesthetics is a science telling us what's beautiful -- almost too ridiculous for words. I suppose it ought to include also what coffee tastes well.

When Rhees asked Wittgenstein about his 'theory' of deterioration (referring to one of Wittgenstein's examples, which was the deterioration of the German musical tradition), Wittgenstein reacted with horror to the word: 'Do you think I have a theory? Do you think I'm saying what deterioration is? What I do is describe different things called deterioration.'

Rather than trying to answer the traditional questions of aesthetics ('What is beauty?' etc.), Wittgenstein gives a succession of examples to show that artistic appreciation does not consist (as one might think from reading some philosophical discussion of aesthetics) in standing before a painting and saying: 'That's beautiful.' Appreciation takes a bewildering variety of forms, which differ from culture to culture, and quite often will not consist in *saying* anything. Appreciation will be *shown*, by actions as often as by words, by certain gestures of disgust or satisfaction, by the way we read a work of poetry or play a piece of music, by how often we read or listen to the same piece, and how we do so. These different forms of appreciation do not have any one thing in common that one can isolate in answer to the question: 'What is artistic appreciation?' They are, rather, linked by a complicated series of 'family resemblances'. Thus:

¹⁶⁰ Note the similarity with Plato's way of articulating the notion of untruth, as applied to the sophist in the *Sophist*.

It is not only difficult to describe what appreciation consists in, but it impossible. To describe what it consists in we would have to describe the whole environment.

(Monk 1990), pp. 404-405

This excerpt from Monk's biography covers a couple themes that are important for our purposes. First, it reminds us of the not necessarily anti-scientific, but at least **non-scientific** nature of LW's approach to aesthetics (as to other topics) and especially the idea underlying the way he framed the whole *TLP*, i.e. that what is really valuable in life inherently falls outside the realm of propositional truth-values. This is immediately linked to -second- the ultimately **non-propositional**, even non-verbal nature of aesthetic appreciation (as other ways in which we give meaning to our world), and -third- his thoroughly **holistic** view of the context that would be relevant to describe how we appreciate things aesthetically.

Eran **Guter's** contribution "'A Surrogate for the Soul": Wittgenstein and Schoenberg' (Guter 2011) offers a number of interesting insights. First of all, Guter shows how LW links musical understanding to our ability to operate with an intuitive sense of human **physiognomy**, in a way that defies any mechanical conception of rule-following, let alone an epistemic grounding in propositional contents ((Guter 2011), pp. 124-125); see also (Guter 2017)).

Guter also insists on the very intricate ways in which LW sees a link between **musical meaning** and the way in which art is embedded in the surrounding culture. Guter's article 'The Good, the Bad, and the Vacuous: Wittgenstein on Modern and Future Musics' (Guter 2015) contains an extensive (but somewhat uneven) analysis of some of LW's remarks on music against the background of **Spengler's** work and musicologist Heinrich **Schenker's** equally (if not more) conservative work on the tonal and harmonic features of classical music.¹⁶¹ Guter emphasizes the point that -according to Spengler and LW- the intelligibility of classical music was the result of the organic relation it had with other aspects of the culture it was a part of.

The following excerpt from one of LW's manuscripts (a notebook that he intermittently used as some kind of a diary in 1930-1932 and 1936-1937, containing miscellaneous remarks, often but not always of a personal nature, and often but not always written in code), and the main

¹⁶¹ The work of Heinrich **Schenker** (1868-1935) represents a vision of what constitutes some of the values appreciated in the tradition of European 'classical' music (in short: a certain form of tonality and harmony), which is a highly reductive approach in that it de facto only applies to European classical music of a very brief period in time, but also in that it completely ignores most of the aspects that are important in most types of music (rhythm, phrasing, timbre, ..), including the European 'classical' music that is its subject matter. This vision was horribly backward even in its own time and especially in the Vienna of the 1920s, with its burgeoning artistic experimentation; on a somewhat larger scale, as a theory of music in general, it is criminally ethnocentric and classist, in that it excludes most music made on the planet, including most music made in Europe, from its scope. For a recent internet controversy involving Schenker's theory, opposing (i.a.) far right internet guru Ben Shapiro and a number of musicologists, see the interesting video 'Music Theory and White Supremacy' on music educator Adam Neely's youtube-channel (Neely 2020).

subject matter of Guter's above-mentioned 2015 article, illustrates this aspect of LW's thinking quite well:

27.

Die Musik aller | | der vergangenen Zeiten entspricht immer gewissen Maximen des guten & rechten der selben Zeit. So erkennen wir in Brahms die Grundsätze Kellers etc. etc. Und darum muß eine gute Musik die heute oder vor kurzem gefunden wurde, die also modern ist, absurd erscheinen, denn wenn sie irgend einer der heute ausgesprochenen Maximen entspricht so muß sie Dreck sein. Dieser Satz ist nicht leicht verständlich aber es ist so: Das Rechte heute zu formulieren dazu ist so gut wie niemand gescheit genug & alle Formeln, Maximen, die ausgesprochen werden sind Unsinn. Die Wahrheit würde allen Menschen ganz paradox klingen. Und der Komponist der sie in sich fühlt muß mit seinem Gefühl im Gegensatz stehen zu allem jetzt Ausgesprochenen & muß also nach den gegenwärtigen Maßstäben absurd, blödsinnig, erscheinen. Aber nicht anziehend absurd (denn das ist das was doch im Grunde der heutigen Auffassung entspricht) sondern nichtssagend. Labor ist dafür ein Beispiel dort wo er wirklich Bedeutendes geschaffen hat wie in einigen, wenigen, Stücken.¹⁶²

(Ms-183,-59-61, d.d. 19310127)

According to LW, it is almost impossible to write -what he considers- good music in the era in which he wrote (an era which -I guess- is still going on, or -then again- maybe not) because good music corresponds to the conception of the good and the right of its time. If someone wrote music that corresponded to the slogans of LW's era (or just before or after), then it has to be trash (LW says "Dreck").

¹⁶² Guter ((Guter 2015), p. 426) gives the following translation: "The music of all periods [the music of the past] always appropriates certain maxims of the good and the right of its own time. In this way we recognize the principles of Keller in Brahms etc etc. And for that reason [good] music, which is being conceived today or that has been conceived recently, which is therefore modern, seems absurd; for if it corresponds to any of the maxims that are articulated today, then it must be rubbish. This sentence is not easy to understand but it is so: no one is astute enough to formulate today what is correct, and all formulations, maxims, which are articulated are nonsense [Unsinn]. The truth would sound entirely paradoxical to all people. And the composer who feels this within him must confront with this feeling everything that is [now] articulated and therefore [his music] must appear by the present standards absurd, timid [blödsinnig]. But not absurd in a dressed-up sense (for after all, this is basically what corresponds to the present attitude) but vacuous [Nichtssagend]. Labor is an example of this where he created something really significant as in some few pieces."

I underlined those words that I object to as **translations** of the German original: "seems absurd" is not the same thing as "has to appear absurd"; "it is so" is not a correct translation of "es ist so" (better: "that's the way it is", "it is true" (referring to (the contents of) the previous sentence), ...); "correct" is a possible translation for "recht", but not if you emphatically and correctly translated a previous occurrence of "das Rechte" by "the right"; "formulations" is not the same thing as formulas, in the linguistic sense of "verbal expressions that are fixed in form", which is clearly what is meant here; "timid" is horribly wrong as a translation of "blödsinnig", which clearly and transparently means something like "feeble-minded" and is unambiguously synonymous with "stupid", not "timid" (one may even argue for "retarded", as an offensive but period-correct translation but that would require some further research); "anziehend" simply means 'attractive' (from the verb 'anziehen' in the sense of 'to attract', also in the physical sense of the word) and has nothing to do with 'dressing up' (Guter's misunderstanding is probably based on a mistaken use of a dictionary: 'anziehen' can also be used in the sense of 'to put on' with a piece of clothing as a direct object, but this usage has nothing to do with the meaning of the adjective "anziehend"); one could make an argument that "Nichtssagend" fits in with a broader semantic field involving 'empty' or 'vacuous' expressions but "vacuous" as a *translation* for a word that transparently means "saying nothing" is objectionable.

It is important to understand that for LW, these aspects apply not only to aesthetic matters, but to meaning in general (see section 2.0.3 below). Throughout LW's philosophical work we find examples that point out analogies between propositional meaning and non-verbal types of meaning, and these analogies go both ways: LW applies concepts like 'gesture', 'sentence' [Satz] to music (as is traditional), but also invokes the use of these terms as applied to music to shed light on verbal meaning; similarly, he talks about the way musical phrases can be felt to follow each other 'logically', and uses that example to shed light on what it means for one proposition to 'follow logically' from another (cf. e.g. section 2.3(F) below).

Thus, *the idea that cultural artefacts are intelligible to the extent that they are an organic expression of the culture that they are part of, also applies to discourse in general and philosophy in particular.* Throughout part 2 of this study (but see especially section 2.2), we will encounter passages in which LW applies exactly this line of thought to mathematical issues.

(A) LW on Mahler

LW's comments on Mahler are particularly revealing for the purposes of this study, so I would like to dwell on those for a moment.

After remarks about perceiving something as something, the following excerpt¹⁶³ is the end for the entry dated 14/1,¹⁶⁴ and the next day, LW returns to his line of thought about perception, so we can read this as a more or less self-contained piece.¹⁶⁵

LW first expresses his belief that Mahler's music is worthless and asks -apparently in all seriousness- what poor Mahler should have done with his obvious talent:¹⁶⁶ should he have written his worthless symphonies and then burnt them? should he have forced himself not to write them? Of course, poor Mahler was -out of vanity- not able to see what LW could see... Let us note the almost incredible confidence (if this is even the right word) that accompanies what ultimately boils down to an expression of personal taste, if not cultural affiliation.

What is interesting here, is how categorical LW's judgment is and how it is not based on any formal features of Mahler's music, but on the idea that *it only appears to be classical music and in the end is not that*.

Equally interesting are the terms that LW uses to articulate his judgement: the only concrete term is '**vanity**' [Eitelkeit]; technical ability is definitely not the problem, nor is any other traditional aesthetic criterion. LW also interprets Mahler's case in very broad quasi-historical (Spenglerian?) terms: Mahler is of a different nature as the great composers of the past, and perhaps the circumstances have changed to such an extent that one can't even begin to compare the value of both types of works.

¹⁶³ | Wenn es wahr ist, wie ich glaube, daß Mahlers Musik nichts wert ist, dann ist die Frage, was er, meines Erachtens, mit seinem Talent hätte tun sollen. Denn ganz offenbar gehörten doch eine Reihe sehr seltener Talente dazu, diese schlechte Musik zu machen. Hätte er z.B. seine Symphonien schreiben & verbrennen sollen? oder hätte er sich Gewalt antun, & sie nicht schreiben sollen? Hätte er sie schreiben & einsehen sollen daß sie nichts wert seien? Aber wie hätte er das einsehen können? Ich sehe es, weil ich seine Musik mit der der großen Komponisten vergleichen kann. Aber er konnte das nicht; denn wem das eingefallen ist, der mag wohl gegen den Wert des Produkts mißtrauisch sein, weil er ja wohl sieht, daß er nicht, sozusagen, die Natur der andern großen Komponisten habe, - aber die Wertlosigkeit wird er deswegen nicht einsehen, denn er kann sich immer sagen, daß er zwar anders ist, als die übrigen (die er aber bewundert) aber in einer anderen Art wertvoll. Man könnte vielleicht sagen: Wenn Keiner, den Du bewunderst, so ist wie Du, dann glaubst Du wohl nur darum an Deinen Wert, weil Du's bist. - Sogar wer gegen die Eitelkeit kämpft, aber darin nicht ganz erfolgreich ist, wird sich immer über den Wert seines Produktes täuschen.

Am Gefährlichsten aber scheint es zu sein, wenn man seine Arbeit irgendwo in die Stellung bringt, wo sie, zuerst von einem selbst & dann, von Andern mit den alten großen Werken verglichen wird. An so einen Vergleich sollte man gar nicht denken. Denn wenn die Umstände heute wirklich so anders sind, als die frühern, daß man sein Werk der Art nach nicht mit den früheren Werken vergleichen kann, dann kann man auch den Wert nicht mit dem eines andern vergleichen. Ich selbst mache immer wieder den Fehler, von dem hier die Rede ist. Unbestechlichkeit ist alles! |

Konglomerat: Nationalgefühl, z.B.

¹⁶⁴ This remark is part of a manuscript that is written 17 years after the ones we will discuss here below, and the thematic continuity is remarkable as such. Note that LW did not necessarily become milder and less trenchant with age.

¹⁶⁵ This is not to say that there is no link between the topic of 'aspect-seeing' and LW's critique of Mahler's music: both topics illustrate the theme of non-propositional meaning, which we have argued to be perhaps the most significant aspect of LW's later work, as compared to his earlier work (cf. section 1.1.1(E) above).

¹⁶⁶ LW has always had a positive opinion towards Mahler's talent as a conductor and in this passage he appears to not even deny his technical ability as a composer either.

MS 154 17v-19r [1931?] [=Zettel, p. 16-17]

In the context of a long string of -in 2022- unpleasantly racist remarks about the “Jewish mind”,¹⁶⁷ LW ends up talking about the derivative nature of his own work, and finally -and in passing- about the -I guess- “Jewish”/“derivative”/ (I would say:) inauthentic nature of Ma[h]ler’s work.¹⁶⁸ Once again, LW applies this criticism to himself, which once again does not excuse its vulgarity and idiocy.¹⁶⁹

Mahler’s work is unlike a classical symphony, in the same way that a **picture** of an apple tree is unlike an apple tree: something altogether different. What is interesting for our purposes is LW’s insistence on “organic”, hard to formalize, aspects at the bottom of his judgement: not only does LW not point at any formal aspect of Mahler’s work at all, he actually points away from the formal aspects, explicitly stating that the difference between Mahler and the classics is at its clearest exactly where he formally does resemble the classics.

Again, LW objects to art that is not properly embedded because it **pretends** to be something else than it is, but it is -again- important to note that this embedding does not necessarily boil down to the absence or presence of this or that formal feature. Thus, LW approves of the simple tonality in (for instance) Josef Labor’s work,¹⁷⁰ but disapproves of the occasional simply tonal passages in Mahler’s work, where they sound somehow **inauthentic** (Guter 2011) p. 233).¹⁷¹

¹⁶⁷ Manuscript 154 contains a lot of remarks about Jewishness, as well as remarks on Brahms, Bruckner, Mendelssohn, ...

¹⁶⁸ Es ist dem jüdischen Geiste typisch das Werk eines Andern besser zu verstehen als der es selbst versteht.

Ich habe mich oft dabei ertappt wenn ich ein Bild entweder richtig hatte rahmen lassen oder in die richtige Umgebung gehangen hatte so stolz zu sein als hätte ich das Bild gemalt. Das ist eigentlich nicht richtig; nicht „so stolz als hätte ich es gemalt“ sondern so stolz als hätte ich es malen geholfen, als hätte ich sozusagen einen kleinen Teil davon gemalt. Es ist so als würde der außerordentliche Arrangeur von Gräsern am Schluß denken daß er doch, wenigstens ein ganz winziges Gräschen, selbst erzeugt habe. Während er sich klar sein muß, daß seine Arbeit auf einem gänzlich andern Gebiet liegt.

Der Vorgang der Entstehung auch des winzigsten & schäbigsten Gräschens ist ihm gänzlich fremd & unbekannt. Das genaueste Bild eines ganzen Apfelbaumes hat in gewissem Sinne unendlich viel weniger Ähnlichkeit mit ihm als das kleinste Maßliebchen mit dem Baum hat. Und in diesem Sinne ist eine Brucknersche Symphonie mit einer Symphonie der heroischen Zeit unendlich näher verwandt als eine Mahlerische. Wenn diese ein Kunstwerk ist, dann eines gänzlich anderer Art. (Diese Betrachtung aber selbst ist eigentlich Spenglerisch.)

Als ich übrigens in Norwegen war, im Jahre 1913-14 hatte ich eigene Gedanken, so scheint es mir jetzt wenigstens. Ich meine, es kommt mir so vor, als hätte ich damals in mir neue Denkbewegungen geboren (Aber vielleicht irre ich mich). Während ich jetzt nur mehr alte anzuwenden scheine.

¹⁶⁹ But that perhaps only means that I do not share these aspects of LW’s Form of Life, just as LW apparently does not share certain aspects of Mahler’s Form of Life.

¹⁷⁰ Labor (1842-1924) was a renowned pianist, organist and piano teacher, and a protégé of the Wittgenstein family. His technically anachronistic compositions, for some reason appreciated by LW, sound highly unremarkable to me and appear to be all but forgotten.

¹⁷¹ The contents of the previous remark are fleshed out by the following quote out of LW’s diaries of the same year: “Wenn die späten unter den großen Komponisten einmal in einfachen klaren harmonischen Verhältnissen schreiben, dann ist es als bekennen sie sich zu ihrer Stammutter. Maler scheint mir gerade in diesen Momenten (wenn die Anderen am stärksten ergreifen) besonders unerträglich & ich möchte dann immer sagen: aber das hast Du ja nur von den Anderen gehört, das gehört ja nicht (wirklich) Dir“. Translation by Klagge and Nordmann: “When for a change the later ones of the great composers write in simple harmonic progressions, they are showing

The very preposterousness of LW's comments on Mahler, the very fact that the worthlessness of Mahler's compositions is a self-evident fact to LW (but not necessarily to us (or to Mahler)), does illustrate one of LW's important philosophical points:¹⁷² these personal, cultural, historical, in any case obviously contingent perceptions, sensations and judgements are at the bottom, the **bedrock** of our worldview or our lived experience. The aesthetic which LW apparently grew up with, was bedrock to him, as much as gravity, air to breath, or $1+1=2$. LW's aesthetic, as discussed here above, shows to what extent LW is the product of an aesthetically rigid culture (especially as compared to the burgeoning experiments happening in the Vienna he knew well). But the philosophical point we may take from this is that for all of us, *at any point in time, the bedrock of our aesthetical appreciations -however open-minded we are- is ultimately as historically contingent, complex, and non-propositional as LW's.*¹⁷³

(B) LW's aesthetics of authenticity

Again, LW's judgment (in this case of Mahler's music) involves some kind of deep and 'organic' embedding of the musical utterance in the culture that it is part of, not conformity to this or that formal criterion. What is objected to is that Mahler –supposedly, i.e. according to LW– in the passages referred to pretends to use certain idioms in a 'classical' way but in fact *only apparently* does so.

It is interesting to observe that LW also objects to the '**modern' music** of composers such as Schönberg that burgeoned in the Vienna that he grew up in (there are many 'external' socio-biographical links between LW and Schönberg, but they do not appear to lead to anything philosophically relevant, except perhaps by contrast ((Guter 2011), p. 209 and passim). It may be interesting to consider how the formalism of Schönberg's serial approach might have been objectionable to LW in the same way that **formalism in mathematics** would irk him. Both are defined by the use of algorithm-like methods. Both break with more or less long-standing practices (or at least are perceived as such). Both are signs of the time, and for LW, as for a number of his contemporaries (most notably Spengler, as discussed in section 2.0.0 above),

allegiance to their ancestral mother. Especially in these moments (where the others are most moving) Mahler seems especially unbearable to me & I always want to say then: but you have only heard this from the others, that isn't (really) yours." LW did write "Maler", without the "h" (I checked the photograph of the manuscript).

¹⁷² If we wanted to be charitable, we could suggest that this is perhaps one of the reasons LW kept indulging in or even developing this type of remarks in the context of the notebooks in which he documented his work.

¹⁷³ One can argue (Jip van Besouw, personal communication october 2022) that other cultural affiliations, less rigid than LW's, might allow for a more eclectic, permeable and changeable aesthetic bedrock. But that does not change the fact that the aesthetic appreciation itself, when it occurs, is bedrock.

this time was not a good time. For all practical purposes, this may also be interpreted as LW simply displaying his early 19th century tastes.¹⁷⁴

LW did not advocate some kind of **functionalist aesthetics** either: his claim is not that form should only follow function and his objections against ornamentation are not based on functionalist concepts¹⁷⁵ but on the idea that -for instance- a bed should look like a bed and that people normally walk around a bed; the very functional idea to put wheels underneath it is ruled out for *that* reason (McGuinness 2002b) pp. 18-20:

The two engineers discussed how Eccles's new house should be furnished and were agreed on the exclusion of ornament. Wittgenstein was, as usual, the critic and adviser: in July 1914 he wrote to Eccles,

I can't see any drawing of a bed; or do you wish to take the one which the furniture manufacturers submitted? If so, do insist that they cut off all those beastly fancy ends. And why should the bed stand on rollers? You're not going to travel about with it in your house! By all means [probably 'At all events' is meant] have the other things made after your design!

[...]

It does not seem that the subordination of design to function, in the sense of intended use, would be an accurate description of Wittgenstein's tastes. These were connected, very typically for him, with his views on the value of abstract education. He used to say that mathematics would promote good taste, 'since good taste is genuine taste and therefore is furthered by whatever makes people think truthfully'.⁴ Speaking to Russell he emphasized construction as the decisive feature. A thing must be fully the thing it

¹⁷⁴ Cf. the following remarks on **African art**, as published in *Lectures and Conversations on Aesthetics, Psychology and Religious Belief*, pp. 8 seqq. (Wittgenstein 1967):

27. [Rhees: Is there tradition in Negro art? Could a European appreciate Negro art?]

28. What would tradition in Negro Art be? That women wear cut-grass skirts? etc., etc. I don't know. I don't know how Frank Dobson's appreciation of Negro Art compares with an educated Negro's. If you say he appreciates it, I don't yet know what this means. He may fill his room with objects of Negro Art. Does he just say: "Ah!?" Or does he do what the best Negro musicians do? Or does he agree or disagree with so and so about it? You may call this appreciation. Entirely different to educated Negro's. Though an educated Negro may also have Negro objects of art in his room. The Negro's and Frank Dobson's are different appreciations altogether. You do something different with them. Suppose Negroes dress in their own way and I say I appreciate a good Negro tunic. Does this mean I would have one made, or that I would say (as at the tailor's): "No ... this is too long", or does it mean I say: "How charming!?"

29. Suppose Lewy has what is called a cultured taste painting. This is something entirely different to what was called a cultured taste in the fifteenth century. An entirely different game was played. He does something entirely different with it to what a man did then.

30. There are lots of people, well-offish, who have been to good schools, who can afford to travel about and see the Louvre, etc., and who know a lot about and can talk fluently about dozens of painters. There is another person who has seen very few paintings, but who looks intensely at one or two paintings which make a profound impression on him. Another person who is broad, neither deep nor wide. Another person who is very narrow, concentrated and circumscribed. Are these different kinds of appreciation? They may be called 'appreciation'.

To the extent that this account is accurate, LW again displays his own embeddedness, but it is interesting to see his insistence on the fact that he doesn't know how other people, with different aesthetico-cultural affiliations, appreciate art, which shows a certain awareness and lucidity that seems to be absent from the world-view of his peers when it comes to non-European art (or European art, for that matter), but also from his own remarks about Mahler.

¹⁷⁵ A lot has been made of the lack of ornamentation in the house that LW designed for his sister. This design feature it shared with contemporary functionalist architecture. However, LW's own justification for this feature appears to have been very different from functionalist discourse.

was; and life must go on around it in the way appropriate to that. Thus Eccles's bed, as we have just seen, was not to have rollers: it was to be a thing around which people moved.

A first important aspect is that functionality is immediately dismissed as a relevant criterion. Next, the sentence "He used to say that mathematics would promote good taste, 'since good taste is genuine taste and therefore is furthered by whatever makes people think truthfully'" illustrates how, for LW, mathematics is fully intertwined with culture in general, as well as with the individual practitioner's ethics and aesthetics. And finally, I would like to highlight "A thing must be fully the thing it was; and life must go on around it in the way appropriate to that": to the extent that McGuiness' interpretation is correct, it illustrates once more the pervasiveness of various avatars the authenticity trope in LW's thought.

So: we have seen that '**fakeness vs. authenticity**' is recurring theme in LW's thoughts about aesthetical matters. This central aspect of LW's aesthetic could tentatively be summarized as follows: *things should look/sound/appear as they are*. Again, it appears that what LW finds most objectionable is that things are fake, *i.e. that they look/sound/appear like A, but actually are B*. This goes for people (see section 2.0.3 above), for art (see here above), this goes also for statements about math (see below).

2.0.3 Epistemic authenticity: nonsense as fake sense and bad faith

Here above I collected circumstantial evidence for the idea that **authenticity** is a core concept when it comes to understanding LW's modes of thought in general, as well as the general cultural milieu from which he emerged. Both in ethical and aesthetical matters LW's stern criticism appears to have been primarily directed against various avatars of **fakeness** (in the very general sense defined above): classical music should sound like classical music and what sounds like classical music should also *be* classical music; a bed should look like a bed and what looks like a bed should *be* a bed; a decent person's behavior should show who (s)he is. In what follows, I will try and formulate how the notions of authenticity/fakeness apply to the epistemic matters that make up the bulk of LW's work, and especially how it relates to his account of meaning, as discussed in the above.

[\(A\) ethical aspects of LW's philosophy](#)

Apart from the anecdotal stories discussed above, there is evidence for an **ethical/aesthetical aspect to LW's philosophy** itself.

The most explicit indication is perhaps LW's famous letter to Ludwig von Ficker, probably written at the end of October or the beginning of November 1919, in which LW states:

Und da ist es Ihnen vielleicht eine Hilfe, wenn ich Ihnen ein paar Worte über mein Buch schreibe: Von seiner Lektüre werden Sie nämlich - wie ich bestimmt glaube - nicht allzuviel haben. Denn Sie werden es nicht verstehen; der Stoff wird Ihnen ganz fremd erscheinen. In Wirklichkeit ist er Ihnen nicht fremd, denn der Sinn des Buches ist ein Ethischer. Ich wollte einmal in das Vorwort einen Satz geben, der nun tatsächlich nicht darin steht, den ich Ihnen aber jetzt schreibe, weil er Ihnen vielleicht ein Schlüssel sein wird: Ich wollte nämlich schreiben, mein Werk bestehe aus zwei Teilen: aus dem, der hier vorliegt, und aus alledem, was ich *nicht* geschrieben habe. Und gerade dieser zweite Teil ist der Wichtige. Es wird nämlich das Ethische durch mein Buch gleichsam von Innen her begrenzt; und ich bin überzeugt, daß es, *streng, nur* so zu begrenzen ist. Kurz, ich glaube: Alles das, was *viele* heute *schwefeln*, habe ich in meinem Buch festgelegt, indem ich darüber schweige. Und darum wird das Buch, wenn ich mich nicht sehr irre, vieles sagen, was Sie selbst sagen wollen, aber Sie werden vielleicht nicht sehen, daß es darin gesagt ist. Ich würde Ihnen nun empfehlen das *Vorwort* und den *Schluß* zu lesen, da diese den Sinn am Unmittelbarsten zum Ausdruck bringen. -

(von Ficker 1988) pp. 196-197

If taken seriously, this should be a key [“ein Schlüssel“, says LW] to his whole philosophy (at least: at this, early, stage): his work consists of two parts: that what’s in the book, on the one hand, everything he did not write, on the other; the second part is the most important part; the ethical is -so to speak- delimited [begrenzt] from the inside by the book; and strictly speaking, that is the only way it can be delimited.

The question is then: how exactly does an ethical and/or aesthetic impetus generate LW’s apparently technical work on meaning in a logical or logical-anthropological sense? I believe this issue should be taken seriously, and also as a ‘technical’ philosophical-analytical matter, not only at the ‘meta-level’ or as a matter of human interest.¹⁷⁶

The letter was intended to sell the *TLP* to a publisher and the way LW formulates the matter here is not that different from what is said in the preface and in the last few pages of the *TLP*: the contents are presented as a definitive solution for all logical problems, but the message is at the same time that ‘not much is done’ by solving these problems (*TLP*, ‘Preface’) and that these solutions are ultimately meaningless and function like a ladder that can be thrown away after one has climbed up on it (§6.54).

It remains to be seen to what extent this applies literally to later stages of LW’s development, but in the light of the analyses below, I believe it is safe to say that even in his later work on PhilMath, LW was only focusing on technical details to the extent that they shed light on a small number of very general topics, of a definitely non-technical nature.

¹⁷⁶ Most of the mainstream literature focusing on LW in connection with ethics does not seriously take into account LW’s claim that the whole of his philosophical endeavor is motivated by ethical concerns. For instance the entry ‘Wittgenstein on ethics’ (Arrington 2017) in the most recent high-profile manual on LW does not mention the issue as to what LW might have meant when he said that the point of *TLP* is an ethical one.

Although most of what LW has to say about 'ethics' in the *TLP* and the *Lecture on Ethics* (1929-1930 (?), published as (Wittgenstein 1965)), is not really relevant for our purposes,¹⁷⁷ it is important to note that LW's use of the term 'ethical' is somewhat idiosyncratic in that, for LW, *ethics is the study of value in general and covers everything that is important in life*:

Now instead of saying "Ethics is the enquiry into what is good" I could have said Ethics is the enquiry into what is valuable, or, into what is really important, or I could have said Ethics is the enquiry into the meaning of life, or into what makes life worth living, or into the right way of living. I believe if you look at all these phrases you will get a rough idea as to what it is that Ethics is concerned with.

(Wittgenstein 1965), p. 4

If we take this sense of the term 'ethics' literally, it becomes clear what LW meant when he wrote to Ficker that the key to his whole philosophy (at the time of the *TLP*) was ethical, i.e. the part that was not in the book: if the *TLP* is only concerned with the study of statements that are either true or false, and the *TLP* shows how little is achieved by 'solving all the issues' in that domain, i.e. how little these things really matter, then it actually makes sense to say that it attracts the attention to all that *does* matter and to the fact that the things that really matter are not reducible to matters of propositional truth or falsehood.

Now, it also follows that presenting things that are **a matter of values** as if they are **a matter of facts** (i.e. inherently bivalent (true or false) propositions about the world) is also a case of **fake sense**: as propositions, these utterances are fake, even if they consist in a (misguided) attempt to express something of value to the speaker.¹⁷⁸

¹⁷⁷ **Sidenote on LW on the transcendental nature of ethics/aesthetics.**

Without going into details, and mostly to preemptively put the issue aside as irrelevant in the present context, I feel I need to mention LW's remarks on the transcendental nature of ethics in *TLP* 6.4 and following subsections: "Es ist klar, daß sich die Ethik nicht aussprechen lässt. Die Ethik ist transzendental. (Ethik und Ästhetik sind Eins.)" (*TLP* 6.421). There are similar lines of thought in his *Lecture on Ethics*. The gist of LW's thoughts on this matter is quite clear: utterances on ethics (or aesthetics) are not propositional in the sense that they say something true or false about the world, and in that sense, they are transcendental (not a fact, therefore not part of the world) but ipso facto they are also nonsense in the quasi-technical sense of the term adopted in the *TLP* (neither true nor false, so not saying anything about the world, therefore nonsense). However, these lines of thought are not immediately relevant to the present subject matter, which is not the logical status of talk about ethics per se, but the actual ethical values that LW appears to endorse and enact in his life and his work. Of course, the apparent tension between LW's views on ethical discourse (especially the early ones) and his own ethical-aesthetical discourse remains an interesting and philosophically relevant issue (cf. the tension between his anti-revisionist views on the aims of philosophy and the critical remarks that are omnipresent throughout his philosophical work; cf. section 0.2(D) above and section 3.1.1(C8) below).

¹⁷⁸ Cf. -of course- LW's famous admonishment to silence at the end of the *TLP*. Even if LW at the time of the *TLP* may have believed one should literally shut up unless one had something factual to say (which is an unteachable position, of course, as the mere existence of the *TLP* self-consciously shows), the more realistic view of what human language use consists of, which LW started developing as soon as he came back to philosophy in the 1920s, implied ipso facto a less narrow-minded view on ethical talk. Already in the *Lecture on ethics* quoted above, LW expressed an opinion in which he backpedals the austerity of the end of the *TLP*: "Ethics so far as it springs from the desire to say something about the ultimate meaning of life, the absolute good, the absolute valuable, can be no science. What it says does not add to our knowledge in any sense. But it is a document of a tendency in the human mind

(B) fakeness as the target of Wittgenstein's philosophical criticism: terms, concepts and argumentative patterns

In section 1.2 above, we pointed out that for LW, meaninglessness/nonsense can be defined in terms of lack of embedding in everyday practices and that gibberish is not a problem in this respect: only things that sound like they make sense but don't, are a problem; fake sense is the problem. The above sheds new light on this notion of 'fake sense', giving it more width and depth than it had in the purely technical context in which we first introduced it: fakeness, pretense, fiction, etc. have become much more central concepts, with a much wider network of connections to other concepts, than they seemed to be, at first.

In sections 2.1 through 2.4 below, I will show how these concepts are systematically exploited in a number of key passages in LW's work on mathematics and how they are key to our understanding of what is at issue in these pages.

Here, I want to offer a quick illustration of **the pervasiveness of terms that denote 'fakeness'** in one form or another within LW's philosophical work. The purpose is to focus the readers' attention to a cluster of terms, concepts and argumentative patterns that will reoccur time and time again in the material I will analyze below.

A nice sample to start with is the passage in *PhU* in which LW is most explicit about the aims and methods of his philosophy (§§89-133 according to the commentary by Baker & Hacker (Baker and Hacker 2005)). Against the backdrop of the above, it is truly remarkable how much of the contents of these paragraphs consists of exactly the preoccupation with fiction, illusion, nebulosity, etc. Let us first analyze a few -for some readers- very familiar passages.

the crystalline nature of logic as a nimbus surrounding thought (*PhU* §97)

97. Das Denken ist mit einem Nimbus umgeben. -- Sein Wesen, die Logik, stellt eine Ordnung dar, und zwar die Ordnung a priori der Welt, d. i. die Ordnung der *Möglichkeiten*, die Welt und Denken gemeinsam sein muß. Diese Ordnung aber, scheint es, muß *höchst einfach* sein. Sie ist vor aller Erfahrung; muß sich durch die ganze Erfahrung hindurchziehen; ihr selbst darf keine erfahrungsmäßige Trübe oder Unsicherheit anhaften. -- Sie muß vielmehr vom reinsten Kristall sein. Dieser Kristall aber erscheint nicht als eine Abstraktion; sondern als etwas Konkretes, ja als das Konkreteste, gleichsam Härteste. (Log. Phil. Abh. 5.5563.)

Wir sind in der Täuschung, das Besondere, Tiefe, das uns Wesentliche unserer Untersuchung liege darin, daß sie das unvergleichliche Wesen der Sprache zu begreifen trachtet. D. i., die Ordnung, die zwischen den Begriffen des Satzes, Wortes, Schließens, der Wahrheit, der Erfahrung, u. s. w. besteht. Diese Ordnung ist eine Über-Ordnung zwischen -- sozusagen -- Über-Begriffen. Während doch die Worte

which I personally cannot help respecting deeply and I would not for my life ridicule it" ((Wittgenstein 1965), p. 12).

“Sprache”, “Erfahrung”, “Welt”, wenn sie eine Verwendung haben, eine so niedrige haben müssen, wie die Worte “Tisch”, “Lampe”, “Tür”.¹⁷⁹

LW evokes the traditional conception of logic as something in common to thought and the world, something a priori to all experience, something simple and pure, the most concrete thing, hard as crystal, and calls this “a nimbus that surrounds thought”. The word ‘nimbus’ evokes something visible but not actually there, not actually real, a **mere appearance**.

Similarly, he evokes the idea that the specialness, the depth, that what is essential to us about logical investigation resides in the ‘incomparable’ nature of language. LW calls this idea an ‘illusion’ [Täuschung].

Note the array of terms denoting what I will call **sensationalism** (and -following the next excerpt- **pathos**): the (fake) **specialness** that pervades the kind of discourse LW is criticizing here (in this case the self-description of logic) gets a negative connotation, in contrast to the **humble, trivial** nature of **everyday** applications of language, which is evaluated positively.

pathos, illusion, fake depth, sensationalism (*PhU* §110-112)

110. “Die Sprache (oder das Denken) ist etwas Einzigartiges” -- das erweist sich als ein Aberglaube (nicht Irrtum!) hervorgerufen selbst durch grammatische Täuschungen.

Und auf diese Täuschungen, auf die Probleme, fällt nun das Pathos zurück.

111. Die Probleme, die durch ein Mißdeuten unserer Sprachformen entstehen, haben den Charakter der Tiefe. Es sind tiefe Beunruhigungen; sie wurzeln so tief in uns, wie die Formen unserer Sprache, und ihre Bedeutung ist so groß, wie die Wichtigkeit unserer Sprache. -- Fragen wir uns: Warum empfinden wir einen grammatischen Witz als tief? (Und das ist ja die philosophische Tiefe.)

112. Ein Gleichnis, das in die Formen unserer Sprache aufgenommen ist, bewirkt einen falschen Schein; der beunruhigt uns: “Es ist doch nicht *so!*” -- sagen wir. “Aber es muß doch *so sein!*”

113. “Es ist doch *so* -- -- --” sage ich wieder und wieder vor mich hin. Es ist mir, als müßte ich das Wesen der Sache erfassen, wenn ich meinen Blick nur *ganz scharf* auf dies Faktum einstellen, es in den Brennpunkt rücken könnte.¹⁸⁰

¹⁷⁹ 97. Thinking is surrounded by a nimbus. a Its essence, logic, presents an order: namely, the a priori order of the world; that is, the order of *possibilities*, which the world and thinking must have in common. But this order, it seems, must be *utterly simple*. It is *prior* to all experience, must run through all experience; no empirical cloudiness or uncertainty may attach to it. -- It must rather be of the purest crystal. But this crystal does not appear as an abstraction, but as something concrete, indeed, as the most concrete, as it were the *hardest* thing there is (*Tractatus Logico-Philosophicus* 5.5563).

We are under the illusion that what is peculiar, profound and essential to us in our investigation resides in its trying to grasp the incomparable essence of language. That is, the order existing between the concepts of proposition, word, inference, truth, experience, and so forth. This order is a *super-order* between a so to speak a *super-concepts*. Whereas, in fact, if the words “language”, “experience”, “world” have a use, it must be as humble a one as that of the words “table”, “lamp”, “door”. |45|

¹⁸⁰ 110. “Language (or thinking) is something unique” a this proves to be a superstition (not a mistake!), itself produced by grammatical illusions. And now the impressiveness retreats to these illusions, to the problems.

111. The problems arising through a misinterpretation of our forms of language have the character of depth. They are deep disquietudes; they are as deeply rooted in us as the forms of our language, and their significance is as

The topic of these paragraphs is the idea that certain phenomena are sometimes conceived of as **unique, special, deep**. LW suggests that these ideas are based on illusions, false appearances, misinterpretations, superstition. We will see further on that this opposition of **special vs. trivial** (typically implying that what is conceived of as special, actually is trivial) is a recurring theme in LW's work.

Very interesting is LW's use of the term '**pathos**' in this context: he calls the idea that language or thought would be something special a 'superstition' based on illusions, and says that the 'pathos' of conceiving of trivial things as special things is based on these superstitious illusions. The term pathos evokes a -typically excessive- emotional involvement and expression,¹⁸¹ which LW then illustrates in §§112-113 by means of the description of someone frantically pondering a 'deep' philosophical question, back and forth. This use of the term 'pathos' reminds us what Sass had to say about **theatricality** (cf. 2.0.1(C) here above), and we will see below that similar terms, related to **ostentatiousness** and **sensationalism**, are applied in other technical contexts.

Luftgebäude (*PhU* §118)

118. Woher nimmt die Betrachtung ihre Wichtigkeit, da sie doch nur alles Interessante, d. h. alles Große und Wichtige, zu zerstören scheint? (Gleichsam alle Bauwerke; indem sie nur Steinbrocken und Schutt übrig läßt.) Aber es sind nur Luftgebäude, die wir zerstören, und wir legen den Grund der Sprache frei, auf dem sie standen.

119. Die Ergebnisse der Philosophie sind die Entdeckung irgend eines schlichten Unsinn und Beulen, die sich der Verstand beim Anrennen an die Grenze der Sprache geholt hat. Sie, die Beulen, lassen uns den Wert jener Entdeckung erkennen.

120. Wenn ich über Sprache (Wort, Satz, etc.) rede, muß ich die Sprache des Alltags reden. Ist diese Sprache etwa zu grob, materiell, für das, was wir sagen wollen? Und wie wird denn eine andere gebildet?

-- Und wie merkwürdig, daß wir dann mit der unsern überhaupt etwas anfangen können!

[...] ¹⁸²

great as the importance of our language. -- Let's ask ourselves: why do we feel a grammatical joke to be deep? (And that is what the depth of philosophy is.)

112. A simile that has been absorbed into the forms of our language produces a false appearance which disquiets us. "But *this* isn't how it is!" a we say. "Yet *this* is how it *has to be!*" |48|

113. "But *this* is how it is -- -- --", I say to myself over and over again. I feel as though, if only I could fix my gaze *absolutely sharply* on this fact and get it into focus, I could not but grasp the essence of the matter.

¹⁸¹ The standard translation 'impressiveness' (retained in (Wittgenstein 2009)) is wrong as it completely misses the relevant connotations of 'pathos'.

¹⁸² 118. Where does this investigation get its importance from, given that it seems only to destroy everything interesting: that is, all that is great and important? (As it were, all the buildings, leaving behind only bits of stone and rubble.) But what we are destroying are only houses of cards, and we are clearing up the ground of language on which they stood.

119. The results of philosophy are the discovery of some piece of plain nonsense and the bumps that the understanding has got by running up against the limits of language. They a these bumps a make us see the value of that discovery.

The first thing of interest to us in this excerpt is the dichotomy between on the one hand “all that is interesting”, i.e. “everything great and important”, vs. on the other hand, the trivial everyday. The context gives a de facto negative connotation to ‘everything great and important’ which turns out to be ‘buildings made of air’ [Luftgebäude],¹⁸³ and a positive evaluation of the trivial, everyday language which is the bedrock on which we stand. By implication the coarseness and materiality of everyday language is evaluated positively. Again, LW opposes (1) the **special** and the **trivial**, and (2) the **solid and real** vs. **mere appearance**.

contradictions (*PhU* §125)

Es ist nicht Sache der Philosophie, den Widerspruch durch eine mathematische, logisch-mathematische, Entdeckung zu lösen. Sondern den Zustand der Mathematik, der uns beunruhigt, den Zustand *vor* der Lösung des Widerspruchs, übersehbar zu machen. (Und damit geht man nicht etwa einer Schwierigkeit aus dem Wege.)

Die fundamentale Tatsache ist hier: daß wir Regeln, eine Technik, für ein Spiel festlegen, und daß es dann, wenn wir den Regeln folgen, nicht so geht, wie wir angenommen hatten. Daß wir uns also gleichsam in unsern eigenen Regeln verfangen.

Dieses Verfangen in unsern Regeln ist, was wir verstehen, d. h. übersehen wollen.

Es wirft ein Licht auf unsern Begriff des Meinens. Denn es kommt also in jenen Fällen anders, als wir es gemeint, vorausgesehen, hatten. Wir sagen eben, wenn, z. B., der Widerspruch auftritt: “So hab’ ich’s nicht gemeint.”

Die bürgerliche Stellung des Widerspruchs, oder seine Stellung in der bürgerlichen Welt: das ist das philosophische Problem.

Although somewhat different from the other excerpts in this series, I decided to include this excerpt here because it also shows a number of features that will come back later.

First of all, the presence of this excerpt in a context in which LW explains the aims and methods of the philosophical project underlying his later work, shows that **mathematics/mathematical logic** is still at the core of this project and that **contradictions** in formal systems are for LW among the core problems philosophy needs to deal with.

120. When I talk about language (word, sentence, etc.), I must speak the language of every day. So is this language too coarse, too material, for what we want to say? *Well then, how is another one to be |49| constructed? a* And how extraordinary that we should be able to do any- thing at all with the one we have!

[...]

¹⁸³ The standard translation ‘houses of cards’ is simply wrong: houses of cards are fragile but they are something, ‘Luftgebäude’ are nothing.

Next, LW also insists on the opposition **philosophy vs. mathematics/mathematical logic**. Philosophy looks at the problem from a ‘civilian’ (non-technical) ¹⁸⁴ point of view and technical solutions to the problem of contradictions are not interesting from a philosophical point of view. Similarly, LW is not primarily interested in a ‘solution’ to the problems related to contradictions but in the ‘unrest’ that these problems cause, i.e. not a problem internal to the game of axiomatic formalisms, but a real-life issue (cf. the notion that philosophy is a kind of therapy, etc.).

In the same vein of thought, LW’s diagnosis of the problem of contradictions as a matter of us not having understood the consequences of **the rules of a game** we invented ourselves: the real problem -for LW, as opposed to most philosophers of mathematics- is a matter of the **pragmatics** of mathematical formalism (I mean: the way axiomatic systems are used in actual practice), not a matter of their syntax or their semantics. This excerpt thus previews one of LW’s more important criticisms that we will study below: that mathematicians/philosophers of mathematics have a tendency to pretend that what is merely an aspect of the rules of a game they invented is instead a deep, mysterious, awe-inspiring **fact of nature** (cf. section 2.(A) below).

the given is trivial (*PhU* §129)

129. Die für uns wichtigsten Aspekte der Dinge sind durch ihre Einfachheit und Alltäglichkeit verborgen. (Man kann es nicht bemerken, weil man es immer vor Augen hat.) Die eigentlichen Grundlagen seiner Forschung fallen dem Menschen gar nicht auf. Es sei denn, daß ihm *dies* einmal aufgefallen ist. -- Und das heißt: das, was, einmal gesehen, das Auffallendste und Stärkste ist, fällt uns nicht auf.¹⁸⁵

In this excerpt, LW points out that the -for his kind of philosophy- most important aspects are ‘hidden’ because they are simple and everyday. I included this excerpt because it is a very clear statement of the core difference between LW’s approach and most contemporary (and present-day) philosophy of mathematics: LW is interested in what is presupposed by what most mathematicians and philosopher of mathematics would simply accept as ‘given’.

(C) LW’s vocabulary and rhetoric of fakeness (summary overview)

Anticipating the analyses in sections 2.1 through 2.3 below, without any claim to systematicity (let alone completeness), and for the purely practical purpose of guiding the readers’ attention

¹⁸⁴ See also *BGM5* §2 / *MS*-126,30-32 (cf. sections 1.2.2(B) and 1.1.1(H) above), for the similar notion “im Zivil”.

¹⁸⁵ 129. The aspects of things that are most important for us are hidden because of their simplicity and familiarity. (One is unable to notice some- thing a because it is always before one’s eyes.) The real foundations of their inquiry do not strike people at all. Unless that fact has at some time struck them. -- And this means: we fail to be struck by what, once seen, is most striking and most powerful.

towards the aspects I want to focus on below,¹⁸⁶ I now present a summary overview of some of the terms, concepts and argumentative patterns that will be omnipresent in the excerpts I will analyze below.

- First, there are numerous occurrences of words that denote the cluster of concepts around **mere appearance, illusion and delusion** quite directly, for instance: Täuschung (*PhU* §80, §§96-97, §110; Ms-117, 105-110 = *BGM2* §19-22, see section 2.1(A) below; MS 136 110b-111a, see section 2.0.2(A) above); vortäuschen (*PhU* §251); vorspiegeln (*PhU* §253); Illusion (*PhU* §311, §362; *BPP2* §268);¹⁸⁷ mystification (*Blue Book* p. 3); *PhU* §270: “[...] daß die Annahme dieses Irrtums nur ein Schein war”); Einbildung (Ms-121,27r-28v, see section 2.2(C))
- In a number of passages, LW speaks of things that appear to be **visible** but are either **not solid objects, or not there at all**: Nimbus (*PhU* §97); Dunstkreis (*PhU* §117); Chimären (*PhU* §94); Luftgebäude (*PhU* §118); Atmosphäre von Gedankennebeln (Ms-113,93r-v, see section 2.2(B) below).
- The notion of illusion can also take the shape of **misunderstandings, misrepresentation, prejudice or superstition**: Mißverständniss/Mißverstehen (*PhU* §90-93; *PhU* §100; *PhU* §120; *PhU* §132); Mißdeutung Ms-113,93r-v, see section 2.2(B) below); Vorurteil (*PhU* §108), Aberglaube/ abergläubisch (*PhU* §110; Ms-118,116r-116v, see section 2.3(B) below; Ms-125,66r-68r, see section 2.3(D) below).
- Quite frequently, LW speaks of **pretense** in technical contexts as well as in everyday conversation (cf. section 2.0.1(C) above):¹⁸⁸ “Die Erklärung des Dedekindschen Schnittes tut so als wäre sie anschaulich” (Ms-106,245-255, see section 2.2(A) below); cf. also the use of “erscheinen lassen” (“make appear”) and the adverb “angeblich” (allegedly, supposedly) in Ms-117, 105-110 (see section 2.1(A) below). LW’s somewhat idiosyncratic but at the same time perfectly apt use of the term ‘**prude**’ in Ms-124,71-74 (section 2.3(C) for mathematical proofs in which one adheres to the strictest formal criteria, but allows complete nonsense fits in as well.
- Sometimes, LW speaks of **fiction**: Fiktion (Ms-126,133-138, section 2.1(C) below); “einen der Mengenlehre zu Grunde liegenden fiktiven Symbolismus” and “In der Mathematik

¹⁸⁶ I could have presented this material after my analyses, as part of my conclusions, but it is more useful here, so I can refer back to it.

¹⁸⁷ Cf. also the title *Insight and Illusion: Themes in the Philosophy of Wittgenstein* (Hacker 1986)).

¹⁸⁸ **Methodological remark**: some of these terms also occur quite often in other functions as well (e.g. terms related to ‘pretending’ occur in a large number of contexts in which LW analyzes language games that involve the concept of ‘pretending’: pretending to be in pain, pretending to read, pretending to play a game, pretending to be (un)conscious or unwell, etc. (e.g. in the context of arguments regarding the privateness of feelings); for typical examples, see e.g. *Z* §§568-571 or *PhPF*, xi, §§352-364. It is therefore impossible to automatize this kind of work. A close reading of the context remains necessary.

können | | dürfen wir alles fingieren nur nicht einen Teil unseres Kalküls“ (Ms-113,93r-v, section 2.2(B) below); “Der gebräuchliche Ausdruck fingiert einen Vorgang eine Methode des Ordners“ (Ms-117, 105-110, section 2.1(A) below). Cf. also the notion of a painting of a thing vs. a real thing: “wie der gemalte Fels die gemalte Burg trägt“ (Ms-124,71-74, section 2.3(C)).

- In some cases, the notion of ‘appearance’ indicates things that look like they are operational or functional, but are not, as in **sham or merely ornamental** parts in architecture (*PhU* §217), or machine building (*PhU* §132, *PhU* §§270-271); cf. section 1.2.1(B)).
- In still other cases, we observe the notion of **trickery and tricks**: Taschenspielerkunststück (*PhU* §308), Hokus Pokus (Ms-117, 105-110 = *BGM2* §19-22, cf. section 2.1(A) below); Kunststückchen Ms-118,116r-116v = *BGM 1*, Anhang III, §18-19, see section 2.3(B) below); I am not sure whether the notion of ‘mathematical alchemy’ (Ms-126,82-83) fits in with this category.
- LW tends to criticize his various targets for being fake, in the sense of **overstating** the importance, interestingness of ultimately trivial things; what appears to irk him most is the unwarranted **theatrical display of emotion**: Pathos (§110);¹⁸⁹ “prahlerisch“ (Ms-117, 105-110 = *BGM2* §19-22, see section 2.1(A) below); “wovor einem schwindlig werden kann“ (Ms-121,60r-64r = *BGM II*, §§40ff., see section 2.1(B) below); “Und dann wundert man sich z.B. darüber, daß...!“ (Ms-106,245-255, see section 2.2(A) below). LW’s mockery of mathematicians “fear and veneration” vis-à-vis contradictions fits in (Ms-118,116r-116v = *BGM 1*, Anhang III, §18-19, see section 2.3(B)), and so does his mockery of Hilbert’s “Cantor’s paradise” (MS-126, 55-56, see section 1.3(C) above). Another rhetorical device that LW’s uses is the mis-en-scène of a panicky interlocutor that displays exactly the theatrical emotions he objects to (see sections 2.3(A), 2.3(D), 2.3(F)).

Besides the above terms that express the ‘fakeness vs. authenticity’ trope as such, it may be worthwhile to briefly also mention the following patterns, which occur frequently in conjunction with occurrences of that trope:

- a persistent dichotomy of **the special vs. the trivial**: discourse invoking the special (i.e. the non-everyday, **interestingness, importance, mystery, depth, sublime**, etc.) is typically *negatively evaluated*, whereas the trivial and everyday is typically *positively evaluated*: Sublimes (*PhU* §89), Tiefe (*PhU* §89, 110, 111; Ms-126,133-138, see section 2.1(C)),

¹⁸⁹ See also Ms-183,228, d.d. 19370330: “Hüte Dich vor einem billigen Pathos wenn Du über Philosophie schreibst! Das ist immer meine Gefahr, wenn mir *wenig* einfällt. Und so ist es jetzt. Ich bin zu einem seltsamen Stillstand gekommen & weiß nicht recht, was ich machen soll.“ Also note the notion of cheapness [billig] in this excerpt; LW uses this adjective quite frequently in this sense but it happens to not occur in the passages that interest us below.

Merkwürdiges/Seltsames vs. Alltäglich (*PhU* §93); Einzigartiges (*PhU* §95, §110); das unvergleichliche Wesen, Kristall vs. niedrig (*PhU* §97); metaphysisch vs. alltäglich (*PhU* §116); Ein “führendes Problem der mathematischen Logik” vs. “ein Problem der Mathematik, wie jedes andere” (*PhU* §124); “interessant & merkwürdig”, “die Geheimnisse der mathematischen Welt” (Ms-121,60r, see section 2.2(B) below; LW’s mockery of Hilbert’s pathos about ‘Cantor’s paradise’ (MS-126, 55-56, see section 1.3(C) above) fits in with this trope;

- a recurring evaluation of the objects of LW’s criticism as **ridiculous, comical or childish**: “Scherzfrage” (Ms-121,27r-28v = *BGM2* §23, see section 2.2(C) below); “komisch” (Ms-118,116r-116v = *BGM* 1, Anhang III, §18-19, see section 2.3(B) below);¹⁹⁰ cf. also the recurrent references to **childrens’ games** (“Daumenfangen” Ms-118, 111v-, see section 1.2.1(C) above and 2.1(A) below; “Fingerhut-Verstecken” (Ms-126,133-138, see section 2.1(C) below. This trope need not merely be a conventional way of being dismissive of an opponents opinion: in many cases, LW’s use of this trope appears to be purposeful, in that it highlights features of the objectionable discourse that -from LW’s point of view- actually are childish and/or absurd.
- an emphasis on **the specificity of philosophy** (or at least LW’s own brand of philosophy) **as opposed to mathematics and logic**: as pointed out elsewhere (cf. section 0.2(D) above and section 3.2.2 below), LW is aware of the fact that his point of view is different from the various stances he interacts with; remarks that refer to this contrast are recurrent features throughout LW’s writings.

(D) epistemic fakeness: epistemic pretense, epistemic bad faith, epistemic bad taste

In what follows I will try to show how the above concepts are omnipresent in LW’s PhilMath, and in key functions, at that. Before embarking on the analyses that will make up the bulk of part 2 of this study, I would like to briefly summarize the conceptual framework I tried to sketch in the above and point out how the concept of fakeness shapes a few modes of thought / argumentative patterns that shape the critical aspect of LW’s PhilMath:

- The key concept in LW’s criticism is (what I call) **epistemic fakeness**:¹⁹¹ some utterances look/sound like they make sense but actually don’t, i.e. ‘fake’ concepts do not really have

¹⁹⁰ Another examples is LW, Ms-119,36: “(Sehr komisch Haldane über die Ewige Wahrheit eines arithmetischen Satzes. ganz ähnlich: die Seelen der Menschen die unsichtbar, also durchsichtig sind (Grabbe))”.

¹⁹¹ In coining these terms, my only purpose is to provide myself and the reader with a shorthand that helps me refer to these patterns without having to resort to cumbersome paraphrasis. I will avoid projecting these terms onto LW’s text during the running commentary below, but will use them freely in my conclusions and the appendix.

a function within a real-life 'everyday' practice and are therefore strictly speaking nonsense / meaningless; this is where we ended up at the end of part 1 (cf. section 1.4), but throughout section 2.0, we have pointed out that this notion of fakeness has many more ramifications than we could expect at the point from which we started out. In some cases, LW calls out **epistemic fiction**, i.e. discourse about math that makes use of concepts that do not correspond to anything in the actual mathematical technique the discourse refers to.

- When there appears to be an active effort to create such an illusion, LW calls out the **epistemic pretense**, often employing a tone of moral indignation. We'll see that many of the critical remarks we'll read below take this shape and use a vocabulary that evokes pretense, illusion, etc. We have seen that this concern is echoed by (rooted in?) a strong dislike for theatricality and pretense in a person's, including his own, behavior.
- Particularly infuriating is (what I call) **epistemic bad faith**,¹⁹² i.e. discourse that makes the effort to satisfy all the formal criteria expected from it, but that is still nonsense, in the sense that it doesn't have a real-life function in a real-life practice. This is exactly what LW blames 'prudish proofs' for: attaching great importance to the syntax of the formalism but allowing complete nonsense in the contents that are expressed by the proof. A similar objection could be made towards Gödel's famous endeavor, when it pretends to be normal arithmetic, whereas the apparatus it uses is clearly designed to do only that particular trick.
- There is an aesthetic aspect to epistemic fakeness, that I call **epistemic bad taste** (or: **epistemic kitsch**): we have seen above (section 2.0.1(A)) that 'style' was important in LW's ethics/aesthetics and LW's dislike for vanity, ostentatiousness, theatricality and inauthentic behavior in general (section 2.0.1(C)) has an epistemic counterpart in his objections to discourse that presents trivial things as interesting, special, mysterious, awe-

¹⁹² Jip Van Besouw (personal communication, October 2022) brings up the matter of intentionality, that seems to be semantically inherent in the concept of 'bad faith'. LW does not thematize intentionality in the excerpts that I analyze here, but the pretense, trickery, etc. do presuppose blameworthiness (if not intentionality per se).

inspiring, ...¹⁹³ We will encounter a few passages in which LW appears to be irked by this kind of kitsch in discourse about mathematics.¹⁹⁴

What is hard to convey -hence the volume of the material referred to in this section- is the lack of separation between the existential/cultural dimension and the epistemic/technical dimension of LW's philosophical concerns: on the one hand, LW's 'technical' philosophical remarks are permeated and motivated by very general, non-technical concerns, incl. a certain existential urgency, but on the other hand, it should be understood that these remarks are still intended to be technical contributions to whatever technical issue is at hand.¹⁹⁵ I hope that my analyses here below will make this clear.

¹⁹³ As pointed out above, LW quite often attaches a positive connotation to triviality and everydayness and a negative one to the special, interesting, etc. I am not claiming that LW is against genuine emotion when faced by phenomena that we experience as 'deep' per se, only that he is particularly annoyed at discourse that tries to conjure up such emotions in a 'cheap' way by misrepresenting things that are ultimately trivial.

For the fact that LW does value genuine enthusiasm, see the fact that he speaks of Ramsey's 'ugly mind' in terms of the fact (?) that Ramsey is unable to be enthusiastic about philosophical matters and that he reduces any philosophical issue very quickly from the paradoxical to the trivial:

Ramseys Geist war mir sehr zuwider. Als ich vor 15 Monaten nach Cambridge kam da glaubte ich, ich würde nicht mit ihm verkehren können denn ich hatte ihn von unserer letzten Begegnung vor etwa 4 Jahren bei Keynes in Sussex in so schlechter Erinnerung. Keynes dem ich dies sagte sagte mir aber er glaube ich sollte sehr wohl mit ihm reden können & nicht bloß über Logik. Und ich fand Keynes' Meinung bestätigt. Denn ich konnte mich über manches ganz gut mit Ramsey verständigen. Aber auf die Dauer ging es doch nicht wirklich gut. Die Unfähigkeit Ramseys zu wirklichem Enthusiasmus oder zu wirklicher Verehrung was das Selbe ist widerte mich endlich mehr & mehr an. Andererseits hatte ich eine gewisse Scheu vor Ramsey. Er war ein sehr rascher & geschickter Kritiker wenn man ihm Ideen vorlegte. Aber seine Kritik half nicht weiter sondern hielt auf & ernüchterte. Der kurze Zeitraum wie Schopenhauer ihn nennt zwischen den beiden langen in denen eine Wahrheit den Menschen, zuerst paradox, & dann trivial erscheint war bei Ramsey zu einem Punkt geworden. Und so plagte man sich zuerst lange vergebens ihm etwas klar zu machen bis er plötzlich die Achsel darüber zuckte & sagte es sei ja selbstverständlich. Dabei war er aber nicht unaufrichtig. Er hatte einen häßlichen Geist. Aber keine häßliche Seele. Er genoß Musik wirklich & mit Verständnis. Und man sah ihm an welche Wirkung sie auf ihn ausübte. Von dem letzten Satz eines der letzten Beethovenschen Quartette den er mehr als vielleicht alles andere liebte sagte er mir er fühle dabei die Himmel seien offen. Und das bedeutete etwas von ihm.

| | wenn er es sagte.
(Ms-183,6-8, d.d. 19260427)

¹⁹⁴ Let me add here one more -very funny- quote from LW, which didn't find a proper place in the above, in which LW writes -without any direct context- that mathematicians always like some kind of "haut-goût" (literally: "high taste", but in German and in English referring to the slight taste of decay that in certain circles is/was appreciated for game meat) about their propositions, which -says LW- in this case as always is the result of rot/putrefaction [Ms-126,105, d.d. 19421125: "Die Mathematiker lieben einen haut-goût an ihren Sätzen, der, wie überall, von der Fäulnis herrührt."].

¹⁹⁵ In the preface of his classic *The Greek Particles* (one of my main tools in a previous life), classical scholar J.D. Denniston writes: "The reader should be able to bathe in examples. If I have selected and arranged mine reasonably well, the mere process of semi-quiet immersion may help him as much as hours of anxious thought" ((Denniston 1954), p. vi)." I believe this is equally true for any endeavor that requires understanding a complex, multifaceted corpus.

2.1 Wittgenstein's critical remarks on math I: diagonal stuff (hocus pocus, cheap thrills, prudish proofs)

The issue of ordering numbers and the use of diagonal methods (the contributions of Dedekind and Cantor immediately come to mind, but LW also quotes other mathematicians in this context) are a recurring topic in LW's remarks on math. In this section, I discuss three passages in which LW focuses on diagonal techniques and the ways in which these are used in the context of various proofs.

(A) *hocus pocus*: Ms-117, 105-110 (= *BGM2* §19-22), not dated, but after 19370911 and before 19380627

Throughout this extended passage,¹⁹⁶ LW isolates the diagonal technique [Kalkül], as a perfectly normal mathematical technique, from its philosophical use. There is for LW no problem with the mathematical technique itself. But he goes to great lengths to point out that this technique is -at least potentially- independent of the way it is used for philosophical (theoretical? ideological?) purposes: one can use it to find a number that is different from all other numbers in a sequence; as such, one can teach it even to children.¹⁹⁷ LW would have no objections to any of this.

¹⁹⁶ The excerpt looks like a more or less well-defined unit. The demarcation of the beginning is not 100% clear, but LW does seem to leave some extra space at the top of the page, before the first words of the excerpt and also stops writing "[Ansätze]" as a heading, which he did from page 97 toll 104 of the manuscript). Still, what precedes is also of interest to our excerpt and the editors' choice to print the remarks in the order in which they occur in MS 117 is a good one. The excerpt clearly ends with the "hocus pocus" sentence (after that LW uses the notebook for drafting a couple prefaces, so that is a clear ending.

However, the way this excerpt is fitted in with a larger unit in the standard edition is a good illustration of the editorial practices that gave rise to the standard editions of LW's work: in the standard edition, these remarks (*BGM2*, §20-22), in which LW calls diagonal procedures "hocus pocus", are followed by (§23) by a remark on the "Krankheit einer Zeit" and how the bad health of our philosophical problems can only be cured by a change in our way of life (see section 2.2(C)). This presentation has no roots in LW's own work, and the "Sickness of our time" and the "Hocus Pocus" remarks are from altogether different manuscripts.

¹⁹⁷ Es ist hier sehr nützlich sich vorzustellen, daß das Diagonalverfahren zur Erzeugung einer reellen Zahl längst vor der Erfindung der Mengenlehre bekannt & auch den Schulkindern geläufig gewesen wäre, wie es ja sehr wohl hätte sein können. So wird nämlich der Aspekt der Entdeckung Cantors geändert. Diese Entdeckung hätte sehr wohl bloß in der Interpretation || neuen Auffassung dieser altbekanntten, elementaren Rechnung liegen können. Die Rechnung || Rechnungsart selbst ist ja nützlich. Die Aufgabe wäre etwa: Schreibe eine Dezimalzahl an die verschieden ist von den || allen Zahlen:

0'1246798

0'3469876

0'0127649

0'3426794

Man denke sich eine lange Reihe.

Das Kind denkt sich: Wie soll ich das machen ich müßte ja auf alle die Zahlen zugleich schauen um zu vermeiden daß ich nicht doch eine von ihnen anschreibe || damit ich nicht doch irgend eine von ihnen aufschreibe. Die Methode sagt nun: durchaus nicht; ändere die erste Stelle der ersten Zahl, die zweite der zweiten, etc. etc. & Du bist sicher eine Zahl hingeschrieben zu haben, die mit keiner der gegebenen übereinstimmt. Die Zahl die man so erhält könnte immer die Diagonalzahl genannt werden.

For LW, the mistake [Fehler] begins when one tries to apply the notion of “ordering things in a series” [Reihe] to the cardinal numbers at large. LW denies categorically that the question “can one order the set R in a series?” has any clear sense at all,¹⁹⁸ and goes on to deny that it is obvious that the diagonal proof proves that “ordering in a series” does not work in this case, arguing instead that the problem can also be viewed as an issue with the concepts involved.¹⁹⁹ So, on pp. 61-64, LW is not contradicting the idea that “one cannot order R in a series”, nor its opposite: he is saying that the question has no clear sense (cf. what he says about “north of the pole”, Fermat’s conjecture, etc.; see sections 1.1.2(B) and 1.2.1(B)). Then, immediately after this, follows a passage that displays a number of **aesthetical and ethical terms**,²⁰⁰ is particularly rich in interesting contrasts, and also shows how the notion of fakeness can take on an almost ‘technical’ function in what remains a discussion about what can be inferred from Cantor’s diagonal argument (as it is usually understood).

LW makes it very explicit that his most important target is when people start to pretend that their preferred theoretical claims follow ‘**naturally**’ from this or that technique:²⁰¹

Das Gefährliche, Täuschende, der Fassung “Man kann die reellen Zahlen nicht in eine Reihe ordnen” oder gar “Die Menge ... ist nicht abzählbar” liegt darin, daß sie das was eine Begriffsbestimmung Begriffsbildung ist als eine Naturtatsache erscheinen lassen.

¹⁹⁸ Der Fehler beginnt damit daß man sagt die Kardinalzahlen ließen sich in eine Reihe ordnen. Welchen Begriff hat man denn von diesem Ordnen? Ja man hat natürlich einen von einer endlichen Reihe, aber das gibt uns ja hier höchstens eine vage Idee einen Leitstern für die Bildung eines Begriffs.) Der Begriff selbst ist ja von dieser & einigen andern Reihen abstrahiert; oder: der Ausdruck bezeichnet eine gewisse Analogie von Fällen & man kann ihn etwa dazu benützen um ein Gebiet, von dem man reden will vorläufig abzugrenzen.

Damit ist aber nicht gesagt, daß die Frage einen klaren Sinn hat: “Ist die Menge R in eine Reihe zu ordnen?” Denn diese Frage bedeutet nun etwa: Kann man mit diesen Gebilden etwas tun was dem Ordnen der Kardinalzahlen in eine Reihe entspricht. Wenn man also fragt: “Kann man die Reellen Zahlen in eine Reihe ordnen?” So könnte die gewissenhafte Antwort sein: “Ich kann mir vorläufig gar nichts Genaueres darunter vorstellen”. – “Aber Du kannst doch z.B. die Wurzeln & die algebraischen Zahlen in eine Reihe ordnen; also verstehst Du doch den Ausdruck!” – Richtiger gesagt ich habe hier gewisse analoge Gebilde, die ich mit dem gemeinsamen Namen “Reihen” benenne. Aber ich habe noch keine sichere Brücke von diesen Fällen zu dem ‘aller reellen Zahlen’. Ich habe auch keine allgemeine Methode um zu versuchen ob sich die oder die Menge ‘in eine Reihe ordnen läßt’.

¹⁹⁹ Nun zeigt man mir das Diagonalverfahren & sagt: “hier hast Du nun den Beweis, daß dieses Ordnen hier nicht geht”. Aber ich kann antworten: “Ich weiß – wie gesagt – nicht, was es ist, was hier nicht geht.” Wohl aber sehe ich: Du willst einen Unterschied zeigen in der Verwendung von “Wurzel”, “algebraische Zahl”, etc. einerseits & “reelle Zahl” andererseits. Und zwar etwa so: Die Wurzeln nennen wir “reelle Zahlen” & die Diagonalzahl, die aus den Wurzeln gebildet ist auch. Und ähnlich mit allen Reihen reeller Zahlen. Daher hat es keinen Sinn von einer “Reihe aller reellen Zahlen” zu reden, weil man ja auch die Diagonalzahl der || jeder Reihe eine “reelle Zahl” nennt. – Wäre das nicht etwas ähnlich, wie wenn man gewöhnlich jede Reihe von Büchern selbst ein Buch nennt & nun sagte: “Es hat keinen Sinn von ‘der Reihe aller Bücher’ zu reden, da jede || diese Reihe selbst ein Buch ist || wäre.”

²⁰⁰ The underlining of terms with an ethical or aesthetical connotation is mine.

²⁰¹ 19. The dangerous, deceptive thing about the idea: "The real numbers cannot be arranged in a series", or again "The set... is not denumerable" is that it makes the determination of a concept--concept formation--look like a fact of nature.

This element will become important later on (I believe it is one of the most important points to be gained from LW's work), so let's emphasize it here: what LW objects to as 'dangerous, misleading' is that it makes what actually is a case of concept-formation, look like **a fact of nature**.

LW then reformulates the conclusion one may want to draw from Cantor's technique:

Bescheiden heißt || lautet der Satz: "Wenn man etwas eine Reihe reeller Zahlen nennt, so heißt die Entwicklung des Diagonalverfahrens auch eine 'reelle Zahl' & zwar eine die 'von allen Gliedern der Reihe verschieden' sei || ist. || & zwar sagt man, sie sei von allen Gliedern der Reihe verschieden.

Modestly, the proposition would sound as follows: "If one calls something 'a series of real numbers', then its expansion via the diagonal procedure could also be called a 'real number', more specifically one that would be different from all the other members of the series".²⁰² This would be a conclusion/interpretation that LW could agree with.

Then, LW explicates what he believes is wrong with the standard interpretation of Cantor's diagonal technique (i.e. the interpretation in terms of the non-denumerability of the reals):

Unser Verdacht sollte immer rege sein, wenn ein Beweis mehr beweist, als seine Mittel ihm erlauben. Man könnte so etwas einen 'prahlerischen Beweis' nennen.

Der gebräuchliche Ausdruck finger einen Vorgang eine Methode des Ordnen die hier zwar anwendbar ist aber nicht zum Ziele führt wegen der Zahl der Gegenstände die größer ist als selbst die der || aller Kardinalzahlen.²⁰³

One should always be suspicious when a proof proves more than its means allow for. Such a proof, LW calls an '**ostentatious**' proof [einen 'prahlerischen Beweis']. Thus, the usual way the conclusion drawn from the diagonal technique is formulated, creates the **fiction** of "a

²⁰² 20. The following sentence sounds sober: "If something is called a series of real numbers, then the expansion given by the diagonal members of the series procedure is also called a 'real number', and is moreover said to be different from all members of the series".

NB: this standard translation is wrong:

(1) 'sober', while emphasizing the contrast with the concept of 'ostentatious' in what follows, misses the ethical connotation of "bescheiden";

(2) the translation not only distorts the grammar of the German text (which need not be a problem), but seems to be based on a misunderstanding of the grammar of the original and thus also distorts the point of the original: the German text says that modestly, the proposition (i.e. the conclusion that could follow from Cantor's diagonal argument) would sound as follows (not that that the following sentence sounds sober); the English translation may be read as suggesting that the sentence 'sounds sober', but perhaps is not actually sober (which is not meant at all), but -more importantly- completely misses the idea that what follows is a more modest reformulation of what precedes.

²⁰³ 21. Our suspicion ought always to be aroused when a proof proves more than its means allow it. Something of this sort might be called 'a puffed-up proof'.

22. The usual expression creates the fiction of a procedure, a method of ordering which, though applicable here, nevertheless fails to reach its goal because of the number of objects involved, which is greater even than the number of all cardinal numbers.

method of ordering that is applicable in this case, but fails to reach its goal, because of the number of objects that is larger than even the number of all cardinal numbers". So, LW believes that the standard interpretation is based on the idea that it is in principle possible to order the real numbers but that we never get there in practice because of the number of objects to be ordered. And that idea he calls a **fiction**: it is clear how we can go about ordering an indefinite amount of integers, and whatever the number of integers, the same method (or methods) will continue to work; but these methods are clearly and obviously not going to work when we are asked to order the reals as a whole; so, from LW's pragmatic point of view, it is literally unclear what 'ordering' could even mean in the case of the reals.

LW then explains what he means, again by contrasting the usual conclusion with an alternative formulation of his own:

Wenn gesagt würde: "Die Überlegung über das Diagonalverfahren zeigt Euch, daß der Begriff 'reelle Zahl' viel weniger Analogie mit dem Begriff Kardinalzahl hat, als man, durch gewisse Analogien verführt, zu glauben geneigt ist" so hätte das einen guten & ehrlichen Sinn. Es geschieht aber gerade das Gegenteil: indem die 'Menge' der reellen Zahlen angeblich der Größe nach mit der der Kardinalzahlen verglichen wird. Die Artverschiedenheit der beiden Konzeptionen wird durch eine schiefe Ausdrucksweise als Verschiedenheit der Ausdehnung dargestellt.²⁰⁴

LW's reformulation is rather drastic: reflection on the diagonal procedure shows you that the concept of 'real number' is much less analogous to the concept of 'cardinal number' than one would be inclined to believe, tempted by certain analogies; this account would have a **good and honest** sense. However, in reality, exactly the opposite happens, in that the 'set' (scare quotes required) of real numbers -allegedly [angeblich]- is compared qua magnitude with the set of cardinal numbers. So, because of a skewed way of expressing oneself, what is actually a difference in kind between conceptions is falsely represented as a difference in extension. The passage ends as follows: "Ich glaube & hoffe eine künftige Generation wird über diesen Hokus Pokus lachen".²⁰⁵ The prediction that future generations will laugh about Cantor's **hocus pocus** has not (not yet?) come true: Cantor's contributions appear to be firmly entrenched in the mainstream mathematical canon. For the recurring theme of ridiculousness in LW's evaluation of mathematical discourses, see also LW's assessment that most

²⁰⁴ If it were said: "Consideration of the diagonal procedure shews you that the concept 'real number' has much less analogy with the concept 'cardinal number' than we, being misled by certain analogies, are inclined to believe", that would have a good and honest sense. But just the opposite happens: one pretends to compare the 'set' of real numbers in magnitude with that of cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension.

²⁰⁵ I believe, and hope, that a future generation will laugh at this *hocus pocus*.

mathematicians' fear of contradictions is ridiculous (Ms-118,116r; see section 2.3(B) below); cf. also LW Ms-113,92r-92v.²⁰⁶

So: LW does not object to Cantor's diagonal technique as such: as a technique to find a number that is different from a list of other numbers, it is perfectly fine. LW does object to its usual prose interpretation as a proof for the non-denumerability of the reals: for LW, the problem with this interpretation is that it *pretends* that the notion of 'ordering' would be applicable to the irrationals in the same way it is applied to the rationals, that the difference is a merely quantitative matter, whereas this is in practice -obviously- not the case: it is not at all clear what 'to order' would even mean in this context. Similarly, the interpretation *pretends* that nothing happened when the notion of 'number' (as in counting) is extended to the 'set' (scare quotes required) of real numbers, as if this extension is not the result of the decision to coin a new concept, but a fact of nature. In a very typical fashion, LW formulates his objections to this apparently 'technical' matter in clearly **ethical/aesthetical terms** (cf. the words that I underlined in the above quotations of the text): LW criticizes the standard interpretation of Cantor's technique because it is misleading and dangerous, lacks modesty, is ostentatious, contains fictitious aspects, lacks honesty, is ridiculous hocus pocus.

(B) against vertiginous imagery: LW, Ms-121,60r-64r, d.d. 19381225 [// BGM II, §§40 ff.]

The following passage, written on Christmas day 1938, also deals with the problem of ordering numbers in a series and especially the idea that "one cannot order all fractions in a series":

25.12.38

"Man kann die Brüche nicht ihrer Größe nach ordnen. – Das klingt vor allem sehr || höchst interessant & merkwürdig.

Es klingt interessant in ganz anderem Sinne || anderer Weise, als, etwa, ein Satz aus der Differentialrechnung. Der Unterschied liegt, glaube ich, darin, daß ein solcher sich leicht mit einer Anwendung auf Physikalisches assoziiert, während jener Satz ganz & gar || einzig & allein der Mathematik anzugehören gleichsam die Physik || die Naturgeschichte der mathematischen

²⁰⁶ Die Ausdrucksweise: $m = 2n$ ordne eine Klasse einer ihrer echten Subklassen || Teilklassen zu, kleidet einen einfachen || trivialen Sinn durch Heranziehung einer irreführenden Analogie in eine paradoxe Form. (Und statt sich dieser paradoxen Form als etwas Lächerlichem zu schämen, brüstet man sich eines Sieges über alte Vorurteile des Verstandes.) Es ist genau so als stieße man die Regeln des Schach um & sagte, es habe sich gezeigt, daß man Schach auch ganz anders spielen könne. So verwechselt man erst das Wort „Zahl“ mit einem Begriffswort wie „Apfel“, spricht dann von einer „Anzahl der Anzahlen“ & sieht nicht daß man in diesem Ausdruck nicht beidemal das gleiche Wort „Anzahl“ gebrauchen sollte; & endlich hält man es für eine Entdeckung daß die Anzahl der geraden Zahlen die gleiche ist wie die der geraden & ungeraden.

Gegenstände selbst zu betreffen scheint. Man möchte von ihm etwa sagen: er führe uns in die Geheimnisse der mathematischen Welt ein. Es ist dieser Aspekt vor dem | | welchem ich warnen will.²⁰⁷

From the outset, LW tells us quite clearly what he finds suspicious and wants to warn us for: the problem is that the sentence “one cannot order all fractions in a series” sounds **interesting and mysterious**, quite different from a formula out of a differential calculus book, as if it concerned “the **natural history** of mathematical objects”.

A few paragraphs further on, LW argues that the image of trying to stuff ever more objects into the same space (implicit in the idea of ordering all fractions) is **sensational** (it “makes our heads spin” [wovor einem schwindlig werden kann]) but also inappropriate, in that the mathematical technique merely shows something **trivial**:²⁰⁸ the issue is not that the technique goes on and on and on, the issue is that the idea of an end is simply not part of this technique: it does not make sense to speak of the “next larger fraction”, the word “end” is simply not applicable (it is meaningless/nonsensical) in this context.²⁰⁹

Then, LW appears to identify the root of the issue in terms of problems with the **expansion** of a practice: when one invents a new technique or a new framework, one needs new images, new means of expression to describe it; it would be absurd to want to describe the new scheme, the new kind of framework by means of the old expressions.²¹⁰ So, when one says “There is no next larger fraction, but there is a next larger cardinal number”, it is not clear

²⁰⁷ 40. "Fractions cannot be arranged in an order of magnitude."--First and foremost, this sounds extremely interesting and remarkable.

It sounds interesting in a quite different way from, say, a proposition of the differential calculus. The difference, I think, resides in the fact that *such* a proposition is easily associated with an application to physics, whereas *this* proposition belongs simply and solely to mathematics, seems to concern as it were the natural history of mathematical objects themselves.

One would like to say of it e.g.: it introduces us to the mysteries of the mathematical world. *This* is the aspect against which I want to give a warning.

²⁰⁸ Wenn ich mir bei dem Satz, die Brüche können nicht ihrer Größe nach in eine Reihe geordnet werden, das Bild einer unendlichen Reihe von Dingen mache, & zwischen je zwei Nachbarbäumen neue Bäume in die Höhe schießen & nun wieder zwischen jedem Baum & seinem Nachbar neue Bäume & so fort ohne Ende, so haben wir hier (sicher) etwas, wovor einem schwindlig werden kann.

Sehen wir aber, daß dieses Bild zwar | | wohl sensationell, aber ganz unzutreffend ist, daß wir uns nicht von den Worten “Reihe”, “ordnen”, “existieren” & anderen fangen lassen dürfen, so werden wir auf eine Darstellung des Sachverhalts zurückgehen, in der alles wieder trivial & gewöhnlich aussieht. so werden wir (wieder) auf die (Darstellung der) Technik des Bruchrechnens zurückgreifen an der nun nichts Seltsames mehr ist.

²⁰⁹ Daß wir eine Technik erfinden | | bilden, in der der Ausdruck “der nächst größere Bruch” keinen Sinn hat, daß wir ihm keinen Sinn gegeben haben, ist nichts Erstaunliches.

Wenn wir eine Technik des fortgesetzten Interpolierens von Brüchen anwenden, so werden wir keinen Bruch den “nächst größeren” nennen wollen.

Von einer Technik zu sagen, sie sei unbegrenzt, heißt nicht, sie laufe ohne aufzuhören weiter – wachse ins Ungemessene; sondern, es fehle ihr die Institution des Endes, sie sei nicht abgeschlossen. Wie man () von einem Satz sagen könnte, es mangle ihm der Abschluß, wenn der Schlußpunkt fehlt oder von einem Spielfeld es sei nicht begrenzt, wenn ihm die Regeln des Spiels keine gezogene Grenze vorschreiben.,

²¹⁰ Eine neue Rechentechnik soll uns ja eben ein neues Bild liefern, eine neue Ausdrucksweise; & wir können nichts Absurderes tun, als dieses neue Schema, diese neue Art von Gerüst, mittels der alten Ausdrücke beschreiben zu wollen.

what the function of this sentence actually is. It is like comparing two board games: certain moves exist in checkers, but not in chess. Similarly, there is something we call “to construct the next larger cardinal number”, but there is nothing we would call “to construct the next larger fraction”.²¹¹

LW applies exactly the same argument we already encountered in section 1.2.1(B), with respect to nonsensical pseudo-expressions (“north of the pole”) in general, and in section 1.1.2(B) with respect to conjectures: the expression “the next larger fraction” has no actual function in an actual practice and is therefore **meaningless**.

LW’s point appears to be the following: the expression “construing the next bigger number” works in the case of integers, but is simply not applicable to fractions; similarly, applying the image of a line to the order of numbers does not always work. Trying to apply the image of a line to represent the order of ‘all fractions’ is a sensational one, it makes peoples’ heads spin, but this is not an appropriate representation of what’s going on: that the image does not work well in this context is a **trivial** fact.

So, the main conceptual opposition he operates with here is between (1) **sensational/vertiginous** vs. (2) **trivial**, and he clearly values (1) negatively. Again, LW has no objection against any technical aspect of the mathematical technique, but objects to the way it is represented in theoretical terms, more precisely to the way the concept of ‘infinite’ is used in a context in which the concept of an ‘end’ is simply not applicable and hence, and similarly the concept ‘next larger’ is not just for reasons of finite time impossible to apply, but because the idea of ‘next larger’ is simply meaningless in this context.

(C) [prude proofs: Ms-126,133-138, d.d. 19411215-19411217](#)

This excerpt is part of what looks like a nice collection of one-liners, which, however, lacks the cohesiveness and cogent development displayed by the other excerpts studied here. The reason why I do include it here is because it takes up a few themes that were highlighted in the previous excerpts and adds some nice new details to the picture, most notably the

²¹¹ Was ist die Funktion eines solchen Satzes wie: “Es gibt zu einem Bruch nicht einen nächst größeren Bruch, aber zu einer Kardinalzahl eine nächst größere”?

Es ist doch gleichsam ein Satz, der zwei Spiele vergleicht; [wie: im Damespiel gibt es ein Überspringen eines Steines, aber nicht im Schachspiel.]

Wir nennen etwas “die nächst größere Kardinalzahl konstruieren” aber nichts “den nächst größeren Bruch konstruieren”.

wonderful notion of a “prudish proof”.²¹² After a first try at articulating this idea on December 15 1941,²¹³ LW writes the following on December 16:

16.12.

Eine Beweisführung ist prüde: wenn man ängstlich die geringste logische Zweideutigkeit vermeidet, aber groben Unsinn duldet.

This notion of prudishness expresses one of the core gestures of LW’s critical attitude towards the mathematical discourses he usually targets (Dedekind’s and Cantor’s diagonal proofs, etc.) quite well: even if a proof is technically correct (even according the strictest formal requirements), what mathematicians say about their proof, how they use the proof within a larger context, still can be nonsensical from a philosophical point of view.

For the purposes of the present study, it would take too long an explanation and too much context to interpret the illustration that immediately follows, so I will skip it here.²¹⁴ However, the next day, LW offers some material that is up our alley:

17.12.

Den Fehler in einem schiefen Rasonnement suchen & Fingerhut-Verstecken.

Man könnte fragen: Was könnte ein Kind von 10 Jahren am Beweis des Dedekindschen Satzes nicht verstehen? – Ist denn dieser Beweis nicht viel einfacher, als alle die Rechnungen die das Kind beherrschen muß? – Und wenn nun jemand sagte: den tieferen Inhalt des Satzes kann es nicht verstehen – dann frage ich: wie kommt dieses Gesetz | | dieser Satz zu einem tiefen Inhalt?

LW starts from a loose comparison between looking for mistakes in arguments and a **children’s game**;²¹⁵ I am not entirely sure what LW means with this analogy, but perhaps he wants to highlight the childishness of trying to be puritanical about logical syntax (no doubt, “while allowing nonsense in the contents of one’s argument” is understood; see above). LW then continues (without much connection, except perhaps the purely associative link between the game of Fingerhut-Verstecken to children in general) by asking what in Dedekind’s proposition would be unintelligible to a 10-year-old, suggesting that that proof is easier than

²¹² I can’t seem find this wonderful idea anywhere in the standard editions.

²¹³ Eine Beweisführung ist prüde, wenn die geringste logische | | , wenn die lässigste logische Zweideutigkeit ängstlich vermieden wird, grober Unsinn aber geduldet. | | vermieden wird, & grober Unsinn geduldet. Die Hauptunklarheit in der Mathematik ist die Unklarheit darüber, was entdeckt & was bestimmt wird.

²¹⁴ Wie, wenn ich sagte, die allgemeine Theorie der reellen Zahlen bereitet eine Phraseologie vor, die dann im besondern Fall von großem Nutzen ist. – Aber indem | | wenn sie diese Phraseologie vorbereitet ist sie entweder ein selbständiges Stück Mathematik, oder sie kann die reellen Zahlen in vager Allgemeinheit durch Beispiele behandeln. Dabei würde natürlich die Exaktheit nichts einbüßen, denn die Anwendung dieser allgemeinen Fingerzeige auf jeden besonderen Fall würde immer wieder vollkommene Bestimmtheit herstellen.

²¹⁵ The game goes as follows. Someone hides a thimble in the room and sits down. The players go and look for the object with their hands in their backs. As soon as someone has found it, (s)he sits down. The game is over when everyone sits down. See <http://www.spieledatenbank.de/spiele/1181.html>.

the calculations the child is supposed to master at that age; if someone would reply that the child would not understand the deeper contents of the proposition, LW would reply: "How does this law/proposition acquire such deeper contents?". We encounter exactly the same move in a similar context at the beginning of section 2.1(A) above. Here, as elsewhere, LW attacks this idea of **depth** by pointing out that all that is essential to the mathematical technique as such, can be understood and applied by a child.

LW then starts a new paragraph, again only loosely connected to the previous one, on the status of irrationals as numbers.²¹⁶ LW first discusses the fact that irrational numbers do not really have a proper numeral (Zahlzeichen, literally "number sign"), a fact not mentioned in the handbook he apparently happens to be reading. Various familiar themes come back:

- a) LW's critique in terms of the **fictionality** of some mathematical claims (cf. section 2.4.2(B) below) applies in this case to the idea that an irrational number would have a sign, but "an infinitely long sign". To LW, calling the rule for the development of the decimal expansion a 'number sign' would make some sense, but the notion of 'an infinite sign', readily accepted by most (?) mathematicians since Cantor, not at all. Of course, from LW's point of view, meaning depends on actual use, and in that case, the signs with which one actually calculates are not infinite. Hence, the 'fictitiousness' of the so-called 'infinite signs'.
- b) LW's very mentioning the issue is an avatar of his idea that **notation** is a central aspect of mathematics, as opposed to mere packaging (cf. 1.1.3(C)); in this case, the lack of a proper sign is -according to LW- indicative of an "infinitely fundamental" difference.
- c) Again (cf. the 'good and honest' alternative in section 2.1(A)), LW reinterprets Cantor's diagonal argument in a decidedly un-Cantor-like way: in a certain sense, Cantor's argument shows that irrational numbers can't have a proper sign.
- d) LW then articulates his criticism in terms of a fundamental disagreement with the mainstream vision of the continuum: the image of the number line is a perfectly natural one up to a certain point, but does not function well as a general theory of the real numbers.

The latter point deserves a more thorough reading on our part. Just like the treatment he gave Cantor's arguments in other contexts that we encountered in paragraphs (A) and (B) here

²¹⁶ Es wird nirgends bei Hardy hervorgehoben, daß die irrationale Zahl nicht in dem Sinne wie die rationale Zahl ein Zahlzeichen besitzt. Die Fiktion ist wohl, daß sie ein unendlich langes hat. Am ehesten könnte natürlich noch das Zeichen der Entwicklungsregel als das Zahlzeichen gelten. - Aber dieses Fehlen des Zahlzeichens bedeutet einen unendlich fundamentalen Unterschied.

Und in gewissem Sinne sagt ja der Cantorsche Diagonalbeweis, daß sie kein Zahlzeichen haben kann. Das Bild der Zahlengeraden ist ein absolut natürliches bis zu einem gewissen Punkt: nämlich, soweit man es nicht zu einer allgemeinen Theorie der reellen Zahlen gebraucht.

above, LW gives Cantor's diagonal argument a severely deflationary interpretation, restoring -as it were- its **triviality**. LW denies that the conclusions that are usually drawn from the argument are valid, and points at a much more simple and 'modest' conclusion: apparently, irrational numbers are fundamentally different from rational numbers, and apparently, the line does not function very well as a representation of the theory of the real numbers in general. So, rather than accepting a dubious expansion of the notion of 'line' for it to apply to the reals in general, he prefers to simply observe that the analogy between the notion of 'line' as we know it and the theory of the reals in general breaks down at a certain point. In other words: LW's account implies that the conceptual problems with the continuum are due to the fact that mathematicians have forced the image of a line onto aspects of number theory that it obviously does not apply to.

This brings us to another fundamental aspect of LW's general attitude towards mathematics: although he doesn't attack the unity of mathematics explicitly in this case, his readiness to abandon the image underlying the idea of a continuum shows that he does not share the **monism** that is presupposed in mainstream PhilMath, as well as mainstream mathematical discourse and practice. As opposed to the '**totalitarian**' tendencies of mainstream math (I mean: the pervasive desire to represent all of math as a single system, whether this vision comes naturally or has to be forced), LW readily abandons the concept of the number line, as soon as it appears to no longer apply as easily.

Finally, I would like to attract attention to the fact that LW evaluates this concept of the continuum in terms of its **naturalness**: he concedes that the analogy with a line is a perfectly natural one up to certain point (which shows that he does not dogmatically object to the notion of 'naturalness' in general), but is quick to specify that this is no longer the case if one tries to use this analogy as a general theory of the reals. So, LW simply denies that the line is a 'natural' representation of the distribution of the reals.

After these remarks, there are no more entries into this notebook for 6 days.

2.2 Wittgenstein's critical remarks on math II: set theory as a sign of the times

Set theory emerges as the main contender in the debates about the Grundlagen in the early 20th century. LW does not appear to object at all against the notion of "set" as such, but has difficulties accepting a number of expansions of this notion, which are essential for it to play

a foundational role: the idea that a line is a set of points, the notion of infinite sets, the idea that the reals can be represented by a line / an infinite set, etc.

The following three passages show a direct connection between LW's critical approach to PhilMath, and the cultural critique of Spengler, Weininger, Kraus as discussed in section 2.0.0 above, in that they directly link then recent developments in mathematics to very broad cultural tendencies.

(A) the pest of the set-theoretical expression + an instinctless time: LW, Ms-106,245-255 [1929, entries not dated]

The following (very early) excerpt occurs after a number of as such interesting remarks that appear to have in common that they were inspired by Brouwer, but it would take us beyond the needs of the present study to try and interpret their mutual coherence. So, let us merely focus on the ethico-aesthetical wording of the most obviously critical parts. LW starts off with one of his harsher claims:

Die Mathematik ist ganz durch die perniziöse mengentheoretische Ausdrucksweise verseucht.

According to LW, set-theoretical parlance has had a thoroughly destructive impact on mathematics. The word "verseuchen" is a very strong term, typically applied to pests, pollution, radiation, or epidemics.²¹⁷ It is interesting to note that LW's blames the set-theoretical "way-of-expressing-oneself" [Ausdrucksweise], not set theory as a piece of mathematical technique, which reminds us of the prose-calculation distinction that we discussed in section 1.2.3(A) above.

LW illustrates this claim by attacking the idea that a line consists of points (is a set of points): this set-theoretical understanding of the concept of "line" is not coherent with the traditional concept of the line.²¹⁸ LW says that a line (in the traditional sense of that word) is a 'law' [Gesetz], a constructive procedure, and therefore doesn't consist of anything. One can presume LW means that a line is defined by the way it is constructed (say: the act of drawing a line). As a drawn line, it can perhaps consist of shorter lines, but not of points. LW is

²¹⁷ <https://www.dwds.de/wb/verseuchen>

²¹⁸ Ein Beispiel dafür ist daß man sagt die Gerade bestehe aus Punkten. Die Gerade ist ein Gesetz & und besteht aus gar nichts. Die Gerade als farbiger Strich im visuellen Raum kann aus kürzeren farbigen Strichen bestehen (aber natürlich nicht aus Punkten). – Und dann wundert man sich z.B. darüber, daß „zwischen den überall dicht liegenden rationalen Punkten“ noch die irrationalen Platz haben! Was zeigt eine Konstruktion wie die des Punktes $\sqrt{2}$? Zeigt sie diesen Punkt wie er doch noch zwischen allen rationalen Punkten Platz hat? Sie zeigt einfach, daß der durch die Konstruktion erzeugte Punkt nicht rational ist.

Und was entspricht dieser Konstruktion & diesem Punkt in der Arithmetik? Etwa eine Zahl, die sich doch noch zwischen die rationalen Zahlen hineinzwängt? Ein Gesetz das nicht vom Wesen der rationalen Zahl ist.

specifically annoyed by the fact that people are apparently amazed [Und dann wundert man sich z.B. darüber, daß...] at the image that between the very dense succession of rationals there still is room for the irrationals. For LW, the whole demonstration only shows the trivial fact that the point that is constructed in this way is not rational; in arithmetical terms, this merely means that irrationals are a law [Gesetz] that is not of the same nature as the one that generates the rationals. For the present purposes, I highlight LW's annoyance at peoples' **wonder** about something he deems **trivial**, a recurrent theme in our explorations.

LW then turns towards the example of Dedekind's cut as a way to construct $\sqrt{2}$. Again, it's not necessary for our purposes to go deep into the technical details, or into the issue as to whether LW is right. Suffice it to observe that LW -again- articulates his criticism in terms of **illusion** and **pretense**. This time he attacks the idea that Dedekind's cut would be an insightful way to introduce the construction of $\sqrt{2}$: rather than being a construction of $\sqrt{2}$, the cut already presupposes the structure of square roots.²¹⁹ The introduction of $\sqrt{2}$ by the cut is therefore **mere appearance** [bloßer Schein] and its transparency [Anschaulichkeit] is mere **pretense** [Die Erklärung des Dedekindschen Schnittes tut so als wäre sie anschaulich].²²⁰ Again, LW does not object to the technique as such, but to the fact that -in this case- Dedekind pretends that it does things that in fact it doesn't.²²¹

²¹⁹ If I read Mancosu correctly ((Mancosu 2003), p. 71), Bernays seems to have similar objections to a similar interpretation of the technique involving cuts: "The first standpoint consists in accepting as a real number anything that is given by a cut (say by the condition $x^3 < 2$). The problem with this first method is that it does not delimit at the outset the domain of the real numbers ("der Begriff der reellen Zahlen wird nicht 'bestimmt' umgrenzt"). For this reason we truly have a vicious circle here, since real numbers are defined by partitions which in turn are defined by reference to what real numbers [partitions] exist possessing a specified property. But, according to Bernays, one does not always follow this standpoint".

²²⁰ Die Erklärung des Dedekindschen Schnittes tut so als wäre sie anschaulich, wenn nämlich gesagt wird: Es gibt nur 3 Fälle entweder hat R ein letztes Glied & L kein erstes oder etc. In Wahrheit läßt sich keiner dieser Fälle denken (oder vorstellen).

Wenn man als Eigenschaft der Ober- & Unterklasse im Dedekindschen Schnitt $x^2 < 2$ und $x^2 > 2$ nimmt, warum nicht gleich $x < \sqrt{2}$ und $x > \sqrt{2}$? Man glaubt durch die erste Fassung einer Schwierigkeit ausgewichen zu sein.

Wenn wir logisch vorgehen so müssen wir die rationalen Zahlen einteilen in solche deren Quadrat größer als 2 ist & solche deren Quadrat nicht größer als 2 ist. (Denn, daß, was nicht größer ist entweder gleich oder kleiner ist sagt die Logik nicht, sondern das sehen wir erst durch Inspektion eines Zahlenverhältnisses.) Gut, ich schneide also: Rechts vom Strich liegen alle Zahlen mit größeren Quadraten, links alle anderen. Aber wer sagt denn, daß das so ist? Das setzt ja eben die Kenntnis der Struktur von x^2 und 2 voraus. Die Einführung der $\sqrt{2}$ durch den Dedekindschen Schnitt ist bloßer Schein, der dadurch zustande kommt, daß der „Schnitt“ eine räumliche Illustration ist der uns die Struktur vor Augen führt, die wir klassentheoretisch – amorph – nicht erfassen können.

²²¹ LW deals with the same material in Ms-126,131-132, d.d. 19421214:

Die geometrische Illustration der math. Analysis ist allerdings unwesentlich, nicht aber die geometrische Anwendung. Ursprünglich waren die geometrischen Illustrationen Anwendungen der Analysis. Wo sie aufhören dies zu sein, können sie leicht gänzlich irreführen.

Hier haben wir dann die phantastische Anwendung. Die eingebildete Anwendung.

Die Idee des 'Schnittes' ist so eine gefährliche Illustration.

Nur soweit, als die Illustrationen auch Anwendungen sind, erzeugen sie nicht das || jenes gewisse Schwindelgefühl, das die Illustration erzeugt im Moment, wo sie aufhört eine mögliche Anwendung zu sein; wo sie also dumm wird.

Immediately after these remarks, we read a paragraph written in LW's usual code, -easy to decipher, but not immediately readable by a casual passer-by:²²²

Ich glaube die Mathematik hat im vorigen Jahrhundert eine ganz besonders instinktlose Zeit gehabt an der sie noch lange leiden wird. Ich glaube diese Instinklosigkeit hängt mit dem Niedergang der Künste zusammen, sie entspringt der selben Ursache.

According to LW, mathematics has had an especially **instinctless** time in the previous century, which will plague it for a long time to come, and this instinctlessness is the result of the same causes that also lead to the **decline** [Niedergang] of the arts. Perhaps, we can try to explicitate what LW may have meant by 'instinctlessness': against the backdrop of those aspects of LW's modes of thought that we highlight in this study, it seems fair to interpret 'instinctlessness' as referring to a lack of connection between mathematical discourse and that what gives mathematics its real-life meaning (the term 'forms of life' was not yet part of LW's vocabulary in 1929, but the biological connotations of 'instinct' do remind us of LW's later work). Also note the negative assessment of the state of mathematics and the arts, which - again- are presented as results of the same underlying historical cause.

(B) set theory's self-misrepresentation + mental fog + fictional symbolism: LW, Ms-113,93r- Ms-113,93v, d.d. 19320508 [= Ts-213 (The Big Typescript), §750]

This excerpt is part of a series of loosely connected paragraphs on various aspects of the set-theoretical approach to the continuum. It doesn't look like there is a particular link to the immediately preceding or following context.

Die Mengenlehre wenn sie sich auf die menschliche Unmöglichkeit eines direkten Symbolismus des Unendlichen beruft führt dadurch die denkbar krasseste Mißdeutung ihres eigenen Kalküls ein. Es ist freilich eben diese Mißdeutung die für die Erfindung dieses Kalküls verantwortlich ist.²²³

²²² Why did LW decide to write this not particularly personal or particularly scandalous remark in code? Good question. No idea. Just to practice?

²²³ When set theory appeals to the human impossibility of a direct symbolization of the infinite it thereby introduces the crudest imaginable misinterpretation of its own calculus. To be sure, it is this very misinterpretation that is responsible for the invention of that calculus. But of course that doesn't show the calculus to be something inherently incorrect (at most it shows it to be something uninteresting), and it's odd to believe that this part of mathematics is imperilled by any kind of philosophical (or mathematical) investigations. (With equal justification chess might be imperilled by the discovery that wars between two armies do not follow the same course as the battle on the chess board.) What set theory has to lose is rather the atmosphere of thought-fog surrounding the bare calculus, that is to say, the references to a fictional symbolism underlying set theory, a symbolism that isn't employed in its calculus, and the apparent description of which is really nonsense. (In mathematics we're allowed²⁷ to make up everything, except for a part of our calculus.) (Wittgenstein 2005)

Let us try and unpack this quite dense paragraph. First, LW refers to the idea of the human incapacity of a direct symbolic representation of the infinite. It is not clear to me if LW intends to refer to a specific passage by a specific author, but the idea that “**the finite human mind is incapable to represent the infinite**” (as well as the follow-up idea of an **infinite mind**) is quite commonly heard even in present-day discussions involving mathematicians.²²⁴ LW claims (or rather: presupposes) that set theory relies on [sich beruft auf] this idea; again, this looks like a quite common idea, then and now: we need the elaborate symbolism of various types of transfinite to calculate with, because we, humans, are incapable of grasping infinity directly. This is where LW’s critical wrath is directed: he calls the idea that set theory as a technique [Kalkül] is motivated by the finitude of the human mind “the crudest possible misinterpretation” of this technique.

LW then stipulates that the fact that this **misinterpretation** is actually what *gave rise to the invention of the technique* does not make the technique incorrect (perhaps *uninteresting*, but not *incorrect*). Like any other part of mathematics, qua technique [Kalkül], this part of mathematics cannot be imperiled by any kind of philosophical (or mathematical) investigations. LW likens this situation with the fact that the game of chess cannot be imperiled by the fact that armies on a battlefield do not behave like the parties in the game.²²⁵ So: LW simply denies that there is a direct link between mathematical technique and the outside world (which coincides with his overall vision of meaning, according to which any link between linguistic expressions and the world is mediated by a practice).²²⁶

If the technique is not the problem, and mathematical or philosophical considerations cannot impact the technique anyway, what is the problem? LW’s answer is this: set theory should get rid of **the atmosphere of mental fog** [die Atmosphäre von Gedankennebeln] that hangs around technique, i.e. the reference to a **fictitious** symbolism that is never actually used in set theoretical technique but still is supposed to underlie set theory, despite the fact that its description is **nonsense**: nobody actually calculates with infinitely expanding numerals and the idea that this due to the finite nature of our minds, as opposed to infinite minds that would be able to calculate with them, sound like gibberish indeed. LW ends this excerpt with one of

²²⁴ Interestingly, a quick internet search shows that this slogan was (is?) also popular in some theological circles. It is tempting to -hypothetically- link the origins of this idea to Cantor, who was notoriously untidy about the boundaries between mathematics and theology.

²²⁵ Aber der Kalkül an sich ist natürlich dadurch nicht als etwas Falsches erwiesen (höchstens als etwas Uninteressantes) & es ist sonderbar, zu glauben daß dieser Teil der Mathematik durch irgend welche philosophische (oder mathematische) Untersuchungen gefährdet ist. (Ebenso könnte das Schachspiel durch die Entdeckung gefährdet werden daß sich Kriege zwischen zwei Armeen nicht so abspielen wie der Kampf auf dem Schachbrett.)

²²⁶ NB: apparent formalism about the technique is not (not necessarily) incompatible with pragmatism.

his oracular sounding sentences (between brackets): “(In math we are allowed to make up [fingieren] everything, but not a part of our technique [Kalkül])”.²²⁷

What does LW mean by the image of ‘the atmosphere of fog’? Fog is in itself not solid at all, but it does prevent us from seeing clearly; around set theory (which, as a technique, is not problematic), there is an ‘atmosphere’ of mental fog, perhaps implying that it is always there, like the atmosphere around the earth (or alternatively, implying a general mood of unclarity?).²²⁸

So, again, the problem is not the actual mathematical technique/Kalkül behind set theory, which is OK, albeit not necessarily interesting. What is problematic to LW is the misrepresentation of set-theoretical technique in mainstream set-theoretical discourse, and specifically the **fictionality** of some of the concepts that are invoked: there is simply nothing infinite in the actual infinitesimal technique, the concept of ‘infinite numerals’ is pure fiction (they are not actually used), and the talk about ‘infinite minds’ comprehending ‘infinite numerals’ is **nonsense** (gibberish as such (wheter in set theory or outside it)²²⁹ and not serving a purpose in the context of set theory).

This excerpt lacks some of the Spenglerian apocalyptic tone of both the previous and the next excerpt, but is perhaps more specific as to the exact technical objections LW has: LW objects to discourse that is disconnected from what is actually done in the technique itself, and in the case of the notion of ‘infinite mind’ also disconnected from anything else.

(C) the illness of an era: LW, Ms-121,27r-28v, d.d. 19380530 [// BGM2,§23]

In the manuscript, the following remarks occur after a three day pause in the writing and do not seem to refer to anything in particular in the previous remarks (short, loose, sparse and not very good remarks about music and architecture, balloons,²³⁰ Fermat,²³¹ and the slogan

²²⁷ Was der Mengenlehre verloren gehen muß ist vielmehr die Atmosphäre von Gedankennebeln die den bloßen Kalkül umgibt. Also die Hinweise auf einen der Mengenlehre zu Grunde liegenden fiktiven Symbolismus der nicht in ihrem Kalkül verwendet wird, & dessen scheinbare Beschreibung in Wirklichkeit Unsinn ist. (In der Mathematik können | | dürfen wir alles fingieren nur nicht einen Teil unseres Kalküls.)

²²⁸ Both meanings of the word ‘Atmosphäre’ were established in German long before LW wrote this paragraph (see <https://www.dwds.de/wb/etymwb/Atmosphäre>).

²²⁹ It takes some heavy formatting, whether within math or within theology, to make this idea even remotely palatable, I guess. A quick survey among some of my acquaintances shows that only people with a background in logic or PhilMath regard it as ‘a thing’, whereas working mathematicians and people with a more applied background may have heard of it, but tend to be dismissive of it.

²³⁰ Das Vergnügen, das wir an einem aufgeblasenen Gummiballon haben. Wir sind nicht gewöhnt mit Körpern zu hantieren, die so groß im Verhältnis zu ihrem Gewicht sind.

²³¹ Es hilft wenn man sagt: der Beweis des Fermatschen Satzes ist nicht zu entdecken, sondern zu erfinden.

“The system of systems is a contradiction”). So, we can assume LW starts a new line of thought at the beginning of our excerpt.²³²

30.5.

Die Krankheit einer Zeit heilt sich durch eine || die Veränderung in der Lebensweise der Menschen & die Krankheit der philosophischen Probleme konnte nur durch eine veränderte Denkweise & Lebensweise geheilt werden nicht durch eine Medizin die ein Einzelner erfand.

Denke, daß der Gebrauch des Wagens gewisse Krankheiten hervorruft oder begünstigt & die Menschheit von dieser Krankheit geplagt wird, bis sie sich, aus irgendwelchen Ursachen, als Resultat irgendeiner Entwicklung, das Fahren wieder abgewöhnt.²³³

LW sounds like Weininger or **Spengler**, including the historical perspective and the negative assessment of the era he lives in. Then, making a direct link between cultural critique in the most general terms possible and an apparently quite technical subject matter in PhilMath, LW illustrates what he means by considering the task “Name a number that is bigger than the number of all numbers”.²³⁴

For starters, LW points out that the math problem [Aufgabe] ‘name a number that is greater than the number of all numbers’ sounds like a **joke**, which is fair: even after more than a century of transfinites, the assignment on its own still sounds distinctively weird. LW then turns to a slightly less weird-sounding exercise, looking for numbers between $1/n$ and $1/m$, which -as we know- may lead to similarly paradoxical results in the context of set theory, but makes sense and is even useful because it is systematically linked to other such problems. It is true that we could easily imagine a use for such an assignment, e.g. in a didactic context. Still, it is also true that the method(s) used to do this assignment would not give rise to a theory

²³² This is good illustration of the pitfalls of the editorial practices that gave rise to the standard editions of LW's work: in *BGM* 2, 23 these remarks follow the remarks in which LW calls diagonal procedures “hocus pocus” (section 2.1(A)). This juxtaposition has a striking effect, and in previous versions of the present study, I got some interpretative traction out of it. I don't exclude that in the course of a process of cutting and pasting of the kind that gave rise to the typescript that was used as a basis for PhU, LW could have made this move. But the thing is: he didn't, and the fact that I was misled by it, shows that it is de facto misleading.

²³³ 23. The sickness of a time is cured by an alteration in the mode of life of human beings, and it was possible for the sickness of philosophical problems to get cured only through a changed mode of thought and of life, not through a medicine invented by an individual.

Think of the use of the motor-car producing or encouraging certain sicknesses, and mankind being plagued by such sickness until, from some cause or other, as the result of some development or other, it abandons the habit of driving

²³⁴ “Nenn' mir eine Zahl, die größer ist, als die Zahl aller ganzen Zahlen!” - diese || Diese Aufgabe hat den Charakter einer mathematischen Scherzfrage.

Welcher Art wäre denn die Aufgabe: “Nenne mir eine Zahl zwischen $1/n$ und $1/m$ ”? Nun es wäre eine Übung in der Bildung solcher Zahlen. Ihre Nützlichkeit liegt darin, daß es hier ein System solcher Aufgaben || Probleme gibt.

Es ist nämlich eine ganz wesentliche Frage: Was ist denn die Anwendung dieses (neuen) Zahlenbegriffs außerhalb der Mathematik. – Denn mit 1, 2, 3, 4 ... kann ich nicht nur Zahlen zählen, sondern auch Äpfel, & wenn nun ein Zahlwort nur in mathematischen Sätzen & in keinen andern vorkommen könnte, oder wir doch nicht wissen, welche Rolle es außerhalb der mathematischen Sätze spielen kann, so weist dies auf eine sehr wesentliche Unklarheit unsererseits hin. Es ist nämlich nicht klar ob wir nicht bloß durch eine Einbildung verführt sind hier den Namen || das Wort “Zahl” zu gebrauchen.

of the order of 'all fractions', as it would inevitably lead to same old problems that always show up in such contexts.

LW then transitions to a simple and plain statement of his doctrine about **applications**: according to LW, the question as to the extra-mathematical use of a new concept of 'number', is an important one: "with the natural numbers, you can't just count numbers, you can also count apples; when a 'number' occurs only in mathematical propositions and couldn't occur in any other sentences, there exists an 'essential unclarity' about these new numbers. It is not clear whether we are not using the word 'number' as the result of an **illusion** [Einbildung].

So: again, LW has no objections to the mathematical technique as such, but he does object to what mathematicians say about their own technique. In this case, he questions two (related) aspects of standard set-theoretical discourse:

- the idea that there is such a thing as infinite number signs (cf. paragraph (B) here above);
- using the word 'number' in such a way that it disrupts the link with counting, as applied in everyday applications (cf. section 1.2.2 above, in which we discussed the passage where LW says that mathematical terms should also be used in civilian clothing).

In the analyses we presented here above, we have explained LW's remarks in terms of 'the expansion' of the use of certain terms, but here LW thematizes the idea of the 'expansion' of a term very explicitly himself and articulates the consequences of the disruption of the connection with everyday practice quite clearly: the problem with expanding the use of the word 'number' in this way is a **fundamental unclarity about its meaning**: as long as our idea of a number remains directly connected to counting and the basic operations of everyday calculation with the integers, the notion of 'number' retains its transparency; once we start using the word 'number' in such a way that this connection gets disrupted, the meaning of the concept also loses its transparency.

This line of thought should remind us of the way LW conceives of meaning in general in terms of embedding in everyday practice (cf. section 1.2 above), but the critical nature of these remarks, in which he condemns real-life contemporary usage for being disconnected from normal, everyday, actual practice (in this case mathematical practice) should remind us of journalist Karl Kraus' criticism in terms of the disruption of the connection between public discourse on the one hand, and contemporary social and political reality on the other (cf. section 2.0.0 above).

The general idea that set-theory had an entirely negative impact on math remained a constant throughout LW's work on mathematics, not only observed in the remarks from 1929, 1932

and 1938 studied above, but in Ms-127,184-187, not dated, but written some time after 19440304, we still read “der Fluch des Einbruchs der math. Logik in die Mathematik”.²³⁵

What is important for our purposes is that this example (the disconnect between set-theoretical lingo and ordinary math) is supposed to illustrate LW’s general assessment of the ‘illness of an era’ (cf. also “eine ganz instinktlose Zeit” (paragraph (A) here above), again in the context of set theory bashing). It shows how LW’s PhilMath is permeated by not only his conception of meaning as embeddedness in everyday practice, but also by the culture-critical concerns that he inherited from his *maîtres à penser* Karl Kraus and Oswald Spengler: it is remarkable to see how these different aspects coincide in this context.

2.3 Wittgenstein’s critical remarks on math III: on paradoxes, on harmless contradictions and (maybe a little bit) on Gödel

In this section, I discuss a few excerpts in which LW discusses contradictions and paradoxes so as to criticize the way axiomatic systems are conceptualized in contemporary contributions to the Grundlagen debates. As many of these excerpts are often interpreted as criticism of Gödel’s work, I will first briefly discuss the topic of ‘LW on Gödel’ in order to make clear what I will and will not deal with in this section.

[introductory remarks: LW on Gödel](#)

LW’s remarks on Gödel (or rather: Gödel-related topics; see below) yielded a large, often somewhat technical literature. There were a few early, mostly dismissive, appraisals (e.g. (Kreisel 1958)²³⁶), but since the 1990s, there has been specialized discussion involving such authors as Juliet Floyd, Mark Steiner, Victor Rodych, and more recently Timothy Lampert ((Floyd 1995); (Floyd 2001); (Floyd 2017); (Floyd and Putnam 2000); (Bays 2004); (Floyd and Putnam 2006); (Steiner 2001); (Rodych 2006); (Sayward 2005); (Rodych 2003); (Rodych 2002); (Rodych 1999); (Lampert 2013); (Lampert 2018)). Most of these contributions try to come to grips with the issue as to what exactly LW says about Gödel’s proof and whether his apparently critical remarks are actually relevant criticism of Gödel’s work or not. It is

²³⁵ It is worth repeating that LW was not the only one to disagree with Cantor’s set theoretical innovation and that Poincaré and Kronecker also strongly disapproved (cf. section 1.2.3 above, where I also quoted Dieudonné, who, as a member of the Bourbaki-collective, was on board with the formalisation of math, but displayed the same very ambivalent attitude towards mathematical logic; for a very interesting remark on Bourbaki’s skepticism towards the foundationalist project (Schroeder (Schroeder 2021) p. 202).

²³⁶ For instance: “Here, for once, Wittgenstein also makes a justified objection (p. 130, 56) among all the wild shots which miss the mark: the one-sidedness of the consistency problem.” ((Kreisel 1958), p. 155).

interesting to note that there is a substantial discussion in the literature about the question as to whether LW actually had read Gödel's famous papers and if so, if he had understood them. In agreement with my general strategy as declared at the outset of this study, I will not engage with the technicalities of this literature and following LW's lead (see below), I have no intention to speak about the technicalities of Gödel's proof: I believe they are perfectly fine, and I believe LW also thought they were perfectly fine. This being said, I also believe that the brunt of LW's critical thought in the context of PhilMath, even if not targeting anything specific to Gödel's proofs, does apply to Gödel's work as much as to any other contemporary exponent of PhilMath. This is what I will try to make clear in the present introductory remarks. Let me first take a step back and look at the material that we are talking about. **Gödel is mentioned in LW's manuscripts** at the following spots:

- Ms-117 (not dated, after august 1938?): Gödel is mentioned twice (p. 147 and pp. 151-152), in very tentative remarks; historically interesting but nothing in it for us;
- Ms-121 (pp. 71r-85v, d.d. 19381228-19390102): Gödel mentioned 7 times between p. 75v and p. 84r, including some interesting material;
- Ms-122 (p. 28v, 19391118): Gödel mentioned in passing;
- Ms-124 (83-96, d.d. 19410702-04): Gödel mentioned 6 times; interesting and coherent material;
- Ms-126 (p. 131, d.d. 19421213): Gödel's casual prefatory proof mentioned in passing;
- Ms-163: 4 mentions between p. 16r and p. 20v, d.d. 19410708, all of which are doublets of material also found in Ms-124; 9 more mentions between p. 24r and p. 42v, d.d. 19410708-11; interesting material, but very messy manuscript.

Surprisingly (?), most of these passages are **not overtly critical** towards anything that is specific to Gödel's work at all. Part of these remarks simply assert that LW does not consider it his task as a philosopher to talk about any aspect of the proofs themselves.²³⁷ Interestingly, LW actually credits Gödel with having invented a situation that creates a problem that makes us change our perspective on math:²³⁸ the proof itself is of no philosophical interest, but the type of proof [Beweisart] is, and the problem it shows is applicable to much more elementary aspects of math as well.²³⁹ To be fair, in other contexts, LW does call Gödel's articles

²³⁷ Ms-124,84 (=Ms-163,16r): "Meine Aufgabe ist es nicht über den Gödelschen Beweis (z.B.) | | , z.B., zu reden; sondern an ihm vorbei zu reden". Ms-163,24r-24v (/ / Ms-124,94): "Man kann mit Recht fragen, welches Interesse | | welche Wichtigkeit Gödels Beweis für unsre Arbeit habe. Denn er kann keines unserer Probleme lösen | | löst keines unserer Probleme. – Die Antwort ist: daß die Situation uns interessiert | | für uns von Interesse ist, in die ein solcher Beweis die Menschen bringt. 'Was sollen sie nun sagen?' – das ist unser Thema". Ms-163,37v-38v: "Nicht der Gödelsche Beweis interessiert mich, sondern die Möglichkeiten auf die Gödel durch seine Diskussion uns aufmerksam macht. / Die math. Tatsache daß hier ein arithmetischer Satz ist, der sich in P nicht beweisen noch als falsch erweisen läßt, interessiert mich nicht".

²³⁸ Gödel would probably agree on the fact that his work operates a change to our perspective on math (it probably was intended to do that from the outset), but he definitely would not endorse the change in perspective that LW has in mind.

²³⁹ Ms-163,39v-40v:

unphilosophical and his concepts slimy (Ms-124,115-119; see section (E) here below) and in another context (Ms-121,81v), he calls a ‘Gödelian reason’ to decide whether the proposition ‘This proposition is self-evident’ is true or not, “dumb” [dumm].

Conspicuously absent from the above list are the so-called “**notorious**” remarks, published as *BGM1*, Anhang 3. The published text was based on Ts-223, in its turn a somewhat sanitized version of Ms-118, 105v-116v, which LW wrote in Norway between 19370922 and 19370924?, in the presence of his lover Francis Skinner. This is one of the most cohesive pieces of prose in the whole Nachlass and a remarkable burst of creativity.²⁴⁰ This is the text that most comments on ‘LW and Gödel’-related topics are based on. What has prompted this text to be systematically interpreted as targeting Gödel is the fact that it discusses the concepts of provability vs. truth in the context of formal systems, but Gödel is not mentioned in these remarks, only Russell.

Of course, everything LW says on contradictions and axiomatic systems in general also applies -I would say a fortiori- to Gödel’s work, but it is important to note that LW does not address most of these remarks to Gödel specifically, and that he quite clearly and repeatedly indicates in other writings that Gödel’s specific proofs and what they prove specifically have no real philosophical interest.

So, I believe it’s fair to say that whereas Gödel’s proof is not really a deep-running theme in LW’s work, **contradictions** and what they show us about axiomatic systems do make up a recurrent and wide-reaching theme (cf. also *PhU* §125, discussed in section 2.0.3(B) above). In the below, I will be focusing on *a selection of passages dealing with paradoxes and contradictions and (following the indications in LW’s text) I will not particularly focus on Gödel*. My emphasis is on how LW’s treatment of these matters is congruent with the rest of his philosophy and how they display the same critical themes and attitudes as the rest of his oeuvre.

Der Gödelsche Beweis bringt eine Schwierigkeit auf, || entwickelt eine Schwierigkeit, die sich auch in viel elementarerer Weise zeigen muß. || die auch in viel elementarerer Weise erscheinen muß. (Und hierin liegt, scheint es mir, zugleich Gödels großes Verdienst um die Philosophie der Math., & zugleich der Grund, warum sein besonderer Beweis nicht das ist was uns interessiert.)

11.7.

_ Ich könnte sagen: Der Gödelsche Beweis gibt uns die Anregung dazu die Perspektive zu ändern aus der wir die Mathematik sahen. Was er beweist, geht uns nichts an, aber wir müssen uns mit dieser mathematischen Beweisart auseinandersetzen.

Trage! Stehst || Stündest Du fest & trägst, so wird es auch dem Andern am meisten nützen. Mach keine Scene, sei nicht ironisch, sei nicht unnatürlich.

Trage!

Also interesting are the interludes in code (italic in the above), in which LW admonishes himself to stand strong and not make scene, not be ironic, not be artificial, so as to be maximally useful to others.

²⁴⁰ For a detailed commentary, see (Kienzler and Grève 2016), including a point of view that is grosso modo congruent with mine here.

(A) contradictions in foundational formal systems are actually harmless: MS-118, 111v- , dd. 19370923
[= BGM 1, Anhang III, §11-13]

A good place to start is LW's claim that whatever problems may show up in so-called 'foundational' axiomatic systems are actually **harmless and irrelevant**. LW starts from the following hypothetical scenario, in a way a particularly naïve reinterpretation of Gödel's famous consistency vs. completeness arguments, though nothing is said about the particular ways in which sentence P ("this sentence is unprovable") has been proven or disproven:²⁴¹

- A has proven the unprovability of P ("this sentence is unprovable") in Russell's system, which means that he has proven P, which means that P belongs and does not belong to the system.
- B: That's what happens when you construct this kind of propositions!
- A: But that is a contradiction!
- B: Yep. We've got a contradiction here. Does it do any harm? ²⁴²

So, basically, LW suggests that it remains to be seen whether a contradiction is harmful or not. LW then asks the same question about **the Liar's paradox**: does it do any harm? ²⁴³ We have seen in section 1.2.1(C) above that LW's answer is that the paradox is obviously harmless and that it resembles pointless games one may play with very small children who have not yet understood that they will never be able to catch that thumb, because it is structurally impossible to do so.

Then, LW makes the following interesting point: despite the obviously **harmless and childish** nature of the paradox, people have been seriously **tormented** by it. Which is true: logicians have been and are still struggling with paradoxes. The philosophical significance of this point is easy to underestimate: it shows that the fact that people are taking a problem seriously does not mean that it is actually a serious problem.²⁴⁴ And this is something that needs to be

²⁴¹ Let me repeat once more that LW makes no claim to talk about Gödel specifically (cf. Ms-163, 39v-40r, discussed above).

²⁴² Nehmen wir an, ich beweise die Unbeweisbarkeit (in Russells System) von P; so habe ich mit diesem Beweis P bewiesen. Wenn nun dieser Beweis einer in Russells System wäre, – dann hätte ich also zugleich seine Zugehörigkeit & Unzugehörigkeit zum Russellschen System bewiesen. – Das kommt davon, wenn man solche Sätze bildet. – Aber hier ist || wäre ja ein Widerspruch! – Nun so ist hier ein Widerspruch. Schadet er hier etwas?

²⁴³ Schadet der Widerspruch der entsteht, wenn Einer sagt: "Ich lüge. – Also lüge ich nicht. – Also lüge ich etc." Ich meine: ist unsere Sprache dadurch weniger brauchbar, daß man in diesem Fall aus einem nach den gewöhnlichen Regeln sein Gegenteil & daraus wieder ihn folgern kann? – Der Satz (selbst) ist unbrauchbar, & ebenso dieses Schlüsseziehen; aber im übrigen kann man es tun, wenn man will. || warum soll man es nicht tun? Es ist (nur) eine brotlose Kunst. || ! – Es ist ein Sprachspiel das Ähnlichkeit mit dem Spiel des Daumenfangens hat. (Dies || Dieses wird so gespielt: Man hält den Daumen der rechten Hand mit der linken, so daß seine Spitze noch oben aus der linken hervorschaut. Nun entzieht man die rechte Hand rasch dem Griff der linken Hand & trachtet die rechte Daumenspitze noch mit der rechten Hand zu fangen, ehe sie sich zurückzieht.)

²⁴⁴ Interesse erhält jener || Widerspruch nur dadurch, daß er Menschen quält || gequält hat; & dadurch || so zeigt, wie die Sprache zu quälenden Problemen führen kann. || was für Dinge uns quälen können. || wie aus der Sprache quälende Probleme wachsen können. || wie aus der Sprache quälende Probleme wachsen können; & was für Dinge uns quälen können. || zeigt, was Menschen quälen kann.

established and understood before LW's arguments even can play a role in a wider PhilMath context.

NB: nowhere it is said that Gödel's stuff -technically speaking- boils down to a paradox like the Liar's (again: Gödel is not mentioned and neither is anything related to the specifics of Gödel's proof), but one can safely infer that -according to LW- in both cases, the question one should ask is: *is the contradiction a problem?* And LW's suggestion is in both cases, that the contradiction is not harmful and for all practical purposes irrelevant, i.e. not really a problem. So: LW does not argue against any technical details internal to any specific argument but attacks one of the presuppositions of the debate: the idea that contradictions are a huge problem, or in other words: that consistency (in the sense of: avoiding contradictions within a formal system) would be a fundamental issue at all.

It is also interesting to see that LW's little dialogue illustrates mostly a difference in attitude between the **pathos** of distress displayed by (the voice I called) A and the "so what?" attitude displayed by B: LW is not disagreeing with the observation that there is a contradiction here, he does not even mention any of the internal features of the proofs involved in his hypothetical scenario, he only attacks the **panicky attitude towards contradictions is general**.

(B) the superstitious fear and veneration for contradictions: Ms-118,116r-116v, d.d. 19370924 [= BGM 1, Anhang III, §18-19]

This excerpt, still from the 'notorious remarks', a few pages down the road from the previous excerpt, represents LW's **anti-foundationalism** at its most focused and most strident.

Just before our excerpt starts we read the comment (between brackets) that "The mathematicians' **superstitious fear** and veneration when faced with a contradiction is very funny",²⁴⁵ which is interesting in different ways: (1) LW identifies the problem at the level of fundamental beliefs and emotions (**pathos**) external to any proofs or arguments; (2) the recurring theme of **ridiculousness** manifests itself. As for (2), the "sehr komisch" disappeared in the typescript, and hence does not occur in the standard editions either; on the one hand, it is easy to understand why LW or his literary executors decided to cut this remark from a more public version of the text (laughing at one's colleagues' opinions has never been acceptable way of formulating criticism, let alone laughing at their fears and quasi-religious beliefs), but on the other hand, the idea that the standard attitude towards contradictions is funny, need not be merely dismissive: from LW's radically pragmatic point of view, this baseless fear and the ensuing behavior is literally comical, in a slapstick kind of way.

²⁴⁵ "(Sehr komisch ist die abergläubische Angst & Verehrung der Mathematiker vor dem Widerspruch.)".

LW is still thinking about the Gödel-like sentence “P is unprovable” and now takes on the hypothetical scenario in which the proposition was false and therefore provable. True to his quasi-dialogical style, LW (?) immediately interjects: “But why do you call it ‘false’? Because you have seen a proof? Or for other reasons? In that case, it’s not a problem”. So, LW is questioning what it could mean for some proposition within a formal system to be false without being proven. He then suggests that ‘false, but not because it’s proven to be false’ could mean a number of different **harmless or irrelevant** things. For instance, one could argue that ‘tertium non datur’ is false, because ‘yes and no’ is heard quite frequently and makes perfect sense in these contexts, or one could say that the idea that ‘the negation of a negation is an assertion’ is false, because people sometimes use double negations as a strengthened negation.²⁴⁶ Both examples are intended to show that you can call many things ‘false from outside the axiomatic system’, without them having any relevance to what can reasonably be done within the system. Again: the message is that *for all practical purposes, the contradiction need not be a problem.*

LW then goes on to tackle the hypothetical -definitely Gödel-like- conclusion “... therefore P is true and unprovable” and suggests that this boils down to just writing “Therefore $\vdash P$ ”. LW then compares the above scenario to the scenario in which someone has deduced from certain principles concerning natural forms and architectural style that Mount Everest, where nobody actually can live, would be an excellent location for a little castle [Schlößchen] in Baroque style.²⁴⁷ The simile of the little baroque castle on Mount Everest is an adequate expression of the idea of a lack of embedding in anything real: you can make formally correct deductions within your formal system all day, but when it comes to real-world conclusions with real-world consequences, the validity of the conclusions will be measured by means of real-world criteria, not by criteria internal to the formalism; if the real-world interpretation of your result is ridiculous (or otherwise undesirable) in real-world terms, it deserves to be rejected.

²⁴⁶ “Aber angenommen, der Satz wäre nun falsch - & daher beweisbar! -” - Warum nennst Du ihn ‘falsch’? Weil Du einen Beweis siehst? - Oder aus andern Gründen? Dann macht es ja nichts. Man kann ja den Satz des Widerspruchs sehr wohl falsch nennen, mit der Begründung z.B., daß wir sehr oft mit gutem Sinn auf eine Frage antworten: “Ja - & nein.” Und ebenso || desgleichen den Satz “ $p \equiv \sim\sim p$ ”: weil wir die Verdoppelung der Verneinung als eine Verstärkung der Verneinung verwenden & nicht bloß als ihre Aufhebung.

²⁴⁷ The standard **translation** ‘chalet’, though funny in its own way, is wrong: (1) it is not clear what ‘a chalet in baroque style’ even could be (a chalet is by definition a rustic type of building that is -at its origins- specifically adapted to alpine circumstances, all of which is incompatible with the very notion of ‘baroque’); (2) it misses the point that LW wants to make completely (whereas a chalet has some functional features that makes it suited for mountainous circumstances (though perhaps still not for the most inhabitable parts of the Everest, before it became part of the tourist industry), a little baroque castle, with its almost exclusively ornamental character, is one of the most incongruous and unlikely choices).

LW then concludes this excerpt by asking the following question: “How could you actually make the assertion [Behauptung] plausible to me, because you can’t actually use it for any other purposes than for this little trick of yours?”²⁴⁸ Let’s first note that “that little magic trick” [jenen Kunststückchen] sounds very dismissive and echoes the recurrent theme of **magic**, **slight of hand** and other types of **illusion**.

This is a very clear articulation of what LW’s criticism of the foundationalist use of formal systems actually consists in: it problematizes the relation between the foundational system and ‘real-life math’ and asserts the primacy of the latter.

This also illustrates what LW meant when he said that he ‘talks past’ the contents of Gödel’s actual proof and says things that are applicable to Gödel’s proof but to much simpler and generic aspects of math as well: the way LW’s argument is formulated is intended (I guess) to apply to Gödel and the dismissive tone of the ‘little magic trick’ remark is perhaps rightly interpreted as a snide remark targeting Gödel (who is -it bears repeating- not mentioned throughout this text), but it is also very clear that what is being attacked here has nothing to do with the internal mechanics of Gödel’s proofs, and applies to any type of proof involving formal systems.

LW’s line of thought in this excerpt, the way we read it here, has an important corollary: if it’s true that contradictions in foundational systems need not be a problem because they are peripheric anyway, then that implies that LW does not believe that math needs to be a unified formal system to be valid.²⁴⁹ This **anti-monism** about math is not overtly expressed in this excerpt, but is one of the things that recur time and time again throughout our analyses (cf. section 1.3(Fd) above and section 2.4.3(C) below), and LW does express it explicitly in connection with Gödel elsewhere (Ms-121,76r):

Gödel zeigt uns eine Unklarheit im Begriff (der) ‘Mathematik’, die darin zum Ausdruck kam, daß man die Mathematik für ein System gehalten hat.

LW says: *Gödel shows us an unclarity in the concept of ‘mathematics’ that found its expression in the fact that one has taken mathematics for a system.* So, LW does not believe that math needs to be a system (this is one of the clearest avatars of the **anti-monism** that we observed also elsewhere

²⁴⁸ Du sagst: “... also ist P wahr & unbeweisbar.” Das heißt wohl: “Also $\vdash P$.” Von mir aus- aber zu welchem Zweck schreibst Du diese ‘Behauptung’ hin? (Es || Das ist, als hätte man || jemand aus gewissen Prinzipien über Naturformen & Baustil abgeleitet, auf den Mount Everest, wo niemand wohnen kann, gehöre ein Schlößchen im Barockstile.) Und wie könntest Du mir die Wahrheit der Behauptung plausibel machen, da Du sie ja zu nichts weiter brauchen || verwenden kannst, als zu jenen Kunststückchen?

²⁴⁹ It is interesting to see to what an extent Gödel and LW take a similar path in this respect: both come to the conclusion that maths’ validity does not depend on whatever is proven in a formal axiomatic system. But then they diverge to an extreme degree in their opinions as to what it does depend on, which is in its turn also very interesting.

in LW's work) and he appears to believe that Gödel's results -whatever Gödel's own thoughts in this regards may be- actually can (or even should) be understood in such a way that they show that math is not a system.

(C) ein guter Engel: LW, Ms-124,71-74, d.d. 19410623-19410624 [// BGM7, §16]

In this excerpt LW asks his main question with respect to the Grundlagen issue at its most plain and simple: why does math need a foundation? And his answer shows how the issue of contradictions is intimately related to this question. For this reason, the excerpt deserves a close and detailed reading on our part.

(1) LW's basic answer is simple: **math does not need a foundation**, no more than that propositions about physical objects or sensorial impressions need an analysis. LW then adds: What they do need, like the other kinds of sentences, is a clarification of their grammar.²⁵⁰ In LW's work, 'grammar' means an account of the meaning of words in terms of the ways in which they are used, of their function within a practice (language game, etc.). So, -rather than participating in the debates about what could serve as a proper foundation for math, he simply denies that that would be a meaningful endeavour and proposes his own form of philosophy as an alternative.

(2) For LW, mathematical problems concerning the so-called 'foundations' are as little fundamental to math as **a painted rock carries a painted castle**.²⁵¹ This implies that for LW formal systems are a mere picture of math, not math itself.²⁵² It also follows that whatever is shown by the use of such formal systems cannot be somehow more fundamental than what could be shown without them.

(3) LW then asks: "But didn't that **contradiction** make **Frege's logic** useless for offering a foundation to arithmetic?". And the reply is: "Sure! But who has said that it had to be useful to that purpose in the first place?".²⁵³ LW refers to Russell's observation that his paradox could be derived from Frege's logistic system. As opposed to Russell, Frege and -as far as I know- anyone else involved in the logistic approach to the Grundlagen, LW simply denies that this is a real problem.

²⁵⁰ Wozu braucht die Mathematik eine Grundlegung?! Sie braucht sie, glaube ich, ebensowenig, wie die Sätze über physikalische Gegenstände oder Sinnesdaten, || Sinnesempfindungen, eine Analyse. || wie die Sätze, die von physikalischen Gegenständen handeln, oder von Sinneseindrücken, eine Analyse. || wie die Sätze, die von physikalischen Gegenständen - oder die, welche von Sinneseindrücken handeln, eine Analyse. Wohl aber bedürfen die mathematischen, sowie jene andern Sätze einer Klarlegung ihrer Grammatik.

²⁵¹ Die mathematischen Probleme der sogenannten Grundlagen liegen für uns der Math. sowenig zugrunde, wie der gemalte Fels einer gemalten Burg. || wie der gemalte Fels die gemalte Burg trägt.

²⁵² Again, a move that is not that different from Gödel's basic move.

²⁵³ 'Aber wurde die Fregesche Logik durch den Widerspruch zur Grundlegung der Arithmetik nicht untauglich? Doch! Aber wer sagte denn auch, daß sie zu diesem Zweck tauglich sein müsse?!

(4) The next day, LW continues to think about the case of **Frege's logic** and -in a familiar fashion- conjures up a scenario in which 'a savage' has been given Frege's logic as a tool to derive arithmetical propositions. This scenario is a slightly mythologized version of the fantasy underlying formalism in general, i.e. that a purely **mechanical** execution of the rules of a formal system could adequately and completely represent (or even replace) mathematical reasoning. In this case we are asked to imagine the formalist's nightmare: suppose now that this 'savage' has derived the contradiction without knowing it is a contradiction, and is now deriving arbitrary true and false propositions.

Someone reacts to this scenario by saying "Up until now, **a good angel** has saved us from going this way". To which, someone else (LW?)²⁵⁴ replies, in familiar fashion: "Well, what more do you want?", after which we read the wonderful comment: "I believe one could say: **a good angel will always be necessary, whatever you do**".²⁵⁵ The reference to the fact that 'a good angel' is always needed, expresses LW's belief that for things to work out *foundations are actually irrelevant*: whether our foundational system turns out to be consistent or not, the successful *application* of our mathematical techniques will not depend on it.

It is important to understand that it is *as a matter of fact* not true that the so-called 'foundations' are what make math reliable: obviously (I would say, but perhaps there are people around that would disagree), elementary mathematics and geometry are -at an intuitive, immediate level, but also as an historical fact- a lot more secure than any set-theoretical (or otherwise foundational) theory -say ZFC or in LW's own experience Russell's *Principia Mathematica* or Frege's *Grundgesetze*- could ever be. Why? Because those applied techniques are basic aspects of our Forms of Life, intertwined with those activities that make up the bulk of our everyday lives (building stuff, buying and selling stuff, etc.), having deep historical, cultural, biological and physical roots, in a way that the 19th-20th century foundational axiomatic systems simply are not. Elsewhere (see section (D) here below), LW even plays with idea that it is perfectly imaginable that people operate with inconsistent systems.

²⁵⁴ This last reply -as opposed to the previous sentence- does not have quotation marks, which suggest that this 'voice' coincides with 'the auctorial voice', if that means anything in the case of LW's notebooks.

²⁵⁵ 24.6.

Man könnte sich sogar denken, daß man die Fregesche Logik einem Wilden als Instrument gegeben hätte, um damit arithmetische Sätze abzuleiten. Er habe den Widerspruch abgeleitet, ohne zu merken, daß es einer ist, & aus ihm nun beliebige wahre & falsche Sätze.

'Ein guter Engel hat uns bisher bewahrt, diesen Weg zu gehen.' Nun, was willst Du mehr? Man könnte, glaube ich, sagen: Ein guter Engel wird immer nötig sein, was immer Du tust.

(D) why couldn't contradictions have a function? (LW, Ms-125,66r-68r, d.d. 19420923 [// BGM4, §59]; LW- Ms-121,74r, d.d. 19381228)

In the following two excerpts, LW takes the idea that contradictions are not necessarily a problem in a slightly different direction (similar to his equally fictional scenarios we discussed in section 1.3), exploring the idea that contradictions could actually serve a function in logic. In the first excerpt (Ms-121,74r, d.d. 19381228), LW proposes to present Russell's paradox/contradiction as '**the rarest, most exotic flower**' of the *Principia Mathematica*.²⁵⁶ A little dialogue, similar to the one we read in section 2.3(A), ensues: (1) a **panicky** interlocutor says "But any proposition follows from a contradiction! What would become of logic?"; and -as was the case in section 2.3(A)- (2) LW's reply is comforting, cool, calm, collected and eminently **pragmatic** (in both the technical and non-technical sense of the word 'pragmatic'): "So, don't derive anything from a contradiction, then...".²⁵⁷ What is interesting here, is that LW explicitly thematizes the pragmatics of using axiomatic systems: there is no reason why anything within the formal system would force one to use it, interpret it, in a certain way: what makes the formal system meaningful, is how it is actually used within the practices in which it occurs. I will develop this idea a little further in Appendix 4.1 below.²⁵⁸

The following somewhat isolated ²⁵⁹ paragraph (LW, Ms-125,66r-68r, d.d. 19420923), shows a similar idea, in that LW explores another -somewhat more radical- way to make

²⁵⁶ Warum sollte die Russellsche || symbolische Logik nicht zu einem Widerspruch führen dürfen? Warum || Ja, warum sollte man dieses nicht als die seltenste Blume dieses Systems empfinden. || sollte man in diesem nicht die seltenste Blume dieses Systems sehen? || sollte man in diesem nicht eine exotische Blume dieses Systems sehen?

²⁵⁷ "Aber aus einem Widerspruch folgt ja jeder Satz! Was würde dann aus der Logik?"

Nun so folgere nichts aus einem Widerspruch!

²⁵⁸ LW tags this excerpt with a semi-comical and all in all rather superficial comment, suggesting that if some mathematicians **superstitiously** fear contradictions as if confronted by the devil, others might want to celebrate 'black masses', indulging in contradictions:

Wenn Mathematiker sich abergläubisch vor dem Widerspruch wie vor dem leibhaftigen Teufel gebärden, warum sollten nicht andere eine Art schwarze Messe feiern (&) sich in Widersprüchen ergehen?

²⁵⁹ The remark that interests us here occurs in the context of a series of reflections on imaginary scenarios in which mathematical results could be available through other means than calculations, e.g. through calculators occurring in nature, as the result of the chemical properties of paper, by making icecubes melt, as the result of unexplicable human behavior. For the present purposes, it is not necessary to delve into this -otherwise appealing- material.

Ms-125,64r:

Wenn Rechenmaschinen in der Natur vorkämen & von den Menschen gefunden & benützt würden, so hätten wir eine Arithmetik ohne Sätze & ohne Beweise.

[...]

Ms-125,66r- Ms-125,67r:

Unsre Rechenmaschine in der wir die Operationen verfolgen können - & eine Rechenmaschine, die auf einem besonderen Papier, worauf wir die Angabe schreiben, durch einen chemischen Vorgang das richtige Ergebnis erscheinen läßt. -

So könnte man den Kubus einer Zahl finden indem man einen Eiswürfel von der betreffenden Kantenlänge abwägt. Und man könnte natürlich unser Rechnen auch als so einen Vorgang betrachten. In diesem Fall wäre die Rechnung ein Nebenprodukt bei der Erzeugung des Resultats. (Wie das Schnurren der || einer Maschine.)

contradictions part of a formal system. In this case, LW proposes to construe Russell's contradiction as something supra-propositional: **the self-contradicting proposition stands above the propositions**, like a **Janus-headed** monument, looking in both directions. One could even start logic with this contradiction and -as it were- descend down from it to the actual propositions.²⁶⁰ The move that LW makes in this excerpt is -in its effects- similar to all the other imaginary scenarios: it serves to make us think about how much is **presupposed / 'given'** before anything meaningful can even occur.

(E) Gödel's unphilosophical paper + the slimy concepts of most mathematicians LW, Ms-124,115-119, d.d. 19440310 (// BGM7, §§32-34):

What is interesting in this excerpt is that LW does name Gödel by name and that LW's overt criticism directly addresses a number of presuppositions that are generally accepted.

The remark about Gödel and the comments on the relevance of contradictions that come with it, occur in the context of a reflection on the question what constitutes a calculation. LW insists on the fact that a lot of different criteria may be involved in determining whether something counts as 'calculating' (training, correctness, practical application, intentionality, ...).²⁶¹ This brings LW to the following remark, directly addressed at Gödel and 'most mathematicians':

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Das Unphilosophische an Gödels Aufsatz besteht | | liegt darin, daß er das Verhältnis der Mathematik & ihrer Anwendung nicht sieht. Er hat hier die schleimigen Begriffe der meisten Mathematiker.

Denke Dir den Fall, in welchem Menschen zwar immer gleiche Endresultate bei einer Rechnung erzeugten aber, sozusagen, unerforschliche Wege zu diesen gingen, d.h. Rechnungen hinschrieben, die wir nicht nachrechnen können & die sie selbst nicht erklären könnten. (Wie es bei schwierigen Problemen oft geschieht.) (Kunstrechnen)

²⁶⁰ Warum sollte man den Russellschen Widerspruch nicht als etwas Überpropositionales auffassen, etwas das über den Sätzen thront & nach beiden Seiten (wie ein Januskopf) | | zugleich schaut. | | nach beiden Richtungen schaut. N.B.: der Satz $F(F)$ - in welchem $F(\xi) = \sim\xi(\xi)$ - enthält keine Variablen & könnte also als etwas Überlogisches, als etwas Unangreifbares, dessen Verneinung es nur wieder selber aussagt, gelten | | dastehen. Ja könnte man nicht sogar die Logik mit diesem Widerspruch auffangen? Und von ihm gleichsam zu den Sätzen niedersteigen.

Der sich selbst widersprechende Satz stünde wie ein Denkmal (mit einem Januskopf) über den Sätzen der Logik.

²⁶¹ These paragraphs contain a lot of interesting contents, which -however- are not immediately relevant to the subject matter that concerns us in this study:

10.3.44.

Man könnte sagen: Experiment - Rechnung sind Pole, zwischen welchen sich menschliche Handlungen bewegen.

Wir konditionieren einen Menschen in dieser & dieser Weise; wirken dann auf ihn durch eine Frage ein; & erhalten ein Zahlzeichen. Dieses | | eine Zahl. Diese verwenden wir weiter zu unsern Zwecken & es erweist sich als praktisch. Das ist das Rechnen. - Noch nicht! Dies könnte ein sehr zweckmäßiger Vorgang sein - muß aber nicht sein, was wir 'rechnen' nennen. Wie man sich denken könnte, daß zu Zwecken denen heute unsere Sprache dient Laute ausgestoßen würden, die doch keine Sprache bildeten.

Zum Rechnen gehört, daß alle die richtig rechnen dasselbe Rechnungsbild produzieren | | erzeugen. Und 'richtig rechnen' heißt nicht: bei klarem Verstande, oder ungestört rechnen, sondern so rechnen.

²⁶² This paragraph has -perhaps unsurprisingly- not been retained in the standard editions, though most of the rest of this excerpt has been retained.

LW says that *what is unphilosophical about Gödel's paper is that he doesn't see the relation between math and its application* and that in this respect, he has the **slimy concepts** of most mathematicians. What does slimy mean, here? At face value, 'schleimig' would indicate a lack of solidity (or: of undue liquidity where solidity is expected), but perhaps there is also a connotation of slickness and obsequiousness (?). What is it that LW calls slimy? Apparently, the lack of clarity that most mathematicians have about the relations between math and its applications is either slimy in the more literal sense, or in the sense of insincere docility.²⁶³

LW then formulates the idea that every proof gives the mathematical construction a new leg, like the leg of a table.²⁶⁴ The link with what precedes is probably that for LW, but not for Gödel and most mathematicians, every proof is equally fundamental, that there are not really 'foundations' that are unequivocally 'underlying' other parts of math.

LW then turns to one of the now familiar themes that he also developed in the extended passage that we analyzed in section 1.3 above: the topic of **demarcation** and the issue of mathematical techniques with **fringe or 'fantastic' applications**.

Previously (see e.g. section 1.3), LW had asked the question as to whether math with a purely fanciful application wasn't math anyway, suggesting that it was. He now formulates a potential objection: don't we call it 'math' only because there are many transitions, bridges from the fanciful to the non-fanciful? Would we still say that people were doing math who only calculated (operated with signs) for occult purposes?²⁶⁵

And -in typical style- he formulates what looks like an objection to the objection: but wouldn't it then [i.e. even if we agree that operations with symbols for purely occult purposes do not count as proper math] still be incorrect to say that it is essential to proper math that it builds concepts?²⁶⁶ LW then comes to the climax of his argument, the point to which this kind of line of questioning apparently always leads:

²⁶³ It could be an interesting exercise to try and think both potential interpretations through. In the case of the literal interpretation, what would the choice for this particular adjective imply: does it mean that they are not solid enough, that they are runny, and adapt their shape to any surface they happen to come into contact with, and/or that they are not fluid enough and stick to the hands of whoever tries to use them? Under the other interpretation, the question could be: towards whom is the obsequiousness of these concepts directed, whom are they supposed to please: religious or political authorities, perhaps? Or perhaps LW means that they avoid all conflict by merely confirming the consensus once it was established.

²⁶⁴ Jeder math. Beweis stellt das math. Regelgebäude || Gebäude auf einen || gibt dem mathematischen Regelgebäude || Gebäude einen neuen Fuß. [Ich dachte an die Füße eines Tisches]

²⁶⁵ Ich habe mich gefragt: Ist Mathematik mit rein phantastischer Anwendung nicht auch wirkliche Mathematik? - Aber es fragt sich: Nennen wir es 'Mathematik' nicht etwa nur darum weil es hier Übergänge, Brücken gibt von der phantastischen zur nichtphantastischen Anwendung? D.h.: würden wir sagen, Leute besäßen eine Mathematik, die das Rechnen, Operieren mit Zeichen, bloß zu okkulten Zwecken benützten?

²⁶⁶ Aber ist es dann doch nicht unrichtig zu sagen: das der Mathematik Wesentliche sei, daß sie Begriffe bilde?

- Denn die Mathematik ist doch ein anthropologisches Phänomen. Wir können es also als das Wesentliche in einem großen Teil | | Gebiet der Mathematik (dessen was 'Mathematik' genannt wird) erkennen & doch sagen, es spiele keine Rolle in anderen Gebieten. Diese Einsicht allein wird freilich nicht ohne Einfluß auf die sein, die die Mathematik nun so sehen lernen. Mathematik ist also eine Familie; aber das sagt nicht daß es uns also gleich sein wird, was alles in sie aufgenommen wird.²⁶⁷

Math is an **anthropological** phenomenon, and like all other anthropological phenomena, it is not a homogeneous thing: what is essential in one area within what we call math, need not play a role in other areas. In other words: math is a family, which -as usual- suggests that it is not a single thing, but several things that more or less resemble each other, but in this case, LW emphasizes the fact that this does not mean that just anything can be accepted into it, either.

LW also formulates an interesting little corollary to this last paragraph: the insight that math is not homogeneous should have a serious impact on those who learn to see math in this way. This remark connects back to the beginning of this excerpt, i.e. the idea that not only Gödel, but 'most mathematicians' have slimy ideas concerning the relation between math and its applications. It is also interesting to compare it to §644 *The Big Typescript*, in which LW also makes a link between how people view math and the way they are indoctrinated to see it through the education they are given (cf. section 3.2.3(C) below).

LW then surmises that one could say that if there was no mathematical proposition that you understood better than the Axiom of Choice, then you didn't understand math at all.²⁶⁸ This makes sense: I guess most people would agree that 'to understand math' would in the first place imply that one master basic arithmetic and geometrical techniques, algebra, trigonometry, calculus..., and even if we end up including set theory, understanding the Axiom of Choice would perhaps not be the most representative thing to focus on.²⁶⁹

LW then comes back to one of his favorite scenarios: what if we deduce propositions from a **hidden contradiction**? This time, he makes us imagine a case in which there are real-life consequences: **a bridge collapses**. What would happen? LW surmises that we would attribute the collapse to other reasons, for instance in religious terms ("It was God's will"). LW then

²⁶⁷ --For mathematics is after all an anthropological phenomenon. Thus we can recognize it as the essential thing about a great part of mathematics (of what is called 'mathematics') and yet say that it plays no part in other regions. This insight by itself will of course have some influence on people once they learn to see mathematics in this way. Mathematics is, then, a family; but that is not to say that we shall not mind what is incorporated into it.

²⁶⁸ Man könnte sagen: verstündest Du keinen mathematischen Satz besser als Du das Mult. Ax. verstehst | | das Mult. Ax., so verstündest Du Mathematik nicht.

NB: "the Multiplicative Axiom" is an older term, used in the *Principia Mathematica*, which LW was familiar with, for what is better known as the Axiom of Choice (cf. Linsky (Linsky 2021), §11).

²⁶⁹ An interesting but rather technical paragraph follows in the manuscript, which would take too much space for too little benefit to comment on in the context of this study.

asks the following, familiar question: is our calculation a mistaken one, or is it not a calculation at all? In an equally familiar way, LW then puts on his imaginary ethnographer's hat and explores what we would say if we observed such people from the outside. We would certainly have to acknowledge the differences with our own way of doing things, but we could not easily deny the fact that these people have some kind of mathematics.²⁷⁰

LW then refers to the classical story of the king who decreed that all visitors to his city must state their business there and would be hanged if they lied, and the case of the visitor that states that he came to be hanged.²⁷¹ The king will try to make sure that this unpleasant situation could no longer occur. LW asks: what kind of measures could the king take? what kind of problem is this? And he suggests that the problem is similar to the question as to how one can change the rules of a game in such a way that this or this situation can no longer occur, and that that is a mathematical problem [Aufgabe].

It is hard not to make a direction connection with Gödel's work: what LW appears to suggest is that -rather than the **dramatic conclusions** that are usually inferred from Gödel's results (to begin with by Gödel himself)- one could also take note of the fact that *this* is the result of constructing *this kind of propositions*, and simply stop constructing this kind of propositions if one doesn't like this kind of results.²⁷² So: whereas Gödel's own interpretation of his famous results remains within (or pretends to remain within) the syntax-cum-semantics of the formalism, LW's approach operates entirely at the level of the **pragmatics** of math.

LW then points out that it would be weird to turn the issue of the **demarcation** of math into a mathematical matter.²⁷³ As LW pointed out earlier, the identity of math is an anthropological matter: its meaningfulness depends on its being deeply embedded in real-life everyday practices, and whether something is considered part of math is not a simple question: it depends on many factors and many different answers are equally possible. It is simply true that Gödel has not understood this. Whereas Gödel and the later LW may -perhaps

²⁷⁰ -Hier ist ein Widerspruch: Aber wir sehen ihn nicht & ziehen Schlüsse aus ihm. Etwa auf mathematische Sätze; & auf falsche. Aber wir erkennen diese Schlüsse an. - Und bricht nun eine von uns berechnete Brücke zusammen, so finden wir dafür eine andere Ursache, oder sagen, Gott habe es so gewollt. War nun unsre Rechnung falsch; oder war es keine Rechnung?

Gewiß, wenn wir als Forschungsreisende nun die Leute betrachten | | beobachten, die es so machen, werden wir vielleicht sagen: diese Leute rechnen überhaupt nicht. Oder: in ihren Rechnungen sei ein Element der Willkür, welches das Wesen ihrer Mathematik von dem der unsern unterscheidet. Und doch würden wir nicht leugnen können daß die Leute eine Mathematik haben.

Was für Regeln muß der König geben, damit er der unangenehmen Situation von nun an entgeht, in die ihn sein Gefangener gebracht hat? - Was für eine Art Problem ist das? - Es ist doch ähnlich diesem: Wie muß ich die Regeln dieses Spiels abändern, daß die & die Situation nicht eintreten kann. Und das ist eine mathematische Aufgabe.

²⁷¹ Cf. a footnote in the standard edition Wittgenstein & von Wright (ed.) & Rhees (ed.) & Anscombe (ed.) 1978(3) -- Remarks on the Foundation of Mathematics, p. 400.

²⁷² In the same way, we stopped writing 0/x. See appendix 4.1(D).

²⁷³ Aber kann es denn eine mathematische Aufgabe sein, die Mathematik zur Mathematik zu machen?

Kann man sagen: "Nachdem dies mathematische Problem gelöst war, begannen die Menschen eigentlich zu rechnen"?

surprisingly?- agree on the fact that math cannot be reduced to its formal representation in an axiomatic system, they are diametrically opposed on what can be concluded from this observation and where to take it from there: for Gödel, the answer had to be the existence of a mathematical universe out there, but for LW, the answer is anthropological: the validity and value of mathematical results of mathematical practices, mathematical concepts, mathematical results are a matter of the way they are embedded in the heterogeneic mess of real-life practices.²⁷⁴

(F) axiomatic formalism as a tumor: Ms-161,59v-63r, not dated, probably 1941²⁷⁵

LW starts from the following imaginary scenario, very similar to the one we encountered in paragraph (E) hereabove: suppose some of the results of our calculations turn out to be based on a **hidden contradiction**. He then asks: well, does that make the results illegitimate? By now, it should be clear to the reader of the present study that this question should be interpreted in the context of LW's 'anthropological' approach to math and that it's perfectly imaginable that people operate with procedures that we would perhaps find inconsistent. This time, LW wants us to explore the case in which someone would want to avoid adopting such results and fears that some of them could sneak through. The reply to that could be: well, that is an idea that can serve as the example for a new technique [Kalkül], the way one can have an idea for a new game.²⁷⁶ So: LW insists on the fact that making sure that the suspect type of propositions no longer seeps through would be a new game, different from the first one, in which the techniques aimed at filtering these out played no role. Again, LW shows that the way we do math is only one out of many different imaginable ways to perform calculations within a practical context and that insisting on consistency is not necessarily at the root of all types of calculation.

This brings LW to the bombshell remark that made me include this excerpt here:

Der Russellsche Widerspruch ist nicht, weil er ein Widerspruch ist beunruhigend, sondern weil das ganze Gewächs deren Spitze er ist gleich einem Krebsgewächs ist welches zweck- & sinnlos aus dem normalen Körper herauszuwachsen scheint.

²⁷⁴ Cf. Floyd & Kanamori 2016 (Floyd and Kanamori 2016), p. 290: "Of course, unbeknownst to Gödel, by 1934 Wittgenstein too had rejected the *Tractatus* idea of a "possible projection" in logical space, and refashioned the idea against the more anthropological backdrop of "language games" and "forms of life". But, as is clear in *Max Phil IX-X*, it is logical, and not anthropological ideas of meaning that interested Gödel."

²⁷⁵ http://wab.uib.no/wab_nachlass-table.page

²⁷⁶ Sagen wir, wir erhielten manche unsrer Rechenresultate durch einen versteckten Widerspruch. Nun – sind sie dadurch illegitim? – Aber wenn wir nun solche Resultate durchaus nicht anerkennen wollen & doch fürchten es könnten welche ent schlüpfen || durch schlüpfen.– Nun dann haben wir also eine Idee die einem neuen Kalkül als Vorbild dienen soll. Wie man die Idee zu einem Spiel haben kann.

LW bluntly claims that Russell's paradox is not disquieting because it is a contradiction but because the whole tumor of which it is the top appears to grow out of the normal body like a **cancer**, without a purpose and without sense. To be clear: the whole axiomatic system is the outgrowth, the cancer, not the tiny little contradiction that is a mere part of it. As in the excerpts we studied previously, LW denies that contradictions are inherently problematic. As in 2.3(A) and 2.3(D), LW gives room to the obvious **panicky** objection: "But this is a contradiction! You can't just let a contradiction stand!". And again, LW's reaction is: "Why not?", after which he points out a few examples of harmless interpretations of what looks like contradictions.²⁷⁷

Immediately after this, LW illustrates his point with a characteristic analogy with **music**: we can sometimes immediately acknowledge that a certain musical phrase 'logically' follows another musical phrase. Just as in the case of the solution to a mathematical problem, we don't doubt that this is the correct solution. Still, it is easy to imagine, at least in the musical case, that other solutions would have worked equally well. Similarly, we can be convinced that two names go well together.²⁷⁸ So, LW highlights the fact that what we accept as a 'logical' sequence need not necessarily be of a propositional nature.

Then, he comes back to the theme of contradictions in formal systems in a way that highlights the **pragmatism** of his approach. LW evokes the following little dialogue:

A: "We make inferences that respect all the rules, but suddenly a contradiction shows up. The conclusion must be that the set of rules is useless, because the contradiction wrecks (literally: topples) the whole game".²⁷⁹

B: "Why do you allow the contradiction to wreck the game?"²⁸⁰

This last reaction to that conclusion is simple but far-reaching in its radical pragmatism (cf. (D) above): it implies that the consequences of a contradiction in a formal system depend on a decision on the part of the user/player, that there is no logical *emustí* here, there is no natural

²⁷⁷ Aber Du kannst doch einen Widerspruch nicht gelten lassen! – Warum nicht? Wir gebrauchen ihn ja manchmal in unsrer Rede, freilich selten – aber man könnte sich eine Technik || Sprachtechnik denken in der er ein ständiges Implement ist.

Man könnte z.B. von einem Objekt in Bewegung sagen es existiere an diesem Ort & existiere nicht an ihm. || es existiere & existiere nicht an diesem Ort; Veränderung könnte durch den Widerspruch ausgedrückt werden.

²⁷⁸ Nimm ein Thema wie das Haydnsche (Choräle S.A.) nimm den Teil einer der Brahms'schen Variationen, die dem ersten Teil des Themas entsprechen & stell die Aufgabe den zweiten Teil der Variation im Stil ihres ersten Teiles zu konstruieren. Das ist ein Problem sehr ähnlich einem mathematischen. Ist die Lösung gefunden, etwa wie sie Brahms gibt so zweifelt man nicht || so ist es uns klar daß dies die Lösung sei || ist. || so zweifelt man nicht – dies ist die Lösung.

Mit diesem Weg sind wir einverstanden. Und doch ist es hier klar, daß es leicht verschiedene Wege geben kann mit deren jedem wir uns einverstanden erklären können, deren jeden wir konsequent nennen können.

Ich könnte mir denken, daß Einer sagte || meinte die Namen 'Fortnum' & 'Mason' paßten zusammen.

²⁷⁹ 'Wir machen lauter legitime – d.h. in den Regeln erlaubte – Schritte, & auf einmal kommt ein Widerspruch heraus.⇒

Also ist das Regelverzeichnis, wie es ist, nichts nutz, denn der Widerspruch wirft das ganze Spiel um.'

²⁸⁰ Warum läßt Du ihn es umwerfen?

link here between anything in the formal system and what you decide to do with it (in Appendix 4.1(D) I will explore this aspect of the primacy of the pragmatics of formalism a little further).

The dialogue continues:

A: "But what I want is that we can continue to mechanically make inferences without ever reaching contradictory results".²⁸¹

B: "Well, what kind of foreseeability [**Voraussicht**] do you want?"²⁸²

This last intervention points out that expectancies can be of different kinds: a formal mathematical proof offers a different kind of foreseeability than a proper understanding of whatever one needs previsibility about. For instance: for most practical purposes, there are no foreseeable problems at all with the existence of Gödel-type inconsistencies.

Let us now briefly reconstruct LW's line of thought throughout this rather complex excerpt, paraphrasing:

- the whole excerpt is about scenarios in which we operate with a hidden contradiction;
- LW points out that it remains to be seen whether such an inconsistency matters and how we best deal with it;
- LW then turns to the example of Russell's paradox and states that it is part of an outgrowth that is not part of the normal functioning of the body, which is good news in that the contradiction as such is harmless, but bad news in that there is a tumor;
- LW then gives a few examples in which things follow each other 'logically' in an organic way, and then addresses the way in which he believes that the formalism that gives rise to Russell's paradox is not organic: for all practical purposes one can just decide to ignore it.

LW then adds the following brief 'meta' observation that is suitable as a conclusion to this section:

| Die philosophische Betrachtung der Mathematik hat eine andere Pointe als die mathematische von math. Sätzen & Beweisen. |

The philosophical approach [Betrachtung] to mathematics has a different point from mathematical approach to mathematical propositions and proofs. This rings true, in several ways:

²⁸¹ Aber ich will, daß man nach den Regeln soll mechanisch weiter schließen können, ohne je zu widersprechenden Resultaten zu gelangen.

²⁸² Nun, welche Art der Voraussicht willst Du?

- LW's approach notices and questions aspects that are simply presupposed and not noticed by mathematicians;
- most mathematicians will continue to not see the relevance of LW's remarks when we point them out to them.

What is it that interests the philosopher but not the mathematician (or any other technician)? I believe all the above is a good illustration of the fact that the philosopher (in this case LW) is interested in what is presupposed by the practice, what is **given** (cf. section 3.3(C) below), whereas the mathematicians are happy to unthinkingly work with whatever is given to them.

2.4 Conclusions to Part 2: LW's critical remarks

The present section 2.4 is the conclusion to Part 2 of this study. Given the fact that Part 3, the general conclusion to the whole study, follows immediately after this section, the scope of section 2.4 will be limited to the following paragraphs:

- in section 2.4.1, I give a summary overview of the trajectory we covered in Part 2 and how it fits in with the overall structure of this study;
- in section 2.4.2, I summarize a few recurrent topics in LW's PhilMath, as they appeared in the above: (A) the idea of expansions beyond everyday mathematical practices and the loss of meaning caused by disconnection from everydayness; (B) fakeness, pretense and fiction in mathematical discourse; (C) demarcation and heterogeneity; (D) set-theoretical discourse as a sign of sick times; (E) paradoxes, contradictions and the function of axiomatic systems;
- in section 2.4.3, I discuss LW's critical agenda as it emerges from our analyses, identifying the following lines of attack: (A) anti-foundationalism; (B) anti-monism; (C) anti-sensationalism; (D) anti-exceptionalism.

It may be useful to repeat that I am in principle dealing with LW's later work, but that the lines of thought that I am focusing on here show remarkable continuity, which made it possible to include passages from different periods in LW's development.

2.4.1 Part 2 (overview)

At the end of Part 1, we had established that for LW meaning/sense could be defined as 'embedding in everyday practices', and conversely nonsense equals a lack of embedding in everydayness. We also pointed out that the only way 'nonsense' can be problematic is when

it appears to make sense, but doesn't (gibberish that sounds like gibberish is never a problem, at least not a philosophical one).

In the case of mathematics, this notion of 'embedding in everyday practice' takes the shape of embedding in what is called "applications", i.e. everyday practices that involve mathematical techniques.²⁸³

The following question immediately imposes itself: what counts as everyday? We observed that, whereas the idea of everydayness has a certain intuitive appeal in terms of the naïvely prototypical (archetypical?) examples it certainly covers (buying apples, building a house, ...), it turns out to be highly problematic at even the most obvious critical scrutiny: why can't - say- metaphysical discourse be part of the philosopher's everyday? why isn't transfinite stuff part of Cantor's everyday? etc. We came to the conclusion that everydayness was not part of LW's results, but part of his agenda.

In section 2.0 we found out that everydayness is part of a cluster of deep-rooted ideas, which helps explain why LW wants to work with this somewhat suspect concept in the first place: LW's concept of everydayness was a core ingredient of the culture-critical agenda that links LW's philosophical work with most notably Spengler, but also with what has been called Lebensphilosophie in general. We pointed out that authenticity (as opposed to fakeness) was a (perhaps 'the') core concept in LW's critical lines of thought, both in his philosophical work, including his PhilMath, and in his private life, as documented in the biographical literature. Authenticity, everydayness and sense/meaningfulness (and their opposites: nonsense, disruption of the link with the everyday, and fakeness) turned out to be structurally related and therefore mutually supportive concepts in LW's critical lines of thought. For an overview of the vocabulary LW uses in this context, see 2.0.3(C).

The rest of Part 2 consisted of running commentaries based on close readings of three series of selected passages taken from LW's manuscripts:

- in section 2.1, we read three passages in which LW discussed diagonal methods, which allowed us to illustrate a number of features of LW's critical remarks at large: the ethical-aesthetical vocabulary, the notions of pretense, fiction and (what I called) bad faith and bad taste; the idea of expanding beyond everydayness and resulting loss of meaning; the anti-monist and anti-foundationalist strands we already observed in section 1.3 (cf. 1.3(F)).
- in section 2.2, we focused on three passages in which LW criticizes set-theoretical parlance; these passages illustrate the Spenglerian strand in LW's philosophy, as well as the degree of harshness his criticism sometimes reaches;

²⁸³ This use of the word 'applications' shows to what extent a de pragmatized view of mathematics is entrenched in the standard idea of math: as if 'pure' math somehow precedes the 'applied' math.

- in section 2.3, we read a series of passages that discuss contradictions in formal systems and other Gödel-related (or seemingly Gödel-related) topics; many of the previously observed strands reappear, but the anti-monist and ‘bad faith’-related strands acquire greater depth in these analyses.

2.4.2 A few recurrent topics in Wittgenstein’s PhilMath

(A) mathematical meaning as embedding in everydayness vs. expansions of mathematics and disconnection from everydayness

One of the core ideas in LW’s work as a whole is the idea that meaning is a matter of embedding in everyday practice, and conversely that discourse that is not well-embedded in everyday practice lacks meaningfulness, i.e. does not make sense. In the case of math, LW starts from the idea that the meaningfulness of mathematics depends on its relation with everyday applications. Conversely, the problem with the use of certain terms in PhilMath is the discontinuity with the everyday use of these terms.

Applied mathematical techniques are straightforwardly unproblematic: their meaningfulness is guaranteed by the fact that they are embedded in everyday practices and are an integral part of the Form of Life of the practitioners. LW goes to great lengths (cf. section 1.3) to point out that even very weird math-like techniques would make sense if they were part of real-life practices. In section 3.2.1(C) below, I argue that the historical and ethnographical records abundantly show that that is actually the case.

Total discontinuity from everyday practice would not be a problem either (it would perhaps be uninteresting, but not wrong).²⁸⁴ Thus, radical formalism would be perfectly fine, as such (perhaps uninteresting, but not wrong). If math was really construed as ‘the study of formal systems’ (to borrow Haskell Curry’s famous definition), the issues that LW objects against would not occur. But in reality, mathematics is not ‘the study of formal systems’: it’s a matter of obvious fact that more “advanced” mathematical techniques and concepts are expansions of more basic techniques and concepts, which in turn are rooted in real-life ‘applications’, and it is understood that anything we would call math should at least cover the natural numbers and basic arithmetic operations on these numbers, as well as basic geometry.²⁸⁵

So, purely formal math would not lead to any problems, and purely applied technique does not either. The trouble begins when one wants to have the cake and eat it, i.e. at the same time

²⁸⁴ Cf. the parallel with nonsense (section 1.2(A)): pure gibberish is not a problem; pseudo-propositional nonsense (gibberish that pretends to mean something) is problematic.

²⁸⁵ It would be interesting to interrogate various philosophers of math and various practitioners of math on the question as to whether they would consider Spencer Brown’s calculus (Spencer Brown 1972) to be a kind of math or not.

(1) redefine mathematical concepts in terms of a newly invented axiomatic formalized system, and (2) maintain that these are still the same concepts (see section (B) here below for a few examples), especially when one wants to (3) present these newly invented axiomatic systems as somehow underlying, or -even worse- as the foundations of, pre-existing techniques. For LW, the main problem is the **pretense**, i.e. the fact that these discourses within PhilMath pretend that these axiomatic formalisms are something they are not (see section (B) here below).

There is a link between this line of thought about expansions as add-ons and LW's anti-monist and anti-foundationalist objection to the claim that math (as it is) by nature is and always has been a single system, such as the ones that the early 20th century foundationalist efforts strove towards. As an alternative to this idea, LW introduces the idea that more advanced math is always an expansion of basic technique (in its turn deeply embedded in applications) and should not be presented as the discovery of the principles that somehow underlie the basic techniques. In other words: expansions are just that: expansions, add-ons, new techniques alongside (as opposed to above or beneath) the old ones.²⁸⁶ This corresponds to *PhU* §124 (already discussed in section 1.2.2(A) above), in which LW states that a so-called 'leading problem of mathematical logic' is for him a problem of mathematics like any other.

This reasoning underlies LW's criticism of set-theoretical verbiage, the interpretation of diagonal methods, but also the way in which the function of foundational systems in general is articulated in mainstream accounts.

(B) fakeness (fictionality, pretense, fake depth, bad faith, ...) in math

The problem with expansions -according to LW's view- occurs when one pretends that what one means by a certain concept hasn't changed although one now attributes attributes [sic, fs] to that concept which are incompatible with what that concept used to mean:

- 'number': for LW, the expansion of the term 'number' stops to be a natural one when it loses its connection with counting and basic arithmetical operations, i.e. when it is forced to include the artificial irrational numbers that only exist because of the theory of the continuum, not because they actually occur in actual calculations;
- 'ordering': similarly, it is misleading to pretend that ordering fractions still means the same thing when one talks about 'ordering all fractions': it should be obvious that there is no such thing as 'ordering all fractions' in the normal sense of the word 'ordering': there

²⁸⁶ LW pushes this idea to its limits - or perhaps beyond the limits of usefulness, when he claims that adding a couple of digits to the decimal development of -say- pi, or any other decimal expansion, is an expansion of math ("So seltsam es klingt: Die Weiterentwicklung einer irrationalen Zahl ist eine Weiterentwicklung der Mathematik" (Ms-126,133, d.d. 19421214)). This idea is coherent but perhaps not useful.

is no possible real-life method to go about this;²⁸⁷ the expression ‘the next bigger fraction’ is simply meaningless as long as it does not correspond to an actual technique;

- ‘line’: LW objects to the set-theoretical idea that a line is a collection of points, as this conception disrupts the link with the constructive procedure (‘law’/‘Gesetz’) that has always been the defining feature of a line;
- ‘set’: there is no a priori problem with expanding the notion of set, but it is disingenuous to pretend that there is nothing weird about an infinite set or the set of all sets, or similar constructions.²⁸⁸

In the same vein, LW also blames mainstream set-theoretical discourse for including completely **fictitious** elements:

- fictitious symbolism: infinitely long numerals are not actually calculated with and are only posited as a way to fill out the theory of the continuum;
- fictitious formal systems / fictitious code: formal systems such as Russell’s (let alone Gödel’s) are too unwieldy to actually be used for the purposes of actually proving actual theorems and are not used for that purpose in actual practice;
- fictitious constructions: according to LW, Dedekind’s cut has never been a way to construct $\sqrt{2}$, in that it presupposes the notion of $\sqrt{2}$ in order to be intelligible; therefore, as a constructive procedure, it is a fiction;
- fictitious methods: a method for ordering infinite sets does not actually exist.

There is something particularly infuriating about the conjunction of the idea of “infinite numerals” and the idea of “infinite minds” (cf. section 2.2(B)): from LW’s point of view, the idea that irrationals are represented by a numeral that happens to be infinite and that we happen to not be able to comprehend because our minds happen to be finite, is gibberish in the sense of mere word salad: it serves no function within the actual technique, and it means nothing outside it, either (except perhaps in the context of very particular **religious** contexts, which would lead us to a whole other can of worms; see Appendix 4.3(B1)).

Finally, LW also objects to the pretense that something is awe-inspiring, mysterious, deep, ... when it is actually trivial, for instance the vertiginous image of adding more and more reals

²⁸⁷ And no, this is not a matter of the finite amount of time we got to do it, or the finitude of our brain power. There simply is no algorithm or other method to go about this, unless one expands the notions of ‘method’ until it doesn’t mean anything.

²⁸⁸ The following examples could be included in this list of objectionable expansions of mathematical concepts, although they happened to not occur in the excerpts we analyzed above: “number”, as applied to strings of symbols in Gödel’s code, and “arithmetic”, as applied to operations with Gödel’s code, are a stretch: it is prima facie plausible that Gödel’s code did not arise from ‘normal’ arithmetical concerns but was invented for the sole purpose of making the logical, meta-mathematical, philosophical point Gödel wanted to make. NB that, in BGM 7 §22, LW problematizes the use of the word ‘number’ for coded strings such as: “But it must of course be said that that sign need not be regarded either as a propositional sign or as a number sign.--Ask yourself: what makes it into the one, and what into the other?”.

in a smaller and smaller interval of the continuum, or the drama around paradox-like contradictions in axiomatic systems. I have pointed out that within the context of LW's oeuvre, this type of objection is an avatar of the same concern about authenticity that also shows in LW's existential aversion of (or struggle with) theatricality and vanity.

(C) demarcation

One of the recurrent rhetorical devices in the passages we analyzed in the above (1.3 and 2.1, 2.2, 2.3) involves the issue of the demarcation of math: LW describes an -often made-up, but not necessarily unrealistic- scenario in which this or that math-like technique is applied in this or that practical context, and then asks the question: 'Is this still math'?

This appears to be a genuinely open question and it doesn't matter much whether each individual reader (or LW himself, for that matter) would be tempted to reply 'yes' or 'no' to it in any individual case. So, the point appears not to be that LW argues for this or that cut-off point, but rather that wherever one chooses to put that cut-off point, it will be a more or less arbitrary, or at the very least contingent, decision, **not a fact of nature**. The heterogeneity of the applications (and of the techniques, for that matter) shows that there is no single, principled, clear-cut, 'natural' way to demarcate math from other practices.

LW's way of conceptualizing this heterogeneity is in terms of his famous notion of 'family' (as in 'family resemblances'): there is no single criterion (or set of criteria) that determines whether a certain item is included under the concept, but that does not mean that just anything can be included either.

(D) set-theoretical parlance as a symptom of sick times

In section 2.0.0 and section 2.0.2 above, we pointed out a few important resemblances between LW's work and Oswald Spengler's culture-critical ideas, whether by direct influence or shared cultural backgrounds.

The most important of these Spenglerian strands is the notion that the dissolution of the organic unity of a culture leads to a lack of intelligibility of the products of the society in question. LW and Spengler shared the idea that Western culture had reached that point in the 19th century: from the early 19th century onwards, Western culture had started to decline into what Spengler called 'civilization', which showed in the arts, in politics, in the sciences, etc. This problem of 'loss of intelligibility' through a loss of organic embedding in a culture was perhaps *the* main problem for LW, both existentially and philosophically.

It is important to understand that for LW, 19th and 20th century mathematics is a case in point and that LW's criticism of certain types of discourse on mathematics, most notably set-

theoretical parlance, but also certain ways to interpret Gödel's results and foundationalist discourse in general (incl. the logistic approach to the Grundlagen issue, in which LW took part himself), should be understood in these terms.

In LW's PhilMath, the Spenglerian strand within his critical approach is articulated in terms of the fact (?) that discourse about math had become disconnected from (1) actual mathematical technique (Kalkül) qua operations with symbols and (2) actual everyday applications. LW did not always clearly distinguish between both, sometimes mentioned only one of both aspects, and sometimes mentioned them both.²⁸⁹

It is also worth mentioning that the Spenglerian strand in LW's PhilMath shows a remarkable continuity: the remarks about set theory that I quoted in section 2.2 date from 1929, 1931 and 1938 respectively (and for those who want to believe that LW grew out of this in later life, we can refer to extremely harsh and obviously Spenglerian remarks about Mahler dating from 1948; cf. section 2.0.2(A) above). And some time after March 4 1944, LW still wrote: "The curse of the invasion of mathematics by mathematical logic ... [der Fluch des Einbruchs der math. Logik in die Mathematik...]" (Ms-127,186 [= BGM 5 §46]).

(E) LW on paradoxes, contradictions and the functions of axiomatic systems (a.k.a. "LW on Gödel")

At the beginning of section 2.3, I felt the need to explain that most of what the literature interprets in terms of LW's critique of Gödel's famous proofs, is -according to LW himself- not that: at no point does LW deal with the technicalities of Gödel's proof and most of the time his criticism is not even directed specifically at Gödel or his work (LW's 'notorious' remarks do not mention Gödel even once, only Russell and Frege) and in those contexts in which LW does mention Gödel, he often repeats that he is not interested in the specifics of those proofs and that the philosophical importance of Gödel's proof is that it attracted the attention towards features of mathematics that are much more general and apply equally well to much more basic areas of math than Gödel's work.²⁹⁰

²⁸⁹ Cf. Severin Schroeder's analysis in terms of 'two strands' (see section 3.1.1(B) below). I am not sure if Schroeder is right in being so categorical about the distinction between these 'strands' and their chronological succession. Nothing much depends on it for the present purposes. I do believe that from early on in his 'intermediate period', i.e. even before his 'anthropological' approach acquired its more mature shape, LW conceived of 'Kalkül' as a real-life human activity, not as an abstract formal process. I mean: it appears to me that the distinction between a 'formal' strand or period and a -what I call- pragmatic strand or period in LW's thought is an artefact of the doxographic approach, i.e. the result of wanting to project conceptual distinctions onto a text in which they do not play a real role.

NB: the undeniable link between LW's criticism of set-theory and the Kraus- and Spengler-related 'lebensphilosophische' strands [sic, fs] in LW's thought may be an argument for the continuity between earlier and later stages in his thought with respect to the 'real-life' embedding of mathematical technique and hence for a non-formalist -or not really formalist- interpretation of 'Kalkül', even early on in his 'intermediate period'. *Pace* not only Schroeder (Schroeder 2021), but also Rodych (Rodych 2018).

²⁹⁰ For a very similar result, see Floyd 2021 (Floyd 2021), pp. 71-72: "Often Wittgenstein is regarded as quarreling with Gödel, but that is because he is too often read as a radical finitist or conventionalist. His remarks certainly

LW does not have a problem with Gödel's technique as such (it is an undeniably virtuoso piece of mathematical logic) and does not attack the internal workings of the proofs.

LW attacks at a very fundamental level, at which the specificity of Gödel's work is not the heart of the matter:

- LW questions the legitimacy of coding back and forth from prose into a formal code and back (and Gödel-code is not necessarily any different from Russell-code in this context; though Gödel's insistence that his code is 'normal arithmetic' is particularly infuriating);²⁹¹
- LW objects against the fictionality of these formalisms: nobody actually uses them to do actual math; the idea that they underlie actual math is a fiction;
- LW questions the importance of contradictions of the type in question (Gödel's result, but also Russell's paradox/contradiction, ...);²⁹²
- LW attacks the idea of the unity of math (monism) which is presupposed in the usual interpretations: if math is not viewed as a single system, concerns about consistency lose their central status and a lot of their urgency and importance;
- LW attacks the very idea that math would need a foundation and points out that whatever foundations are proposed, they would always be less secure than the everyday applications that they grew out of.

LW comments on the relationship between formalism and practice in general, and the relation between the practice and the prose that surrounds it. His point is that the problem is never located in the syntax of the formalism, not even in its semantics, but in the pragmatics of the formalism, more precisely the way one chooses (yes, chooses) to integrate it in one's practices and the consequences one chooses to attach to it.²⁹³ In the same way that the Liar's paradox appears to not fundamentally (or rather: not at all) imperil the usability of natural language and for all practical purposes can safely be ignored as irrelevant and therefore harmless,²⁹⁴ it remains to be seen whether a contradiction in an axiomatic system actually has any consequences for real-life working mathematicians' math at all, and LW suggests that for most

struggle to place Gödelian incompleteness into his way of thinking. But if we take seriously his Later views, we see what the struggle is about, and it is not about refuting Gödel". I come back to this quote in section 3.1.1(B) below.

²⁹¹ Ms-124,89 : 'Der Satz sagt, daß diese Zahl aus diesen Zahlen auf diese Weise nicht erhältlich ist.' – Aber bist Du auch sicher, daß Du ihn recht ins Deutsche übersetzt hast? Ja gewiß, es scheint so. – Aber kann man da nicht fehlgehen?

²⁹² Cf. also LW (and Turing) on the nonsensicality of paradoxes: LFM lecture 20, pp. 206–7 quoted in section 1.2.1(C) above.

²⁹³ It's like proofs of the existence of god: nobody ever has been convinced by one of those either. It's perhaps not a coincidence that Gödel was interested in this, as well. See (Floyd and Kanamori 2006); (Park 2018).

²⁹⁴ It is worth repeating that *LW does not say that Gödel's proof somehow involves the Liar's paradox*. The analogy is that both are not an integral part of the normal functioning of the language.

purposes Gödel's results can safely be ignored.²⁹⁵ Gödel's results are also similar to paradoxes in that neither has a function in real-life practice: the Liar does not serve any real purpose (except perhaps to entertain and/or annoy); Gödel's stuff did not emerge from proper arithmetic but is a little magic trick that serves only one purpose.

LW's main point is not that there are or aren't actual contradictions in this or that axiomatic system, formally speaking; LW's point is that nothing that could occur in formal axiomatic systems, not even contradictions, could ever have any impact on what he considers the core of proper mathematics, i.e. the hurly-burly of actually applied techniques. It is important to understand that *it is a fact, a hard empirical, anthropological, fact that nothing important needs to depend on problems with axiomatic systems*: no set-theoretical problem will ever have any impact on basic (and not so basic) arithmetic, geometry, trigonometry, calculus, etc., not even contradictions in set-theory.

And this is where LW *does* attack Gödel: for LW, the dramatic, far-reaching consequences that Gödel attaches to his results are unwarranted, in that they show a lack of understanding of the (relatively peripheral) status of axiomatic systems in general, as well as of the (central) role of applications. In one of the few passages in which he directly addresses Gödel (see section 2.3(E) above), LW's main objection is that Gödel's conceptualization of what his own work is about, does not take into account -what I call- the pragmatics of math, i.e. how math is defined by the way it is embedded in real-life practices, which leads him (KG) to wrongly consider math a single system and to a wrong appreciation of the importance of contradictions.²⁹⁶

Thus, LW was not in the first place concerned with any formal proof (which is just a piece of Kalkül like any other piece of Kalkül, and thus without particular philosophical importance); for LW, the problem is that it is not clear to what extent what *pretends to be* a proof of a theorem concerning the completeness of consistent formal systems (assuming that these terms mean what they always mean), is not *actually* a revolutionary reinterpretation of what the concepts 'consistency', 'proof', 'theorem', etc. traditionally mean.²⁹⁷

²⁹⁵ Except for limited applications in informatics and cryptography, it can be expected that Gödel's stuff never will have any consequences indeed.

²⁹⁶ In a certain sense, LW's alternative views on the functions of 'Widerspruch' or his alternative interpretations of Gödel-like endeavors could be interpreted like his other made-up examples, in the same way as what he does with mathematical applications: in the end, he only tries to attract our attention to the given: to what and how much is already given, before we even start to evaluate the truth of our theoretical conceptions. So, there is an intrinsic link between the critical aims behind his approach, his use of made-up examples and his emphasis on alternative, even fringe applications,

²⁹⁷ Viewing a proof as an object is an example of a radical departure from the everyday meaning that proof had in earlier stages of the practice.

Dutilh Novaes 2012 (Dutilh Novaes 2012), p. 68: quoting Netz 1999 (Netz 1999), on the importance of persuasion at the heart of ancient Greek math and logic; similarly, p. 78: "In a slogan, an argument, proof, or demonstration is a discourse; a calculation is a procedure."

It is important -for the purpose of this study- to keep in mind that LW's concern with aesthetic/ethical authenticity ("things should be what they appear to be", see sections 2.0.2(B) and 2.0.3(C)-(D) above) is one of the main points of LW's 'notorious' remarks. LW words his criticism in moral terms: he repeatedly seems to feel genuine moral indignation towards what he considers bad faith arguments: the idea that contradictions within axiomatic systems are presented as a genuine threat to the foundations of mathematics looks like a childish little trick to LW and he can't believe that it is actually taken seriously by those who propose it.²⁹⁸ As opposed to most commentators, I take LW's word on his attitude towards Gödel seriously and literally: I believe he is genuinely ²⁹⁹ not interested in the specifics of Gödel's proof. So: even if it were true that LW has not understood (perhaps not even read) KG's work, it doesn't follow that the above objections are ipso facto irrelevant for what they are: they are not intended to attack any of the specifics of Gödel's work anyway. And from that, it does not follow that LW's remarks do not apply to Gödel's work, on the contrary: they apply to Gödel's work in the same way they apply to Frege's work or Russell's work or to any other work in mathematics that operates with a formal axiomatic system. And of course, they may still apply more clearly to Gödel's work than to any other work; after all, that was Gödel's main merit, according to LW: to have created a situation in which this aspect of formalism in general became obvious.

2.4.3 The critical agenda underlying Wittgenstein's Philosophy of Mathematics

Across the various topics discussed here above (as well as in sections 1.3), there are a small number of recurring lines of thought that systematically attack the same targets, and in that sense embody the agenda that underlies LW's PhilMath.

(Berto 2009) Berto 2009 -- The Godel Paradox and Wittgenstein's Reasons on 'naïve proof' and truth, p. 212:

My strategy exploits an idea proposed by Richard Routley and Graham Priest in various influential essays,⁶ which allows us to interpret Gödel's proof precisely as a *paradoxical* derivation. The core thought is to see what happens when one applies G1 to the theory that captures *our intuitive, or naïve, notion of proof*. By 'naïve notion of proof' Routley and Priest mean the one underlying ordinary mathematical activity: 'proof, as understood by mathematicians (not logicians), is that process of deductive argumentation by which we establish certain mathematical claims to be true' [Priest, 1987, p. 40]. Since Hilbert, formal logicians treat proofs as purely syntactic objects. However, proving something, for a working mathematician, amounts to establishing that some sentence is *true*. When we want to settle the question whether some mathematical sentence is true or false, we try to deduce it, or its negation, from other mathematical sentences which are already known to be true.

²⁹⁸ LW's diagnosis in terms of -what I call- bad faith is in a way corroborated by what has become known on the psychodrama of Gödel's development: --> see appendix

²⁹⁹ And even if it could be proved that there is an apologetic strand in what LW says (i.e. if he actually didn't understand Gödel's work and said what he said as a cover-up or justification), this would not in fact change anything.

(A) anti-foundationalism

LW very explicitly opposes the idea that the efforts of his contemporaries and the generation that came just before him (Russell, Frege, Hilbert, ...) -and let's not forget that his own early philosophical activity under (or perhaps rather 'with') Russell was an integral part of that effort- to ground math in a single coherent system were not only unsuccessful but also fundamentally misguided. LW frontally attacks the idea that axiomatic systems play a foundational role at all (this is presented as a matter of fact):³⁰⁰

- (1) De facto, axiomatic systems do not actually offer a foundation for actual mathematical technique, as applied in actual practice, nor does it actually unify the heterogeneous collection of mathematical techniques into a single system (cf. section (B) here below, on LW's anti-monism).
- (2) De facto, mathematical techniques don't need foundations; in actual fact, applied mathematical technique does not become less or more valid, legitimate or secure by having or not having such a 'foundation' (for things to go right, we need 'a good angel' anyway, says LW).³⁰¹

LW's anti-foundationalism has been picked up on by most if not all commentators (Schroeder (Schroeder 2021), §3.6; Rodych (Rodych 2018), §2.5.1 et passim). My only contribution here is to emphasize the links between this aspect on the one hand and (a) LW's underlying philosophical agenda in terms of authenticity and fakeness and the links with the culture-critical strands inspired by most notably Kraus and Spengler, as well as (b) LW's pervasive pragmatism, on the other hand. I also think the importance of this stance appears to be underestimated or not taken seriously enough in the literature, in that it is not a separate strand within LW's work that can be isolated from other strands, but an omnipresent part of what motivates LW's philosophical activity, even beyond his work on mathematics. Thus, for instance, LW has nothing against axiomatic systems qua technique per se (in other words: he is not an anti-formalist per se), his objections only target what is being said about their foundational function.³⁰²

³⁰⁰ Cf. Schroeder, title of §3.6: "Even if we assume (for argument's sake) that all arithmetic could be reproduced in Russell's logical calculus, that would not make the latter a foundation of arithmetic".

³⁰¹ LW's anti-foundationalism ties in with his pragmatism, structuralism and holism: if practice is ontologically irreducible, and if no dimension within the internal structure of practice has primacy over the other dimensions, it does not make sense to look for foundations.

³⁰² Of course: if one rejects the foundational status of these formal systems, they may lose a lot of their appeal (cf. paragraph (C) below on 'sensationalism' vs. triviality).

(B) anti-monism

Several lines of thought in LW's work ultimately boil down to anti-monism.

heterogeneity

LW points out that as a matter of historical and anthropological fact, math is inherently heterogeneous. He does not believe that math needs to be a unified formal system to make sense; on the contrary, the various techniques one calls 'applications' do perfectly fine in isolation and -in principle, but also in actual fact- they can operate even if they are incompatible with other applications or techniques.

LW's preoccupation with demarcation also emphasizes heterogeneity and multiplicity, both implicitly through the abundance of examples that shows that the cut-off between what still counts as math and what doesn't is inherently unstable and more or less arbitrary, and explicitly, by pointing out that no single criterion or set of criteria really does the trick, a situation he conceptualizes in terms of the concept of a 'family' of techniques.

LW's view of the continuum (?) of the reals is a case in point: LW denies that the mainstream set-theoretical conceptualization of the continuum as a line (and the line as a set of points) even makes basic sense. For LW, the continuum-approach to the reals does not represent a single system, but is a failed attempt to unify irreducibly heterogeneous techniques.

axiomatic systems as post hoc add-ons

LW also simply and explicitly denies that math is a systematic whole (this is again presented as a matter of fact).³⁰³ LW insists on the fact that axiomatic foundational systems are not underlying the heterogeneous mess of actual mathematical techniques, but should be viewed as add-ons, as separate techniques alongside the old ones. According to LW, axiomatic systems also fail to really undo the underlying heterogeneity: even if one codes all the different techniques into a single formal system, they retain their identity (trigonometry, ...).

consistency / contradictions

LW's approach to consistency and contradictions also shows his fundamental anti-monism about math: he argues repeatedly and extensively that the presence of contradictions need not be a problem: whether the contradiction has an impact or not remains to be seen on a case by case basis. This attitude boils down to a de facto pluralism about math: consistency is an important "foundational" issue if (and only if) you believe that math is and has to be a unique

³⁰³ Cf. Ms-121,76r: Gödel zeigt uns eine Unklarheit im Begriff (der) 'Mathematik', die darin zum Ausdruck kam, daß man die Mathematik für ein System gehalten hat.

and unitary system; the idea that contradictions can be dealt with locally and need not have consequences for math as a whole implies that math is not viewed as a single system.³⁰⁴

(C) anti-sensationalism, anti-pathos, anti-kitsch

Throughout our analyses, we encountered a positive valuation of the trivial vs. an aversion towards sensationalism, ostentatiousness, pathos and vertiginousness, which is clearly related to the emphasis on style in LW's personal ethics, which we focused on repeatedly throughout section 2.0.

Thus, we pointed out several cases in which LW blames mathematicians and/or philosophers of mathematics for suggesting depth in their discourse that is actually not there, for presenting what are unforeseen complications with the rules of a game we invented, as awe-inspiringly deep facts of nature, and for going for cheap thrills by invoking mystery by the use of (inappropriate) vertiginous imagery, etc. LW often portrays these aspects of mathematical discourse as cheap, childish and ridiculous.

The harsh, indignant terms that LW uses to express his irk with these phenomena can be understood if we take into account the ethical-aesthetical values that are at the bottom of LW's philosophical drive. LW blames the founding fathers of set-theory not merely for being childish and tasteless, but also for their willing participation in the decline of western culture.

(D) anti-exceptionalism

The later LW (on whom we focus here) argues extensively against the special (crystalline, pure, unique, unitary, interesting, mysterious, ...) status of math and logic which was prevalent in the logistic framework within which he worked when he started out. The alternative that the later LW offers is to view math as -in his own words- an "anthropological phenomenon", which implies that mathematical practices are like any other practices, intertwined with non-mathematical practice in real life; that mathematical words are like any other words, mathematical symbols are like any other symbols, mathematical agents are like any other agents.³⁰⁵

LW also points out the reason why the outlook of 'most mathematicians' is so different ('unphilosophical'): their acceptance of prejudices, superstitious presuppositions about their own trade, enforced by their training.

³⁰⁴ Unless, of course, one is ready to let go of consistency as such. I don't seem to recall any such argument on the part of LW.

³⁰⁵ I am tempted to add: "And mathematical objects are like any other kind of objects" (cf. section 1.1.2(C)), but this is still supposed to be a conclusion to Part 2 of this study and no such claim was made in the passages from LW's work analyzed above.

Part 3. General conclusions

It bears repeating that we have to make a clear distinction between the following two questions: (1) what is LW actually saying? and (2) is he right / do I agree? The polarizing nature of LW's legacy, especially in PhilMath, has for a consequence that Wittgenstein-scholarship is not very good at distinguishing between (1) and (2). Those who have a vested interest in LW's status as a great philosopher will probably not be inclined to spend time studying aspects of LW's work that they are not ready to defend; those who dislike LW a priori will be quick to dismiss whatever aspects that are easy to dismiss. I tried/try to point out the internal coherence of LW's lines of thought for what they are, not necessarily even bothering with the second question, trying to be charitable, but certainly not assuming that LW necessarily has to say things that I can agree with, let alone things that are readily acceptable to a 21st century readership versed in present-day PhilMath.

In the present 'General conclusions', as opposed to the conclusions to Part 1 and Part 2 in which I mostly remained focused on LW's actual text for its own sake, I attempt to articulate how (my reading of) LW's work may contribute to a few somewhat broader topics:

- in section 3.1, I discuss the potential contribution of my work to various issues in Wittgenstein exegesis;
- in section 3.2, I articulate a few reflections on the fact that LW should be viewed as a precursor of present-day PhilMathPract and on the ways in which his work can still contribute to that field, as well as to PhilMath at large, especially by the way he conceived of the role of PhilMath and its relation to mathematics itself and the history of mathematics, as well as of its place in society at large;
- in section 3.3, I briefly discuss the very general question as to whether and to what extent Wittgenstein's philosophy of mathematics is inherently critical, suggesting that LW's philosophy, including and perhaps even especially his PhilMath, is a critique in the Kantian sense of the word, i.e. an approach that focuses on the question as to what is 'given', 'before' anything meaningful can even occur;
- in section 3.4, I give a very brief summary of the main points I made in this study.³⁰⁶

³⁰⁶ In this draft version, the distinction between the present Part 3 and the Appendices was never sharp, and many lines of thought presented here could have ended up in the Appendices, and vice versa.

3.1 Issues in Wittgenstein exegesis

My interaction with the literature has been somewhat minimal throughout my analyses of LW's text, partly because of the unfinished nature of this draft, but also because of the specific focus of my analyses, which is quite different from most authors interested in PhilMath. In this section, I explore a few of the ways in which my contribution interacts with Wittgenstein-scholarship at large.

3.1.1 Remarks on exegesis: interpretation vs. use: "was LW an X-ist"?

(A) introductory remarks

When philosophers read philosophical texts they may want to do either or both of the following:

- (1) try and understand what the author's text means on its own terms;
- (2) see what we can do with the text, how we can learn from it, how we can use it for our own purposes.

I believe there is nothing wrong with using an author's work for one's own purposes: an author can only hope that his work will be used and not just 'interpreted'.³⁰⁷ However, I don't think that it is -at least in principle- controversial to say that in order to be able to do (2), it is a good idea (but not necessarily necessary) to first do (1). In any case, understanding a text does not merely imply understanding the words that are said (the semantics of the text), but also why they are said, for what purpose, serving what agenda (if any), in what polemical context (if any), etc. (in other words: one also needs to understand the pragmatics of the text). One of the more insidious ways in which one can distort the intentions behind an author's text is to make it speak out on issues it does not actually deal with. It appears that this happens a lot in LW exegesis, and particularly the exegesis of LW's remarks on mathematics.

(B) the present study and existing scholarship on LW's PhilMath

As already stated above, for various reasons my interaction with the literature has been somewhat problematic throughout this study, and not only due to the unfinished state of this manuscript.

Let us first briefly look at **Victor Rodych's** entry 'Wittgenstein's Philosophy of Mathematics' in *The Stanford Encyclopedia of Philosophy* (Rodych 2018). A number of differences between Rodych's account and mine here are immediately noticeable:

³⁰⁷ I am referring to the distinction between 'use' and 'interpretation' in (Biletzki 2003)).

- (1) Rodych's account is **doxographic** (which is of course inherent to the genre of the encyclopedia entry), it is mainly a list of opinions on various topics (a list of 'isms'); therefore, it ignores completely LW's claim that that is not what he does; it also ignores the inherently polyphonic, exploratory style of LW's way of doing philosophy; for instance, concerning Gödel-like topics, Rodych simply assumes that LW is commenting on Gödel's famous proof, thereby completely ignoring LW's own claims to the contrary; the same goes for Rodych's account of LW's finitism, formalism, etc. I don't think it is actually true that LW is an anti-finitist, in that he doesn't object to Cantor's Kalkül: he only objects to what is being said about that Kalkül;
- (2) Rodych's account is **chronological**, an aspect that is almost completely lacking in my account (mostly because this is outside the scope of my study): I focus mainly on LW's later work, and my account emphasizes the continuity between the later work and earlier work; I believe Rodych's account -by virtue of its doxographical design- inevitably underestimates the continuity in LW's work: what I see as various ways in which LW explores certain topics over time, will inevitably appear as totally different successive or competing opinions in a doxographical account;
- (3) Rodych's account does not really show how **the specifically mathematical aspects** of LW's work are a part of **his philosophy at large**, whereas my account emphasizes that the main themes in LW's PhilMath are the same themes he also addresses elsewhere.

Next, let's look very briefly at **Juliet Floyd's** recent, short but dense, monograph *Wittgenstein's Philosophy of Mathematics* (Floyd 2021), which gives an account in which the evolution in LW's PhilMath is studied from the perspective of his views on aspect-seeing, a topic that is not limited to PhilMath, but reoccurs in different contexts within his work. From the even more general perspective developed here, aspect-seeing is one of the many avatars of LW's continuing focus on aspects of meaning that are not easily reducible to propositional truth, which is in its turn part of his Kantian preoccupation of what constitutes the 'given' for mathematics. This focus on the non-propositional aspects of the 'given' inherently ties in with his critique of the autonomy of math and his inherently anti-foundational stance in the Grundlagen-debates, which in turn is an expression of an underlying agenda with respect to the problem of meaning, in its turn a product of a culture-critical concern with authenticity. This does not mean that my reading of individual passages has to differ that much from Floyd's. Thus, for instance, I would agree with most of what Floyd says in the following passage about LW's reading of Gödel ((Floyd 2021) 2021, pp. 71-72):

Often Wittgenstein is regarded as quarreling with Gödel, but that is because he is too often read as a radical finitist or conventionalist. His remarks certainly struggle to place Gödelian incompleteness into his way of thinking. But if we take seriously his Later views, we see what the struggle is about, and it is not about refuting Gödel. There is a richness to the ways we may articulate arithmetic, as Gödel proved, and a richness to the idea of “follows from a set of axioms in a formal language,” as Church and Turing showed. Conventionalists, formalists, and logicians should admit and learn from this. For the mature Wittgenstein, the multidimensional play comes out in our articulations of mathematics in everyday phraseology and its embedding in life, in the techniques we establish and share. This is not an alternative to formalizing theories where we can, but it is not wholly reducible to that activity.

Floyd arrived to this conclusion from extensive research of LW’s PhilMath, working her way out to the wider context, whereas I worked my way into the PhilMath from the outside in. It should be reassuring that both trajectories ended up not too far from each other, at least on this point.

Another case in point would be Chapter 5 “The two strands in Wittgenstein’s later philosophy of mathematics” of Severin **Schroeder’s** 2021 monograph *Wittgenstein on Mathematics* (Schroeder 2021), in which Schroeder speaks of “two strands” in LW’s view of math (“math as calculus” vs. “math as grammar”), between which he sees that there is some tension (as well as a chronological aspect: the intermediate period mostly representing the “math as calculus” doctrine, and the transition towards “the math as grammar” doctrine occurring between 1937 and 1939). I have very little to say on this, except that if those are strands, then there are many more strands in LW’s thinking on math, with a lot of tensions between them, and would suggest -again- that both occur in the context of a research activity with a lot more continuity in its biases and agendas than Schroeder’s account suggests. On the specific issue of Schroeder’s two strands, I would like to insist that LW’s ‘math as Kalkül’ idea should not be read in a way that ignores the general backdrop of LW’s developing ‘anthropological’ approach, his developing everydayism, his developing pragmatism, etc.: for LW, Kalkül has never been an abstract, formal process, but has always been something that actual agents do, and need not exclude the mature LW’s full blown pragmatism; similarly, LW’s “math as grammar” can be read as one of the ways in which LW’s tried to articulate the ultimately non-propositional nature of whatever grounds the meaningfulness of math (see section (C3) below on LW’s ‘normativism’).

These very summary comparisons, as well as the remarks that follow below, illustrate the fact that the present study is not very topical within the context of the current scholarship on LW’s PhilMath: there is very little overlap between my approach and the literature and even in

those cases in which the same passages are discussed, there is not much for me to comment on, except for a few quite generic observations.

(C) ismism: 8 isms

If we look at those contributions in the literature that aim to give an overview of LW's PhilMath as a whole (e.g. Frascolla (Frascolla 1994), Rodych (Rodych 2018), Schroeder (Schroeder 2021), Floyd (Floyd 2021)), as well as the literature we've already quoted in the above, we encounter **an impressive array of -isms**: logicism and anti-logicism, formalism and anti-formalism, revisionism and anti-revisionism; inferentialism, verificationism and empiricism; intensionalism and extensionalism; (anti-)Platonism, Cartesianism and Kantianism; finitism; (social-)constructivism, constructionism and conventionalism; realism, anti-realism and anti-anti-realism;

It may be tempting to mock or otherwise criticize this **ismsism**³⁰⁸ and the decontextualization and simplification it entails, but the device is not only convenient but also requires for the commentators to commit to their interpretations, which is a good thing. So, my point is not to launch a frontal attack on the 'ismist' / doxographical approach as such, but to articulate a few concrete remarks on the ways in which the attribution of isms to LW impacts our understanding of the body of work under scrutiny and how the readings I present in the above may contribute to reframing this understanding.

(C1) logicism - anti-logicism

Logicism is one of the isms that is obviously relevant to understanding LW's PhilMath (cf. Schroeder, chapters 2 and 3: "Logicism" resp. "Wittgenstein's critique of logicism"). It is uncontroversial that there is a chronological aspect here: LW changed his mind on the viability of the logic-based framework within which he had worked up until the *TLP* and said so explicitly and unambiguously in his later work, most notably in the preface to his *PhU* (published in 1953, but largely written before the end of 1937, cf. e.g. Ms-117,110 ff. d.d. 19370627).

Still, my contribution reminds us of the fact that there is a lot of continuity in LW's approach, as well. The **language-critical and culture-critical** aspect was there from the outset: even the *TLP* is primarily a critique of senseless verbiage, as is LW's later work. As we have repeatedly emphasized in the above, this critique of certain types of language use was deeply rooted in LW's modes of thought, and in those of many of his contemporaries. As far as direct influences

³⁰⁸ By the ad hoc term 'ismism', I simply mean "the practice of using the suffix -ism", and explicitly not anything else.

are concerned (cf. section 2.0.0 above), the cultural criticism of journalist Karl Kraus was an early and determinant influence on LW; and even if LW read Spengler later in life, he said that Spengler articulated many thoughts that he had entertained on his own. In this sense, both LW's early logicism and his later anti-logicism are avatars of the same underlying agenda.³⁰⁹ Even if admirers of his early 'logistical' work (Russell, Carnap, ...) didn't see it this way and lost interest in LW's later work,³¹⁰ which seemed alien to them, and even if some later scholars miss this point as well, the later LW is -in a certain sense- still a logician, still concerned with the same foundational (?) questions, still struggling with the problem of meaning vs. nonsense.

(C2) formalism - anti-formalism

On the one hand, LW has been read as a formalist, esp. his 'middle/intermediate period' (for instance, Rodych (Rodych 2018), §2.1 "Wittgenstein's Intermediate Constructive Formalism"; (Ferreirós 2016) pp. 89-90). Presumably, these readers take LW's insistence on 'Kalkül' -which in LW's work by default refers to actual calculations done by actual people, as opposed to, for instance, idle philosophical 'prose'- for some kind of anti-conceptual and therefore 'formalist' stance. However, even if in his 'intermediate period', LW hadn't yet developed as rich a conceptual apparatus as he did later on, I do believe his notion of mathematical technique as a rule-based activity referred to what practitioners of mathematical techniques actually do, not to abstract formal systems in the formalist sense of formalism [sic! fs]. One may want to call that formalism (and in that case LW was a formalist about small-talk, or any other natural-language practice, as well), but I don't think that is what formalism usually means in PhilMath.

On the other hand, LW's insistence on the embedding of Kalkül/mathematical technique in actual, real-life practice, and especially the idea that the meaningfulness of Kalkül ultimately

³⁰⁹ Cf. Sass's -at least formally- similar claim that LW's early logicism and his later -what I called- everydayism are avatars of the same underlying psychologically/existentially uneasy relation to the everyday (Sass 2001 (Sass 2001), p. 122; see also section 2.0.1(C) above). I am not very comfortable with this parallelism between my work and Sass's, but it's there.

³¹⁰ Cf. Russell's scathing remarks in *My philosophical development*:

I have not found in Wittgenstein's *Philosophical Investigations* anything that seemed to me interesting and I do not understand why a whole school finds important wisdom in its pages. Psychologically this is surprising. The earlier Wittgenstein, whom I knew intimately, was a man addicted to passionately intense thinking, profoundly aware of difficult problems of which I, like him, felt the importance, and possessed (or at least so I thought) of true philosophical genius. The later Wittgenstein, on the contrary, seems to have grown tired of serious thinking and to have invented a doctrine which would make such an activity unnecessary. I do not for one moment believe that the doctrine which has these lazy consequences is true. I realize, however, that I have an overpoweringly strong bias against it, for, if it is true, philosophy is, at best, a slight help to lexicographers, and at worst, an idle tea-table amusement. ((Russell 1959), pp. 216-217).

depends on its actually being applied in actual everyday applications, is incompatible with formalism as it is usually understood in the PhilMath of the heyday of the Grundlagen-debates. I believe I have shown that this central aspect of LW's work was as much opposed to formalism (in the sense of the belief that math's validity can be (or should be) guaranteed by the fact that it can be presented as a formal system) as Kurt Gödel was, though for completely different reasons.

Of course, there may be a chronological aspect to this matter (cf. Rodych (Rodych 2018)): LW's earlier work is easier to interpret as somehow formalist than LW's later work, in that LW's later work emphasizes the real-world, real-life, everyday character of the relevant encompassing structures that give meaning to math, more clearly.

The general impression that emerges from the readings presented above is that LW does not argue against the use of formal systems at all: the invention of new mathematical techniques is not a problem per se; what he does do is (1) problematize the semantic relation between the formal system and whatever the system is supposed to formalize or encode, and (2) emphasize the pragmatics of math, i.e. the actual techniques as performed by actual practitioners and their embedding in real-life applications.

Again, the scope of the research that I present here -both the relative width of the overall topic of the encompassing research project ("practice and related concepts") and the relative narrowness of the specific focus of this study on a small number of lines of thought within LW's oeuvre- does not allow me to comment on the details of the more technical or more chronological aspects of these accounts. My only contribution is to point out that *any interpretation of the chronology of LW's evolving views on this topic should take into account the links between the topic at hand and LW's culture-critical views on the fragmentation of modern society, everydayness, authenticity, etc.* (these ideas are perhaps also evolving, but not that much...): for instance, it would not make sense to attribute a fully formalist stance (in the sense of the belief that math's validity is ultimately grounded in its being a formal system) to LW, if it can be shown that he held strong 'everydayist' beliefs at that time.

(C3) normativism

Some commentators have emphasized LW's 'normativism' about math (Schroeder (Schroeder 2021), Chapter 6 "Mathematics as Grammar"), referring to the prolonged period in which LW experimented with the idea that mathematical statements were not propositions (i.e. statements referring to facts about the world) but '(grammatical) rules' determining how the terms involved were to be used.

I am -of course- not arguing against the undeniable fact that there has been a period in LW's use of the term 'grammar', as applied to math, was very prominent and that LW more or less lost interest in this term in favor of other lines of thought. But I think it is important to understand that this 'grammar' theme fits in with LW's continuing concern with the embedding of math in encompassing structures: as soon as LW understood that more could be said meaningfully than he had assumed in the *TLP* and that meaning is ultimately not a matter of propositions expressing facts, this boosted his interest in the non-propositional aspects of meaning, including the idea that language use is always part of encompassing practices (i.e. what I called LW's pragmatism and his holism and structuralism about practices). LW's (idiosyncratic) use of the concept of 'grammar' was one of the ways in which he tried and conceptualized the idea that words are meaningful by virtue of them being part of larger behavioral patterns rather than in terms of a correspondence with an outside world (cf. section 1.1.1(A)).

Similarly, LW's use of the concept of 'grammar' should not be separated from his evolving views on rule-following in general: again, the main point appears to be that rule-following can't be understood if one takes the rules as the ultimate ground for this kind of behavioral pattern; instead, the mature LW presents a holistic account of rule-following as part of an irreducibly complex practice.

In other words: LW's normativism about math should be understood as one of the concepts he experimented with within the context of his continuing concern with the idea that meaning cannot be reduced to propositional truth (i.e. reference to the world) and his gradually more coherent articulation of a holistic account. Again, it would not make sense to discuss LW's 'math as grammar' idea in isolation from his ideas on meaning as grammar and on rule-following in general. And the topic rule-following in its turn should be read in connection with LW's developing holism.

(C4) constructivism / social-constructivism / conventionalism³¹¹

LW often points out the cultural contingency of mathematical practices, sometimes invoking the practices of imaginary tribes or alternative decisions within an otherwise 'normal' math (cf. e.g. section 1.3 above, but also several examples throughout section 2.3). This has often been interpreted as "constructivism" and in a certain sense perhaps rightly so. But LW does not argue that practitioners could freely or arbitrarily choose to adopt any number of

³¹¹ I want to point out that the 'social' part of 'social-constructivism' is not necessarily self-evident either: whereas LW does often refer to communities, tribes, cultures etc., I don't think one can interpret these as the ultimate locus for meaning, as is often done: cf. what I had to say about LW's holism in section (cf. section 1.1.2(A1) above and 3.2.1(B) below).

mathematical alternatives anytime: he is very much aware of the phenomenon of the logical ‘must’ and appears to suggest over and over again that a lot is given at any time in our history, even if this ‘given’ is the result of an accumulation of contingencies. Cf. (Steiner 2009); (Bangu 2006); (Bangu 2019)). Furthermore, social and cultural aspects make up only one dimension amongst many more within LW’s holistic conception of practices (Language Games, Forms of Life): cognitive, biological, physical aspects are equally primordial, all of which constitute very strong constraints on any arbitrariness or freedom one may want to attribute to practitioners.³¹²

So: as an interpretation of -at least the later- works of LW, ‘conventionalism’ would be incompatible with LW’s holism, but more importantly, it doesn’t account for LW’s focus on the fact that at any given time and place, an accumulation of contingencies does constitute an ultimate given (cf. Bangu’s term “contingently necessary” (Bangu 2019), p. 19).

(C5) finitism

In the above we have referred several times to LW’s qualms against various uses of the concept of ‘infinity’, which -I guess- ipso facto makes him side with the cause of finitism.

However, the readings presented above suggest that LW’s finitist arguments are not finitism for the sake of finitism: the problems with certain uses of the concept of infinity that LW points out, fit in with a much broader program. LW doesn’t say that one should not calculate with infinite numbers for reasons X, Y or Z.

LW does object to certain cases in which ‘infinity’ is used within discourse about math in such a way that it doesn’t correspond to anything in the actual math. A case in point is the frivolous use of the word ‘infinite’ in the context of ‘infinite minds calculating with infinite numerals’ (cf. section 2.2(B) above). The present study emphasizes the not math-internal aspect of this criticism: the use of the term ‘infinite’ is criticized for its lack of embedding in practice, in exactly the same way one could criticize various ‘metaphysical’ ways of using common terms.

³¹² In that context, I believe it’s worth pointing out that ‘conventionalism’ (insofar as it literally implies that a group of people have made a decision together) is not a viable account for mathematical meaning, let alone meaning in general. All the arguments against the idea that the ‘community’ would be the proper locus for the foundation of meaning (cf. section 1.1.2(A1) above and section 3.2.1(B) below), apply a fortiori to the idea of a convention: a group of people literally coming together (“convening”) to decide over a certain matter is a very specific way of coming to an agreement and it is simply not the case that the kind of wide-reaching agreement that underlies our ability to participate in collective practices is the result of any decision-making process of the kind that we could call a ‘convention’ without the term losing all of its specific meaning. Note that similar arguments can be / should be (and have been) brought in against ‘social contract’ theories in political philosophy and related fields.

Since a few decades, Philosophy of Science and related disciplines have made a “naturalistic turn”, in that it is understood that the claims of these branches of philosophy (insofar as they have made this ‘turn’) should “at least be aligned with, if not explicitly grounded in, scientific practice” (Nersessian and MacLeod 2022).

Even if some lines of thought in LW’s work can be recuperated for the case of naturalism against Platonism, it could be argued that LW was as much an anti-anti-Platonist as an anti-Platonist: LW clearly acknowledged the intuitions that are expressed by Platonism;³¹³ LW would also agree -in some sense and to a certain extent- with, for instance, Gödel’s Platonist arguments against formalism: for LW, axiomatic systems clearly do not exhaust math either, though of course he strongly disagrees with the validity of the consequences drawn by Platonists.

Maddy’s “second philosophy” does recuperate a number of aspects of LW’s work, especially the apparently anti-Platonist strands, for the naturalist cause. However, our reading of LW suggests that her approach to philosophy as a prolongation of science, is not compatible with LW’s overtly non-scientific view of philosophy:

- (1) LW’s own vision of his task as a philosopher (therapy, critique, ...) are diametrically opposed to the scientism that underlies Maddy’s naturalist stance (cf. also section 0.2(B) above, as well as section 3.2.2 below for the way LW differentiates philosophy from other approaches);
- (2) LW’s own holism is directly opposed to the reductionism inherent in most types of scientism.

Of course, this need not be a problem per se: “use” of a certain text for one’s own purposes is OK, and “inspiration” works in mysterious ways. But this approach can’t be sold as Wittgenstein-scholarship, i.e. it would be disingenuous to project these concerns onto LW’s text. The issues are illustrated quite well in the confrontation between Lynch and Bloor: Lynch argues quite convincingly that Bloor’s recuperation of LW for his own scientific sociological approach deviates in important ways from LW’s own outlook on the aims and methods of his philosophy (see (Lynch 1992), (Lynch 1997), (Kusch 2004b), (Kusch 2004a), (Bloor 2004), (Kusch 2006)).

³¹³ I would personally argue -from a point of view that I would qualify as generally Wittgensteinian- that there is nothing wrong with talk about mathematical objects from a pragmatic point of view (if something functions as an object within a practice, it is an object within that practice); in that regard, mathematics is -again- like any other practice: there is no problem with the notion of mathematical objects, provided one thoroughly relativizes the notion of object, the way one would any other type of object; note that this does not preclude skepticism about the claims to the ‘objectivity’ of mathematics (cf. sections 1.1.2(C) above and 3.2;1(B) below).

The materials collected and processed in the present study make us land squarely on the non-naturalistic, anti-scientistic side of Wittgenstein-scholarship, and both Bloor's and Maddy's positions (however attractive they may be in the context of some of the debates they were devised to contribute to) do not appear to be viable as scholarly interpretations of LW's work. Especially the ways in which LW explicitly opposes his own work on mathematics to other approaches precludes any ambiguity in this regard and LW's Kantian focus on 'the given' (cf. section 3.3 below) suggests a deep continuity with 'philosophia prima'.³¹⁴

(C7) skepticism vs. anti-skepticism

In the aftermath of Kripke's seminal work ((Kripke 1982)) on skepticism about rule-following, including the private language argument, LW's relation to skepticism has been at the forefront of Wittgenstein-scholarship: some authors claim that LW's arguments target skepticism; other authors claim that LW's argument are ultimately skeptical themselves (for an overview, see Kusch (Kusch 2006), already referred to above in connection with scientism and the debates between Bloor and Lynch).³¹⁵

I am ready to believe that Kripke's topic was an important one and that skepticism -in the general sense of asking the question as to whether the meaningfulness of our discourse is ultimately guaranteed or not- is a valid way to articulate a central preoccupation in LW's work, from the *TLP* (in which LW concedes the relative viability of solipsism, a related position) to *ÜG* (cf. the riverbed analogy in *ÜG* §§94-99, quoted in section 3.3(C) below), the last paragraphs of which were written in the very last weeks of LW's life: the problem of 'meaningfulness vs. nonsense' that runs throughout the present study coincides more or less completely with Kripke's 'skepticism' issue. In the context of LW's own development, the topic probably even originated in his involvement with the Grundlagen-issue in *PhilMath*: what is it that guarantees the reliability of math?

Whereas LW started out within a logistic approach to the problem, the later LW's response to this question was that math is based in a messy hurly-burly of everyday practices, which most of the time does not lead to any problems at all. Among the passages studied in the above, one of the more interesting was the one in which LW pointed out that -with or without foundations- 'a good angel' [ein guter Engel] is always needed in order for things to turn out

³¹⁴ Of course, 'naturalism' is a label that can cover a very wide range of positions, some of which are very weak, and some of the above may not apply to some positions that do claim to be naturalistic. It is very hard to make any general arguments without arguing about semantics (in the vulgar sense of the word). For instance, the way the term 'naturalism' is applied to LW's work in McGinn 2021 (McGinn 2021), especially Chapter 7, to my mind has very little to do with 'naturalism' as intended in the present section.

³¹⁵ NB: from the point of view developed here, Kripke's appeal to "community" as a substitute 'foundation' is simply wrong, both as an interpretation of LW's work and as such (cf. section 1.1.2(A1) above).

OK (cf. section 2.3(C)). In other words: nothing actually guarantees that things will work out and there is nothing we can do about it, but this need not be a real problem in actual practice, and it typically isn't.

Thus, the core of LW's anti-foundationalism about math coincides with the issues underlying the debates about skepticism and LW's PhilMath offers us a very clear way of approaching this issue:³¹⁶ the skeptic may be literally speaking 'right' within his own game, but his position is ultimately unhelpful and irrelevant for all practical purposes, in that everyday practice is always the ultimate criterion for meaningfulness, even in the case of math, despite the common claims to a special status of math on the part of practitioners. In other words: skepticism ultimately cannot be upheld in good faith.

(C8) revisionism/criticism vs. anti-revisionism / "leave everything as it is"

Many commentators (among many others: Maddy (Maddy 1993); Dawson (Dawson 2015); Scheppers (Scheppers 2017) , Chapter 3, §1) have noted a paradoxical tension between (1) LW's outspoken anti-revisionism ("leave everything as it is") and (2) the critical aspects of LW's work in general, and the critical remarks that we focused on in Part 2 of this study in particular.

Thus, Maddy says the following in her 1993 article "Wittgenstein's Anti-Philosophy of Mathematics" ((Maddy 1993), p. 55):

Surely, one cannot deny the law of the excluded middle or rule out non-constructive existence proofs and at the same time leave "mathematics as it is". But what is the motivation for this prohibition? If philosophy provides compelling reasons to abandon the Platonist picture, if current mathematical practice is based on that picture, why shouldn't the result of philosophical analysis be allowed to reform that practice? Mightn't Wittgenstein's reluctance be a form of false modesty? [³¹⁷]

This reading of Wittgenstein's late views uncovers a tension between the upshot of his philosophical views and his insistence that philosophy alters nothing.⁽⁵⁾ It tempts us to downplay the non-interference remarks in favor of the presumed payoffs of his contentful philosophical conclusions. A directly opposed approach - my focus in this paper - would give pride of place to the non-interference claims and adjust the reading of the rest to match.

So, as pointed out by Maddy, there are two ways of dealing with this tension:

³¹⁶ As I will point out in section 3.3(C) below, the Grundlagen issue (and the skepticism issues, for that matter) ultimately boils down to a Kantian problem: what is given? what is bedrock?

³¹⁷ I find the idea that anyone could attribute 'modesty' - whether false or genuine- to LW funny.

- deflating the non-revisionist claim and accepting the consequence of the apparent criticism, for instance that according to LW, set theory is not proper math ((Maddy 1993); (Steiner 2009));
- accepting non-revisionism as central to LW's purpose and trying to interpret the apparent criticism in that light (Dawson 2015).

Our interpretation of the function of the everyday in LW's thought actually may offer a kind of solution to the underlying paradox: everydayism is a basic presupposition of LW's work, not a result; in that case, i.e. if we accept the distinction between the everyday and its opposite, there need no longer be a tension: *LW leaves the everyday as it is (because it is ipso facto meaningful) and only criticizes the non-everyday (because it is by definition fake)*. Again: LW's problem ultimately boils down to the disruption of the link with the everyday and the lack of authenticity that results.

LW's remarks on mathematics actually illustrate this general aspect of his outlook quite well: count as 'everyday' those mathematical discourses that are embedded in actual practices using actual mathematical techniques. We have seen in the above that LW systematically criticizes various kinds of contemporary (set-theoretical or otherwise foundational) verbiage, while explicitly not criticizing any actual mathematical technique: LW does not criticize the Kalkül/mathematical technique, only what some people say about it. Cf. also the idea that for LW, all techniques are similar from a philosophical point of view: there is no such thing as 'a leading problem of mathematical logic' (*PhU* §124).³¹⁸

Of course, the very idea of "the everyday" is in its turn a very problematic concept that we may not necessarily want to accept, but it is one that is deeply rooted in LW's outlook on the world and has deep connections to other conceptual clusters (Lebensphilosophie, authenticity, pessimism about European civilization, ...) that LW shared to various extents with such contemporaries as Kraus, Spengler, Weininger, but also Heidegger.

(D) final remarks about LW's alleged -isms

So: focusing on those aspects of LW's work that appear to have some technical interest from the point of view of mainstream PhilMath at large tends to lead the interpreter away from LW's philosophical agenda at large and prevents us from understanding the remarks on this or that aspect as part of its wider context, including the objectives of philosophy according to LW.

³¹⁸ It is interesting to note that in the 1936 manuscript which already resembles *PhU* in many ways, LW mentions Ramsey as the source of the idea he criticizes (Ms-142,108, not dated, but after November 1936: "Ein "führendes Problem der mathematischen Logik" (Ramsey) ist ein Problem der Mathematik, wie jedes andere"). For LW's negative opinion of Ramsey as a philosopher see also section 3.2.2(B) below.

The reader's point of view / approach also impacts the issue of **chronology vs. continuity**: depending on one's methodology, one is bound to emphasize one or the other: chronological accounts will inherently emphasize discontinuity.³¹⁹

What I have attempted to do in this study is to read LW's PhilMath in terms of the **agendas and biases** that underlie his philosophy as a whole.³²⁰ In the above, I identified a number of such biases and agendas and suggested that the apparently technical issues should be read in the light of those biases and agendas.³²¹ The picture that emerges is one in which LW's contribution to PhilMath is critical at a very fundamental level, attacking aspects that are still prevalent in PhilMath today,³²² and viewing math as much more closely intertwined with other aspects of culture and society than is fashionable in present-day PhilMath.³²³

3.1.2 Reading Wittgenstein's philosophy of mathematics in the context of his work at large

Throughout this study, I have read LW's work on mathematics as an integral part of his oeuvre as a whole, and have pointed out to what extent the work on math is also a central part of his oeuvre, displaying all the major themes, including the culture-critical ones, and in many case, earlier and/or more incisively than in other parts of his oeuvre.³²⁴

(A) LW's aims, methods and style

As pointed out in my introduction to the present study (section 0.2 above), the relative lack of success of LW's PhilMath within the mainstream of PhilMath at large, is probably due to a

³¹⁹ Chronological accounts also inherently imply a larger corpus.

³²⁰ This focus on ultimately non-propositional aspects that underlie and give meaning to LW's philosophical discourse, is a very Wittgensteinian move. You are what you eat.

³²¹ Schroeder's "two strands" (cf. section 3.1.1(B)) are a case in point.

³²² However, in my, admittedly anecdotal, experience, these opinions are not necessarily as prevalent with working mathematicians and even less so with engineers or others who use mathematical technique in a more applied fashion, although most of the latter have been 'indoctrinated' with the monist doctrine throughout their studies. There is also an interesting generational aspect: those who learnt most of their math starting from a set-theoretical framework, vs. those who first learnt the techniques separately on their own terms and only later were taught set-theory.

Perhaps someone should run a **sociological survey** investigating the attitudes of engineers, various kinds of working mathematicians and philosophers of math towards mathematical monism.

³²³ I hope the above also helps explaining why my interaction with the literature is rather limited: the above has very little to add to the technical details that are the main focus in the literature. Of course, I could (and maybe will) include many more references to various texts I have read, pointing out that this or that aspect of my text corresponds to this or that aspect of the other text (for instance, Chapters 11 and 12 of Schroeder's recent monograph could be referred to a lot in section 2.3 above), but I don't think that would be a good use of my time at this point.

³²⁴ It may appear somewhat ironic that after arguing against the projection of -isms onto LW's text in the previous section, I will introduce a few -isms of my own, but -as stated previously- the device is a convenient one and my point has never been that I am against -isms in general. Furthermore, within the context of this study I would like to commit to my interpretations by labeling them.

lack of common ground as to the aims of philosophy, its methods and the style in one can and cannot write philosophical texts.

style

One of the difficulties that any commentator on LW's writings has to deal with, is the fact that the texts seem to jump from one topic to the next in an apparently haphazard way, only to come back to the same topic later on, over and over again.³²⁵ In the case of the manuscripts on mathematics-related topics (with a few exceptions), it seems to be impossible to give a systematic account of any of the topics at hand without jumping back and forth within and across several manuscripts.

However, from the point of view developed here, this apparent lack of structure becomes less of a problem: for LW, the 'invention vs. discovery' theme, the 'surveyability of proof' theme, the 'infinite sets' theme, the 'diagonal proof' theme, the 'truth and provability' theme, etc. are not separate topics that are studied for their own sake; on the contrary, the reason why these themes recur throughout LW's work is that they all illustrate the major concerns that underlie his philosophical work in general. As discussed above, it is important to take the texts for what they are, i.e. a rather direct reflection of the actual process of a philosopher at work, and not try to make the texts answer questions that are alien to it.

aims & methods: criticism & non-scientism

It is a well-known fact that LW formulated the aims of (his) philosophy in terms of it being a **therapy** (etc.), as opposed to the accumulation of propositional knowledge. Despite the fact that this is a well-known aspect of LW's work, it is rarely taken seriously in the literature on LW's PhilMath, which remains mostly doxographic: despite the fact that LW repeatedly points out that he does not intend to articulate propositional truths about his subject matter,

³²⁵ LW gives an adequate description of this in the *Vorwort* of his *PhU*:

Nach manchen mißglückten Versuchen, meine Ergebnisse zu einem solchen Ganzen zusammenzuschweißen, sah ich ein, daß mir dies nie gelingen würde. Daß das Beste, was ich schreiben konnte, immer nur philosophische Bemerkungen bleiben würden; daß meine Gedanken bald erlahmten, wenn ich versuchte, sie, gegen ihre natürliche Neigung, in einer Richtung weiterzuzwingen. -- Und dies hing freilich mit der Natur der Untersuchung selbst zusammen. Sie nämlich zwingt uns, ein weites Gedankengebiet, kreuz und quer, nach allen Richtungen hin zu durchreisen. -- Die philosophischen Bemerkungen dieses Buches sind gleichsam eine Menge von Landschaftsskizzen, die auf diesen langen und verwickelten Fahrten entstanden sind.

Die gleichen Punkte, oder beinahe die gleichen, wurden stets von neuem von verschiedenen Richtungen her berührt und immer neue Bilder entworfen. Eine Unzahl dieser war verzeichnet, oder uncharakteristisch, mit allen Mängeln eines schwachen Zeichners behaftet. Und wenn man diese ausschied, blieb eine Anzahl halbwegser übrig, die nun so angeordnet, cut down, in order to give the viewer an idea of the landscape. So this book is really just an album.

the scholarship on LW's PhilMath often still takes the shape of an inventory of such opinions.³²⁶

We have also seen in the above that LW repeatedly opposes his own 'anthropological' approach 'looking in from the outside' to the mathematical approach to mathematical issues, which is important in the following ways:

- LW's approach is inherently **comparative** in that it studies mathematical practice alongside other practices and in contexts in which mathematical and non-mathematical practices are intertwined;
- by virtue of being comparative, LW 's approach also opposes mathematical **exceptionalism**, i.e. the idea that mathematics is unlike any other human endeavor (see section 3.2.2 below), almost universally presupposed in PhilMath.

I believe I have shown that these aspects should be taken seriously, as well: the critical stance towards discourse on mathematics is a fundamental aspect of LW's work and fits in with his approach to philosophy in general. In what follows, I will make an attempt at showing that present-day PhilMath, and especially PhilMathPract, would still learn something from LW's work in this regard.

(B) LW's conceptual apparatus: meaning as embedding in everyday practice

Part 1 of this study consisted mostly of an overview of the conceptual apparatus that the later LW developed and the way this translates into his PhilMath. So as to not repeat basically the same summary over and over again, I refer to section 1.4 above or section 3.4 below for a

³²⁶ Within the passages I focused on for the purposes this study, I encountered several occasions on which LW reminded himself to stay away from dogmatism, for instance in Ms-122, 68r-88r, discussed in section 1.1.3 above: "Immer bin ich hier zum Dogmatismus geneigt!". From the same manuscript: Ms-122,27r: "(In dieser ganzen Untersuchung fühle ich mich nicht wohl: mir scheint, ich bin dogmatisch.); Ms-122,83v: "(Ich habe das bestimmte Gefühl, daß ich sehr unvorsichtig bin. Also irgendwie im seichten Wasser des Dogmatismus herumschwimme.)". Similarly: Ms-117,192: "(Sei aber hier nicht dogmatisch. Es gibt Übergänge, die die Betrachtung erschweren.)"; Ms-130,53: " "Es muß sich doch so verhalten" ist kein Satz der Philosophie. Dogmatismus."; Ms-163,55r-55v: "Wenn der Diagonalbeweis etwas tut, so ist es, daß er unsern Begriff vom System ändert. || so ändert er unsern Begriff vom System. Hier muß man aber unterscheiden zwischen dem Begriff in der Math. & außerhalb der Math. Nur von diesem müssen wir sagen er habe sich geändert. [Furchtbar unklar!] Hier darf man nicht dogmatisch sein wollen: Von manchem neuen Beweis wird man zu sagen geneigt sein, er ändere unsern Begriff, von manchem – sozusagen trivialen – nicht. Aber für uns ist gerade der Übergang zwischen der Geneigtheit, das eine, & der, das andere zu sagen, das Wichtige || wichtig." See also Ms-142,111-112 (also interesting because Spengler is mentioned):

Nur so nämlich können wir der Ungerechtigkeit – oder Leere unserer Behauptungen entgehen, indem wir das Vorbild als das, was es ist, als Vergleichsobjekt – sozusagen als Maßstab – hinstellen; & nicht als das Vorurteil, dem die Wirklichkeit entsprechen müsse. (Ich denke an die Betrachtungsweise Spenglers.) Hierin nämlich liegt derjenige || ein gewisser Dogmatismus, in den unsre Philosophie so leicht verfallen kann.

Es ist wahr: eine Maßeinheit ist gut gewählt, wenn sie viele der Längen, die wir mit ihr messen wollen, in ganzen Zahlen ausdrückt. Aber der Dogmatismus behauptet, jede Länge müsse ein ganzes Vielfaches der || unserer Maßeinheit sein.

relatively brief overview. Suffice it here to merely remind the reader of the key concepts in my account:

- LW's **pragmatism**, i.e. the fact that the locus of meaning is real-life practices, which implies that the relation between a symbolic system and "the world" is always mediated by a practice; in the case of math, this implies a focus on (1) mathematical technique [Kalkül] and (2) applications from which such techniques emerge and in which such techniques occur;
- LW's **holism** about practice: a practice is a real-life multidimensional structure, involving not only the cognition and action of the agent, but also the cultural, biological and physical aspects of the context/world in which it takes place; this implies that propositional truth is ultimately rooted in non-propositional practices, which in its turn implies that LW does not share the idea of the ultimateness of the axioms, not even the ultimateness of propositional truth;³²⁷
- LW's **structuralism**: for LW, the relations between the epistemic, the linguistic, the cognitive, the physical, the cultural, etc. dimensions of practice are internal to the structure of that practice, which means that it does not make sense to try and isolate -for instance- the epistemic aspect as meaningful in its own right, let alone as underlying the other aspects of practice;³²⁸
- LW's **everydayism**: LW's insistence on the everyday (as opposed to whatever he finds undesirable) is a bottom-line aspect of his philosophical and existential outlook (as opposed to a result of his analyses) and is deeply intertwined with the culture-critical (Spenglerian) strands in his worldview.

(C) philosophy as criticism: ethics & aesthetics: authenticity

Throughout Part 2, I have illustrated the claim that LW's philosophy is essentially and pervasively critical and that 'authenticity' (as opposed to fakeness) is the core value that is at issue in this aspect of his work. To a certain extent I have been able to sketch the relation between the concept of everydayness and the concept of authenticity, and between LW's

³²⁷ Interestingly, LW shares with Kurt Gödel the idea that math cannot be reduced to an axiomatic system, but both authors then completely diverge on where this observation leads us: whereas Gödel interprets evidence for the limits of formalism as evidence for the existence of a Platonic mathematical universe 'out there', LW interprets the very same results in a deflationary way as evidence for the primacy of everyday practice as the irreducible locus of the meaningfulness of math.

³²⁸ At this point, it may be interesting to point out that there is a close conceptual **link between LW's structuralism and his anti-foundationalism**: if the epistemic aspect of a practice is inherently only one of several irreducible aspects of a practice, this obviously precludes any attempt to make any propositional content foundational.

everydayism, LW's pragmatism and LW's 'meta-philosophical' ideas (therapy, etc.) as these apply to LW's PhilMath.

Despite the relatively clear and consistent account of these aspects achieved in the above, a number of **loose ends** remain: some work remains to be done in order to clearly articulate the links between (1) the above cluster of concepts, (2) LW's Spenglerian outlook,³²⁹ (3) the fragmentation of the everyday into a hurly-burly of practices, and (4) LW's opinions on authenticity/fakeness and everydayness as compared to Martin Heidegger's work involving the very same concepts.³³⁰

So as to avoid too many redundancies within the present Part 3 of this draft, I refer to section 3.3 for more substantial remarks on the critical nature of LW's philosophy in general and his PhilMath in particular.

3.1.3 Reading Wittgenstein's philosophy of mathematics in its biographical and historical context (incl. the Grundlagen-debates in contemporary philosophy of mathematics)

(A) cultural & biographical background

In section 2.0 we explored a few aspects of the culture-historical and the biographical context from which LW's philosophical work emerged, specifically the importance of a number of issues of **authenticity and fakeness**.

Following Janik & Toulmin's *Wittgenstein's Vienna* (Janik and Toulmin 1973), I pointed out that LW's sensibilities with respect to meaningful vs. meaningless language use were from early on informed by journalist **Karl Kraus'** analysis of the problems with never saying what is actually going on politically, socially and culturally in the last decades of the Habsburg regime ("Kakania").

I also pointed out that LW's negative view of his own era and the 100 years immediately preceding it, which we encountered as an ingredient of LW's PhilMath in section 2.2, was also part of the cultural ambience in which LW grew up, and we can point at **Oswald Spengler** as

³²⁹ One of the problems I am not clear about is the reason why the deeply flawed ramblings of an amateur half-wit like Spengler can have made such an impression on people like LW and Martin Heidegger, both obviously of an altogether different intellectual caliber.

³³⁰ I already had the intention to write a separate paper on the issues surrounding everydayness and authenticity (cf. [McGinn 2021](#), Chapter 3), but the present study made an unexpected contribution to, and substantially enriched my understanding of the issues. One of the results of this process was that it actually became less clear to me in what ways LW's views on the fragmentation of the everyday were different from those of -say- Heidegger, or most Nazis for that matter.

a direct influence on LW's thought in this regard, even if LW read Spengler after he already had developed similar lines of thought on his own.

Furthermore, I referenced LW's existential concerns with **vanity and theatricality** and how they were in line with the concern with authenticity (as opposed to fakeness, pretense, bad faith, theatricality, vanity, ...) as an omnipresent aspect of the cultural ambience from which LW emerged: for LW, things should look/sound/appear the way they are.

We focused -amongst other things- on LW's preposterous remarks on Gustav Mahler as a composer (cf. section 2.0.2) and highlighted the fact that LW's problem was not a matter of Mahler's not conforming to formal criteria. LW's most scathing objections against Mahler's music are directed against those moments in which Mahler does sound like old-timey tonal classical music, but -according to LW- is not really old-timey tonal classical music.

I introduced the terms **epistemic fakeness** (epistemic pretense / epistemic bad faith and epistemic bad taste / epistemic kitsch, etc.; my terms). I avoided these terms in my running commentaries in sections 2.1, 2.2 and 2.3 so as not to project them onto LW's text, but they will come handy here and in my own work building on LW's work.

This is perhaps the right time for me to express the fact that I am somewhat **uncomfortable** with the role that these **biographical and otherwise circumstantial materials** play within the present research. In a perhaps similar fashion, Sass feels inclined to justify his own use of these materials and his psychological interpretation in a very long apologetic note ((Sass 2001), fn. 31, pp. 144-145), which -in my opinion- makes things worse.³³¹ So, I hope it will suffice for me to simply state that what I'm trying to point out is not that something that is not philosophical (e.g. psychological or sociological) should somehow explain the philosophy; what I am pointing out is:

1. that there are a number of structural patterns that occur within both LW's private discourse and his work, and that the biographical material illustrating these patterns is part of the philosophically relevant context within which we should read LW's text; this case is strengthened by the fact that the lack of separation between philosophical work and private life in LW's case is well documented (cf. "logic and my sins", as well as the fact that there is considerable overlap between personal notes and philosophical remarks

³³¹ Similarly, I am not at all convinced by the romantic narrative Monk tries to impose on his material in *The Duty of Genius* ((Monk 1990)). Monk's biography of Bertrand Russell ((Monk 1996); (Monk 2000)) has attracted some flack for its flagrant negative bias towards its subject matter, but I am not sure if I am not at least as uncomfortable with Monk's more subtle (but unabashedly positive) biases towards LW.

in LW's manuscript, etc.), as well as the fact that his conception of philosophy does not separate it from existential issues;

2. that the philosophically relevant context for LW's PhilMath does not coincide with the math-internal, technical issues that are *specific to PhilMath* (LW says so repeatedly and explicitly) and that therefore the *not specifically mathematical aspects of LW's work in PhilMath* should not artificially be sanitized from it.

(B) LW & the Grundlagen-debates

One of the points I have repeatedly made throughout this study is that many (if not all) of the mathematical topics that LW focuses on are dealt with as examples of his main interest in the Grundlagen-related issues. Many of the **central themes in LW's philosophy at large** can be linked directly to the issue as to what grounds math and his reaction to the answers that were proposed in contemporary debates:

- the problem of **meaning vs. nonsense**: even if related culture-critical strands (most notably influenced by Karl Kraus' work) predated LW's career as a professional philosopher, this topic became a proper philosophical topic for LW to do work on, in the shape of the question as to what guaranteed mathematics' validity;
- the **discrepancy between formal correctness and authentic meaningfulness**: this discrepancy was not only an important aspect of the Krausian, culture-critical strand in LW's work, but is ultimately also what is at stake in the Grundlagen-debates;³³²
- a preoccupation with vanity, insincerity and **inauthenticity** in general: this apparently driving factor behind LW's private and philosophical thought is illustrated most clearly by his irritation with the bad faith he thought to discern within some of the prevalent mathematical discourse of his time;
- **pragmatism** and **everydayism**: these central aspects of LW's philosophical approach are not also an avatar of Lebensphilosophie in general but can also be read fruitfully as reactions to logicism, intuitionism and Platonism in PhilMath.

In any case, LW's contribution to (or interaction with) the foundational issues operate at a very fundamental [unfortunate pun!] level, as I pointed out repeatedly in the above:

- anti-monism: LW confronts the prevalent monism (or perhaps rather 'totalitarianism', in the relevant sense of 'the tendency to try and encompass everything within a single system') about math head-on;

³³² Again: Gödel's contribution can be interpreted as highlighting the very same issue.

- anti-foundationalism: LW thinks that the idea that math would need 'foundations' is wrong to begin with.

In other words: rather than endorsing one of the available positions within the debate (say: Platonism, formalism, intuitionism), LW contributes to the debate by attacking/questioning the prepositions that those positions all share. For the link between the Grundlagen-related aspects in LW's PhilMath and the relation between the critical aspects of LW's philosophy and the Kantian tradition, I refer to section 3.3 here below.

One of the -for me- surprising results of the work presented in the above is the remarkable **continuity** between LW's early work and his later work with respect to (1) the extent to which the culture-critical strands intertwine with the more technical strands, and (2) its focus on the issues that defined the Grundlagen-debates. LW stuck with what interested him when he started out in philosophy: the problem of what ultimately grounds math. One should not forget that, even if he distanced himself somewhat from mainstream PhilMath in later life, LW was there as a player, in close contact with Russell, for a number of years at the beginning of his career. As a matter of fact, Russell appears to acknowledge LW's influence and various historians attribute this or that aspect of Russell's work to young LW's influence.³³³ Interestingly, Kurt Gödel appears to have attributed what he doesn't like about Russell's later work to the nefarious influence of LW:

Though Gödel (rightly) blamed Wittgenstein for causing Russell's retreat from the unified approach to truth as correspondence he pursued in *Principia*- as well as for the more constructivistic treatment of orders in its second edition- yet, having been so influenced by Russell's original treatment of "judgments of perception" and the MRTJ [= "multiple relation theory of judgment"] in *Principia*, as well as in Russell's subsequent publications, Gödel was, from 1932 onward, fascinated by Russell's long-lasting ambition to analyze the notions of *experience* and *belief*.

³³³ An interesting example is the following (Floyd & Kanamori (Floyd and Kanamori 2016), p. 270):

In (1918: IV) Russell attributed the "discovery of this fact" about a "map-in-space" to Wittgenstein. Colorfully putting the point about "judgments of perception" in terms of his zoological metaphor, Russell wrote, of judgments involving willing, wishing, and so on that (1918: IV§3): "I have got on here to a new sort of thing, a new beast for our Zoo, not another member of our former species but a new species". Wittgenstein, of course, would have rejected this zoological gloss on his idea, for -as Gödel knew from reading the *Tractatus* during the heyday of its influence on the Schlick circle in Vienna-his point to Russell had been, not that there was a new form in view, but, rather, that a recasting of the whole notion of form as the *possibility* of structure would be needed, and this is just what Wittgenstein worked out in the *Tractatus* (1921: 2.033). Russell saw logic as a "skeletal" enterprise of classifying and identifying bones beneath beliefs, a framework "within which the test of coherence applies" (1912: ch. XII), whereas Wittgenstein took logic to be itself a "scaffolding" to be taken up and taken down, an aid in the construction of a true depiction of reality, but not itself in the business of studying what actually is (1921: 3.42, 4.023, 6.124). Russell's effort to eliminate "possibility" in favor of actual correspondence failed in the face of Wittgenstein's new conception of logic. And Russell granted this fact, ever after, not only for logic, but even for mathematics.

Up until the *TLP*, LW operated within a logistic framework (Russell's), which he conceived of as transcendental, in the sense that the framework itself was not part of the world that the propositions expressed in it referred to. Interestingly, already in the *TLP*, LW expressed his conviction that this approach was ultimately unable to deal with anything of real importance. When LW came back to philosophy in 1929, he had already let go of his earlier framework, but continued to ponder over basically the same issue: if math is ultimately not grounded in an axiomatic system, what does justify the validity that we attribute to it?

Even if my research cannot really (or rather: not directly) contribute to the issues that most scholarship is mainly focused on, there is one point I can make that may indirectly contribute to the field: *if it's true that LW's PhilMath is closely connected to his culture-critical and ethical concerns (and I believe I have shown that), any exegesis of LW's PhilMath (incl. chronological approaches) should take this connection into account.* Thus, for instance, the undeniable facts about LW's early fascination with Kraus and the problems of disconnection from everydayness should inform any interpretation of LW's PhilMath of the so-called intermediate period and would make any overly formalist interpretation ipso facto unlikely.

3.2 Wittgenstein as a philosopher of mathematical practice

Although LW's contributions may have been more provocative in the context of the Grundlagen-debate of the first half of the 20th century than they are now (esp. in PhilMathPract circles),³³⁴ it is interesting to note that many of his main points still can sound fresh, new, and even controversial (in some -if not most- circles), 70 years later. In any case, the main points of LW's work on math appear to also be one of the main points of the emerging research tradition of PhilMathPract, as represented -for instance- by (Van Kerkhove and van Bendegem 2007), (Mancosu 2008), (Van Kerkhove 2009), (Ferreirós 2016). For a historical overview, see (van Bendegem 2014). I want to make two separate points concerning the relation between LW's work and PhilMathPract:

- (1) that LW's work foreshadows the themes and attitudes that define PhilMathPract as it is practiced now in 2022;³³⁵

³³⁴ Or perhaps the opposite is the case: maybe some of the presuppositions that were still somewhat liquid back then, have solidified since... For instance, as noted elsewhere, learners up until the 1970s learnt their mathematical techniques separately and were introduced to set-theory only later (if at all), whereas from the mid 1970s (depending on one's country) many have started out within a set-theoretical framework from almost the very beginning of their education. This evolution must have had an impact on the extent to which the framework was felt to be a natural one.

³³⁵ For an apparently identical claim but with remarkably little overlap with my argument, see Pérez-Escobar 2022 (Pérez-Escobar 2022).

- (2) that the lack of a specifically philosophical attitude in present-day (“naturalist”(?)) or more generally empirically-minded) PhilMath in general and PhilMathPract in particular is not necessarily a good thing and that LW’s critical approach may offer venues for further development.

3.2.1 Wittgenstein and present-day PhilMathPract

LW was doing PhilMathPract long before it was called that. On the one hand, LW was an indirect influence on the present generation of practitioners, in that he was one of the initiators of the practice turn, i.e. his work was part of the philosophical roots of work based on the concept of ‘practice’ in general, whether on mathematics or on other topics (cf. (Schatzki, Cetina, and von Savigny 2001); (Soler et al. 2014)), but on the other hand, his work on math also directly foreshadowed a lot what is going on in PhilMathPract today.

(A) focus on actual practices

The obvious way in which LW foreshadows present-day PhilMathPract is that the later LW persistently emphasized the actual, real-life contexts in which mathematical phenomena occur, as the proper locus for mathematical meaning. This shows in the following aspects:

- a focus on what practitioners of mathematics actually do and on practical applications: LW distinguished systematically between what practitioners of math actually do and what is said about their practice and technique; he also emphasized the fact that calculations ultimately originate in, and derive their meaningfulness from, their embedding in real-life practices
- a focus on variability in mathematical practice, the accent on historicity, variation, social aspects, etc.: this aspect was being highlighted in the passage containing a long list of fringe applications that we analyzed in section 1.3, but also in the alternative mathematics imagined by LW in some passages analyzed in section 2.3;
- the inclusion of cognitive, biological, physical aspects in one’s perspective on math finds a clear precursor in LW’s holistic conception of what is a practice, a language game, a form of life.

(B) towards a more radically pragmatic concept of practice

As compared to neighboring fields such as Philosophy of Science, Science and Technology Studies, or Integrated History and Philosophy of Science,³³⁶ PhilMathPract has been relatively

³³⁶ (Soler et al. 2014); (Schatzki, Cetina, and von Savigny 2001); (Schatzki 2015); (Hui, Schatzki, and Shove 2017)

slow in its exploitation of the potential of a practice-based approach and LW's work may serve as a reminder that much more radical implementations of the basic concepts are available.

Let me briefly enumerate a few ways in which average standard present-day PhilMathPract has not picked-up on conceptual resources available in LW's work:

(1) *practice-based ≠ agent-based / community-based:*

Many scholars who operate with the concept of 'practice' appear to construe practices as ultimately reducible to agents and/or communities of agents, conceived of as the proper locus for meaningful behavior. However, this conception of practice is not compatible with LW's holistic view of practices / forms of life, etc., and it can be argued that much of the potential of a practice-based approach gets lost this way (see section 1.1.2(A1) above).³³⁷

(2) *pragmatic vs. epistemological perspectives:*

Whereas more resolutely practice-based approaches immediately lead to a displacement of the central role of epistemological concepts (knowledge, truth, ...), most approaches within PhilMathPract continue to try and maintain not only a basically epistemological perspective, but also (and this is important!) an epistemological agenda, while incorporating practice-related notions in a piecemeal and eclectic fashion. However, from a somewhat more coherently pragmatic perspective, one should be able to show to what extent 'knowledge', 'truth', etc. actually play a role in actual mathematical practice; as LW points out, one can easily imagine math-like techniques being executed without any reference to propositional truth whatsoever (see section 1.1.2(A2) above).³³⁸

(3) *objects, objecthood and objectivity:*

In section 1.1.2(C) above, I have argued that, whereas PhilMath, incl. PhilMathPract, for the most part continues to obsess over the traditional idea of the "unique" ontological status of mathematical objects, from a pragmatic point of view, objecthood is always equally problematic or unproblematic and the relation between mathematical objects and

³³⁷ Apart from this fundamental issue, there is also a more superficial, but not less important, methodological argument to be made against the idea that a community is the default format for collective practices (this argument is not directly related to our reading of LW's work, so I don't come back to it here and refer back to section 1.1.2(A1) above).

³³⁸ In light of the fact that a genuinely pragmatic approach tends to render traditional epistemological concerns marginal, I would suggest that the answer to van Bendegem's question "Foundations of Mathematics or Mathematical Practice: Is One Forced to Choose?" (van Bendegem 1989) should be a resounding "yes". Unless one is ready to construe the term 'foundations' in such a way that it allows for irreducible contingency, plurality, fuzziness, etc., which would make it lose its original charm, I believe.

mathematical practices is not different from the relation between other objects and the practices they occur in; as a corollary, the issue of the apparent ‘objectivity’ of mathematical objects could be reframed as well (cf. section 1.1.2(C) above).

(4) *the heterogeneity of mathematical techniques and practices vs. abstraction and stratification as the mechanism behind the history of math:*

Encompassing grand narratives about the history of mathematics, the proposal to conceptualize the relation between ‘less advanced’ to ‘more advanced’ mathematics (incl. the evolution from pre-mathematical techniques to proper math) in terms of the single mechanism of ‘abstraction’ (Ferreirós (Ferreirós 2016)), can be misleading in that they tend to obscure the heterogeneity and contingency of the ‘buntes Gemisch’ and ‘Gewimmel’ (‘hurly-burly’) of practices co-present at each stage of the history; LW’s image of a hurly-burly is thus directly opposed to Ferreirós idea of a stratification of practices; again: if there is such a thing as a neat stratification in any particular case, then that should be shown, not presupposed (see section 1.1.3(D) above).

(5) *intuition, conceptual thought, understanding:*

Whereas the tradition of PhilMath ever since Plato typically conceives of intuition and understanding as irreducible sources of knowledge, from a pragmatic point of view, intuitions, concepts, and understanding are part of a practice, and not (epistemologically or ontologically) prior with respect to practices: they are practice-specific, the result of training, in many cases historically and geographically variable, ...

(C) [LW’s exotic examples and the ethnographical and historical record](#)

In section 1.3, but also throughout section 2.3, we observed how LW employed one of his signature techniques: setting up a thought experiment by evoking a scenario in which mathematical or math-like techniques are used in contexts that are subtly or markedly different from ours.³³⁹ It is interesting to observe that LW’s made-up examples are not necessarily more exotic or ‘fringe’ than what the ethnographic and historical records show.³⁴⁰ Suffice it here to briefly enumerate a few examples:

³³⁹ For an account of this technique, see also Schroeder 2015 (Schroeder 2015).

³⁴⁰ Similarly, LW’s holism about practices also anticipates the idea that the biological constitution of humans and their biological and physical context are philosophically relevant to PhilMath (cf. the fact that various cognitive approaches show the neurobiological roots of numeracy and geometry, cf. (Dehaene 2011) and (Dehaene and Brannon 2011); (Bangu 2018)).

- While remaining within an all-in-all very traditional, epistemological framework (see paragraph (B) here above), José Ferreirós' historical work on early set-theory (Ferreirós 2007) and on the irrationals ((Ferreirós 2016), Chapter 8, 'The invention of the reals') is documenting and paying attention to the contingent aspects of how mathematics is being shaped by the events that make up its history, and could give rise to much less conservative accounts than the one the author chose to present to his audience.
- Jens Høyrup has made a career out of collecting a huge treasure trove of bizarre mathematical and math-like practices (see i.a. (Høyrup 1990); (Høyrup 2001); (Høyrup 2006); (Høyrup 1983); (Høyrup 2008); (Høyrup and Damerow 2001)); for instance, Høyrup 2009 describes a number of historical algebraic (?) practices which one could be tempted to call 'fake math' (Høyrup 2009).
- Similarly, Karin Chemla's work on ancient Chinese mathematics shows a number of practices that are subtly or not so subtly different from ones we may be more familiar with ((Chemla 2012); (Chemla 2006); (Chemla and Shuchun 2004); (Chemla, Chorlay, and Rabouin 2016)), which invites us to reflect on what makes these practices what they are, not unlike LW's thought experiments, as discussed in section 1.3 above.
- Pythagorean numerology, despite its alleged importance for the emergence of mathematics as a theoretical discipline, for a long time operated with criteria that are definitely at odds with what we would consider proper mathematics, and even with contemporary non-Pythagorean practices. A case in point is Philolaus' music theory (Burkert (Burkert 1972), pp. 386-400), which -for example- held on to the Pythagorean tenet "the whole tone / octave cannot be dissected", which is nonsense from a mathematical point of view, but not so from the point of view of the numerical symbolism that underlies Pythagoreanism.
- Ethnographic research shows the rich variety in the human conception of quantity and other pre-mathematical concepts, which relativizes the universality of even counting, and hence the natural numbers (purportedly given by the godhead, though apparently not to everyone); for references, see for instance (Pinxten and François 2011), (François and Vandendriessche 2016), (Watson 1990), (Moltmann 2013);
- Ethnomethodological work ((Livingston 1986) (Livingston 2015)) shows to what an extent actual mathematical practice (1) looks different from what is talked about in mainstream PhilMath and from the impression one gets from looking at the finished products of

mathematicians' activities (articles, textbooks, etc.)³⁴¹ and (2) looks similar to other human practices, if one actually looks at what mathematicians do from up close.³⁴²

I have two observations regarding the relation between the research mentioned above and LW's work:

- (1) LW did not have access to any of these types of research and he may or may not have used it if he had;³⁴³
- (2) for LW's own purposes, real examples need not necessarily have much of an added value, though -of course- for a 21st century audience, in the present 'naturalist' climate, the fact that most of the examples have real-life counterparts may add to the persuasiveness of the argument.

As for (2): LW did speak out on this, in a way that will not help to endear him to your average 21st century philosopher of mathematical practice: in Ms-116,247, LW ridicules philosophers who collect empirical facts 'as if the factuality of these things was important to us' (cf. also Ms-120,73r):

Da wir in diesen Untersuchungen immer fragen: "was müßte | | sollte man sagen, wenn ...", so genügt uns die Varietät der uns in der Wirklichkeit bekannten Fälle nicht | | der in der Wirklichkeit existierenden Fälle nicht, sondern wir müssen eine Mannigfaltigkeit von Sachverhalten in die Erwägung ziehen, gleichgültig, ob sie wirklich oder erdichtete sind. Daher berührt es komisch, wenn wir einen Philosophen mit der Miene eines Naturforschers nach einzelnen entlegenen Fakten (seltsamen Geisteskrankheiten z.B. | | etwa) fischen sehen. Als wäre das faktische dieser Dinge für uns von Wichtigkeit.

In the end, it is not that important what LW's eccentric opinions on those matters were; what is much more important is the fact that the historical and ethnographic record that was developed in the last few decades, offers a lot of material for philosophers to work with and that LW's remark reminds us of the fact that, despite a burgeoning research activity, *very little is being done with this material at the properly philosophical level* (see section 3.2.2 here below).

³⁴¹ An important aspect is the focus on how these 'finished products' are used in actual practice and how they relate to the real-life research activities in which they originate.

³⁴² I would be interested in reading up on a number real-life 'fringe' applications: the use of math and math-like techniques in various practical settings (rules of thumb, shortcuts for calculations, ...); symbolic systems that are distinctly not mathematical; the historical interactions between mathematics and astrology, numerology, and other no-longer-academic disciplines; etc.

³⁴³ One could think of LW's interaction with Frazier's *Golden Bough* as an example of how LW could have reacted to the type of material under discussion, but also as an illustration of what he meant by "as if the factuality of these things was important to us".

3.2.2 Wittgenstein and the identity of Philosophy of Mathematics and Philosophy of Mathematical Practice

The biggest difference between recent work in PhilMathPract and LW's work is the complete lack of a critical attitude toward mathematics and mathematicians in the former, whereas the above suggests that LW's PhilMath is *essentially* critical. By virtue of the very fact of this difference, LW's work urges us to ask ourselves what the role of PhilMath actually is and/or should be. In this section, I try to articulate how LW's work can/could help present-day philosophers of mathematics to reconsider their identities as philosophers and their relationship with mathematics itself.

(A) LW on his position vis-à-vis the mainstream of PhilMath

LW was very aware of and quite outspoken about the **differences** between his own outlook on the role of philosophy and the one prevalent in mainstream PhilMath. Thus, he repeatedly and explicitly highlights the following features of his own (later) approach:

- LW approaches math **from the outside looking in**, whereas most PhilMath is practiced as a **prolongation of mathematics** itself. This insiders' perspective fosters **exceptionalism** about math (it is easy to maintain that something has unique (incomparable!) attributes if one never actually compares it to anything else). A consequence of this exceptionalism is the fact that -as opposed to LW's essentially critical stance- most PhilMath is highly deferential towards math and mathematicians.
- LW's own approach is **anthropological**, as opposed to the logical and/or epistemological approaches that are prevalent among mathematicians and mainstream practitioners of PhilMath, which means that he views math as a human endeavor among all other human endeavors. LW's anthropological approach of math as a practice among other practices, is ipso facto **comparative**, as opposed to the exceptionalism of the mainstream.
- PhilMath is mostly practiced within the framework of the quasi-religious (or in some cases straightforwardly religious) ideology of unity and uniqueness (what I called **monism**) prevalent among mathematicians. LW's anthropological approach emphasizes the **contingency and variability** of math, thus directly opposing this monism.
- Mainstream PhilMath mostly operates within the conceptual framework within which mathematicians operate themselves; to this insiders' approach (see above), LW opposes what he considers to be a proper 'philosophical' approach, which focuses on the **presuppositions** underlying this conceptual framework (cf. paragraph (B) here below, for

LW's criticism of Ramsey in these terms; see also section 3.3 below for the 'critical' nature (also in the Kantian sense) of LW's philosophy).³⁴⁴

LW was also very conscious of the **historicity** of his work. Thus, LW says in Ms-126,133, d.d. 19411215-19411217 (the same passage in which he introduces the notion of a 'prudish proof', see section 2.1(C) above):

Wir kämpfen jetzt gegen eine Richtung. Aber diese Richtung wird sterben, durch andere Richtungen verdrängt. Und dann wird man unsere Argumentation gegen sie nicht mehr verstehen; nicht begreifen, warum man all das hat sagen müssen.³⁴⁵

It is perhaps interesting to observe that LW's criticism -if anything- applies even more directly to present-day PhilMath than it did in its own time: even the 'naturalist turn' in PhilMath (which does not represent a majority position in the field anyway) has not changed much about those aspects of PhilMath that LW was most critical of. In the same way, we read in section 2.1(A) about LW's hope and expectation that later generations will laugh at Cantor's "hocus pocus":

Ich glaube & hoffe eine künftige Generation wird über diesen Hokus Pokus lachen.
(Ms-117,110)

Generally speaking, this prediction has not really come true, on the contrary: the set-theoretical outlook on math has become part and parcel of how people are taught math and this seems to have solidified and generalized this perspective, even with practitioners who in actual practice have no need for the axiomatic framework of set theory. Of course, LW wrote not even 100 years ago, and perhaps the emergence of PhilMathPract and the fact that LW's writing on PhilMath appears to be in fashion in the 2020s may be indications that his prediction will have a little more chance of coming true than appeared to be reasonable a decade ago.³⁴⁶

³⁴⁴ A current case in point would be Ferreirós declaring his need to 'rescue objectivity' from the pluralism his own work appears to lead towards ((Ferreirós 2016), Chapter 9). My question is: why would a philosopher want to 'rescue' mathematicians' misguided claims?

³⁴⁵ We now fight against one school of thought. But this school of thought will die, supplanted by other school of thought. And then one will no longer understand our argumentations; no longer comprehend why one had had to say all this. [quick translation fs]

³⁴⁶ For indications that set-theory may be on its way out again in math education (at least in Belgium), see Bock & Vanpaemel (Bock and Vanpaemel 2019).

LW is also very much aware of the inherently **polemical relationship** between his philosophy and the mainstream. In the following paragraph (Ms-113,117r-v d.d. 19320517 = “Big Typescript” (1933) §644), he acknowledges that mathematicians must be horrified at him:

Den Mathematiker muß es bei meinen mathematischen Ausführungen grausen, denn seine Schulung hat ihn immer davon abgelenkt sich Gedanken & Zweifeln, wie ich sie aufrolle, hinzugeben. Er hat sie als etwas verächtliches ansehen lernen & hat um eine Analogie aus der Psychoanalyse (dieser Absatz erinnert an Freud) zu gebrauchen einen Ekel vor diesen Dingen erhalten, wie vor etwas Infantilem. D.h., ich rolle alle jene Probleme auf, die etwa ein Knabe | | Kind beim Lernen der Arithmetik, etc. als Schwierigkeiten empfindet & die der Unterricht unterdrückt ohne sie zu lösen. Ich sage also zu diesen unterdrückten Zweifeln: ihr habt ganz recht, fragt nur, & verlangt nach Aufklärung! ³⁴⁷

As opposed to the prevalent idea that mathematical truths are objectively and atemporally out there, LW points out that math as it is now is the result of a long sequence of historical decisions, educational formatting,³⁴⁸ etc.³⁴⁹ The way we are formatted as practitioners of math all along our educational trajectories teaches us to suppress certain lines of questioning as irrelevant. LW allows these questions to come up again.

This should hit home with all of us, qua philosophers: to what extent has the consensus about -say- monism been manufactured? In other words: whether one agrees or not with LW's stance or the attitude underlying his stance, the following question really needs an answer, especially in the light of the results of the last couple of decades of research in PhilMathPract and the history and ethnography of mathematics: *why would one want math to be unique and unitary, complete, objective, atemporal, ... etc.?*

(B) the identity of PhilMath

The **critical attitude**, or even the **critical agenda** (incl. the focus on presuppositions), that was underlying LW's philosophical work, incl. his PhilMath, remains unexplored and unparalleled in contemporary philosophy, esp. contemporary PhilMath. Perhaps it is about

³⁴⁷ A mathematician is bound to be horrified when faced with my mathematical remarks, since his schooling has always diverted him from giving himself over to thoughts and doubts of the kind that I am bringing up. He has learned to regard them as something contemptible and, to use an analogy from psychoanalysis (this paragraph is reminiscent of Freud), he has acquired a revulsion against these things as against something infantile. That is to say, I'm bringing up all of those problems that a child learning arithmetic, etc., finds difficult, the problems that classroom instruction suppresses without solving. So I'm saying to those suppressed doubts: You are quite right, go ahead and ask – and demand clarification! [Translation quoted from (Wittgenstein & Luckhardt (ed.) & Aue (ed.) 2005 -- The Big Typescript TS 213, German-English Scholars' Edition)]

³⁴⁸ For the educational aspect in LW's text, see section 3.2.3(C) here below.

³⁴⁹ This 'etc.' could include the relations between math and its 'applications' as well as surrounding religious and philosophical discourses.

time for philosophy to reclaim that ‘critical’ aspect of its traditional role in academia (and society at large, for that matter).

Although the arguments of scholars like Penelope Maddy against the speculative tradition of ‘*philosophia prima*’ certainly have had their merit in the context in which they intervened, and although various empirical approaches to mathematics as a phenomenon have yielded important results, it is worth paying attention to LW when he points out the specificity of philosophy, as opposed to empirical approaches,³⁵⁰ as well.

As pointed out above (section 3.2.1(C)), LW explicitly states numerous times (perhaps most explicitly Ms-116,247) that:

- *history/ethnography/sociology ≠ philosophy:*

The factuality of our case studies is not what makes them philosophically relevant; I am not insisting that one should follow LW in his anti-empiricism in this regard, but rather that the empirical approaches are not ipso facto philosophy and that LW reminds us that the specifically philosophical approach is worth preserving.

- *math ≠ philosophy:*

As opposed to current practice, LW repeatedly defines his own “from the outside in” approach in opposition to the approach to math displayed by mathematicians. In section 2.3(E), we even encountered the claim that most mathematicians have ‘slimy’ ideas about math.

This fits in with LW’s views on **the role and the status of a philosopher** in general: “a philosopher is not a citizen of a community of thought; that’s what makes him a philosopher” [Ms-112,72r “(Der Philosoph ist nicht Bürger einer Denkgemeinde. Das ist, was ihn zum Philosophen macht.)”]. The context of this remark is interesting for its very heavy-handed criticism of Frank **Ramsey** (Ms-112,70v-71r, d.d. 19311101).³⁵¹ What is interesting here, is that LW’s negative assessment of Ramsey’s thought gives us a clear insight in what LW saw as the proper way to be a philosopher, which is clearly on the opposite side of the spectrum from the insiders’ perspective that prevails in current PhilMath: for LW, Ramsey’s insistence on sticking with the state/community he happened to be part of, unable or unwilling to think

³⁵⁰ Which -by the way- he disparages as unphilosophical; see section 3.2.1(C) above.

³⁵¹ 1.11.31.

(Ramsey war ein bürgerlicher Denker. D.h. seine Gedanken hatten den Zweck die Dinge in einer gegebenen Gemeinde zu ordnen. Er dachte nicht über das Wesen des Staates nach – oder doch nicht gerne – sondern darüber wie man diesen Staat vernünftig einrichten könne. Der Gedanke daß dieser Staat nicht der einzig mögliche sei beunruhigte ihn teils, teils langweilte er ihn. Er wollte so geschwind als möglich dahin kommen über die Grundlagen – dieses Staates nachzudenken. Hier lag seine Fähigkeit & sein eigentliches Interesse; während die eigentliche || eigentlich philosophische Überlegung ihn beunruhigte bis er ihr Resultat (wenn sie eins hatte) als trivial zur Seite schob.)

[...]

(Der Philosoph ist nicht Bürger einer Denkgemeinde. Das ist, was ihn zum Philosophen macht.)

about what is presupposed, disqualifies him as a philosopher. An interesting detail is Ramsey's tendency -according to LW- to push aside any properly philosophical reflection as 'trivial', which reminds us of the fact that LW appears to have systematically embraced triviality in a number of contexts (cf. section 2.0.3).³⁵²

For LW, philosophy is essentially concerned with focusing on presuppositions / the given; in this sense LW remains fully within the Kantian tradition (see section 3.3 below).

3.2.3 Towards a critical agenda for Philosophy of Mathematical Practice

In this section, I attempt to articulate a number of ways in which our reading of LW's PhilMath could inspire present-day practitioners of PhilMathPract to adopt a more critical attitude towards their subject matter.

(A) against whiggism and exceptionalism in PhilMath

I would like to especially accentuate that a lot of what we are learning now from the history and anthropology (incl. sociology, ethnography, ethnomethodology, etc.) of mathematical practices is about *the contingency and variability of mathematics*. LW was exploring the philosophical importance of these aspects a long time (70+ years) before anyone else.³⁵³ This chronological priority is only a minor scholarly detail, hardly worth mentioning, but what is important is that, whereas LW's work on the contingency and variability of mathematics was properly philosophical and was intended to contribute to the then current Grundlagen-debates, *the philosophical consequences of the now abundant observations and insights in these matters are currently not being investigated at all*.

LW's 'anthropological' approach approaches math in the context of human activity at large, which implies that it is an inherently comparative and 'neutral' approach: successful and unsuccessful, present and past, normal (?), exotic and fringe, official and unofficial, etc.³⁵⁴ practices are approached in the same way as mathematical phenomena and as historical data.³⁵⁵ LW's approach is thus in direct opposition to the exceptionalism and whiggism that overtly or covertly prevails in mainstream mathematical discourse.

Thus, LW's work on math reminds us of the fact that, methodologically speaking, **whiggism and exceptionalism** are problematic:

³⁵² NB: LW's work has also been dismissed as trivial or otherwise irrelevant.

³⁵³ For the fact that Spengler did emphasize the historicity and variability of math and the similarities and differences with LW's work, cf. Appendix 4.2.

³⁵⁴ What about 'fictional', 'counterfactual', 'impossible', ... ? Is this a good question? I don't think so.

³⁵⁵ We have seen that LW presents Cantor's way of presenting certain set theoretical notions in the same ballpark as prototypical crank discourse.

- in order to be able to speak of the ‘uniqueness’ of math (exceptionalism), one needs comparison, a broader, anthropological, perspective, without which the claim of uniqueness is a mere profession of faith;
- in order to focus ‘only on successful math’ (whiggism) in a meaningful way, one needs to be able to put successfulness in general and successful math in particular in the context of all math-like alternatives and what it took for them to become successful: on the one hand, even ridiculously exotic practices can be very successful for centuries, for instance: various brands of numerology have been successful for thousands of years and continue to be successful; grossly imprecise rules of thumb and unsystematic ways of measuring and counting continue to be functional in a wide variety of technical contexts, etc.; on the other hand, i.e. conversely, those practices that made it as part of present-day academic math, are the result of an accumulation of highly contingent social/historical circumstances and processes (see below).

To me this suggests -again- that the important question to be asked is: *why are whiggism and exceptionalism -despite their self-evident and often commented on methodological and epistemological flaws- so popular among practitioners of PhilMath?*

A similar lack of perspective shows in the prevailing deference for mathematicians in general and **the great geniuses** in particular. For instance, José **Ferreirós** strikes a remarkably apologetic tone on pp. 96-97 of his (Ferreirós 2016), in which he points out that “Careful logicians such as Frege and Dedekind were very unhappy with the dots “. . .” in the expression $\{0, 1, 2, \dots\}$.” He then explains:

From our multilayered standpoint, which insists on the interplay of different practices and strata of knowledge, there is a natural way of understanding the epistemic role that the dots “. . .” play. I suggest understanding them as indicators of a systematic link, of an interplay within the web of practices. They indicate a systematic connection with perfectly well-known, antecedent practices: we know how to count since preschool, [...]. The receiver of that information may also have good knowledge of the systematic role that the successor function plays in a deductive presentation of arithmetic.

In the context of Ferreirós’ framework, this is a perfectly fine explanation, more or less in line with a Wittgensteinian one as presented in the above.³⁵⁶ But then Ferreirós says the following in footnote 12, attached to the above excerpt:

³⁵⁶ I disagree with the notion of ‘strata of knowledge’ (cf. 1.13(D)), as well as with the idea of an ‘interplay’ of practices (as if practices themselves have agency), but these aspects are not what is being discussed here. What we’re focusing on here is what Ferreirós concludes from his explanation with respect to PhilMath in general and his distinctively and explicitly uncritical attitude towards his subject matter.

Needless to say, no criticism of Dedekind and Frege is implied: for their project of a reconstruction of arithmetic *ab ovo*, purely from logic (and set theory), it was necessary to try what they did.

On the basis of essentially the same kind of reasoning that Ferreirós presents in the above excerpt, LW criticizes the lack of understanding on the part of ‘most mathematicians’ (and that includes Dedekind) of exactly the kind of links that Ferreirós points out, and calls it “slimy” (cf. section 2.3(E) above). Note that Ferreirós’ own explanation implies a direct and fundamental criticism of the very “project of a reconstruction of arithmetic *ab ovo*, purely from logic (and set theory)”. I am not necessarily arguing in favor of LW’s crude way of expressing this idea (if he had prepared that remark for publication, he probably would have chosen a different wording), but I do wish to point out that the hyperbolic deference towards the great geniuses in the history of math, as illustrated by the above quotation, deserves to be questioned. Major mathematicians can have truly ignorant opinions about their own field.³⁵⁷ The difference in basic attitude between mathematics and philosophy, as pointed out by LW, may go a long way in explaining this.

A case in point is the great **Dieudonné** (of Bourbaki fame)’s remarks on the social aspects of mathematics. The following quote from Dieudonné (as quoted in van Bendegem 2014 (in Soler & al. (eds.) 2014), p. 215) was intended as an example to show that “sociological reasons [des raisons sociologiques]” never yield anything convincing (“Je veux bien, mais je n’ai jamais rien vu de très convaincant dans ce sens-là”) and it is representative for the disdain towards sociological, and otherwise empirical, approaches in more traditional branches of PhilMath:

To the person who will explain to me why the social setting of the small German courts of the 18th century wherein Gauss lived forced him inevitably to occupy himself with the construction of a 17-sided regular polygon, well, to him I will give a chocolate medal. (Dieudonné, 1982, p. 23)³⁵⁸
(Dieudonné 1982)

Dieudonné’s demand, as he formulates it in the above quote, is not so much unfair as it is misguided. There would be a lot of interesting things to say about how the construction of polygons ended up in the repertoire of mathematical problems and how Gauss’s activity with respect to polygons fitted in with the context in which it occurred. The relevant context would

³⁵⁷ Cf. Rota 1997 (Rota 1997), p. 107: “Like artists who fail to give an accurate description of how they work, like scientists who believe in unrealistic philosophies of science, mathematicians subscribe to a concept of mathematical truth that runs contrary to the truth.”

³⁵⁸ Celui qui m’expliquera pourquoi le milieu social des petites cours allemandes du XVIIIe siècle où vivait Gauss devait inévitablement le conduire à s’occuper de la construction du polygone régulier à 17 côtes, eh bien, je lui donnerai une médaille en chocolat.

not necessarily be limited to the courts in which Gauss circulated, but could include the people he corresponded with, what exactly his reputation and payment depended on, how exactly Gauss's and his contemporaries' research agenda in general was determined by its relevance to which stakeholders, etc. Thinking, writing, teaching, being taught, buying and selling books, and being paid for doing math are social phenomena, and as such have a lot in common with any number of non-mathematical social phenomena. I am not necessarily an advocate of sociological approaches to epistemological issues; I only wish to point out that Dieudonné's argument is a good example of exceptionalism and whiggism, and ultimately incoherent (see below).

In the same article (pp. 30-31),³⁵⁹ Dieudonné offers more opinions on the social aspects of mathematics. First, he denigrates the practice of working on marginal topics that are not motivated by a proper research agenda (he calls this "non-motivated math" or "waffling" [délayage])³⁶⁰ and explains this practice in social terms, as a result of the pressure to publish in order to have an academic career. What interests us here is the lack of symmetry or neutrality in Dieudonné's discourse: he applies 'sociological reasons' only to the cases of math he dislikes, and not the cases he does like, as if these were not equally social. Dieudonné seems to believe -as many others do- that only -what he considers- failures require social explanations, but -what he considers- successes are sufficiently explained by the fact that they are -what he considers- right.

³⁵⁹ (Dieudonné 1982), pp. 30-31: Alors quand, après de longues années de patientes études, on arrive enfin à une théorie bien faite, bien enseignable, bien utilisable, il semble que les choses devraient s'arrêter là. Mais non ! Cela ne s'arrête pas, parce que certains gens, pour des raisons variées, sociologiques ou autres, se disent : « Que se passerait-il si l'on modifiait l'un des axiomes de cette théorie ? ». Et les voilà à modifier l'axiome trente-six bis, ce qui à la fin produit une nouvelle théorie. Quand on leur en demande les raisons, ils répondent : « Comme ça ! Pour écrire un papier ». Si j'ai parlé de raisons sociologiques, c'est qu'il y a des pays, et il y en a de plus en plus, où la promotion d'un universitaire se fait au poids du papier. Alors, bien entendu, il faut en produire, et quand il n'y en a pas, on se met à modifier l'axiome trente-six bis. Quoiqu'il en soit, voilà ce qui se passe. C'est ce qu'on peut appeler les mathématiques non motivées ou le délayage. On m'objectera que, peut-être, l'axiome trente-six bis modifié sera un jour aussi fondamental que la notion de groupe. Effectivement, ce n'est pas exclu, et j'ai vu dans ma vie deux ou trois cas où une théorie considérée comme totalement dénuée d'intérêt s'est brusquement trouvée accrochée à quelque chose qui vous faisait comprendre le fond des choses. Mais c'est tout à fait exceptionnel et le reste n'est que du délayage qui s'accumule dans les innombrables papiers qu'on écrit, qu'on publie, dont on fait même des comptes rendus et dont personne, par la suite, ne parle plus jamais, sauf ceux, bien entendu, qui délayent ces délayages, ce qui, apparemment, se prolonge indéfiniment.

Enfin, il existe des théories qui s'étiolent progressivement, qui se meurent doucement, non pas que les mathématiciens deviennent moins ingénieux - au contraire, ils le sont peut-être plus -, mais parce que les problèmes traités s'amenuisent, deviennent de plus en plus spéciaux, s'isolent et finissent par ne plus avoir de relation qu'avec la théorie elle-même. Alors que ce qui excite beaucoup les mathématiciens, c'est le fait qu'un problème ait des relations avec d'autres théories.

³⁶⁰ Whereas in the present paragraph I emphasize an aspect of Dieudonné's text that illustrates a typical anti-pragmatic aspect of standard mathematical discourse, there are other passages in the same article that are remarkably close to a Wittgensteinian point of view. Thus, Dieudonné's criticism of "empty" mathematical work that ticks all the boxes for formal correctness but is nonetheless still worthless as mathematics, bears a strong resemblance to the 'bad faith' (my term) that LW blames Gödel for (cf. section 2.3 above).

Whereas this kind of whiggism has been and continues to be addressed as a problem in most neighboring disciplines, such as Philosophy of Science or Science and Technology Studies (for a recent contribution from a leading scholar, see Chang 2021 (Chang 2021)), mainstream PhilMath and History of Mathematics continue to be whiggish to a remarkable extent: many heart-felt convictions about the universality and objectivity of math should shatter in the face of the historical, ethnographic and ethnomethodological record. And this should bring us - again- to the question as to *why these whiggish and exceptionalist ideas are worth defending for those who defend them.*

Thus, what LW's remarks can do for us is remind us of the importance of a **critical attitude**, which may sometimes require a polemical relation with mainstream discourse: rather than merely assuming that what practitioners say is ipso facto right, we can conceive it as our tasks as philosophers to hold mathematicians and fellow-philosophers accountable for what they say and the consequences of what they say about their own activity, its products and their role in society at large. From the above, a number of philosophically interesting questions have emerged: what exactly is specific to math, and what aspects does it share with other practices? what is the epistemic status of mathematical monism, foundationalism and exceptionalism? to what extent are monism, foundationalism and exceptionalism specific to math, to what extent do they participate in broader philosophical, religious, ideological discourses? what drives the predilection for monism (etc.) in math? what does history teach us in this regard? etc.

(B) against epistemic bad faith and epistemic bad taste

Throughout Part 2 of this study, I pointed out that LW systematically uses ethical and/or aesthetical vocabulary referring to various avatars of the concept of **inauthenticity** (delusion, fraud, hocus pocus, pathos, ...) to criticize various aspects of mathematical discourse he objects to.

LW's criticism of math (or rather: discourse about math), formulated in ethical and/or aesthetical terms, does not target bad things done by mathematicians in the margins of their mathematical work: it's not about (at least not directly about) sexism and/or ageism in the context of the recruitment and career management of researchers, or in the context of peer reviews; nor about bullying of the less gifted in an educational context or on the work floor; it's not about acting irresponsibly while working in data-management or applied artificial

intelligence; etc.³⁶¹ It's about problems with mathematical discourse, the ways in which mathematicians conceive of and speak about the nature of the discipline. Examples are:

- Gödel presents his completeness/consistency thing as a result in arithmetic and acts as if this is independent evidence for his Platonism, whereas in actual fact it is almost certainly the other way around (see below);
- the unity of math is fake: de facto, math is not a unitary but a family of quite heterogeneous techniques, which retain their identities even if they are integrated within a superposed axiomatic formalism;
- presenting Dedekind's cut as a construction, whereas it is not, is fake;
- the drama about contradictions in axiomatic systems is fake in that it always remains to be seen whether they have any impact at all;
- foundationalism is fake: de facto, mathematical technique is not founded by 'foundations': it is grounded (?) in an irreducibly complex, contingent, variable, messy web of practices;
- some of the core images that are used to make the set-theoretical concept of the continuum seem intuitive and/or interesting, are actually fake.

For LW, the notion of authenticity has thus a direct bearing on the technicalities of his account of meaning and is therefore not 'external to' logic and epistemology: it operates at the core of the rationality that logic and epistemology are supposed to embody. It points out that the very meaningfulness of mathematical concepts depends on the way they relate to the way they actually operate within mathematical practice and that the meaningfulness of mathematical practices is fundamentally linked to the more basic "applications" that they evolved out of.

I have pointed out that this is coherent with the fact that he views the problems with these types of discourse as part of a larger cultural problem: LW views Cantor's verbiage as symptom of the "illness of our time", or -to use Spengler's phrase- the decline of the West (see also section 3.3(B) below).

I believe no present-day philosopher should try and defend a Spenglerian view of culture and history (though many are already doing so and many more probably will in the near future),³⁶²

³⁶¹ Of course, these are aspects of mathematical practice that also deserve our attention, as philosophers. Cf. the contributions of Maurice Chiodo and the Cambridge Ethics in Mathematics group, e.g. (Chiodo and Bursill-Hall 2019), (Chiodo and Bursill-Hall 2018), (Chiodo and Vyas 2019). Cf. also the emerging field of virtue ethics in PhilMath, e.g. (Rittberg, Tanswell, and Van Bendegem 2018); (Tanswell and Rittberg 2020); (Aberdein, Rittberg, and Tanswell 2021).

My point is that even among those philosophers of mathematics and philosophically inclined mathematicians that are willing to engage with the ethical or political issues involved in mathematical practice, many are not ready to accept the idea that there are ethical, political or more generally ideological issues at the core of mathematics itself. LW's work suggests that these lines of thought deserve to be explored.

³⁶² Let us not forget that not so long ago, Spengler-like arguments actually gave rise to such concepts as healthy Aryan science, which was differentiated from unhealthy, rootless, cosmopolitan science by being rooted in the

but the way issues with the conceptual core of mainstream PhilMath are related to core ethical/aesthetical and societal/political issues in LW's work, does at least suggest that we ask ourselves whether there is still room for such an ethical-aesthetical angle within present-day philosophy.³⁶³

(C) on math education

I would also like to briefly focus on the importance of **math education** with respect to the potential agenda of PhilMath and PhilMathPract in a Wittgensteinian vein.³⁶⁴ Let us come back to the following paragraph (Ms-113,117r-v d.d. 19320517 = "Big Typescript" (1933) §644, already quoted in section 3.2.2(A) above), in which LW mentions the fact that what we experience as 'given' heavily depends on our training, and that in the case of mathematics, the way we are taught at school, we may have been trained to actively shun the philosophically most interesting questions about mathematics:

Den Mathematiker muß es bei meinen mathematischen Ausführungen grausen, denn seine Schulung hat ihn immer davon abgelenkt sich Gedanken & Zweifeln, wie ich sie aufrolle, hinzugeben. Er hat sie als etwas verächtliches ansehen lernen & hat um eine Analogie aus der Psychoanalyse (dieser Absatz erinnert an Freud) zu gebrauchen einen Ekel vor diesen Dingen erhalten, wie vor etwas Infantilem. D.h., ich rolle alle jene Probleme auf, die etwa ein Knabe | | Kind beim Lernen der Arithmetik, etc. als Schwierigkeiten empfindet & die der Unterricht unterdrückt ohne sie zu lösen. Ich sage also zu diesen unterdrückten Zweifeln: ihr habt ganz recht, fragt nur, & verlangt nach Aufklärung!

LW's remark reminds us of the fact that education shapes the 'given' within which mathematicians will operate. This emphasis on formatting through math education is an important aspect, not only at the philosophical level, but also for its potential application to policy. In the context of the present discussion, it is interesting to see that LW explicitly complains about the fact that math education actively stifles any critical attitude that pupils may have.

It is interesting to note that some of the more interesting developments concerning the emerging research topic "Ethics in Mathematics" (EiM) developed as an educational issue (e.g. (Chiodo and Bursill-Hall 2019); (Chiodo and Vyas 2019)). Within the context of the present study, the following quote stands out:

everyday life of the healthy German Volk, and that the present-day political circumstances are sufficiently reminiscent to make it plausible that something similar may happen again.

³⁶³ In Appendix 4.3, I would like to offer a few brief suggestions as to how LW's critical attitude could still apply to contemporary mathematical discourse, despite the fact that I may not be the right person to actually do the work.

³⁶⁴ Within the cluster of sub-topics and sub-disciplines that makes up PhilMathPract, a niche developed which focuses on math education (see, for instance, (François and van Bendegem 2010); (Ernest 2018)).

One reason mathematicians shy away from ethical discussions is that mathematics seeks timeless, absolute truths. The apparent perfection of mathematical truth can be its primary attraction. But ethics doesn't have the same binary clarity or timelessness. Different people may come to different conclusions or hold different moral values which are all reasonable, and mathematicians facing profession-specific ethical challenges have no universally-agreed ethical framework to use, because there isn't one. Unsurprisingly, suggesting that mathematicians need to be aware of ethical issues sometimes gets the response that ethics is imperfect and a matter of opinion, and moreover "*Whose ethics?*" which we would answer with "*Yours!*" We do not suggest that teaching EiM should give all the answers to ethical problems, but we do suggest that it is our duty to educate our students about it. The hard work of solving the questions remains and is an individual's social responsibility. The political debate that follows is part of what informed citizens frequently do.

(Chiodo & Bursill-Hall 2019 -- Teaching Ethics in Mathematics, p. 40)

Chiodo & co's main problem is a straightforwardly educational one: how come so many mathematicians appear to behave irresponsibly in the professional contexts in which they end up, and shouldn't the educational institution that train mathematicians do something about this? These remarks are not articulated by professional philosophers but by working mathematicians. Still, these remarks point at a link between (1) actual real-life problems and (2) the problems with some core concepts in mainstream PhilMath that I -following LW- have been pointing at throughout this study. In other words, there may be a link here between Chiodo & Co's remarks on the lack of moral awareness on the part of mathematicians and the fact some of the core aspects of the practices they are educated in, are encouraging an anti-humanist world-view (?), in which human responsibility (incl. individual responsibility) is de-emphasized in favor of transcendental crystalline logic (?). Our argument makes a link between the inherent features of mathematics as they are presented by the mainstream (its autonomy, its uniqueness, its unity, ...) on the one, and the ethical aspects on the other hand. If you teach people that what they are studying is the language of God, that it is completely free of the sublunar vagaries and concerns, that their talent is unlike any other talent, then it should not come as a surprise that you end up with a bunch of dangerous fools as soon as you ask them to function in the context of the complexities of real life. The point is that presenting math as it is presented in mainstream math education is not innocent (see (Ravn and Skovsmose 2019), Part IV "How Good is Mathematics", and paragraph (D) here below).³⁶⁵

³⁶⁵ Although **education policy** is not part of the subject matter of the present study, the above does suggest that policy choices with respect to math education have complex ramifications, both with respect to the ideological baggage that comes with the content conveyed, and with respect to the consequences of the curriculum in shaping the views and attitudes of future practitioners and the public at large. For instance, in the light of the above, one may want to think about including notions of the history of mathematics in the curriculum so as to mitigate the fundamentalism inherent in the monism and foundationalism of the standard approach to math education.

(D) final remarks on the need for a critical approach to PhilMath and PhilMathPract: why would bad faith and bad taste in PhilMath matter?

LW is part of a long tradition of philosophy conceiving of itself as critical, both in the sense of taking an outsider's point of view with respect to the society/networks within which they operate, as well as in the Kantian sense of being interested in the pre-meaningful presuppositions (see section 3.3 below). At the same time, LW's PhilMath emphasizes the link between the more technical aspects of PhilMath and very broad cultural and societal issues: LW's criticism was rooted in Kraus' social/political criticism and a Spengler-like vision of the decline of Western culture due to its fragmentation, and remained essentially a kind of cultural critique.

One of the immediate implications of focusing on the pragmatics of math, which could be the most important contribution of LW to PhilMath, is that it provides strong evidence against the autonomy (its 'freedom', as Cantor called it) of math: in actual fact, (1) mathematical technique, from its inception and throughout its history, has been rooted in, and intertwined with, non-math-specific applications, and (2) mathematical discourse has been shaped by equally non-math-specific philosophical, religious, etc. concerns.

Furthermore, whatever mathematicians may say about their own field,³⁶⁶ it does function within a wider social context, in multiple ways: math is an important component of various educational systems; practitioners in many other fields need to learn mathematical technique up to various levels; professionals with mathematical training end up in many different domains, in which they will bear real-life responsibilities. For further reflection on how things go wrong when mathematicians hit the road of real-life, I refer to Chiodo and friends (quoted in section 3.2.3(C) here above) and Part IV "How Good is Mathematics" of Ravn & Skovsmose's 2019 *Connecting Humans to Equations. A Reinterpretation of the Philosophy of Mathematics*.

With this out of the way, the question remains as to **why epistemic kitsch and epistemic bad faith would be important**. My claim is that even scientific kitsch, as a way of presenting one's activity, does do damage. Epistemic kitsch contributes to the demise of certain norms, values and criteria attached to scientific/academic practice, which may be inherent to the operability of rationality at large.³⁶⁷ Epistemic kitsch is detrimental in that it encourages

³⁶⁶ For examples of mathematicians claiming absolute innocence, see Ravn & Skovsmose 2019 (Ravn and Skovsmose 2019), pp. 133-135 on the "Thesis of isolation" and the "Thesis of neutrality".

³⁶⁷ Of course, there are circumstances in which one can responsibly work towards the demise of certain norms, perhaps in order to promote other values and norms. But in that case one should be able and willing to explain how what one is doing is a good thing, in its intentions and in its effects. If one is not able or -more plausibly- not

the public to evaluate science by improper criteria, thus creating unreasonable and (which is worse) inapplicable expectations, which in its turn inevitably contributes to a loss of credibility of academia and thus creates room for really ugly alternatives and devalues the educational value that the sciences, but especially math, claim for themselves.³⁶⁸ If a substantial part of the public feels that their pre-existing distrust of intellectuals and science is vindicated when it turns out that science sometimes gets it wrong or when scientists are waffling on TV or they hear that Heisenberg has proven that “scientific knowledge is never certain” or “Gödel has proven that even math is not always true”, this shows that science has communicated a fundamentally misleading picture of how it operates. The current political climate in Europe and the U.S.A. fosters anti-scientific sentiments and improper communication (incl. education) about how science/research actually works has to be part of the cause.

In that sense, whiggist or otherwise unrealistic representations of the workings of science do serious damage to the credibility of academia, by creating not only unrealistic, but outright misleading expectations on what science does: the emphasis should not be on the ultimate truth of the results (not even, or perhaps especially not, in terms of an ‘approximation of the truth’) but on the process of careful examination and permanent re-evaluation of the empirical data and reformulation of the models and theories involved, as well as on the inherently collective (incl. polemical and/or competitive) and historical nature of the long-term endeavor and its record of success, not only in its technological applications but also as an ideological crucible.³⁶⁹

Perhaps even more importantly, epistemic kitsch appears to have for a function to distract the attention from real issues and does impede the kind of critical reflection that is being encouraged in the above.

All of this goes for academia as a whole but perhaps the problem is more acute in the case of math: if math is where we investigate certain core aspects of our rationality and our interaction with nature, at their purest, if math is supposed to set the standards of intellectual rigor, both

willing to discuss what motivates decisions that have an impact on the outcome of one’s work, as it is presented to the public, how can that be a good thing?

³⁶⁸ In the same way that math has been presented as promoting rationality and even “good taste” (incl. by LW), epistemic kitsch has the opposite effect of destroying the potential hygienic/therapeutic value of rationality by replacing it with fantasy.

³⁶⁹ The study of the relations between science qua body of knowledge on the one hand and its technological ‘applications’ on the other, as well as of the practicalities of agenda setting, financing, etc., as conducted with great success in the 20th century in various academic fields (Philosophy and History of Science, Science and Technology Studies, etc.) and the resulting awareness of the fact that science/knowledge is a deeply social/societal phenomenon, should not obscure the historical importance of science as a model for our relationship as humans to our environment and as model for our ability to deal with conflicting opinions in a rational and productive way, i.e. as an *ideology*.

within the purely academic/scientific tradition and as an integral part of the educational system at large, if math wants to continue to claim to offer an educational standard for rationality, then we can't allow bad faith and bad taste to fester at its core, especially now that 'bad faith' (i.e. checking the boxes of formal procedures while bypassing what is their purpose) has become a major societal problem, in that it affects the rationality embodied in some of the core political and judicial institutions.³⁷⁰

This being said, it bears repeating that these issues should not be viewed as external to actual math: not only is the demarcation between what is math-internal and what is math-external not that sharp once one adopts a pragmatic point of view, but -as we argued here above- the societal position of math and the way math construes its own identity as a practice are closely related aspects.

I have argued that the critical aspect that is at the core of LW's PhilMath, is almost entirely lacking in present-day PhilMathPract and I have attempted to demonstrate that there should still be room for a properly critical and properly philosophical approach to PhilMath, not only at the margins of PhilMath but even -in a Wittgensteinian vein- at the core of what makes math math.

3.3 Wittgenstein's philosophy of mathematics as criticism and critique

The specific aim of this study was to focus on a few lines of thought within LW's PhilMath that are not often focused on in the literature, especially the critical remarks that nobody seems to like and makes most of us cringe (cf. section 0.1(B)). It is worth noting that – apart from the aesthetical notes partly published in *Culture & Value* (cf. our analysis of LW's remarks on Mahler in section 2.0.2(A)) and perhaps a number of private occasions – LW's most vehemently moral criticisms can be found in the context of his work on mathematics.³⁷¹ Looking at this specific corpus has uncovered a few strands within LW's philosophy at large for me, of which I didn't previously realize the importance.

³⁷⁰ It would be interesting to further reflect on the origins of logic (and formal reasoning in general) as part of ancient Greek legal practices (cf. Dutilh Novaes (Dutilh Novaes 2012) for the idea that formalism is a way to democratize knowledge - an idea that in its turn deserves further scrutiny) and -for that matter- the relation between formal reasoning and bad faith. On the one hand, philosophical logic appears to originate in Plato's attempt to make sense of the bad faith of the sophists (for an analysis of the passage in Plato's *Sophista* that dramatizes this event, see Hoekstra & Scheppers (Hoekstra and Scheppers 2003)). On the other hand, the above suggests that formalism generates its own kind of bad faith.

³⁷¹ LW's objections to the inauthentic do not only refer to the ethical but also to the aesthetical side of 'fakeness'. Aesthetical inauthenticity recalls cheesiness, kitsch, 'special effects' and ornamentations (for the significance of the notion of 'ornamentation' with respect to both LW's life and his philosophy, see Brian McGuinness' 'Asceticism and ornament' (McGuinness 2002b)).

(A) a list: what is LW objecting against?

I thought it would be fun to try and list up the various things LW is known to have objected to in one context or another:

- hypocrisy, vanity, theatricality, pretense, lack of sincerity... in human behavior (other peoples' or his own)
- the idea that Gödel's 'numbers' are actually numbers
- the idea that a line is a set
- Cantor's extensionalism about infinite sets
- proofs that do not actually convince
- 'metaphysical' questions like "what is time?"
- 'modern' classical music, but especially Mahler's traditional-sounding parts
- Russell's later, vulgarizing and/or political work in general, and his 'shilling-shocker' in particular³⁷²
- a bed on wheels
- diagonal techniques as proofs or 'theories'
- the image of more and more things being crammed in an increasingly tiny space
- anachronistic ornamentation in architecture and interior design
- the claim that formal systems can actually be used to convince one of the reliability of basic arithmetic
- the idea that there can be something sensational about mathematical results
- the idea that contradictions in a formal system are the end of the world

This list -as well as most lists- is -as I said- 'fun', but it is also interesting as an illustration of the following important facts:

- (1) there is a common thread in LW's evaluations, which I summarized under the heading "authenticity";
- (2) the technical-philosophical aspects and the more personal-existential aspects of LW's thought converge in this regard.

(B) recap: from nonsense to fake sense to pretense

The problem of meaningfulness (as opposed to nonsense) has been a constant in LW's philosophical work from the beginning. As pointed out in section 1.2.1, nonsense/meaninglessness/senselessness was one of the central issues dealt with in the *TLP*.

³⁷² LW did maintain until the end that Russell in his prime was very good in conversation (Bouwsma 1999).

We also saw that, insofar as nonsense is a problem, nonsense is fake sense, something that looks like it is meaningful, but isn't, and in section 2.0, we saw that this problem of fake meaning was a problem LW inherited from -most notably- the culture-critical journalism of Karl Kraus.

What did change in LW's later work is how meaning/meaningfulness was construed, conceptually:

- in LW's early work, meaningfulness was construed in terms of being a picture of reality and anything that could not be analyzed as a combination of elementary bivalent (true or false) propositions, was discarded as meaningless;
- in his later work, meaningfulness was construed in terms of embedding in everyday practices / our everyday lives: any discourse that has a function within a real-life everyday practice is ipso facto meaningful; discourse that fails to be embedded in that way is meaningless.³⁷³

This evolution is also reflected in LW's work on math. Apparently, what brought LW to philosophy, was his interest in the issue of the foundations/Grundlagen of mathematics: what is the hard bedrock underneath math? In line with his development as a philosopher in general, LW's approach to this basic issue changed dramatically:

- at first, LW participated in the logistic movement (?) that sought to find bedrock in the crystalline and unified system of logic, as represented by formal logistic systems;
- then, LW evolved towards an 'anthropological' approach, in which mathematics was no longer a unitary propositional system but a heterogeneous bunch of techniques, as applied in an even messier hurly-burly of everyday practices; in this context, the validity and meaningfulness of math (or any other type of practice) is not necessarily problematic; however, what mathematicians and philosopher say about math is often not embedded in everyday practice and (as we have seen throughout Part 2) LW does not hesitate to criticize this type of discourse in very harsh terms.

It is important to understand that 'everydayness' is a key concept in all of this: de facto, all and every discourse is embedded in a practice: Cantor's talk about math is deeply embedded in long-standing traditions of religious and philosophical discourse and so is the metaphysical talk LW is supposed to have railed against as nonsensical. I have repeatedly pointed out this inherent weakness in the concept of everydayness: what counts as 'everyday'? why would Gödel's stuff, or Cantor's stuff, or Heidegger's not be 'normal'? isn't their work part of their everyday? why would one choose to make that distinction? why wouldn't these practices

³⁷³ The chronology of LW's work on math may turn out to be a crucial ingredient if one wants to reconstruct the chronology of LW's philosophy at large.

sufficient to give meaning to the discourses that come with them? what's the problem with these supposedly 'non-everyday' practices? The Spenglerian notion of the 'organic unity' of a healthy culture is politically suspect and descriptively not viable in that it is in direct contradiction with the obvious diversity (fragmentation?) of the everyday in general.³⁷⁴

At the end of Part 1 of this study, we concluded that 'everydayness' was not a result of LW's work on mathematical practice but part of an agenda underlying his investigations.³⁷⁵ This observation offers us a solution to the paradoxical tension between LW's stated anti-revisionism and his blatantly revisionist critical remarks: LW lets *everyday* talk be and objects to talk that was not embedded in *everyday* practice. Whatever we may think of this concept ourselves, it is undoubtedly an inherent, bottom-line feature of LW's outlook.

In section 2.0, I pointed out how 'everydayness' was part of a deep-rooted (?) and wide-ranging ideological (?)³⁷⁶ construction that LW shared with many of his contemporaries:

- **authenticity** (vs. fakeness) is a core concept, applicable at the existential level, but also at the societal/political level, as well as -importantly- at the epistemic level;
- authenticity is related to **everydayness** in that, at least for LW,³⁷⁷ the difference between the authentic/meaningful and the inauthentic/meaningless appears to coincide with the difference between the everyday and the non-everyday;
- embeddedness in everydayness is in its turn related to a Spenglerian vision on the decline of culture through **fragmentation** (?): in a healthy culture, all aspects of society form an organic whole; the disconnect between mathematical discourse and the techniques and applications that make math meaningful, or the (alleged) disconnect between modern music and the rest of the culture, are for LW symptoms of the 'illness of our era'.

In section 1.3 and throughout Part 2, I pointed out that LW's PhilMath consists essentially of a fundamental critique of what philosophers and mathematicians say about mathematics. LW's criticism targets the following concepts:

³⁷⁴ NB that Heidegger appears to view the fragmentation of the everyday as a reason to see it as the cause for inauthenticity, which I completely disagree with, but which is straightforwardly coherent with his Spenglerian views on culture, but not at all with the phenomenology of the everyday he starts out from in *Sein und Zeit* (Heidegger 1967); cf. Scheppers (Scheppers 2017), Chapter 2). As already mentioned above, LW's processing of the same basic ingredients is much more tension-ridden than Heidegger's, in that LW's attitude vis-à-vis the diversity and fragmentation (?) that is characteristic of the everyday is harder to pinpoint.

³⁷⁵ Elsewhere ((Scheppers 2017), Chapter 2, §4), I argued that a very similar phenomenon can be observed in Martin Heidegger's work.

³⁷⁶ It is unfortunate that the term 'ideological' has gotten a negative connotation and if another term with less baggage was available I would have adopted it gladly, but the core meaning of the term fits exactly what I mean.

³⁷⁷ As mentioned before, I am not clear yet about the similarities and dissimilarities between LW outlook, and some of his contemporaries (Spengler, Heidegger, ...).

- **foundationalism:** the idea that the validity of mathematical technique ultimately depends on its being integrated into a foundational framework
- **monism:**
 - the unity of mathematics: the idea that math, as it is, is (and has to be) a single system;
 - totalitariness/completeness: the idea that everything mathematical (including all that mathematicians will come up with in the future, or anything that is deemed or will be deemed relevant to our understanding of math) can and should be integrated in a single system;³⁷⁸
 - the naturalness / objectivity of mathematics: the idea that mathematical results constitute facts about the world (which includes the Platonist variant of objectivism, according to which this world/nature is separate a realm of reality);
- **exceptionalism:** the idea of the uniqueness / specialness of mathematics, epistemologically (as a body of knowledge), ontologically (as a separate realm of reality), pragmatically (as a human endeavor).

It should be clear that these aspects are (1) definitely at the core of the standard view of mathematics, but (2) at the same time definitely go beyond the math-specific: monism, naturalism etc. are not ideas that are specifically mathematical and should not be treated that way.

Interestingly, LW also repeatedly insists on the fact that the main difference between him and most mathematicians / philosophers of mathematics is that the latter don't focus on the presuppositions³⁷⁹ that underlie their mathematical discourse but simply remain within and perpetuate the framework they were trained in, whereas for LW, focus on what is presupposed in these frameworks needs to be the main focus of PhilMath (LW's remarks on Ramsey, quoted in section 3.2.2(B) above, illustrate this quite explicitly). In other words, for LW the identity of philosophy (as opposed to e.g. historical or intra-mathematical approaches to math) crucially depends on its focus on presuppositions.

So: LW's work is **inherently critical**, not only in the 'vulgar' sense of 'expressing ethical/aesthetical judgments' (which we have seen he does do in his PhilMath), but also in

³⁷⁸ Case in point: the tendency for math to try and include 'meta'-aspects (i.e. aspects that need not be specific to math) within its own system, such as proof-theory, model-theory, category-theory, ... My claim is that these aspects are better understood outside the formal system, in that they are in common between math and other endeavors and including them within mathematics destroys all possibility of articulating the commonalities between math and other human activities in a meaningful way.

³⁷⁹ Sass 2001, p. 120: "Indeed, he doubted that anyone who lacked this capacity for getting outside normal presuppositions could really be called a philosopher at all." [with a reference to the paragraph about Ramsey, quoted here above].

that some of its core conceptual resources (pragmatism, everydayism, holism) are motivated by ethical/aesthetical or even cultural/political considerations. One of the claims I want to derive from the work presented above (perhaps overly polemical for the present climate, especially in PhilMath, as well as in Wittgenstein-scholarship) is that even those philosophical claims within LW's work that don't look overtly critical, cannot be properly understood without understanding the conceptual links between (1) central elements in his conceptual apparatus and (2) his critical agenda.

If I am right, this context also explains why LW's criticism of apparently math-specific discourse takes the shape of genuinely ethical indignation: for LW, the problem with the philosophical/mathematical ideas he disagreed with was not that they were technically incorrect, but that they contributed to the decline of to the cultural values he affiliated with.

(C) LW as Kantian critique: "bedrock" and "the given"

LW's work is also 'critical' in a perhaps 'deeper' (?), more technical, Kantian, sense. LW's major concerns are an avatar of Kant's major concerns with the nature of 'the given'. The question as to what counts as 'given', what is our 'bedrock', continues to be LW's main problem, despite the spectacular evolution in LW's views on what constitutes ultimate bedrock. Again, I want to emphasize the continuity rather than the chronological succession³⁸⁰ and I have no intention to do proper historiographical work here,³⁸¹ my only point being that LW's conception of what a philosopher is supposed to do, is -in a certain sense- very much in the tradition of Kant's concept of 'critique'. Let's start by recapitulating a few elements that we encountered in the above:

- LW evaluates mathematical discourse in terms of the meaningfulness of what is being said, not only (and in many cases: not even) in terms of the correctness of what is being said;
- LW criticizes the agenda's underlying the Grundlagen-debates, Gödel's contribution, etc., rather than the technicalities of the arguments within the debates themselves;

³⁸⁰ Some authors downplay the Kantian aspect of LW's work, for more or less superficial reasons (Steinvorth (Steinvorth 1979), Sass ((Sass 2001), pp. 116-117), Egan ((Egan 2019) , p. 150). I am not interested in disputing these arguments, however important they may be from different perspectives. Some of the apparent differences may also be purely semantic. Peneolope Maddy's interpretation of LW's work in terms of 'anti-philosophy', 'naturalism' and 'second philosophy' ((Maddy 1993), (Maddy 2007)), would be an extreme example of downplaying LW's belonging to the philosophical tradition.

³⁸¹ For a number of years, Robert Hanna was the go to reference for the topic of the relations between Wittgenstein and Kant (e.g. (Hanna 2017); (Hanna 2007); (Hanna 2001)). There is now an emerging body of recent work dealing on the topic (see e.g. (Pier 2022), (Ritter 2020), (Waxman 2019)). Again, for the purposes of the present study, my interaction with, let alone contribution to, this body of work will remain minimal.

- LW explicitly distinguishes his own philosophical approach from other approaches by emphasizing his interest in what is 'given' / presupposed before anything meaningful is being said.

LW's work is Kantian in the sense that it distinguishes between the contents of any type of discourse and the 'given' that makes it meaningful and sets the limits of its meaningfulness.³⁸² For LW, as for Kant, philosophy is ultimately all about what counts as 'the given', as 'bedrock', as 'transcendental' with respect to what is meaningful to us, i.e. what we perceive, say and do. The Kantian notion that philosophy should focus on the pre-meaningful, presupposed, transcendental (?) remains central to LW's conception of the aims of his philosophical activity, but how this pre-meaningful 'given' was conceived of, evolved significantly throughout his development:

- in the *TLP*, LW discusses the function of his work in very overtly Kantian terms: (abgrenzen, etc.):³⁸³ tautologies are presented as not saying anything about the world; therefore, logic is considered transcendental;

³⁸² Let me briefly engage with the following excerpt from Egan 2019, p. 150 on LW and Kant:

The interlocutor of Z §351 might well think she shares Wittgenstein's thought that concepts are only at home in the language-games in which they are used. But for Wittgenstein, this way of characterizing the relationship between concepts and language-games is still too loose because it supposes that the concepts are intelligible at all apart from the language-games in which they have a use. Although he clearly has many affinities with Kant and the post-Kantian tradition, it would miss the mark to characterize Wittgenstein as delineating the conditions for the possibility of the concepts that we have precisely because such delineation engages with the question of the conditions under which we *could* have such concepts.

Also interesting is footnote 13:

We find a similar remark at *PI* §142, where Wittgenstein imagines lumps of cheese regularly growing or shrinking without obvious cause. In such cases, rather than saying that we *could not* measure the weight of cheese as we currently do, we should say simply that we *would not*: our current language-game of measuring cheese would simply have no point in such circumstances.

Egan is right in pointing out that LW is not delineating conditions of possibility for concepts, but what he does have in common with any other Kantian project, is that he is engaging with the notion of what is given and the idea that meaningfulness is determined within the framework of what is given.

By the way, a similar problem occurs within Martin Kusch's interpretation of LW's rule following stuff in terms of 'assertability criteria'. To me the term "conditions" sounds bizarre in this context, as if actions could occur without having "a point", as if pointless sentences are filtered out post hoc according to these criteria. Would one be similarly inclined to speak about "feasibility constraints" on non-verbal action? I think not. From a pragmatic point of view, the meaning of utterances should be viewed in the same way as the 'sense' of other actions (this is exactly what I meant when coining the 'Pragmatics first' slogan; cf. section 1.1.1(E) above).

³⁸³ Contra Sass, who appears to claim that LW's early work was not Kantian, but Cartesian Sass (Sass 2001, pp. 116-117): "What now become central are not Cartesian issues and themes – the sense of inwardness involved in identifying with a mental core set apart from body, the emotions, and the external and material world – but, rather, the sense of removal and remoteness that derives from adopting the position of an external observer who exists somewhere outside both the self and the entirety of its world. Such an observer presumes to be able to adopt a totalizing or transcendental stance in which it is possible to know, describe, or somehow intuit the knowable world as a whole in its most fundamental relationship with the human mind. These issues have a more Kantian or post-Kantian flavor, for they pertain to the issue of limits, to questions about the nature and the knowability of the boundaries of possible experience or of sensible discourse itself. Such issues were, in fact, central throughout the entire course of Wittgenstein's philosophical career. By considering them, it is possible to show that Wittgenstein, both early and late, was always driven both to express and simultaneously to deny his schizoid inclinations. In this paper, however, I will focus on the earlier period."

- somewhat later, grammar becomes LW favorite conceptualization of what precedes meaningfulness: within this framework, grammatical rules are ‘transcendental’ with respect to meaningful language use;
- in LW’s mature work, this role was taken over by such concepts as Language Games, Forms of Life, ‘our lives’, etc. , i.e. the kind of holistic structures that I subsumed under the umbrella term ‘practices’.

Interestingly, but perhaps not surprisingly, one of the more explicit articulations of LW’s engagement with the notion of ‘the given’, deals with math: in *PPF* §341-345,³⁸⁴ LW reflects on the fact that people in general agree on the results of calculations, as the basis of what is called ‘**mathematical certainty**’. In typical fashion, LW introduces a hypothetical circumstance in which people do often come to disagreements about calculations and points out that if such disagreements occurred more or less systematically (for instance, because people thought the signs on the paper changed, or their minds slipped, ...), mathematical certainty would not exist. LW refuses to offer a quasi-/pseudo-causal explanation and uses this example to drive home -what I called- the holism and the structuralism of his approach: even it is true that it would be impossible to use certain (unreliable) types of paper and ink as a support for calculations, it would still not be correct to say that mathematical certainty depends on the reliability of the paper and ink: the reliability of the support to that purpose only makes sense if one is already familiar with what a reliable calculation, using a reliable support, would be.

This reasoning leads up to the following seminal one-liner:³⁸⁵

³⁸⁴ 341. Es kann ein Streit darüber entstehen, welches das richtige Resultat einer Rechnung ist (z. B. einer längeren Addition). Aber so ein Streit entsteht selten und ist von kurzer Dauer. Er ist, wie wir sagen, ‘mit Sicherheit’ zu entscheiden.

Es kommt zwischen den Mathematikern, im allgemeinen, nicht zum Streit über das Resultat einer Rechnung. (Das ist eine wichtige Tatsache.) -- Wäre es anders, wäre z. B. der Eine überzeugt, eine Ziffer habe sich unvermerkt geändert, oder das Gedächtnis habe ihn, oder den Andern getäuscht, etc., etc., -- so würde es unsern Begriff der ‘mathematischen Sicherheit’ nicht geben.

342. Es könnte dann noch immer heißen: “Wir können zwar nie wissen, was das Resultat einer Rechnung ist, aber sie hat dennoch immer ein ganz bestimmtes Resultat. (Gott weiß es.) Die Mathematik ist allerdings von der höchsten Sicherheit, -- wenn wir auch nur ein rohes Abbild von ihr besitzen.”

343. Aber will ich etwa sagen, die Sicherheit der Mathematik beruhe auf der Zuverlässigkeit von Tinte und Papier? Nein. (Das wäre ein *Circulus vitiosus*.) -- Ich habe nicht gesagt, warum es zwischen den Mathematikern nicht zum Streit kommt, sondern nur, daß es nicht zum Streit kommt.

344. Es ist wohl wahr, daß man mit gewissen Arten von Papier und Tinte nicht rechnen könnte, wenn sie nämlich gewissen seltsamen Änderungen unterworfen wären, -- aber daß sie sich ändern, könnte ja doch nur wieder durch das Gedächtnis und den Vergleich mit andern Rechenmitteln sich ergeben. Und wie prüft man diese wieder?

345. Das Hinzunehmende, Gegebene -- könnte man sagen -- seien Lebensformen.

³⁸⁵ Boncompagni (2011 (Boncompagni 2011); 2022 (Boncompagni 2022)) makes a lot of this excerpt, but -as far as I can see- she does not acknowledge any Kantian connotations. For an insightful account that does make the link with the Kantian tradition in terms of the given and the giving, see Hurley ((Hurley 1998) chapter 6; cf. Scheppers 2017 (Scheppers 2017), chapter 1, §5). To avoid any confusion, it is perhaps necessary to clarify the fact that I am NOT employing the term “the given” in exactly the same sense as in the phrase “the myth of the given”, as discussed in Hurley’s chapter).

It is important to understand the dynamics, the 'information flow', in this one-liner. It occurs in the context of the question as to 'what grounds (?) mathematical certainty'. Various possible answers have been excluded. The question is now reformulated in the first part of this sentence (in linguistic terms the 'topic'): "what is it that we have to take along for the ride? what is it that is the given to us?". The answer is the second part (the 'comment'): "Forms of life".³⁸⁷

In other words, the message appears to be: there is no answer to the question of the given, there is nothing to explain, beyond the Forms of Life. There is nothing to reduce them to: the very givenness, the sheer facticity, of Forms of Life is irreducible, i.e. it cannot be explained by reducing it to the physical aspects of the objects involved, nor to any cultural aspects of the practice, nor to the cognitive or biological aspects of the agents: what is given, are the Forms of Life in all their irreducible multidimensionality and complexity.

Perhaps the most accomplished articulation of LW's mature views on the given is the wonderful image of the river and its bedrock in *ÜG* §§94-99,³⁸⁸ in which LW expresses his

³⁸⁶ The translation given in (Wittgenstein 2009) is: "345. What has to be accepted, the given, is -- one might say -- forms of life." A number of things can be said about the language of this one-liner and its translation:

- the translation has the copula "is" in the singular, whereas the German original has the copula "seien" in the plural (this difference is probably a mere matter of idiom);
- the copula in the German is also in the 'conjunctive', probably triggered by the modal marker 'könnte man sagen'/'one could say', giving the original a less affirmative flavor than the translation;
- more importantly, whereas 'to accept' certainly can be a translation for 'hinzunehmen', it is also potentially misleading in the present context: the verb literally means "to take along, along with other things", it does not necessarily imply an approval or even a choice; as a matter of fact, I believe it should be obvious from the context, that the idea of 'choosing to accept' would be completely impossible: one obviously does not choose to accept 'one's Form(s) of Life' (including the writing technology or the calculation techniques that are available, etc.) or not (for the fact that such interpretations do exist, cf. Boncompagni 2022, pp. 42-43; for the record: note that I would not necessarily disagree with those who object to LW's 'quietist' (?) politics, only that the term 'hinzunehmend' in this passage is not a case in point).

³⁸⁷ I am using the term topic-comment and not subject-predicate, because grammatically speaking, Lebensformen is the subject. Just saying.

³⁸⁸ 94. Aber mein Weltbild habe ich nicht, weil ich mich von seiner Richtigkeit überzeugt habe; auch nicht, weil ich von seiner Richtigkeit überzeugt bin. Sondern es ist der überkommene Hintergrund, auf welchem ich zwischen wahr und falsch unterscheide.

95. Die Sätze, die dies Weltbild beschreiben, könnten zu einer Art Mythologie gehören. Und ihre Rolle ist ähnlich der von Spielregeln, und das Spiel kann man auch rein praktisch, ohne ausgesprochene Regeln, lernen.

96. Man könnte sich vorstellen, daß gewisse Sätze von der Form der Erfahrungssätze erstarrt wären und als Leitung für die nicht erstarrten, flüssigen Erfahrungssätze funktionierten; und daß sich dies Verhältnis mit der Zeit änderte, indem flüssige Sätze erstarrten und feste flüssig würden.

97. Die Mythologie kann wieder in Fluß geraten, das Flußbett der Gedanken sich verschieben. Aber ich unterscheide zwischen der Bewegung des Wassers im Flußbett und der Verschiebung dieses; obwohl es eine scharfe Trennung der beiden nicht gibt.

98. Wenn aber Einer sagte »Also ist auch die Logik eine Erfahrungswissenschaft«, so hätte er unrecht. Aber dies ist richtig, daß der gleiche Satz einmal als von der Erfahrung zu prüfen, einmal als Regel der Prüfung behandelt werden kann.

99. Ja, das Ufer jenes Flusses besteht zum Teil aus hartem Gestein, das keiner oder einer unmerklichen Änderung unterliegt, und teils aus Sand, der bald hier, bald dort weg- und angeschwemmt wird.

view of the given by means of a comparison with the flow of water in a riverbed: there is at each point in time a difference between the bedrock and the river; there is also a difference between the flow of the water and the changes in the bedrock; even if the difference between the flow of the water and the change in the riverbed is not always very sharp, it still would be wrong to say that logic (qua study of the bedrock) is an empirical science (i.e. the study of the water).³⁸⁹

Of course, the way in which the problem of the given is formulated, has been displaced in LW's (later) work as compared to Kant's, in several ways, and it could be argued that this is what determines LW's place in the history of philosophy. From a birds-eye, broad-brush, historical perspective, LW's specific contribution, beyond the original Kantian position, is to point out that what is 'given' in any given case is characterized by the following features:

- holism: the given can no longer be conceived of as a relatively simple set of features of our cognition or of the world itself ('categories', etc.), nor as the logical structure of propositions, but only as an open-ended hurly-burly of irreducibly complex, multidimensional practices;
- contingency, variation and change have been introduced at the heart of the given: practices are inherently and irreducibly variable and historical;
- deflation of transcendence: to the extent that the traditional, Kantian use of the term 'transcendental' still carried an (undesirable?) metaphysical weight (and that remains to be seen), Wittgenstein's implementation of the concept of transcendence has no longer any connotation of belonging to a 'supernatural realm';³⁹⁰
- as a result, the given is no longer articulated in terms of conditions for meaningfulness, but in terms of the irreducible facticity of meaningfulness: the ultimate ground for meaning, is that things -as a matter of fact-³⁹¹ do make sense to us within the context of our everyday practices.

The way the problem of the given first manifested itself to LW, was probably the way it was presented in the context of the Grundlagen-debates concerning math, and LW's contribution to these debates has been to point out that there is an iceberg of given stuff beneath every propositional system. Gödel may actually agree with this idea, put this way, but unlike Gödel and other Platonists, LW appears to not only oppose formal systems as the ultimate ground

³⁸⁹ Cf. Floyd's recent monograph on LW's PhilMath, in which she characterizes the last period in LW's work as "Fluid Simplicity". Also note that LW apparently continues to view what he does as 'logic'.

³⁹⁰ I don't know enough about Kant to make any claims regarding Kant, but I am not sure that it would be charitable to Kant to attribute such 'metaphysical heaviness' to his conception of the 'transcendental' (did he coin the term to distinguish it from 'transcendent'?).

³⁹¹ I would personally be ready to call this a 'metaphysical fact'. The working title for my long-term project is 'The metaphysics of doing the dishes'.

for math, but any type of propositional truth, however conceived, at all: LW argues that the given underlying math is not a crystalline mathematical universe, but a messy bunch of ultimately non-propositional, heterogeneous, contingent and fluid human practices.

(D) criticism, critique and therapy

LW's interest in philosophy originated with the problem of the foundations of mathematics. Here above we have argued that the way LW conceived of this issue was steeped in the Kantian tradition, in that he framed the problem in terms of an investigation into the pre-meaningful ('transcendental') 'given'. His answer to the question ended up non-Kantian: instead of a neat closed set of categories, we are presented with an open-ended, variable, contingent mess of everyday practices as the irreducible 'given' ground (?) for meaning. We have also seen that there is an inherent link between (1) central elements of LW's conceptual apparatus (pragmatism, holism, everydayism), (2) his critical agenda (ultimately in terms of 'authenticity vs. fakeness') and (3) Spengler-like ideas about the decline of the culture he affiliated with. In the context of his work on math (as well as elsewhere), LW repeatedly articulated these links, quite explicitly.

All of this is at the service of a culture-critical approach to meaning and nonsense: the loss of meaning in certain types of discourse is due to a lack of embedding in everyday practice, which is in turn a symptom of the disintegration of the culture of which these discourses are a part. The way LW formulates his critique involves the systematic use of ethical and aesthetical vocabulary, as well as the idea that epistemic issues are ultimately a matter of lifestyle and hygiene.

And this brings us back to the stated 'medical' aims of LW's philosophy at large: philosophy is supposed to be a kind of therapy for various linguistic, conceptual or otherwise cultural illnesses. I believe that -if anything- the above shows how LW's conception of philosophy can be made to make sense for us: what I have called 'a critical approach' in the above, is indeed -in a certain sense-³⁹² a matter of **epistemic hygiene**.³⁹³

³⁹² NB that the application of medical vocabulary to various cultural phenomena, especially as an expression of criticism or condemnation, is again a sign of the times: another not necessarily desirable (insalubrious?) feature that LW had in common with his contemporaries, including the Nazi ideologues.

This remark illustrates a more general problem: how to deal with *the fact* that a number of concepts (authenticity, hygiene, the everyday, nature, ...) are systematically linked with a political ideology that is undesirable (at least to me), whereas one still would like to be able to point out that -for instance- certain types of behavior are in an important sense 'inauthentic' (bad faith, etc.)? This issue (including a reflection on the differences in the ways in which the same cluster of concepts is put to work by LW, the upper-class; politically ambiguous, conservative, and Martin Heidegger, the revolutionary Nazi) will be the topic of further research on my part.

³⁹³ NB that none of this sounds like naturalism or Maddy's 'second philosophy'; on the contrary, LW's philosophy, in which his PhilMath occupies a central place, appears to be firmly rooted in 'philosophia prima', *both* in its focus on what is presupposed ('the given'), *and* its stated aims, including the fact that for LW, philosophy should operate at the same time at the existential level and at the level of society at large.

3.4 Summary

(A) Approach

This study focuses on LW's PhilMath and more specifically on a number of critical remarks targeting various aspects of early 20th century mathematics and PhilMath. Due to fact that this study is part of an encompassing research activity focused on the concept of 'practice', I was most interested LW's later work, but many of the lines of thought I focused on here, showed remarkable continuity, which made it possible (and also natural) for me to include passages from LW's earlier work.

As for the corpus of texts I focused on, I mostly started from a close reading of a number of excerpts that are not often focused on:

- in section 1.3, I analyzed a prolonged excerpt from MS-126, mainly consisting of a long list of often made-up examples of fringe applications;
- in sections 2.1 through 2.3, I focused on a number of excerpts in which LW objects against generally accepted aspects of mathematics.³⁹⁴

I read these selected passages from LW's work against the background of the following three aspects:

(1) LW's oeuvre as a whole, characterized by the following features:

- **pragmatism:** meaning is conceived of in terms of embedding in practices; in the case of mathematics, this implies the primacy of applications, out of which mathematical techniques emerged and within which they function in a straightforwardly meaningful way;
- **holism:** practices and forms of life are multidimensional structures involving not only the verbal and non-verbal activities of the agents, but also the biological, physiological and cognitive features of these agents, as well as the physical properties of the world they

Let me add a final corollary aimed at Wittgenstein-scholars who want to participate in present-day PhilMath. It is tempting to want to recuperate LW as part of a quasi-empirical approach to math and mathematical practice (I have indulged in that in my own work in linguistics and will certainly indulge in that again, if I ever get to publish the research on the concept of practice that this study is a part of). However, the concept of 'everydayness' and the inherent value-judgment that is implied by it (let alone the Spengler-like strands), are an awkward fit with such an approach. Those of us who feel called to represent a Wittgensteinian voice within current debates within PhilMath have to own up to LW's own mission, as he appears to have understood it himself, or to make a clear distinction between their own agenda and LW's.

³⁹⁴ In order to avoid misunderstandings due to the sanitizing editorial practices of LW's literary heirs (of which we encountered a few examples in the course of our explorations), I read all these excerpts in the version directly taken from the manuscripts, as published in the Bergen online edition of the Nachlass.

operate in; as a corollary to this holism, it follows that the propositional cannot be isolated from the non-propositional, which in its turn implies that no practice can be conceived of as mainly (let alone exclusively) epistemic;

- **structuralism**: the idea that the relations between the various dimensions that make up a practice or form of life are internal to the structure of these practices or forms of life, i.e. that their identity is determined by their place within that structure; in the case of math, a few of LW's more weird-sounding claims start to make sense if we take into account his structuralism: if it is literally true that the meaning of a mathematical term equals the way it is related to an actual mathematical technique with a proper real-life application, then a conjecture about which one has no idea how to prove it, literally has no meaning;
- **everydayism**: the idea that meaningfulness ultimately depends on embedding in everydayness is a fundamental component of LW's agenda (as opposed to a result of his analysis);
- **criticism**: a critical attitude and a critical agenda drives LW's philosophy; I identified the agenda underlying LW's criticism in terms of 'authenticity vs. fakeness' (see section (B) here below).

(2) the early 20th century Grundlagen-debates in which LW started his career (?) in philosophy:

the immediate context for LW's philosophy is the hotly debated issue as to what makes mathematics valid and it is hard to overestimate the central role of the various aspects involved in these debates for LW's work, not only his work on math and logic, but his philosophy at large.

(3) the biographical and cultural context from which LW's work emerged:

- biographical data shedding light on patterns of thought that were predominant in both LW's private life and his work (perhaps harder to separate in LW's case than in other cases), most notably his aversion for vanity, theatricality, insincerity and other avatars of inauthenticity;
- the general cultural context in which LW grew up, incl. the problematization of inauthenticity in terms of a lack of embedding in an organically united culture and the negative evaluation of the perceived dissolution of Western culture since the 19th century (as most famously articulated by Oswald Spengler); this aspect was a major ingredient in (at least some of) LW's lines of thought on math that we analyzed above.

[\(B\) LW's critical PhilMath](#)

I believe I have shown at least the following:

1. LW's criticism of certain aspects of Cantor's, Dedekind's, Russell's or Gödel's work cannot fruitfully be separated from the rest of his PhilMath; it is a central part, perhaps the very core of his work on mathematics;
2. LW's criticism must be understood within the context of the foundational issues that attracted LW to philosophy in the first place, i.e. LW's main concern remained the issue as to what makes mathematics valid, reliable, meaningful;
3. LW's criticism is consistent, in the sense that it always follows the same pattern, but also in the sense that it is coherent with other aspects of his PhilMath and his philosophy at large, fits in with patterns of thought that are also omnipresent in other aspects of LW's life, and -beyond that- in the cultural milieu from which he emerged;³⁹⁵ for the sake of ease of reference, I have used the terms 'fakeness vs. authenticity' to cover the concerns underlying LW's criticism;
4. LW's criticism is closely related to -what I called above- his pragmatism and his everydayism, in that he criticizes those types of discourse that don't make sense (suffer from a lack of meaningfulness), because the connection between them and everyday practices has been disrupted; for LW, the worst sins (?) committed by philosophers of mathematics are (1) that they try to sell in the end trivial conceptual problems with games that they have created themselves, as awe-inspiringly deep facts of nature, whereas (2) they are completely oblivious to (a) the relevance of the iceberg of not strictly mathematical and outright non-mathematical presuppositions that underlies and gives meaning to mathematical discourse, and most notably (b) the hurly-burly of everyday practices and the mathematical techniques that emerged from those practices;
5. LW's criticism of discourse about math operates at a very fundamental level, not at the level of technical details:
 - a. LW opposes the **monism** (totalitarianism/unitarianism) that characterizes most of mainstream PhilMath, i.e. the idea that mathematics consists in a single system/universe (formalizable or not), pointing out that mathematics is in actual practice heterogeneous and fragmented; in the same vein, he also opposes the idea that mathematical theorems are **facts of nature** rather than the consequences of the rules we set ourselves for some of the mathematical/logical games we play;
 - b. LW frontally attacks **foundationalism** as such, by pointing out that the techniques that make up real-life mathematics, as they have been applied in everyday practical situations, do not need any axiomatic systems to ground them; as a matter

³⁹⁵ I have tried to be careful not suggesting that psychological or socio-cultural somehow explain LW's philosophy (pace scholars such as Sass (Sass 2001) or even Monk (Monk 1990)).

of hard empirical/anthropological fact, they are much more secure than foundational systems ever could be; the axiomatic systems that are presented as foundational are actually at best new pieces of math, peripheral add-ons to the existing techniques;

- c. LW profoundly dislikes the **sensationalism** (I call it 'epistemic kitsch') that consists in presenting trivial consequences of the way our language games function, as something deep, mysterious, awe-inspiring ... or otherwise interesting; this dislike coincides with the ethico-aesthetical values that LW advocated in his everyday life;
 - d. many of the aspects of mainstream PhilMath that LW objects to ultimately boil down to its **exceptionalism**, i.e. the prevalent but baseless idea that mathematical practices are different from any other kind of practice, that mathematical objects are fundamentally different from any other kind of object, etc.
6. LW's critical remarks on mathematical topics are indistinguishable from the Spenglerian culture-critical strand in his work: his critique of set-theoretical parlance boils down to the idea that it participates 'the decline of western culture' by virtue of its being disconnected from the organic unity of that culture;
 7. LW's criticism is also a Kant-style critique, in that it is ultimately concerned with what is 'given', what is presupposed before anything can be meaningful; this is also the point at which LW situates the main difference between his own philosophical approach and the approach that mainstream PhilMath takes vis-à-vis math (cf. also item 4 in this list);
 8. I have argued that it would be desirable for present-day PhilMath (and especially present-day PhilMathPract) to reconnect with the critical role that LW attributes to philosophy at large and PhilMath in particular; in any case, it would be disingenuous to claim to represent a Wittgensteinian voice in PhilMath without owning up to the inherently critical nature of LW's work on math.

(C) Final remarks

Though this study started out aiming to show how LW's critical remarks on math illustrate LW's general approach to meaning and nonsense in terms of embedding in the everyday, and how the critical aspect coincides with his general outlook on philosophy as therapy etc., this research ended up substantially enriching my understanding of these more general aspects, especially by highlighting the importance of the Spengler-like strands in LW's thought and the tension-filled nature of LW's views on the fragmentation of the everyday.

The net result of reading my study may be that those who didn't like LW to begin with have even less sympathy for his work, and perhaps also that those readers who started out with a

basically sympathetic attitude towards LW, encountered aspects of his work that they find alien (or even repulsive) to be much more important than they previously thought.³⁹⁶ On the other hand: sooner or later, a critical approach to mathematical monism and exceptionalism should be put on the agenda, with or without LW.

In any case, I think have shown that LW's PhilMath should be read as an integral part of his oeuvre as a whole, both ways:

- on the one hand, LW's PhilMath can't be understood outside the critical agendas that drive his philosophy as a whole (incl. the uncomfortable Spenglerian strands);
- on the other hand, one should not forget how central math was to LW as a philosopher: not only did his interest in philosophy emerge from the Grundlagen-issue, but some of the main themes LW encountered as part of the Grundlagen-debates remained his main focus throughout his oeuvre.

³⁹⁶ I am sure I already owned a hardcover copy of the *TLP* with W.F. Hermans' Dutch translation in 1986-1987 and I got my first copy of *Über Gewißheit* around the same time; I bought the whole *Suhrkamp Werkausgabe* around 1990 and I have been reading this or that part of LW's oeuvre ever since, while being very aware of a great cultural gap between the author and myself and without ever having the slightest urge to become a Wittgenstein-scholar, although I reluctantly admit that Part 2 of this study could count as Wittgenstein-scholarship. I must say that I feel even less kinship with LW after writing the present study, but also that I have gained perhaps even more respect for the technical rigor of the philosopher. The man and the philosopher had to deal with a lot of tensions (more than can be seen if one focuses on only a few isolated topics) and manages to retain a remarkable level of coherence and integrity through it all. All this being said, I want to repeat that I deeply disagree with many aspects of LW's outlook on the world.

Part 4: Appendices

Whereas the main body of this manuscript consists of materials that could/should in the relatively short run give rise to a scholarly publication focusing on Wittgenstein's work on mathematics, the present appendices contain materials that emerged from the same research activity as the above, but that are not necessarily directly about Wittgenstein's work ³⁹⁷ and are not intended to be included in that publication, or in some cases, even published at all.

As these lines of thought have not been written out with a clear picture of their destination in mind, they will be even more programmatic, even more the result of cannibalism, from an even wider set of (my own) source texts, an even rougher draft. I sometimes adopt a tone and phrasing that came to me naturally at the moment I first wrote out my notes, but that I would not necessarily maintain should I rework this material for publication.

4.1 Notes on the pragmatics of formal systems

(A) Formalism as desemantization/depragmatization

(A1) spectrum of formalism

Formalism can be viewed as a spectrum (pace Dutilh Novaes) from unregulated, everyday use of natural language, over merely regulating the use of key words by means of a definition (what Dutilh Novaes would call a regimented language (Dutilh Novaes 2012, p. 58)), over a semi-formal axiomatized system like Euclid's, through a fully symbolic formalism like Principia Mathematica, to a system implemented on a computer.

But even the computer exists in a physical context that determines its workings but is outside the formalism, and for the computer to embody a formalism it needs to be used as a formalism, i.e. in the meaning-giving context of an actual mathematical practice. LW argues that it would not be correct to equate what happens in the computer with math. And not because of the

³⁹⁷ Some of the lines of thought that I develop here do originate in my reading of LW's work, but I am no longer focusing on the way they function within the context of LW's work and in their historical context, but on how they still may interact with present-day (2022) Philosophy of Mathematical Practice and Philosophy of Mathematics at large, and how they contribute to my own long-term research project on the concept of practice. Note that I do not necessarily share LW's agenda(s) either: e.g., the everydayness theme and the prose-calculus distinction are crucial to, and deep-rooted in, LW's thought, whereas I am very critical of these aspects of LW's heritage. Other lines of thought developed here have no direct relation to LW's work at all.

human vs. non-human distinction, but according to the meaningful-meaningless distinction, in terms of embedding vs. non-embedding in a practice (cf. section 1.1.1(H) above).

(A2) desemantization

One of the defining aspects of ‘formalism’ is that it is the result of ‘desemantization’ (in the sense of Dutilh Novaes 2012).³⁹⁸ In a formal language, symbols are manipulated without taking into account any meanings that may have been associated with them. So: one construes a system of symbols that are manipulated without -in principle- paying attention to any type of semantics.

We have a more or less clear idea of how it is possible to manipulate items according to rules, without attaching any meaning to the items (think of chess). To the extent that we operate with truly formal systems in a truly formal fashion in math, that would be a case of desemantization. However, it remains to be seen whether that is what actually happens in the case of mathematical formalism (see below).

(A3) de pragmatization?

There may also be an intuitive appeal to the notion of de pragmatization in this context: desemantization (in the above sense) is ipso facto de pragmatization, in the sense that the symbols are no longer seamlessly integrated in the practice they originally belonged in. In other words, the symbols that used to be informally used in deeply embedded practices, are taken out of these practices and are contemplated as stand-alone objects. In a way, LW thought in those terms: due to his inherent everydayism, LW did not make a clear distinction between everyday practice and practice in general.

However, this intuitive notion of ‘de pragmatization’ is inherently flawed: it is true that symbols in a formalized language are divorced from the particular practice in which they may have occurred having normal semantics and a normal use as a symbol; but the reason why symbols got de pragmatized and/or desemantized can only be understood in terms of a very concrete and specific practice, which is in its turn embedded in web of concrete and historically determined encompassing practices.

(A4) desemantization-resemantization and de pragmatization-repragmatization: problematic concepts

So: desemantization and de pragmatization are always immediately resemantization and repragmatization: as soon as a desemantized set of symbols are used, they ipso facto acquire

³⁹⁸ Alongside computability, a concept that need not directly concern us in this study (but cf. the issue of ‘dead vs. alive signs’ and machine-generated math in section 1.1.1(H)).

a new 'semantics': they may not continue to stand for the same things they used to stand for, but they will by virtue of the fact that they are manipulated, used, and talked about ipso facto acquire object-status and become meaningful in their own way within the practice they occur in.

Similarly, depragmatization is ipso facto re pragmatization: rather than having lost their function within a practice, they have acquired a different function than the one they had in the practice they originally belonged in. For instance: rather than being simply used as a straightforward tool, they have become an object of study. However, the fact that the tool acquires a new function does not mean that it is no longer a tool.

The function of a formal system in mathematics can be compared to the function of an experimental set-up in physics or chemistry. In both cases, we have constructed objects that serve the specific purpose of being 'observed' within a highly specific practical context. These are perhaps complex tools, but still tools; and tools are objects.³⁹⁹

Thus, depragmatization is an inherently problematic concept: in actual practice, all apparently de pragmatized activities (say: ritual behaviors) do have clear pragmatic functions, constitute clearly recognizable action types, are embedded in a Form of Life. What makes them qualify as special, must be a value judgement in terms of normality and/or everydayness, whether negatively as nonsense, or positively as sacred. Which makes sense.

In other words, "resemantization" (in a more general sense than the very narrow one used by Dutilh Novaes) and "re pragmatization" may be misleading terms: desemantization is a sensible practice and at no point are symbols ever 'out there on their own'.

(A5) Dutilh Novaes on the functions and origins of desemantization

Perhaps the most important idea that is developed in Dutilh Novaes' Chapter 2 is that formal languages are a technology.

p. 61:

The basic idea is that formal languages can be fruitfully conceived of as a technology. Of course, this is not very informative unless we can provide a more precise meaning to the rather vague term 'technology'. As a first approximation, a technology can be described as a specific method, material, or device used to solve practical problems. Formal languages as such are not a method by themselves, but they are devices that allow for the implementation of certain methods.

More precisely, formal languages are a cognitive technology

p. 64:

³⁹⁹ A taxonomy of objects, taking into account their function within practices, is one of the major contributions of Heidegger's early work (cf. section 4.1(B1) below).

formal languages are a technology that allows us to reason in ways that are fundamentally different from how we spontaneously reason in more mundane circumstances.

Specifically, they “allow us momentarily to ‘turn off’” our tendency (our “computational bias”) to automatically bring into play prior, contextual, knowledge whenever we try to solve a problem.

It would be interesting to further reflect on the origins of logic (and formal reasoning in general) as part of ancient Greek legal practices.⁴⁰⁰

According to Dutilh Novaes, one of the functions of desemantization is that it gives a degree of epistemic freedom, i.e. that one can manipulate symbols without interpretation, which allows for knowledge how without knowledge why. This in turn -according to Dutilh Novaes- also allows for a certain ‘democratization’ of knowledge in that the notation allows for solving problems by non-experts. Of course, one could also easily argue the opposite: depending on contexts (I mean: this would be an empirical/historical matter), the mastery of an abstract notation and the operations that allow for its actual application, require specialized training.⁴⁰¹

(A6) *the history of formalism and bad faith*

We saw that for LW, nonsense / inauthentic use of language consists in divorcing utterances from their ‘everyday’ context (for instance: “metaphysical” language). This is also the problem with formalisms of all kinds, which -by definition- are ‘desemanticized’ to a certain degree.

⁴⁰⁰ Cf. Dutilh Novaes (Dutilh Novaes 2012), p. 68: quoting Netz 1999, on the importance of persuasion at the heart of ancient Greek math and logic ; similarly, p. 78: “In a slogan, an argument, proof, or demonstration is a discourse; a calculation is a procedure.”

⁴⁰¹ p. 200:

a characteristic of processes of de-semantification is a certain degree of ‘metaphysical freedom’: one is allowed to use and manipulate signs even if it is not clear whether they in fact stand for any existing ‘thing’. [...]

Besides metaphysical freedom, de-semantification also seems to entail a certain degree of what could be described as ‘epistemic freedom’. As noted by Krämer, and discussed in Chapter 3,

signs can be manipulated without interpretation. This realm separates the knowledge of how to solve a problem from the knowledge of why this solution functions. (Krämer 2003: 532)

Of course, this separation of knowledge-how from knowledge-why may give rise to suspicions concerning surveyability and reliability – recall the need for epistemic justification of the notational techniques developed within the abacus tradition discussed in Heeffner 2007 and mentioned in Chapter 3. But, despite these legitimate concerns, if the notation somehow manages to establish itself as reliable, then its application typically represents a *cognitive boost* for the agent, precisely in the senses often discussed in the extended cognition literature. Moreover, notice that an effective and reliable calculating notation may also represent a *democratization* of knowledge: cognitive tasks which would otherwise only be carried out by experts can now be carried out by a wider range of agents.

p. 202:

Long before the computer became a universal medium and a programmable machine, we developed the computer ‘in ourselves’, which is understood here as the cognitive use of algorithmic sign-languages that are freed of the constraints of interpretation. (Krämer 2003: 534; emphasis added)

p. 202:

the view defended here is that de-semantification is neither a necessary nor sufficient condition for the cognitive boost effect, but it may greatly enhance it (for reasons which will become clear shortly).

Formalism starts with the idea that form can be separated from meaning. I have argued elsewhere (Hoekstra and Scheppers 2003) that the precise moment in history when this happened, is staged in Plato's *Sophista* 261c-262e: here, the Stranger points out that words come in different kinds and that some fit and some don't, and that the ones who fit mean something, and the others don't. All of this is news to Theaetetus, the Stranger's interlocutor. Let's also note that this revolution also deeply impacted the issues surrounding the notion of truth:

- In the pre-formalist situation, the main problem was: how is falsehood possible?
- After the formalist revolution, the problem became: how is truth possible?

So, initially, the *sui generis* relation between form and thing was simply a given and it was understood that -by default- speech 'spoke what is' (the direct object of verbs meaning 'to say' referred to -what we would call- reality. In this context, it is not so hard to understand that falsehood was a problem: after all, phenomenologically speaking, lies in the end do the same thing as truthful statements.

Soon enough, i.e. as soon as the bottom-up view of meaning became standard (up till now, but things might be changing), the leading problem became the opposite: how is it possible that words actually say something about the world? All the problems related to formalism start when you look at words in isolation and try to describe their meaning bottom-up, starting from their own semantics, rather than originating in their pragmatic function ("point"), "in context". This development culminated in Frege and in LW's *TLP*, in which truth value and meaning ended up coinciding.⁴⁰²

There is also a lot to say about the relation between formal reasoning and bad faith:

- on the one hand, philosophical logic appears to originate in Plato's attempt to make sense of the alleged bad faith of the sophists (how is it possible to lie?);
- on the other hand, the above suggests that formalism generates its own kind of bad faith.

As for (2) bad faith generated by the use of formal systems, the main problem is the relation between what happens inside the black box of the formal system on the one hand, and its interpretation in everyday prose on the other: what is the criterion to determine whether the outcome of operations in a formal system, once translated back into normal prose, is acceptable? ("the computer says yes!"). The classical formulation of this issue is LW's account

⁴⁰² I already quoted Ferreirós history of set theory (Ferreirós 2007) elsewhere: Russell's peculiar conflation of syntax and semantics has the effect that his work is dealing with philosophical logic, and even metaphysics, throughout", with in footnote: "The strange features of the famous *Tractatus* by his student Wittgenstein [1921] are thus more a symptom than a deviation." (footnote 1 on p. 332). The above suggests that this conflation of syntax and semantics (and metaphysics, for that matter) is not so much a peculiar feature of the approach of Russell and his pupil, but a logical (?) outcome of the evolution of logic.

of what is important about Gödel's incompleteness results (see section 2.3 above; see also appendix 4.3(B2) below).

(B) Mathematical formalism in practice / the pragmatic functions of formal systems

(B1) formal systems as objects

If one is interested in the ontological status of mathematical objects and/or the epistemological status of formal systems, a good starting point would be observing the ways in which such objects are actually manipulated by actual mathematicians in their actual everyday work (cf. ethnomethodological work such as Livingston's (Livingston 1986); (Livingston 2015)). This seems to be the only reasonable way to achieve a viable account of their place in a taxonomy of objects.

Martin Heidegger shows in §§15-10 of his seminal *Sein und Zeit* that 'Zuhandenheit' (i.e. being available in the context of an everyday practice), is the default way for things to 'be'. Cf. Scheppers 2017, Chapter 2, §2. Thus, one can envisage a taxonomy of objects along those lines, perhaps starting from broad categories such as these:

- tools: more or less permanent object that serve at executing an activity but aren't its outcome;
- ingredient: objects that serve as input to an activity and are transformed by it;
- products: objects that are the intended outcome of an activity;
- infrastructure: permanent objects that operate at the background of an activity.

Ultimately, it remains to be seen to what extent it is even possible to formulate any regularities beyond the individual practice.

So, a sensible answer to the question as to how desemanticized systems make sense may be: they are objects with a specific role within that practice. One can compare them with experimental set-ups in chemistry or mechanics: they are constructed and then used in very specific ways; in this case, the activity is 'to observe' (in the specific sense of the particular scientific practice at hand) and the experimental set-up has for a function to be observed (in that specific way).

It is important to note that the ontological status of mathematical objects in general is not problematic at all, not in actual mathematical practice, and not from the pragmatic point of view adopted here, either. What is a problem from our point of view, is Platonic talk about mathematical objects as belonging to 'a separate realm', not because the 'separateness' of the 'realm' (this could be an innocent terminological choice for distinguishing the 'worlds' that go with practices), but because of the implication that there is only one, eternal world of

mathematical objects that is not only separate from other realms (?) but also from mathematical practice.

Thus, whenever mathematicians create and start using/studying a new formalism, a new ontology arises, a new set of types of objects that make sense as such and it remains to be seen if each of these types are more like ingredients, or more like products or more like tools or more like infrastructure (furniture), or sui generis in the processes they are involved in, but there is no reason to a priori proclaim their unique nature.

(B2) formalism and conceptuality / meaning are not opposites / alternatives / complementary

Starting point is Ferreirós (Ferreirós 2016), Chapter 4, in which he defends the thesis of the “complementarity of symbolic means and thought”. My main argument is the idea that “conceptual thought” and formal systems are complementary ways to deal with the same subject matter is based on incoherent premises.

The problem formulated in terms of conceptual thought vs. formal systems is framed the wrong way: formalism is NOT a different way of making sense of ‘the same things’, it is NOT an alternative to ‘conceptual understanding’.

I am not sure whether what Ferreirós calls “conceptual thought” is a viable concept for explaining the meaningfulness of human experience, but this is not a point I want to argue here. What I do want to argue is: whatever the functions of conceptual thought may be in non-formalist contexts (whether non-formalist mathematical practices or any other practice), it plays exactly the same role in formalist mathematical practices.

Whatever one wants to make of “conceptual thought”, it can't be construed as having a similar function as, let alone as an alternative to a formal system. It should be understood that conceptual thought plays exactly the same role in mathematical practices involving formal systems as in mathematical practices that don't, as in doing the dishes, as in watching Bugs Bunny, as in discussing Heidegger.

One may object that the specific problem dealt with in this context, only arises within the specific context of 20th century PhilMath (and couldn't have arisen elsewhere in that specific way): mathematical formalism (i.e. the manipulation of symbols according to explicit syntactic rules, without reference to models outside the syntax) is part of a very specific set of practices (incl. discourses) that occurred at a precise point in the history of mathematics and as such has its very specific, historically contingent set of features.⁴⁰³

⁴⁰³ Larvor 2016 (p.c.) during a session of the Reading Group on Ferreirós' *Mathematical Knowledge and the Interplay of Practices* (Brussels, V.U.B., CLPS).

This historical remark is of course correct but does not imply that the tools used in the analysis have to be ad hoc, on the contrary, meaning in contemporary math has exactly the same infinitely complex contingency as any other practice: this specific practice -as a human practice- shares with all other human practices that it makes sense to practitioners and observers in exactly the same ways as any other practice.⁴⁰⁴ In this sense, formalist types of mathematics have their own 'conceptual thought', as much as non-formalist ('conceptualistic') mathematics. The presence or absence of 'sense' is not what makes them different from each other. So: it would not make sense to try and solve the issue using only ad hoc, math-related resources.⁴⁰⁵

(B3) thought experiment: a purely formal language

Imagine that agent Z construes a purely formal language, and then a number of expressions/strings in it. Imagine that Z insists that his strings have no meaning yet. Then, he plays around with his language and adopts formal criteria which allow him to distinguish strings of expressions that are 'plurp' from expressions that are 'vlarve'. In what circumstances would we admit that Z's plurp & vlarve formalism makes sense? Perhaps it would suffice that Z seems to find this practice worthwhile and Z is an otherwise 'normal' person? Perhaps we would discover that the practice has its own internal coherence, which other people appear to be able to understand if they take the time to learn it...

In any case, "to make sense" implies a relation to an encompassing context (a practice), but that relation and that context can be anything, incl. ludic or aesthetic functions, as well as industrial applications. This is exactly the point of LW's thought experiments involving fringe applications (cf. section 1.3 and section 3.2.1(C) above).

However, such a purely formal language is ipso facto not mathematics as it is practiced now: what makes math math (as we know it) presupposes a link with counting, adding, subtracting, multiplying, dividing, etc. , as well as a link with elementary geometry, surfaces and circumferences, etc. It is easy to imagine the existence and successful application of these techniques without any axiomatization, even without any propositionalization (so without the notion of 'truth'), but we would not be ready to call whatever practice that doesn't have a strong link with the numerals we use in counting "mathematics". This is basically LW's main argument against the whole foundational endeavor that dominated the PhilMath of his era.

⁴⁰⁴ This argument is very similar to the one against math's 'totalitarian' tendency to want to absorb not specifically mathematical aspects within its own formalism (e.g. model theory):

⁴⁰⁵ By the way, this is especially true if one claims to adopt an agent-centered, cognitive, etc. approach!

(B4) *two types of uses*

More concretely: it has been said that there are basically two ways in which you can use a formal system within the context of mathematics or something like mathematics:

1. as a formal representation of a pre-existing informal model: you operate it as a formal model and study it as such, but your intentions remain in actual practice connected to the underlying model;
2. as a stand-alone system: you can start to manipulate the symbols freely and creatively without regard to their representational potential; we could call this type of use “poetic” (see below; cf. also the idea of ‘epistemic freedom’ in Dutilh Novaes 2012, p. 200).

Both types of use appear to be present within mathematical practice.

(C) On poetic practices

Everyday human activities are eminently purposeful, i.e. steered/driven by pragmatic coherence. For instance, an agent walks toward the cupboard, to take out a plastic tile with detergent and sponges inside, in order to the dishes...

However, some of our actions do not seem to obey the same kind of purposefulness (cf. moving one's body to go from A to B in order to get object C for the purpose of using in as a D within the context of practice E vs. moving one's body as part of a sport, a dance, or a game). A number of our activities appear to be gratuitous and will be conceptualized in a variety of ways (NB: these options are not mutually exclusive):

- as ritual behavior, including sacred contexts (Staal (Staal 1996); (Staal 1979));
- as ludic behavior (Huizinga (Huizinga 1938); Netz (Netz 2009)), as in games and sports;
- as art, as in music, dance and poetry;⁴⁰⁶
- or sometimes also as a symptom of mental illness.

Thus, the desire to play around with symbolic systems is not unique to math: this aspect of math is not that different from the way in which natural languages are used in poetry, i.e. by exploiting the formal characteristics of symbols (their ability to rhyme or their ability to form meters) and viewing the meaning as emerging from that play; the parallel is interesting.

An example would be repetitive rhythm, alliteration or rhyme, which can give rise to a specialized, skillful technique (poetry), but which is also found abundantly in spontaneous conversation (Sacks, Tannen, Jefferson (Tannen 1989); (Jefferson 1996)) and in the speech of schizophrenics and other mentally ill ((Cardella 2018); see also the literature on glossolalia

⁴⁰⁶ Cf. also ‘talim’: formal codes as instructions for weaving complex patterns in shawls and carpets. Interestingly, the word talim has a similar etymology as mathematics: item of teaching.

(Bonfim 2015)).⁴⁰⁷ These observations are relevant here, because they suggest that, perhaps, the impulse to produce repetitive forms is a basic human-reflex and not a cultural artefact that is developed on top of 'normal' language use.⁴⁰⁸

An attractive way to model (?) in cognitive terms what happens when people appear to produce poetic effects is in terms of activation: when one item (say: a word) is activated neighboring words are activated as well. And neighboring may refer to several levels:

- /screen/ is semantically a neighbor of /computer/, /television/, /cinema/, etc.
- /screen/ is phonetically a neighbor of /scene/, /seen/, /mean/, /scream/, etc.
- pragmatically, /screen/ is in my present case a neighbor of /wine/.

Normal pragmatics inhibits these associations. Poetry turns them loose, and then repragmatizes them: select random words (e.g. to fit a meter or rhyming pattern); then,

⁴⁰⁷ For ludic' language use in schizophrenics, see Cardella 2018 (Cardella 2018), pp. 93?-95?
p.93?:

Even when they do not end up idolising words, schizophrenics often deconstruct the constitutive elements of language and string ideas together based on formal associations (rhymes, assonances, etymology, and so forth). In other words, they seem to *play* with language, slipping among different language levels. For example, asked to define contentment, a patient answers:

Contentment? Well, uh, contentment, well the word contentment, hav- ing a book perhaps, perhaps your having a subject, perhaps you have a chapter of reading, but when you come to the word 'men' you wonder if you should be content with men in your life and then you get to the letter T and you wonder if you should be content having tea by yourself or be content with having it with a group and so forth.

(Lorenz, 1961: 604)

p. 94?:

As observed by Pennisi,

the schizophrenic language is the elective ground of a continuous slippage among confused and overlapping metalinguistic levels. Each word can be- come the door of a parallel dimension which, usually inviolable, is made accessible on the ground of analogies only recognisable in different and distant universes of discourse.

(1998: 228, author's translation)

The following are examples of this *ludic* use of language:

I was looking at you, the sweet boy that does not want sweet soap. You always work Harvard for the hardware store. Neatness of feet don't win feet, but feet win the neatness of men. Run don't run west, but west runs east. I like west strawberries best. Rebels don't shoot rebels at night.

(Kraepelin, 1913: 39)

[How are you?] To relate to people about new-found...talk about statistical ideology. Er, I find that it' like starting in respect of ideology, ideals change and ideals present ideology and...new entertainments...new, new attainments...

(McKenna and Oh, 2005: 43)

Does water saunter? As to protein, might one tote-it-in? Is it a hydro-car- boat or a carbohydrate? As to any vitamin, might one invite-them-in? Is the dinner-all there with mineral? Is the bulk cellulose or the hulk swell- you-host? Might the medicine have met-us-some? Is it a platypus or ad- ipose? Is the seasoning pleasing? Is food reserved to be preserved? Is one glad-to-give an additive?

(McKenna and Oh, 2005: 49)

⁴⁰⁸ For the the impulsive ('pulsional') character of the poetic urge, see Julia Kristeva's early work on poetry.

optionally, see what they could mean.⁴⁰⁹ Thus, meaning-deficient behavior in humans typically results from over-structuration, not from under-structuration.

It is also interesting to observe that this type of ludic and/or ritual behavior is often conceived as coming from an outside source, not subjected to the will or the intensions of the agent (cf. the notion of inspiration' as an outside, involuntary (depragmatized) source; cf. the Muses or possession by spirits). **Oracular practices** are interesting, that way. For instance, in the case of the Delphic oracle, a woman in an 'altered state of consciousness' produced inspired sounds, that were then transmitted by the priests in the form of (mostly hexametric?) verse.

In the case of **poetry**, one could interpret the exploitation of this reflex as a skillful way to give the pre-semantic impulses free reign and allow the semantics to be an emerging side-effect of the production. A case in point would be the following verses from a song written by Serge Gainsbourg:

Aucun Boeing sur mon transit
Aucun bateau sous mon transat
Je cherche en vain la porte exacte
Je cherche en vain le mot exit

Je chante pour les transistors
Ce récit de l'étrange histoire
De tes anamours transitoires
De Belle au Bois Dormant qui dort

In this case, it is very obvious that the semantics are a pure side-effect of the extremely rich formal play;⁴¹⁰ as a matter of fact, the semantics of the text are very tenuous, i.e. as a narrative it doesn't make sense at all, and even as a lyrical expression of emotion the meaning is precarious. However, as an anagrammatic play on both the phonology (the rhythm and the rime scheme), the morphology (trans-), and the lexical meaning of the words (Boeing-bateau; porte-exit), it is very attractive indeed. Similar effects can be studied in the context of oracular language and other types of sacred language use (mantra's etc.).

⁴⁰⁹ For the suggestion of a link between oracles and the way Ramanujan's work was received, see van Bendegem in Rittberg & al. 2018 (Rittberg, Tanswell, and Van Bendegem 2018), §5.

⁴¹⁰ A particularly interesting case is 'songwriter's yoghurt', i.e. the improvised sounds that singers or songwriters produce before the lyrics of a song are written, as a placeholder for actual lyrics, and quite often the first stage of the process of writing actual lyrics.

This secondary activity of making sense of sounds that were not primarily meant to make sense also seems to be a human reflex.⁴¹¹ This is also reminiscent of what is going on in the case of the ambiguity about mathematicians' relationships with their formalisms: they want them to be 'free', purely formal, but then again, they can't help themselves and remain semi-attached to the initial standard model.

NB: ludic, ritual, poetic practices are obviously also practices, in the full sense of the word: they are perceived as meaningful by practitioners, they require training, there are criteria for success (one can do it right and one can do it wrong), they are social institutions and integral parts of the culture they occur in, etc.

Our provisional conclusion is: math is -in its mainstream interpretations- not interpreted as a poetic practice, but to the extent that it is a purely formal language, it is the result of a human reflex that manifests itself in other ritual/ludic activities as well. Formalism, poetry, and certain (other) types of ritual behavior have in common that they lack or bypass the standard/normal semantic, pragmatic and real-life primacy to focus on the formal features of the behavior itself.⁴¹²

(D) Meaning in a formal system: syntax, semantics and pragmatics

Let's focus on a slightly mythological account of what happens if someone discovers that she/he/they can write $x/0$. Imagine we start from our usual way of calculating with fractions: some of us learn the technique and use it for the purpose of accounting, some of us use it as part of certain engineering practices, etc.

Now let's suppose that someone starts to play around with the symbols and stumbles upon the fact that he/she can write $10/0$:

$$\langle 10/10, 10/9, 10/8, \dots 10/1, 10/0 \rangle$$

⁴¹¹ "Gewöhnlich glaubt der Mensch, wenn er nur Worte hört es müsse sich dabei wohl auch was denken lassen" Goethe, *Faust I*, 2565-2566, quoted at the initium of Høyrup 2007 - Generosity: No Doubt, but at Times Excessive and Delusive.

⁴¹² Interestingly, Netz traces formal, theoretical math back to what he calls ludic math (Netz 2009), i.e. a way of doing math that identified itself explicitly as a game for an elite, distinct from applied techniques in 'geometry' (in the original sense of 'measuring land'), accounting, etc. This suggests that the problematic relation between the freedom of 'elite'/'pure' math and its roots in applications dates back to at least Ancient Greek practices.

Of course, Netz may be fundamentally underestimating the permeability of mathematics, i.e. overestimating the impermeability of the boundaries between mathematics as a 'ludic', 'elite', intellectual endeavor on the one hand, and applied geometrical and arithmetic (accounting) practices on the other: the elites that could afford to engage in ludic (i.e. non-professional, non-applied) math were typically also landowners and must have had at least a passive acquaintance with professional geometry (in the etymological sense of the word) and accounting.

Another line of thought that would be worth pursuing in this context is the Pythagorean/Platonic connection: in the schools that claim lineage to Plato and Pythagoras, mathematical or quasi-mathematical practices and doctrines coexisted with ritual, ascetic and theurgical ones.

It is important to note that *this would only occur in the context of an already 'ludic'/'poetic' practice*: in accounting, one never needs to divide a sum of money among zero beneficiaries; even in a scenario in which there happen to be no beneficiaries when a certain amount needs to be distributed, *nobody would think of using a fraction with denominator 0 in order to know what to do*. Perhaps the same person would notice soon enough that this new symbol $10/0$ is slightly more problematic than the other ones in the initial series. Perhaps, that person would start from the observation that the smaller the denominator in a fraction is, the bigger the result is, and conclude as follows:

$$\begin{aligned} &<4/2 = 2; 4/1 = 4; 4/0,5 = 8; \dots; 4/0 = \infty> \\ &<1000/1000 = 1; 1000/100 = 10; 1000/10 = 100 = 1000/1 = 1000; 1000/0 = \infty> \\ &\text{etc.}^{413} \end{aligned}$$

And eventually, one would also stumble on this sequence:

$$\langle 4/4=1; 3/3=1; \dots; 0/0 = 1 \rangle$$

Once one has stumbled on this new type of symbol $\langle x/0 \rangle$, one may want to decide what to do with it, choosing between any of several options, including the following two very general types of options:

- (1) give it a proper place in your practice,
- (2) stop using it altogether.

Option (1) could go different paths: one could accept $\langle 4/0 = \infty \rangle$ but not $\langle 0/0 = 1 \rangle$, $\langle 0/0 = 1 \rangle$ but not $\langle 4/0 = \infty \rangle$, or both together. In the latter case, one may stumble on the following:

$$\begin{aligned} &\langle 10/0 = \infty; 9/0 = \infty; 8/0 = \infty; \dots 1/0 = \infty; 0/0 = \infty \rangle \\ &\langle 10/10 = 1; 9/9 = 1; 3/3 = 1; 2/2 = 1; 1/1 = 1; 0/0 = 1 \rangle \end{aligned}$$

And then one has to decide what to do about this. Again, one has different options: one could decide that this is an argument for (a) having to choose between the different, apparently mutually incompatible, ways to use these symbols, but one could also use this as (b) a (rather convincing?) argument for mathematical pluralism.

⁴¹³ There are historical instances of this choice, e.g. (Martin & Roitman 2014, p.46):

In 625 CE the great mathematician Brahmagupta knew quite a bit about negative numbers, and even considered division by 0, claiming that $n/0$ is infinity, since "In this quantity consisting of that which has zero for its divisor, there is no alteration, though many be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth."

But, apparently, mainstream Western (?) math has eventually decided for option (2), i.e. to stop using the form $x/0$. This option can be implemented in different ways:

- A. at the level of the syntax: you can modify your syntax by introducing rules that -directly or indirectly- exclude $x/0$ as not well-formed.
- B. at the level of the semantics: you can decide that nothing in your semantics corresponds to this type of sign.
- C. you can decide to do nothing about the problem at the level of your formal system, but solve the problem at the level of what you do with it, for instance:
 - C1. you can decide to simply stop using the sign and explain: “this is useless nonsense”, or: “there’s nothing interesting there”;⁴¹⁴
 - C2. you can quarantine the sign by interpreting it (outside the formal system) as “the secret of Hermes Trismegistus of Lourdes”, and explain that its use by humans would lead to a cataclysm, or make it taboo in any other way.

In other words, one can say plenty of different things when confronted with such a problem:

- “there is no solution to this”
- “the solution is not a number”
- “this proves the existence of spinal numbers”
- “this proves mathematical pluralism”
- “this proves the existence of the devil”

The above argument based on ‘imagined’ or ‘imaginable’ cases are good enough to make our philosophical point (cf. section 3.2.1(C) for LW’s opinion in this regard), but for those of us who prefer empirical stuff, the history of math provides us with materials that correspond to all of the above.

The point is -of course- not that what was ultimately decided was wrong: it can easily be argued that what was chosen turned out to be a fruitful move. The point is: there was no necessity to choose that direction / make that move rather than any other direction/move at any point. Also note that it is perfectly imaginable that different policies coexisted, depending on their usefulness (or lack of usefulness) in different practical contexts. And as pointed out before: the example would fit perfectly within an argument for pluralism.

⁴¹⁴ Case C1 may be indistinguishable from case B in actual practice.

What is important as a conclusion from this, is the understanding that the links made between a formal system and a model are always a matter of pragmatics, and therefore always motivated by stuff that lies outside both the syntax and the model. What makes the relation between a formal system and a model meaningful is always non-formal and always lies outside that relationship. This relationship is always inherently contingent. This also means that these decision-making procedures are necessarily not math-specific.

4.2 Notes on Spengler

Interestingly, the first chapter of the Volume 1 of *Decline of the West*, only preceded by a longish introduction, is titled 'On the meaning of the numbers' [Vom Sinn der Zahlen] and deals with mathematics. It has to be said that Spengler did study math at a university level, enough to be licensed to teach math (among other subjects) in secondary schools. In this chapter, Spengler mostly compares the history of the mathematics of two 'cultures', the one of Ancient Greece (mostly from Pythagoras to Archimedes), and the one of the West (mostly from Descartes to Riemann), briefly mentioning other cultures (most notably -what he calls- "Arabic culture", which includes the latter part of the Roman empire).

Spengler's text shows a number of features that we also encountered in LW's PhilMath:

- Unsurprisingly, Spengler systematically links (supposed) features of the mathematics of a culture with (supposed) features of the artistic production of that culture (this is -after all- one of the key characteristics that make up Spengler's reputation), claiming that what defines (e.g.) Ancient Greek conception of number, also defines its sculpture. In the case of Antiquity, this would be a feeling of everything having clear limits [Begrenzung], of everything being accessible to perception, which shows in the concept of number essentially being a matter of physical size [Größe] and relations between numbers are mostly a matter of proportions, in the sculptural representation of the human body being the main artistic expression, and political structures being limited to no more than what one can see from the top of the hill on which a typical polis is built. Similarly, the classical period of Western culture is characterized by an increasingly algebraic and functional math, in which the link with the ideas of size and proportion are no longer relevant and numbers are conceived of as relative positions (functions), which coincides with the victory of music (the most abstract of arts) over oil-painting.⁴¹⁵ NB that the picture as a

⁴¹⁵ On pp. 111-112, Spengler summarizes this line of thought as follows:

Der antike Mathematiker kennt nur das, was er sieht und greift. Wo die begrenzte, begrenzende Sichtbarkeit, das Thema seiner Gedankengänge, aufhört, findet seine Wissenschaft ein Ende. Der

whole may sound appealing at first sight, but that many of the individual claims that make up this picture are very shaky on their own.

- Spengler strongly emphasizes non-propositional aspects and is ready to go very far into that direction, very often speaking of a 'feel' or a 'feeling' as constitutive of this or that aspect of a culture. Not only does he insist on the fact that math cannot be reduced to mathematical theory (something LW -and many of us, I guess- would have agreed with), but he goes as far as saying that number [Zahl] -as all other aspects of a culture- is ultimately grounded in a 'primary/primeval feel(ing)' [Urgefühl]:

Die geschriebene Mathematik repräsentiert so wenig wie die in theoretischen Werken niedergelegte Philosophie den ganzen Besitz dessen, was im Schöße einer Kultur an mathematischem und philosophischem Blick und Denken vorhanden war. Es gibt noch ganz andere Wege, das den Zahlen zugrunde liegende Urgefühl zu versinnlichen.

- Like LW, Spengler also points out the plurality and variability of math. In Spengler's case, the emphasis is on the fact (?) that every culture has its own math, but also on the existence of different 'styles' and fashions within math.

As a whole, Spengler's prose is a continuous (seemingly endless) stream of interesting-sounding polymathic babble, rich in quasi-insightful analogies (no bibliographical references, though)⁴¹⁶ and unverifiable factoids (not necessarily made-up, but hardly ever well-established), but completely devoid of scholarly or philosophical skill or even -I guess- ambition. What is missing is any friction between the discourse and not only empirical fact, but also alternative concepts, arguments or views. (we have seen that LW's polyphonic style/method is in that regard at the opposite end of the spectrum).

For the purposes of the present study (on LW's work on math), I was particularly interested in Spengler's views on the relation between theoretical math and everyday application, as

abendländische Mathematiker begibt sich, sobald er von antiken Vorurteilen frei sich selbst gehört, in die gänzlich abstrakte Region einer unendlichen Zahlenmannigfaltigkeit von n - nicht mehr von 3 - Dimensionen, innerhalb deren seine sogenannte Geometrie jeder anschaulichen Hilfe entbehren kann und meistern muß. Greift der antike Mensch zu künstlerischem Ausdruck seines Formgefühls, so sucht er dem menschlichen Körper in Tanz und Ringkampf, in Marmor und Bronze diejenige Haltung zu geben, in der Flächen und Konturen ein Maximum von Maß und Sinn haben. Der echte Künstler des Abendlandes aber schließt die Augen und verliert sich in den Bereich einer körperlosen Musik, in dem Harmonie und Polyphonie zu Bildungen von höchster „Jenseitigkeit“ führen, die weitab von allen Möglichkeiten optischer Bestimmung liegen. Man denke daran, was ein athenischer Bildhauer und was ein nordischer Kontrapunktist unter einer Figur versteht, und man hat den Gegensatz beider Welten, beider Mathematiken unmittelbar vor sich.

See also the table on p. 124.

⁴¹⁶ Another point in common with LW. But Spengler had his PhD thesis refused on that basis, LW's (the *TLP*) was accepted anyway.

well as in the question as to whether he had any Wittgenstein-like criticism of 19th and 20th century math in that regard. Somewhat disappointingly, the chapter on the numbers, which I am discussing here, shows a decidedly un-wittgensteinian point of view. For Spengler, the number concept of Antiquity is basically the one that emerges from everyday practice, and the math used in everyday practice (“by children and untaught people”) remains at that stage (?), even when the ‘really-important’ math has moved on:

pp. 101-102:

Die Funktion ist nichts weniger als die Erweiterung irgend eines vorhandenen Zahlbegriffs; sie ist dessen völlige Überwindung. Nicht nur die euklidische und damit auch die „allgemein menschliche“, auf alltäglicher Erfahrung beruhende Geometrie der Kinder und Ungelehrten, sondern auch die archimedische Sphäre des elementaren Rechnens, die Arithmetik, hört damit auf, für die wirklichbedeutende Mathematik Westeuropas Wert zu haben. Es gibt nur noch eine abstrakte Analysis. Für den antiken Menschen waren Geometrie und Arithmetik in sich geschlossene und vollständige Wissenschaften von höchstem Range, beide anschaulich, beide mit Größen zeichnerisch oder rechnerisch verfahren; für uns sind sie nur noch praktische Hilfsmittel des alltäglichen Lebens.

Like LW, Spengler observes that modern era Western math, evolves away from this number-as-size concept, but unlike LW, who sees this as a cause of loss of meaningfulness, he believes that the fact that modern/Western mathematicians remained somehow attached to this ancient conception, was not a good thing and, if anything, actually held modern math back from reaching its full potential and its mature expression:

pp. 102-103:

Jede der tiefsinnigen Schöpfungen, welche von der Renaissance an rasch aufeinander folgen, [...] sind ebensoviel Siege über das populär-sinnliche Zahlengefühl in uns, das aus dem Geiste der neuen Mathematik heraus, die ein neues Weltgefühl zu verwirklichen hatte, überwunden werden mußte. Es gab bisher keine zweite Kultur, welche den Leistungen einer andern, längst erloschenen, so viel Verehrung entgegenbrachte und wissenschaftlich so viel Einfluß gestattete, wie die abendländische gerade der antiken. Es dauerte lange, bevor wir den Mut fanden, unser eignes Denken zu denken. Auf dem Grunde lag der beständige Wunsch, es der Antike gleichzutun. Trotzdem war jeder Schritt in diesem Sinne eine tatsächliche Entfernung von dem erstrebten Ideal. Deshalb ist die Geschichte des abendländischen Wissens die einer *fortschreitenden Emanzipation* vom antiken Denken, einer Befreiung, die nicht einmal gewollt, die in den Tiefen des Unbewußten erzwungen wurde. *So gestaltete sich die Entwicklung der neuen Mathematik zu einem heimlichen, langen, endlich siegreichen Kampf gegen den Großenbegriff*

So: whatever else LW may have learnt from Spengler (or otherwise has in common with him), the crucial combination of pragmatism and everydayism that informs LW’s critical stance towards contemporary math, he does not share with Spengler.

A quick preliminary search also shows that Spengler seems to not share with LW the positive valuation of everydayness. For instance, on p.180 he operates with the distinction between “the everyday person [der alltägliche Mensch]” vs. “the significant person [der bedeutende Mensch]”, in a way that reminds us of Heidegger, but not at all of LW:

(p. 180)

Der alltägliche Mensch sämtlicher Kulturen bemerkt von der Physiognomie allen Werdens, seines eignen und dessen der lebendigen Welt rings um sich, nur den unmittelbar greifbaren Vordergrund. Die Summe seiner Erlebnisse, der inneren wie der äußeren, füllt als bloße Reihenfolge von Tatsachen den Lauf seiner Tage. Erst der bedeutende Mensch fühlt hinter dem volkstümlichen Zusammenhang der historisch-bewegten Oberfläche eine tiefe Logik des Werdens, die in der Schicksalsidee hervortritt und die eben jene oberflächlichen bedeutungsarmen Bildungen des Tages als zufällig erscheinen läßt.

4.3 Notes on epistemic bad faith and epistemic bad taste

Throughout Part 2 of this study, I pointed out that LW systematically uses ethical and/or aesthetical vocabulary referring to various avatars of the concept of **inauthenticity** (delusion, fraud, hocus pocus, pathos, ...) to criticize various aspects of mathematical discourse he objects to. In this section, I would like to offer a few brief suggestions as to how LW’s critical attitude could still apply to contemporary mathematical discourse (despite the fact that I may not be the right person to actually do the work), by developing the concepts ‘epistemic bad taste’ (‘epistemic kitsch’) and ‘epistemic bad faith’, introduced in section 2.0.3, a little further.

(A) epistemic kitsch

Epistemic bad taste / epistemic kitsch (my terms) occurs when the importance (‘depth’, ‘mystery’, ...) of a claim is proclaimed by appealing to the reader’s emotions or ideological convictions, using cheap rhetorical devices, imagery or analogies, thus bypassing both empirical fact and rational thought.⁴¹⁷

In section 2.0.3, I referred to a number of cases in which LW objected to pathos, excitement, sensationalism, vertiginous imagery, theatrics, in various contexts, and throughout Part 2, we observed that this basically aesthetic evaluation was a recurrent theme in LW’s criticism of set-theoretical verbiage, but also of the standard interpretation of paradoxes and

⁴¹⁷ According to Roger Scruton (Scruton 2014), “Kitsch is fake art, expressing fake emotions, whose purpose is to deceive the consumer into thinking he feels something deep and serious.” (also quoted in <https://en.m.wikipedia.org/wiki/Kitsch>). This definition is exactly up the alley of what is meant here by ‘epistemic kitsch’.

contradictions in formal systems. In many cases, LW emphasized the ultimately trivial nature of the mathematical technique or result itself. Examples are:

- dramatizing the consequences of paradoxes in mathematical formalisms (for instance the Gödel-like ones), whereas their actual status within actual math is often marginal at best;
- the ‘head-spinning’ image of more and more reals crammed into a smaller and smaller space as an illustration of a theory of the reals, whereas one could also point out the trivial fact that the notion of ‘ordering’ on a line does not work for the reals in the same way that it works for natural numbers, or that the non-rational reals are not ‘numbers’ in the same sense as rationals, i.e. that rational and non-rational numbers have less in common with each other than one might think; etc.;
- the fake depth attributed to diagonal arguments in standard mathematical discourse, as opposed to the trivial fact that even children can easily learn and comprehend the diagonal technique itself;
- the outlandishness of Hilbert’s hyperbolically exalted notion of ‘Cantor’s paradise’, to which LW replied with the sardonic idea that that it would be equally legitimate to view Cantor’s contributions as a satirical joke.

Let me briefly demonstrate how doubtful aesthetics are very much part of present-day discourse about math by taking a brief look at a random example.⁴¹⁸ Chapter 1 of J.L. Schiff’s 2020 popularizing book *The Mathematical Universe. From Pythagoras to Planck* starts with the title “The Mystery of Mathematics”, under which he prints the following two mottos: “*Pure Mathematics is religion... - Philosopher Friedrich von Hardenberg*” and “*It is impossible to be a mathematician without being a poet in soul...*” - Mathematician Sofia Kovalevskaya”. The title of Chapter 2, “From Here to Infinity”, is followed by “*The Infinite! No other question has ever moved so profoundly the spirit of man... - Mathematician David Hilbert*” and “*The interior of our skulls contains a portal to infinity... - Writer Grant Morrison*”. In none of the quoted cases, the exalted formulation contributes in any way to a proper understanding of the subject matter, on the contrary, the quotes are systematically mystifying the subject matter they are supposed to contribute to:

- One may agree with Kovalevskaya’s idea that a mathematician is -in a way- like a poet but one wonders what Kovalevskaya thinks a poet does: she would probably not agree

⁴¹⁸ Another spectacular example is Hoffmann 2017(2) on Gödel’s famous results ((Hoffmann 2017), p. 318): “Betrachten wir Gödels Satz VII im Lichte des Hauptresultats, so können wir daraus ein atemberaubendes Ergebnis ableiten. Aus ihm folgt, dass unentscheidbare Formeln keine scheuen Wesen sind, die sich ausschließlich in den schattigen Winkeln einer praxisfremden Mathematik tummeln. Das Gegenteil ist der Fall: Wir finden sie im Herzen der Mathematik, inmitten der elementaren Zahlentheorie”. Whatever one may think of the merits of Gödel’s results, it would be hard to maintain that Gödel’s undecidable formula occurs “at the heart of math, in elementary arithmetic”.

that whatever happens in the poet's soul, (s)he is always first and foremost a craftsman, using a certain set of techniques to achieve a certain product that is supposed to have a certain function within the lives of those who consume the product, not unlike a baker, or -indeed- a mathematician. Perhaps we should conclude that mathematicians and poets resemble each other in that both crafts have a penchant for surrounding themselves with obscurantist quasi-religious mystification.

- Why would pure mathematics be religion? What kind of religion would mathematics be: a set of rituals to ensure cosmic continuity, a way towards personal salvation, a non-empirical account of the workings of the universe, ... ? If true, would it be a good thing that mathematics is religion or is that understood as self-evident, and if this is understood as self-evident, why is that the case?
- Similarly, I am not convinced at all that infinity is a question that has actually moved many spirits, let alone profoundly (by the way: is infinity a question?),⁴¹⁹ but even if 'infinity' in some sense did move the spirits of certain people (perhaps in the context of cosmological or theological speculation?), it is not at all self-evident that the concept of infinity as it is used in mathematical contexts is the same concept at all (cf. LW's critique of -what he considered- illegitimate uses of the concept of infinity, see section 2.2(B)).

The question is: what is the function of these platitudes? why would a book about the relation between math and physics need such dubious literary devices anyway? why are these ugly, hollow, sentimental tropes so ubiquitous in the literature about math, much more so than in neighboring fields.

The ways in which mathematical ideas are vulgarized and are received in **popular culture** offer many more illustrations of epistemic kitsch. For instance, exactly the same things that LW blamed Mahler's music for, apply to Hofstadter's *Gödel Escher Bach* (Hofstadter 1979): brilliant talent, excellent technician, great artistic achievement, but his amalgamation of the mathematical concept of recursion with physical feedback loops and Zen koans is ultimately nonsensical and therefore irresponsible in that it sells cheap analogies as genuine insights. Perhaps more pernicious is the popular idea that Gödel has proven that mathematics is riddled with uncertainty (NB that Gödel thought to have achieved the opposite, i.e. that his results show that mathematical truth is independent of any formal mathematical results), often mixed in with an equally inaccurate popular reinterpretation of Heisenberg's famous results, especially when this cocktail of semi-literate half-truths is then deployed for the

⁴¹⁹ I think the need for food and shelter would be a more accurate candidate for the most universally relevant soul-moving topic, with sexual desire perhaps a good second.

purposes of “post truth”-styled attacks on science and/or academia as a whole. For the potential dangers of epistemic kitsch in science communication, see section 3.2.3(D).

A particularly interesting case for further discussion (because it is not only part of the vulgarizing literature) would be the verbiage surrounding **the applicability of math to physics**, as introduced in modern PhilMath by Wigner (Wigner 1960), which has been called ‘mysterious’, ‘unreasonable’, ‘awe-inspiring’, ‘a miracle’, etc. (see also Hacking (Hacking 2014) for some nice examples). Of course, the point is not to deny the existence of the phenomenon of awe and wonder, or to argue that it shouldn't exist. The point is that this kind of emotion is not an argument for anything: people get emotional for the silliest reasons. The emotion is a fact, it can serve as “raw materials” for philosophy to deal with, but it is as such no argument at all, let alone for the ontological status of mathematical objects or the epistemological status of math as such.⁴²⁰ And if it is used as a cheap rhetorical device without any connection to what scientists and mathematicians actually do,⁴²¹ it should be dismissed for what it is: epistemic kitsch.

(B) epistemic bad faith

Epistemic bad faith / epistemic pretense occurs *when someone apparently conforms to (or pretends to conform to) formal criteria for acceptable discourse and at the same time does not participate in those aspects of the encompassing practice that makes (or would make?) such discourse meaningful.*

⁴²⁰ For this notion of “raw materials for philosophy”, see Ms-124,35-36:

Das Piedestal der Mathematik, ist die Rolle | | ist eine bestimmte Rolle, welche ihre Sätze in unsern Sprachspielen spielen.

Die Sätze, welche Hardy in seinem - elenden - Buch, “Apology of a Mathematician”, als Ausdruck seiner Philosophie der Mathematik hinstellt, sind noch gar nicht Philosophie, sondern können wie alle ähnlichen Ergüsse, als Rohmaterial des Philosophierens dienen, & sollten dann nicht in der Form von Meinungen, Feststellungen, oder Axiomen, ausgesprochen werden, sondern in der Form: “Ich bin geneigt zu sagen: ...”, “Ich möchte immer sagen: ...”. Worauf das Philosophieren erst beginnen soll, (um) uns diese seltsame Neigung zu erklären. | | ; uns diese | | sondern können - wie alle ähnlichen Ergüsse - Rohmaterial des Philosophierens sein; & sollten ...

See also (Floyd 2012) Floyd 2012, p. 245, ftn. 33: “Hardy 1940 is mentioned at MS 124, p. 35—a draft remark for PI §254—where Wittgenstein writes that “the sentences that Hardy sets forth as expression of his philosophy of mathematics in his miserable book ‘Apology of a Mathematician’ are in no way philosophy, but could—like all similar outpourings—be conceived as raw material of philosophizing.””.

⁴²¹ Does anyone actually argue that actual **mystical practices** are essential to mathematics? I would not be surprised that this was actually the case in the context of the original Pythagorean/Platonist school(s), but are there any equivalents in modern mathematics? However interesting it would be to be able to study such individual cases (which must exist...), my main point still stands: for most authors indulging in (pseudo-)mystical talk, this does not correspond to anything practical, and mainstream math is not presented (let alone taught) as a mystical practice. Most importantly, no mystical insights are necessary to calculate and to apply calculations, or mathematical technique in general, in science, engineering, accounting, etc.

(B1) Bad faith 1: Cantor on freedom

A first obvious interesting topic is the relation between **Cantor's** work in math and his well-documented theological endeavours (for his correspondence with theologians including the Pope see e.g. Dauben 1977 (Dauben 1977); and Newstead 2009 (Newstead 2009)). It is interesting to observe that even an apparently conservative and prudent (see section 3.2.3(A) above) historian like José Ferreirós admits that Cantor's transfinites perhaps might not have existed (ever), if not for Cantor's intervention: "[...] there is good reason to think that the results would have been established even if Cantor had never lived. Developments could have been much slower, and we may doubt whether the transfinite ordinals would have been introduced, but I do not see reasons to have similar doubts concerning the results in Cantor (1874)." ((Ferreirós 2016) Ferreirós 2016, pp. 256-257).

Cantor's transfinites are not his only contribution that is a case in point of definitely non-mathematical aspects determining apparently intra-mathematical decisions: the origins of his famous proclamation of the '**freedom of mathematics**' is equally confusing. Interestingly, and perhaps somewhat bizarrely or at least ironically, Cantor's idea of the freedom of mathematics, meaning that the only criterion for the acceptance of mathematical theory should be its self-consistency, is a good example of the non-autonomous nature of mathematical practice and discourse in that it was at its origins a more political than an epistemological idea. According to Dauben 1989, Cantor admitted that when he first formulated his idea of 'mathematical freedom', he was also thinking of the oppression he experienced on the part of his nemesis Kronecker, who criticized Cantor's revolutionary contributions in the strongest possible terms (cf. Dauben 1989: " [...] the essence of mathematics is exactly its freedom. This was not simply an academic or philosophical message to his colleagues, for it carried as well a hidden and deeply personal subtext. It was, as he later admitted to David Hilbert, a plea for objectivity and openness among mathematicians."). The idea retained its existential and political connotations throughout Cantor's career. Cantor's activities in academic politics included his involvement in the foundation of an independent union of mathematicians, which he intended to offer an open platform in which all mathematicians could 'freely' discuss mathematical results, without any political pressure from any academic establishment, and the notion of freedom retained its ambiguity as both an epistemological idea and a political/existential one:

Applications might eventually determine which mathematical theories were useful, but for mathematicians, Cantor insisted that the only real question was consistency. This of course was just the interpretation he needed to challenge an established mathematician like Kronecker. Cantor clearly felt obliged, early in his career, to plead as best he could for a fair hearing of his work. So long as it was self-

consistent, it should be taken as mathematically legitimate, and the constructivist, finitist criticisms of Kronecker might be disregarded by most mathematicians for whom consistency alone should be the viable touchstone.

It would be interesting to analyze Cantor's discourse on freedom, both as a meta-mathematical concept and as a political concept, using the tools used to analyze political discourse in general. What is particularly bizarre/ironic in this case, is that the principle of mathematical freedom, which proclaims that the only "viable touchstone" in math is internal consistency, is itself the product of the author's frustrations at the interpersonal and political level.

Furthermore, Cantor's major contributions were definitely and demonstrably not a good example of the autonomy of math either: as mentioned above, Cantor himself famously saw his transfinite stuff as a discovery with theological implications and considered its promotion as a mission from his god. Cantor's Nachlass should be a fruitful field to study the exact ways in which he negotiated the technical requirements of the mathematical format and his religious impulses. Thus, Dauben mentions the following:

For example, as early as his *Grundlagen* of 1883, Cantor referred to collections that were too large, he said, to be comprehended as well defined, completed, unified entities. Unfortunately, he wrote obscurely, with references to absolute sets in explicitly theological terms, explaining that "the true infinite or Absolute, which is in God, permits no determination."

Later on, Cantor found another way to deal with the paradoxes, which was to simply not include the offending collections in set theory:

By the mid-1890s Cantor could no longer be so vague about absolute entities, and was forced to be much more explicit about the paradoxes resulting from consideration of the sets of all transfinite ordinal or cardinal numbers. The solution Cantor then devised for dealing with such mathematical paradoxes was simply to bar them from set theory. Anything that was too large to be comprehended as a well defined, unified, consistent set was declared inconsistent. These were "absolute" collections, and lay beyond the possibility of mathematical determination. This, in essence, is what Cantor communicated first to Hilbert in 1897, and somewhat later to Dedekind in his letters of 1899.

This narrative illustrates quite aptly the idea that decisions about mathematical innovation are never internal to the formal mathematical technique itself, and always a matter of the pragmatics, i.e. the way in which the technique is embedded in a practice, and beyond individual practices, large-scale, cultural, social, biological and/or physical structures

(“Forms of Life”, “our lives”, ...). The fact that Cantor came back on an earlier decision and chose for another option at his disposal, is particularly revealing for our purposes.⁴²²

(B2) Bad faith 2: Gödel's little magic trick

LW's criticism of **Gödel's Kunststückchen** ['little magic trick'] (see section 2.3(B)) is a good example: LW objects to KG's bad faith in presenting his completeness-consistency proof as a normal result in normal arithmetic, whereas it is almost obvious that the only reason one would want to even start trying to prove this, is in order to make the philosophical point that Gödel actually did make (I mean: Platonism). Historiographical work ⁴²³ shows that Gödel's account of his own intellectual biography is far from transparent, most notably when it comes to the origins of his Platonist convictions and the relationship between the latter and his work in mathematical logic: if it is true that Gödel contradicted himself about the chronological order and the direction of the causal relation between his philosophical-religious convictions and his work on the completeness and consistency of formal systems, a case can be made for Gödel's bad faith in a literal and very concrete sense. However, as is often the case with autobiographical accounts, it is not clear to what extent Gödel was strategic about the way he presented his development at various moments in his career, and to what extent he was genuinely enacting a psychodrama of which he was perhaps not really aware himself. Now that his notebooks have (partially) become available for further study, it has already become clear to what extent Gödel's work was driven by religious-philosophical-theological concerns and further study of this material will undoubtedly give rise to a better understanding of the issues at hand.

Whatever the conclusions may turn out to be with respect to the biographical aspect of Gödel's particular case (however interesting they may be, not only for their own sake, but also because of the historical importance and iconic status of Gödel's work), the properly epistemic aspect of epistemic bad faith is what is of interest here: how should we deal with contents that are not incorrect, according to established (formal) criteria of correctness, but nonsensical in that they have no real connection with anything that mathematicians actually *do*, as mathematicians? or: why is it that mathematicians / philosophers of mathematics feel the need to make claims that do not follow from what they do and at the same time refuse to look into the obvious extra-mathematical ideological connections of their claims?

⁴²² Cf ; what we said about the various options available to mathematicians if they stumble on undesirable situations/results.

⁴²³ For instance, even the materials collected by Floyd & Kanamori on their own would suffice to make the point; see (Floyd and Kanamori 2016), pp. 259-260, incl. footnotes 32 and 33.

LW appears to suggest that the problem with “most mathematicians’ slimy concepts” (see section 2.3(E) above) is their lack of understanding of the pragmatics of math, i.e. the fact that they seem to not understand the fact that the meaningfulness of their discourse depends on how it is embedded in their practice (and how their practice is embedded in their culture at large).

(B3) Bad faith 3: The Gödel Program

On the basis of the recurrent arguments in LW’s work against the pervasive and unexamined monism (my term) in mathematical discourse, I would like to suggest that it would be interesting to investigate to what extent the above analysis can be extended to -what I would like to call- **the Gödel Program** at large,⁴²⁴ by which I mean the persistent and systematic defense of monism against pluralist threats throughout the 20th and early 21st centuries. Ever since formalism became an option as a theoretical approach to the Grundlagen-issue, a number of intra-mathematical developments offered challenges to mathematical monism:⁴²⁵

- the study of non-standard models of standard axiomatic systems ever since Skolem’s famous contributions;
- the debates concerning the status of the Continuum Hypothesis, the various independence results, the playing around with alternative axiomatic systems that resulted from it, and the success of forcing as a mathematical technique;
- reverse mathematics, an approach in which one starts from a theorem and works one’s way up to the axioms that are needed to prove it.⁴²⁶

These developments have in common that they exploit the possibility to tinker with expansions of, and variations on, axiomatic systems beyond the standard systems and their standard models, which ipso facto boils down to a de facto pluralism. The Gödel Program, and perhaps the Grundlagen-debates in general, can be understood as a reaction to -what

⁴²⁴ The term was apparently coined by Yourgrau ((Yourgrau 2006) - (Stachel 2007)), as follows: “Overarching much of his research in philosophy and logic was the ‘Gödel program’, the investigation of the limits of formal methods in capturing intuitive concepts” (p. 182; see also pp. 114, 127). This definition of the term is sympathetic and makes the endeavor look more innocent than it is. I will use a somewhat more polemical characterization of essentially the same thing: *the Gödel Program is the systematic attempt at enforcing mathematical monism (especially Platonism) in the face of the challenges it faced since the beginning of the 20th century, for instance by drawing philosophical conclusions from -what looks like- the study of formal systems.*

⁴²⁵ NB that full-blown formalism would yield straightforward pluralism.

⁴²⁶ One may want to consider the potential of ‘**perverse reverse math**’, i.e. an approach to reverse math that actively tries to free itself from monism, by completely dropping the idea of ‘completeness’ and investigating the functionalities (and breakdown) of deliberately non-standard or incomplete axiomatic systems. NB that this approach would embody Cantor’s idea of freedom (as he defined it, not as he intended it) in much more coherent way than is the case in mainstream approaches to the foundations of math. Also note that this would make math the ‘study of formal systems in general’ in a way that it is definitely not at this moment.

many mainstream mathematicians and philosophers felt as a threat to their monist convictions.

An interesting case in point would be the relatively recent debates about **pluralism vs. non-pluralism (a.k.a. the multiverse vs. the universe) in set-theory** (for a nice overview, see e.g. Koellner (Koellner 2013b); (Koellner 2013a)). On the one hand, pluralism is de facto possible: one can start and play around with an indeterminate number of different mathematical systems, pluralists actually do so. On the other hand, many leading mathematicians appear to object to the idea, mostly in the name of convictions that can be reduced to the cluster of concepts that I called 'mathematical monism'.

I believe a 'pragmatic' (as opposed to a 'semantic') approach to meaning in general can contribute to a better understanding of mathematical meaning. and thus may help shed light on the debates surrounding the pluralist vs. non-pluralist conceptions of set theory, by focusing on (1) ways in which a certain awareness of pragmatic / non-semantic aspects of mathematical meaning already play a role in these debates (focus on such concepts as 'choice', 'freedom', ... vs. 'intuitively correct', 'natural', ...); (2) how a systematically pragmatic rephrasing of the problem yields a better understanding of what is at stake in these debates. As suggested in the above, the problem cannot be understood from inside the axiomatic systems, not even in terms of the relation between a formal system and its intended model(s). What counts as a desirable solution is a matter of pragmatics and -apparently- crucially involves extra-mathematical concepts, values and/or practices.

Again: these issues cannot easily be dismissed as external to mathematics in that they concern a core issue in theoretical math and the very identity of the field.

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