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Jánossy densities and Darboux transformations for the Stark and cylindrical KdV equations

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Abstract

We study Jánossy densities of a randomly thinned Airy kernel determinantal point process. We prove that they can be expressed in terms of solutions to the Stark and cylindrical Korteweg–de Vries equations; these solutions are Darboux transformations of the simpler ones related to the gap probability of the same thinned Airy point process. Moreover, we prove that the associated wave functions satisfy a variation of Amir–Corwin–Quastel’s integro-differential Painlevé II equation. Finally, we derive tail asymptotics for the relevant solutions to the cylindrical Korteweg–de Vries equation and show that they decompose asymptotically into a superposition of simpler solutions.

AMS SUBJECT CLASSIFICATION (2020): 35Q53, 60G55, 41A60

KEYWORDS: cKdV, Darboux transformations, Jánossy densities, Riemann–Hilbert problems

1 Introduction

The cylindrical Korteweg–de Vries equation admits a family of solutions which are expressed in terms of Fredholm determinants involving the Airy kernel operator [41, 42, 14]. These solutions have an interesting probabilistic interpretation, as they are gap probabilities for random thinnings of the Airy point process [14], and they are therefore connected to important problems in integrable probability, such as the extreme value statistics of finite temperature free fermions [31, 37, 38, 35], the distribution of the narrow wedge solution of the Kardar–Parisi–Zhang equation [36, 43, 2], the edge eigenvalue statistics in the complex elliptic Ginibre Ensemble at weak non-Hermiticity [9], and multiplicative statistics of Hermitian random matrices [21].

In this work, we show that Darboux transformations of such solutions also enjoy a probabilistic interpretation: they are Jánossy densities of random thinnings of the Airy point process. We investigate the integrable structure of these solutions, and show how they are connected to the Stark equation and to an integro-differential Painlevé II equation. In this way, we reveal a remarkable connection between the Airy point process and scattering theory for solutions of the cylindrical Korteweg–de Vries equation. Moreover, we show that their tail asymptotics can be described as a superposition of simpler solutions. This soliton-like behavior finds its origin in the fact that certain conditional ensembles related to the Airy point process decorrelate asymptotically.

To set the ground and give motivation for this work, we start by recalling results about the Fredholm determinant

$$j_\sigma(s) := 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \rho_n^{\text{Ai}}(\lambda_1 + s, \dots, \lambda_n + s) \prod_{i=1}^n \sigma(\lambda_i) d\lambda_i, \quad (1.1)$$

where $\sigma : \mathbb{R} \rightarrow [0, 1]$ is a function satisfying Assumption A below, $s \in \mathbb{R}$, and

$$\rho_n^{\text{Ai}}(\lambda_1, \dots, \lambda_n) := \det \left(K^{\text{Ai}}(\lambda_i, \lambda_j) \right)_{1 \leq i, j \leq n} \quad (1.2)$$

with K^{Ai} the *Airy kernel*

$$K^{\text{Ai}}(\lambda, \mu) := \int_0^{+\infty} \text{Ai}(\lambda + \eta) \text{Ai}(\mu + \eta) d\eta = \frac{\text{Ai}(\lambda) \text{Ai}'(\mu) - \text{Ai}'(\lambda) \text{Ai}(\mu)}{\lambda - \mu}, \quad (1.3)$$

Ai and Ai' being the Airy function and its derivative, respectively. We will consider functions σ satisfying the following properties.

Assumption A. *The function $\sigma : \mathbb{R} \rightarrow [0, 1]$ is smooth and there exists $\kappa > 0$ such that $\sigma(\lambda) = O(|\lambda|^{-\frac{3}{2}-\kappa})$ as $\lambda \rightarrow -\infty$.*

Remark 1.1. *For $j_\sigma(s)$ to be well defined, we need the trace condition $\int_{\mathbb{R}} \sigma(\lambda) K^{\text{Ai}}(\lambda, \lambda) d\lambda < \infty$. The decay in Assumption A is slightly stronger than this requirement. This will allow us to control the $s \rightarrow +\infty$ behavior of certain objects, see in particular Section 3.3.*

As we shall prove in Lemma 2.1, it follows from Assumption A that $0 < j_\sigma(s) \leq 1$ for all $s \in \mathbb{R}$. Moreover, as proved in [1, 7, 8, 14], introducing

$$v_\sigma(s) := \partial_s^2 \log j_\sigma(s), \quad (1.4)$$

one has

$$v_\sigma(s) = - \int_{\mathbb{R}} \varphi_\sigma(\lambda; s)^2 \sigma'(\lambda) d\lambda, \quad \sigma'(\lambda) := \frac{d\sigma(\lambda)}{d\lambda}, \quad (1.5)$$

where $\varphi_\sigma(\lambda; s)$ solves the *Stark equation* $(\partial_s^2 + 2v_\sigma(s) - s) \varphi(\lambda; s) = \lambda \varphi(\lambda; s)$ ¹. More precisely, $\varphi_\sigma(\lambda; s)$ is the unique solution to the *Stark boundary value problem*

$$(\partial_s^2 + 2v_\sigma(s) - s) \varphi_\sigma(\lambda; s) = \lambda \varphi_\sigma(\lambda; s), \quad \varphi_\sigma(\lambda; s) \sim \text{Ai}(\lambda + s), \quad s \rightarrow +\infty. \quad (1.6)$$

(See Proposition 3.10 for the proof of the boundary values under our current assumptions on σ). In particular, it follows by combining (1.5) and (1.6) that φ_σ solves the *integro-differential Painlevé II equation* of Amir, Corwin, and Quastel [1]

$$\partial_s^2 \varphi_\sigma(\lambda; s) = \left(\lambda + s + 2 \int_{\mathbb{R}} \varphi_\sigma(\mu; s)^2 \sigma'(\mu) d\mu \right) \varphi_\sigma(\lambda; s). \quad (1.7)$$

It is worth observing that the relation (1.5) is the analogue, for potentials with linear background, of the classical *Trace Formula* obtained by Deift and Trubowitz [20, Equation (1)_R, page 183] in scattering theory for the Schrödinger equation for potentials with zero background.

A one-parameter family of isospectral deformations of (1.6) can be constructed as follows. Let $T > 0$ and introduce

$$J_\sigma(X, T) := 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \rho_{X, T; n}^{\text{Ai}}(\lambda_1, \dots, \lambda_n) \prod_{i=1}^n \sigma(\lambda_i) d\lambda_i, \quad (1.8)$$

where

$$\rho_{X, T; n}^{\text{Ai}}(\lambda_1, \dots, \lambda_n) := \det \left(K_{X, T}^{\text{Ai}}(\lambda_i, \lambda_j) \right)_{1 \leq i, j \leq n} \quad (1.9)$$

for a shifted and dilated Airy kernel

$$K_{X, T}^{\text{Ai}}(\lambda, \mu) := T^{-\frac{1}{3}} K^{\text{Ai}}(T^{-\frac{1}{3}}(\lambda + X), T^{-\frac{1}{3}}(\mu + X)). \quad (1.10)$$

¹The Stark equation is nothing else than the Schrödinger equation $(\partial_s^2 + 2u(s)) \varphi(\lambda; s) = \lambda \varphi(\lambda; s)$, for a potential $u(s) = v_\sigma(s) - s/2$ with linear background $-s/2$.

It is readily checked that $J_\sigma(X, T) = j_{\tilde{\sigma}}(XT^{-\frac{1}{3}})$, where $\tilde{\sigma}(\lambda) := \sigma(T^{\frac{1}{3}}\lambda)$, implying that

$$V_\sigma(X, T) := \partial_X^2 \log J_\sigma(X, T) = T^{-\frac{2}{3}} v_{\tilde{\sigma}}(XT^{-\frac{1}{3}}) \quad (1.11)$$

satisfies

$$V_\sigma(X, T) = -T^{-\frac{1}{2}} \int_{\mathbb{R}} \widehat{\varphi}_\sigma(\lambda; X, T)^2 \sigma'(\lambda) d\lambda \quad (1.12)$$

in terms of the function $\widehat{\varphi}_\sigma(\lambda; X, T) := T^{-\frac{1}{12}} \varphi_{\tilde{\sigma}}(\lambda T^{-\frac{1}{3}}; XT^{-\frac{1}{3}})$. The latter is also equivalently characterized as the unique solution to the boundary value problem

$$\begin{cases} \mathcal{L} \widehat{\varphi}_\sigma(\lambda; X, T) = \lambda \widehat{\varphi}_\sigma(\lambda; X, T), \\ \widehat{\varphi}_\sigma(\lambda; X, T) \sim T^{-\frac{1}{12}} \text{Ai}(T^{-\frac{1}{3}}(\lambda + X)), \quad X \rightarrow +\infty, \end{cases} \quad (1.13)$$

where $\mathcal{L} := T \partial_X^2 + 2T V_\sigma(X, T) - X$. Moreover, it is proved in [14, Theorem 1.3] that $V = V_\sigma(X, T)$ solves the *cylindrical Korteweg-de Vries* (cKdV) equation

$$\partial_T V + \frac{1}{12} \partial_X^3 V + V \partial_X V + \frac{1}{2T} V = 0. \quad (1.14)$$

More precisely, the variables $x \in \mathbb{R}, t > 0$ of loc. cit. are related to $X \in \mathbb{R}, T > 0$ of this paper as $x = -XT^{-\frac{1}{2}}, t = T^{-\frac{1}{2}}$, such that the KdV equation for $u = u(x, t)$

$$\partial_t u + \frac{1}{6} \partial_x^3 u + 2u \partial_x u = 0 \quad (1.15)$$

[14, equation (1.7)] is equivalent to the cKdV equation (1.14) for the function

$$V(X, T) := T^{-1} u(x = -XT^{-\frac{1}{2}}, t = T^{-\frac{1}{2}}) + \frac{1}{2} XT^{-1}. \quad (1.16)$$

In the language of integrable PDEs, this implies that the Fredholm determinant $J_\sigma(X, T)$ is a *tau function* of the cKdV equation. In particular, $J = J_\sigma(X, T)$ solves the *bilinear form* of the cKdV equation

$$\partial_X J \partial_T J - J \partial_X \partial_T J - \frac{1}{4} (\partial_X^2 J)^2 + \frac{1}{3} \partial_X J \partial_X^3 J - \frac{1}{12} J \partial_X^4 J - \frac{1}{2T} J \partial_X J = 0. \quad (1.17)$$

The direct and inverse scattering transform for the cKdV equation has been established in [44, 45, 28, 29] for smooth and decaying initial data.

Example 1.2. *The simplest situation occurs when $\sigma = 0$. Then,*

$$J_0(X, T) = 1, \quad V_0(X, T) = 0, \quad \varphi_0(\lambda; X, T) = \text{Ai}(T^{-\frac{1}{3}}(\lambda + X)). \quad (1.18)$$

Example 1.3. *The function $\sigma = 1_{(0, +\infty)}$ does not satisfy Assumption A, but it is nevertheless an instructive degenerate situation. (The present setting could be extended to include such case, cf. Remark 1.8 and Section 3.8.) The integro-differential Painlevé II equation reduces to the Painlevé II equation, and (1.5) is the celebrated Tracy–Widom formula. The cKdV tau function and solution are given by*

$$J_{1_{(0, +\infty)}}(X, T) = F_{\text{TW}}(XT^{-\frac{1}{3}}), \quad V_{1_{(0, +\infty)}}(X, T) = -T^{-\frac{2}{3}} y_{\text{HM}}(XT^{-\frac{1}{3}})^2, \quad (1.19)$$

where F_{TW} is the Tracy–Widom distribution [48] and y_{HM} is the Hastings–McLeod solution to Painlevé II [25]. The asymptotics of the Hastings–McLeod solution imply that (see Figure 1)

$$V_{1_{(0, +\infty)}}(X, T) \sim \begin{cases} \frac{1}{2} XT^{-1}, & XT^{-\frac{1}{3}} \rightarrow -\infty, \\ -T^{-\frac{2}{3}} \text{Ai}(XT^{-\frac{1}{3}}), & XT^{-\frac{1}{3}} \rightarrow +\infty. \end{cases} \quad (1.20)$$

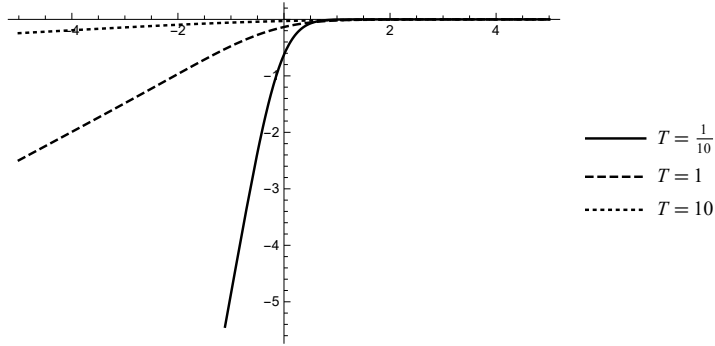


Figure 1: The solution $V_{1(0,+\infty)}(X, T)$ as a function of X for some values of T .

Remark 1.4. *The KdV and cKdV equations, (1.15) and (1.14), respectively, are completely equivalent from an algebraic point of view, since the transformation (1.16) defines a one-to-one correspondence of solutions; e.g., if V is bounded then u is not, and vice versa. In view of the analytic properties of the solutions under consideration, we find it more natural to work with the cKdV equation; moreover, the relevant Riemann–Hilbert problem in our analysis formally matches with the one for the inverse scattering theory of the cKdV of Its and Sukhanov [28, 29], cf. Section 3.5.*

To explain the probabilistic meaning of these cKdV solutions, let the shifted and dilated Airy point process be the determinantal point process [47] on the real line induced by the correlation kernel $K_{X,T}^{\text{Ai}}$ given in (1.10). Equivalently, it is the probability distribution (parametrically depending on $X \in \mathbb{R}, T > 0$) on the space of locally finite configurations of points on the real line characterized by its m -point correlation functions $\rho_{X,T;m}^{\text{Ai}}$ defined in (1.9). More explicitly, this means that for all disjoint Borel sets $B_1, \dots, B_\ell \subseteq \mathbb{R}$ and for all integers $k_1, \dots, k_\ell \geq 1$ summing up to $\sum_{j=1}^\ell k_j = m$, the expected number of m -tuples of points in a random configuration of which k_1 lie in B_1 , k_2 in B_2 , \dots , is

$$\frac{1}{k_1! \cdots k_\ell!} \int_{B_1^{k_1} \times \cdots \times B_\ell^{k_\ell}} \rho_{X,T;m}^{\text{Ai}}(\lambda_1, \dots, \lambda_m) d\lambda_1 \cdots d\lambda_m. \quad (1.21)$$

Then, the relation to the cKdV tau functions $J_\sigma(X, T)$ is expressed by the identity

$$J_\sigma(X, T) = \mathbb{E} \left[\prod_{j \geq 1} (1 - \sigma(\lambda_j)) \right] \quad (1.22)$$

where the expectation on the right involves the particles $\lambda_1 > \lambda_2 > \dots$ of the shifted and dilated Airy process. (It is well-known that the Airy process has almost surely infinitely many particles and a largest particle, see Remark 2.2.) The identity (1.22) follows from the general theory of determinantal point processes, cf. [5, eq. (11.2.4)].

The σ -thinned shifted and dilated Airy point process is obtained from the shifted and dilated Airy point process by removing each particle in a configuration independently with (position-dependent) probability $1 - \sigma$, thus retaining it with probability σ . This point process is also determinantal, with a correlation kernel given by $\sqrt{\sigma(\lambda)} K_{X,T}^{\text{Ai}}(\lambda, \mu) \sqrt{\sigma(\mu)}$, cf. [34, eq (2.5)]; it follows that $J_\sigma(X, T)$ is the *gap probability* of the σ -thinned process, i.e., the probability that a random configuration in this process is empty. Moreover, if $\sigma : \mathbb{R} \rightarrow [0, 1]$ decays sufficiently fast at $-\infty$ (for instance, if it satisfies Assumption A) the σ -thinned shifted and dilated Airy point process has almost surely a finite number of particles (Remark 2.2). In such a case, we can define the global *Jánossy density* of order $m \geq 0$, denoted $J_\sigma(X, T | \nu_1, \dots, \nu_m)$. These quantities are characterized by the property that for all disjoint Borel subsets $B_1, \dots, B_\ell \subseteq \mathbb{R}$ such that $\sqcup_{j=1}^\ell B_j = \mathbb{R}$ and all integers $k_1, \dots, k_\ell \geq 1$ summing up to $\sum_{j=1}^\ell k_j = m$, the probability that a configuration in the σ -thinned shifted and

dilated Airy point process contains exactly m particles, of which k_1 lie in B_1 , k_2 in B_2 , \dots , is

$$\frac{1}{k_1! \cdots k_\ell!} \int_{B_1^{k_1} \times \cdots \times B_\ell^{k_\ell}} J_\sigma(X, T | \nu_1, \dots, \nu_m) \prod_{j=1}^m \sigma(\nu_j) d\nu_j. \quad (1.23)$$

It is known [47, eq. (1.38)] that Jánossy densities can be expressed as

$$J_\sigma(X, T | \nu_1, \dots, \nu_m) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \rho_{X, T; n+m}^{\text{Ai}}(\lambda_1, \dots, \lambda_n, \nu_1, \dots, \nu_m) \prod_{i=1}^n \sigma(\lambda_i) d\lambda_i. \quad (1.24)$$

For brevity, we will collect the *distinct* real numbers ν_i into a vector $\underline{\nu} := (\nu_1, \dots, \nu_m)$ and denote $J_\sigma(X, T | \underline{\nu}) := J_\sigma(X, T | \nu_1, \dots, \nu_m)$. To be consistent with the notation previously introduced, we have $J_\sigma(X, T) = J_\sigma(X, T | \emptyset)$.

As in the case $m = 0$, it is interesting to study the situation when T is kept constant first. This is sufficient to disclose the relation to Stark boundary value problems and to a generalized integro-differential Painlevé II equation. In such case, T can be set to 1 without loss of generality because

$$J_\sigma(X, T | \underline{\nu}) = T^{-\frac{m}{3}} J_{\tilde{\sigma}}(XT^{-\frac{1}{3}}, 1 | T^{-\frac{1}{3}} \underline{\nu}), \quad \tilde{\sigma}(\lambda) := \sigma(T^{\frac{1}{3}} \lambda). \quad (1.25)$$

Accordingly, we will formulate our results in which T is constant more concisely in terms of

$$j_\sigma(s | \underline{\nu}) := J_\sigma(s, 1 | \underline{\nu}), \quad s \in \mathbb{R}. \quad (1.26)$$

Our first result on the integrable structure of the Jánossy densities is their expression in terms solely of the eigenfunctions of the Stark operator.

Theorem I. *Let σ satisfy Assumption A and let $\varphi_\sigma(\lambda; s)$ be the unique solution to the Stark boundary value problem (1.6). For all $s \in \mathbb{R}$ and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$, we have*

$$j_\sigma(s | \underline{\nu}) = \det(L_s^\sigma(\nu_i, \nu_j))_{i,j=1}^m j_\sigma(s | \emptyset), \quad (1.27)$$

where

$$L_s^\sigma(\lambda, \mu) = \int_s^{+\infty} \varphi_\sigma(\lambda; r) \varphi_\sigma(\mu; r) dr = \frac{\varphi_\sigma(\lambda; s) \partial_s \varphi_\sigma(\mu; s) - \partial_s \varphi_\sigma(\lambda; s) \varphi_\sigma(\mu; s)}{\lambda - \mu}, \quad (1.28)$$

and

$$j_\sigma(s | \emptyset) = \exp\left(-\int_s^{+\infty} (r-s) \left(\int_{\mathbb{R}} \varphi_\sigma(\lambda; r)^2 \sigma'(\lambda) d\lambda\right) dr\right). \quad (1.29)$$

The proof is given in Section 3.4.

Remark 1.5. *The kernel $L_s^\sigma(\cdot, \cdot)$ induces a determinantal point process (depending parametrically on $s \in \mathbb{R}$) which is obtained via a conditioning of the shifted Airy point process, in the following sense: assign independently to each point λ in a random configuration mark 1 with probability $\sigma(\lambda)$ and mark 0 otherwise, then condition the resulting marked shifted Airy point process on the event that no points have mark 1. The conditional point process obtained in this manner is determinantal, and has correlation kernel $L_s^\sigma(\cdot, \cdot)$, see Section 2.2 and [17] and [10, 11, 12, 21] for details. The factorization (1.27) receives an interesting probabilistic interpretation: it is the product of an m -point correlation function in this conditional determinantal point process and of the gap probability of the σ -thinned shifted Airy point process.*

Example 1.6. *It is instructive to consider again the case $\sigma = 0$, in which case $\varphi_0(\lambda; s) = \text{Ai}(\lambda + s)$, and so (1.28) reduces to the shifted Airy kernel $L_s^0(\lambda, \mu; s) = K^{\text{Ai}}(\lambda + s, \mu + s)$, cf. (1.3). In this sense we can regard φ_σ as a generalization of the Airy function, and L_s^σ as a generalization of the shifted Airy kernel; it is interesting to check that several properties of the Airy function and the shifted Airy kernel are preserved by this generalization. See for instance Proposition 3.10 and Corollary 3.11.*

Example 1.7. The choice $\sigma = 1_{(0,+\infty)}$ is again not admissible in view of Assumptions A, but with $v_\sigma(s) = \varphi(0; s)^2 = -y_{\text{HM}}(s)^2$, the degenerate case of (1.5), the Stark boundary value problem (1.6) still makes sense, and the kernel $L_s^{1_{(0,+\infty)}}(\lambda, \mu)$ is defined. Moreover, this kernel has appeared in the soft-to-hard edge transition in random matrix theory [18, Theorem 1.3]. In terms of the notations used in [18, Theorem 1.3], we have

$$\begin{aligned}\varphi_{1_{(0,+\infty)}}(\lambda; s) &= -\frac{1}{\sqrt{2\pi}}f_0(-\lambda; s), \\ \partial_s \varphi_{1_{(0,+\infty)}}(\lambda; s) &= -\frac{1}{\sqrt{2\pi}}(g_0(-\lambda; s) + p_0(s)f_0(-\lambda; s)),\end{aligned}$$

and

$$L_s^{1_{(0,+\infty)}}(\lambda; \mu) = \mathbb{K}_0^{\text{soft/hard}}(-\lambda, -\mu; s). \quad (1.30)$$

Then we can identify the Stark boundary problem (1.6) with [18, Equations (1.12)–(1.15)]. Furthermore, $L_s^{1_{(0,+\infty)}}$ is the kernel of the determinantal point process obtained by conditioning the shifted Airy point process on absence of particles on $(0, \infty)$. Recall also that $j_{1_{(0,+\infty)}}(s|\emptyset) = j_{1_{(0,+\infty)}}(s) = F_{\text{TW}}(s)$ is the Tracy–Widom distribution in this case.

Next we give a second expression for the Jánossy densities which is more directly parallel to equations (1.5), (1.6), and (1.7) for the case $m = 0$. This expression involves eigenfunctions of the Stark operator with a modified potential.

Theorem II. Let σ satisfy Assumption A. For all $s \in \mathbb{R}$ and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$, we have

$$\partial_s^2 \log j_\sigma(s|\underline{\nu}) = \int_{\mathbb{R}} \varphi_\sigma(\lambda; s|\underline{\nu})^2 \left(-\sigma'(\lambda) + \sum_{i=1}^m \frac{2(1 - \sigma(\lambda))}{\lambda - \nu_i} \right) d\lambda \quad (1.31)$$

where $\varphi_\sigma(\lambda; s|\underline{\nu})$ solves the Stark equation

$$(\partial_s^2 + 2v_\sigma(s|\underline{\nu}) - s)\varphi_\sigma(\lambda; s|\underline{\nu}) = \lambda\varphi_\sigma(\lambda; s|\underline{\nu}) \quad (1.32)$$

with potential

$$v_\sigma(s|\underline{\nu}) := \partial_s^2 \log j_\sigma(s|\underline{\nu}). \quad (1.33)$$

Moreover, $\varphi_\sigma(\lambda; s|\underline{\nu})$ can be expressed in terms of $\varphi_\sigma(\lambda; s|\emptyset)$ and $\partial_s \varphi_\sigma(\lambda; s|\emptyset)$ as

$$\varphi_\sigma(\lambda; s|\underline{\nu}) = \frac{\det \begin{pmatrix} \varphi_\sigma(\lambda; s|\emptyset) & L_s^\sigma(\lambda, \nu_1) & \cdots & L_s^\sigma(\lambda, \nu_m) \\ \varphi_\sigma(\nu_1, s|\emptyset) & L_s^\sigma(\nu_1, \nu_1) & \cdots & L_s^\sigma(\nu_1, \nu_m) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_\sigma(\nu_m, s|\emptyset) & L_s^\sigma(\nu_m, \nu_1) & \cdots & L_s^\sigma(\nu_m, \nu_m) \end{pmatrix}}{\det \begin{pmatrix} L_s^\sigma(\nu_1, \nu_1) & \cdots & L_s^\sigma(\nu_1, \nu_m) \\ \vdots & \ddots & \vdots \\ L_s^\sigma(\nu_m, \nu_1) & \cdots & L_s^\sigma(\nu_m, \nu_m) \end{pmatrix}}, \quad (1.34)$$

so that, in particular, $\varphi_\sigma(\lambda, s|\underline{\nu})$ has zeros at $\lambda = \nu_1, \dots, \nu_m$.

The proof is given in Section 3.4. In the case $m = 0$, the eigenfunctions $\varphi_\sigma(\lambda; s|\emptyset)$ of this theorem reduce to what we denoted just $\varphi_\sigma(\lambda; s)$ up to this point; for the sake of clarity, we will from now on use the notation $\varphi_\sigma(\lambda; s|\emptyset)$. Similarly said for the notation $v_\sigma(s) = v_\sigma(s|\emptyset)$.

In other words, Theorem II states that the Stark equation with potential $\partial_s^2 \log j_\sigma(s|\underline{\nu})$ is obtained by Darboux transformations (in the original spirit of Darboux [19]) of the Stark equation with potential $\partial_s^2 \log j_\sigma(s|\emptyset)$.

Moreover, by the asymptotics $\varphi_\sigma(\lambda; s|\emptyset) \sim \text{Ai}(\lambda + s) = \varphi_0(\lambda; s|\emptyset)$ as $s \rightarrow +\infty$, one obtains that the appropriate boundary condition for the solution to the Stark equation (1.32) is $\varphi_\sigma(\lambda; s|\underline{\nu}) \sim \varphi_0(\lambda; s|\underline{\nu})$ as $s \rightarrow +\infty$. The function $\varphi_0(\lambda; s|\underline{\nu})$ is explicit in terms of the Ai and Ai' functions, by (1.34) and $L_s^0 = K_s^{\text{Ai}}$.

It follows by combining (1.31), (1.32), and (1.33), that $\varphi_\sigma(\lambda; s|\underline{\nu})$ satisfies a deformation of the integro-differential Painlevé II equation:

$$\partial_s^2 \varphi_\sigma(\lambda; s|\underline{\nu}) = \left(\lambda + s + 2 \int_{\mathbb{R}} \varphi_\sigma(\mu; s|\underline{\nu})^2 \left(\sigma'(\mu) - \sum_{i=1}^m \frac{2(1 - \sigma(\mu))}{\mu - \nu_i} \right) d\mu \right) \varphi_\sigma(\lambda; s|\underline{\nu}). \quad (1.35)$$

A general framework for studying integro-differential equations related to a class of Fredholm determinants was developed in [32].

Remark 1.8. *Theorems I and II hold true more generally for all σ satisfying the decay condition of Assumption A even if they are only piecewise smooth with a finite number of jump singularities. In this case, one has to add to $\sigma'(\lambda) d\lambda$ a discrete measure supported at the singularities of σ . The extension to this case can be done following [14], as we will discuss in Section 3.8.*

Remark 1.9. *Jánossy densities in general carry important information about a point process, but the global Jánossy densities of a determinantal point process are only defined if the associated kernel defines a trace class operator. This is not the case for the Airy kernel operator. To remedy this, one commonly considers Jánossy densities of determinantal point processes restricted to bounded sets B [6]. It is less customary to consider Jánossy densities of thinned determinantal point processes, as we do, but this has the advantage that, for a suitable class of thinning functions σ , global Jánossy densities exist. In the degenerate case $\sigma = 1_B$, we recover the Jánossy density of the determinantal point process restricted to the set B.*

The description of Jánossy densities of determinantal point processes on \mathbb{R} , restricted to bounded intervals B , in terms of solutions to certain differential equations has been recently developed in [40]. The author there proved that for determinantal point processes defined through kernels satisfying the Tracy–Widom criteria [49], the Tracy–Widom method allows to express not only the gap probability (as proved in [49]) but also the Jánossy densities of the process restricted to a bounded interval B , in terms of solutions to a system of differential equations in the endpoints of B . For kernels also enjoying the integrable structure of Its–Izergin–Korepin–Slavnov [27] (e.g., Airy, Bessel, sine kernels), the gap probability can be characterized by a Riemann–Hilbert (RH) problem. This provides an alternative approach to study underlying integrable differential equations, and a powerful tool to tackle their asymptotics. In this work, we extend the RH approach to study Jánossy densities of the thinned shifted Airy point process. We are confident that our method can also be applied to other determinantal point processes with integrable structure, like the ones associated to Bessel and sine kernels.

To construct a family of solutions to the cKdV equation in terms of the Jánossy densities, we restore the full dependence on X, T and we introduce

$$V_\sigma(X, T|\underline{\nu}) := \partial_X^2 \log J_\sigma(X, T|\underline{\nu}), \quad X \in \mathbb{R}, T > 0. \quad (1.36)$$

Theorem III. *For all σ satisfying Assumption A and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$, the function $V = V_\sigma(X, T|\underline{\nu})$ solves the cKdV equation,*

$$\partial_T V + \frac{1}{12} \partial_X^3 V + V \partial_X V + \frac{1}{2T} V = 0, \quad \text{for all } X \in \mathbb{R}, T > 0. \quad (1.37)$$

The proof is in Section 3.7, see Corollary 3.13.

Example 1.10. *When $\sigma = 0$ we have $j_0(s|\emptyset) = 1$ and $L_s^0(\lambda, \mu) = K^{\text{Ai}}(\lambda + s, \mu + s)$, such that, according to (1.27),*

$$j_0(s|\underline{\nu}) = \rho_m^{\text{Ai}}(\nu_1 + s, \dots, \nu_m + s) \quad (1.38)$$

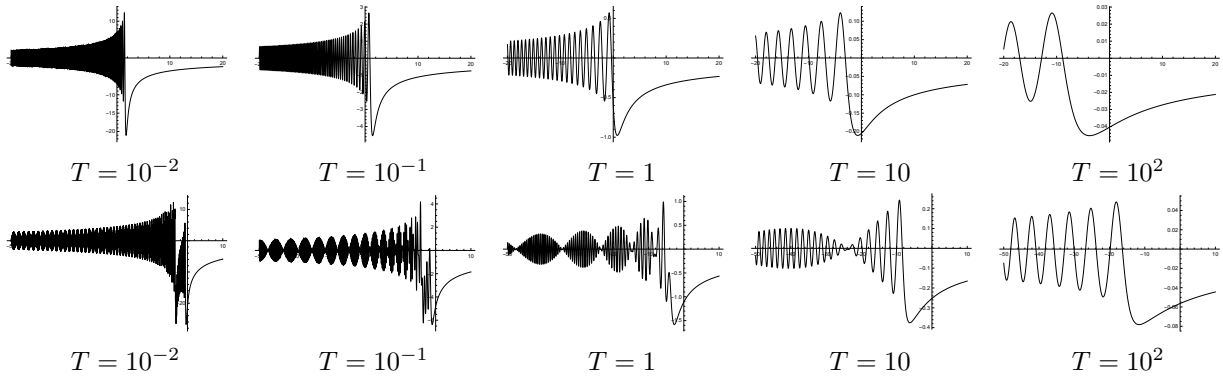


Figure 2: First line: 1-soliton cKdV solution $V_0(X, T|\underline{\nu})$ with $\underline{\nu} = (-2)$ as a function of X for various values of T . Second line: 2-soliton cKdV solution $V_0(X, T|\underline{\nu})$ with $\underline{\nu} = (0, 3)$ as a function of X for various values of T .

is the m -point correlation function in the shifted Airy ensemble (1.2). The corresponding cKdV tau function is

$$J_0(X, T|\underline{\nu}) = \det(K_{X,T}^{\text{Ai}}(\nu_i, \nu_j))_{i,j=1}^m. \quad (1.39)$$

The associated cKdV solution $V_0(X, T|\underline{\nu}) = \partial_X^2 \log \det(K_{X,T}^{\text{Ai}}(\nu_i, \nu_j))_{i,j=1}^m$ is a special case of soliton-type solution [39], cf. Figure 2, exhibiting right tail decay, and left tail rapid oscillations with decaying amplitude. It is straightforward to verify, using the asymptotics for the Airy function and its derivative, that for any $T_0 > 0$,

$$V_0(X, T|\underline{\nu}) \sim -\frac{m}{\sqrt{XT}}, \quad \text{as } XT^{-\frac{1}{3}} \rightarrow +\infty, \text{ uniformly for } T \geq T_0. \quad (1.40)$$

Indeed,

$$K_{X,T}^{\text{Ai}}(v, w) \sim \frac{1}{8\pi X} e^{-\frac{2}{3}T^{-\frac{1}{2}}(X+v)^{\frac{3}{2}}} e^{-\frac{2}{3}T^{-\frac{1}{2}}(X+w)^{\frac{3}{2}}} \quad \text{as } XT^{-\frac{1}{3}} \rightarrow +\infty, T \geq T_0, \quad (1.41)$$

and this allows to prove by induction on m that

$$\det(K_{X,T}^{\text{Ai}}(\nu_i, \nu_j))_{i,j=1}^m \sim \frac{1}{(8\pi X)^m} e^{-\frac{4m}{3}T^{-\frac{1}{2}}X^{\frac{3}{2}}}. \quad (1.42)$$

Since these asymptotics are moreover valid uniformly for complex X with $|\arg X| < \delta$ and $\delta > 0$ small, we can differentiate the logarithm of this expression twice which yields (1.40).

Similarly, after straightforward computations involving the asymptotic behavior of the Airy function and its derivative, we obtain for the left tail if $m = 1$, as $XT^{-\frac{1}{3}} \rightarrow -\infty$, $T \geq T_0$,

$$J_0(X, T|\nu) = \frac{1}{\pi} \sqrt{\frac{|X| - \nu}{T}} \left(1 - \frac{\sqrt{T}}{4} (|X| - \nu)^{-\frac{3}{2}} \cos \left[\frac{4}{3\sqrt{T}} (|X| - \nu)^{\frac{3}{2}} \right] + O(X^{-3}T) \right), \quad (1.43)$$

$$V_0(X, T|\nu) = \frac{1}{\sqrt{T|X|}} \cos \left[\frac{4}{3\sqrt{T}} (|X| - \nu)^{\frac{3}{2}} \right] + O(|X|^{-\frac{3}{2}}T^{-\frac{1}{2}}). \quad (1.44)$$

Remark 1.11. The function σ and the parameters $\underline{\nu}$ can be understood as scattering data for the cKdV solution under consideration. In analogy with [23, 24], we could interpret σ as a function describing a gas of solitons.

Our final result concerns the asymptotic behavior of the Jánossy densities and associated cKdV solutions when $T \rightarrow +\infty$, uniformly in $X \in \mathbb{R}$. To this end, we formulate some stronger conditions on the function σ , cf. [14, 15].

Assumption B. The function $F := \frac{1}{1-\sigma}$ extends to an entire function. Moreover:

- $F' \geq 0$ and $(\log F)'' \geq 0$ on the real line;
- $F(\lambda) = 1 + c'_- e^{-c_-|\lambda|}(1 + o(1))$ as $\lambda \rightarrow -\infty$, $F(\lambda) = c'_+ e^{c_+\lambda}(1 + O(e^{-\epsilon\lambda}))$ as $\lambda \rightarrow +\infty$, for some $c_\pm, c'_\pm, \epsilon > 0$;
- $F(\lambda) = O(e^{c_+\operatorname{Re}\lambda})$ as $\operatorname{Re}\lambda \rightarrow +\infty$.

In particular, the second assumption implies the strong decay $\sigma(\lambda) = c'_- e^{-c_-|\lambda|}(1 + o(1))$ as $\lambda \rightarrow -\infty$ and $\sigma(\lambda) = 1 - \frac{1}{c'_+} e^{-c_+\lambda}(1 + O(e^{-\epsilon\lambda}))$ as $\lambda \rightarrow +\infty$ (for the same constants $c_\pm, c'_\pm, \epsilon > 0$).

The reader may want to keep in mind the prototype example of an admissible function σ given by $\sigma(\lambda) = \frac{1}{1+e^{-\lambda}}$, such that $F(\lambda) = 1 + e^\lambda$.

Theorem IV. Let $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$.

(i) Let σ satisfy Assumption A. For any $T_0 > 0$, there exists $c > 0$ such that

$$J_\sigma(X, T|\underline{\nu}) = \det(K_{X,T}^{A_i}(\nu_i, \nu_j))_{i,j=1}^m \left(1 + O(e^{-cXT^{-\frac{1}{3}}})\right) \sim (8\pi X)^{-m} e^{-\frac{4m}{3}X^{\frac{2}{3}}T^{-\frac{1}{2}}}, \quad (1.45)$$

$$V_\sigma(X, T|\underline{\nu}) = V_0(X, T|\underline{\nu}) + O(e^{-cXT^{-\frac{1}{3}}}) \sim -\frac{m}{\sqrt{XT}}, \quad (1.46)$$

uniformly in $T \geq T_0$ as $XT^{-\frac{1}{3}} \rightarrow +\infty$.

(ii) Let σ satisfy Assumption B. For any $T_0 > 0$, we have

$$J_\sigma(X, T|\underline{\nu}) \sim \frac{|X|^{\frac{m}{2}}}{\pi^m T^{\frac{m}{2}}} J_\sigma(X, T|\emptyset) \prod_{j=1}^m \frac{1}{1 - \sigma(\nu_j)}, \quad (1.47)$$

$$V_\sigma(X, T|\underline{\nu}) = V_\sigma(X, T|\emptyset) + \frac{1}{\sqrt{|X|T}} \sum_{j=1}^m \cos\left(\frac{4|X|^{\frac{3}{2}}}{3T^{\frac{1}{2}}}(1 + A_{X,T}) - \frac{2|X|^{\frac{1}{2}}}{T^{\frac{1}{2}}}\nu_j(1 + B_{X,T}(\nu_j))\right) + O(|X|^{-1}), \quad (1.48)$$

uniformly for $T \geq T_0$ as $\frac{X}{T \log^2 |X|} \rightarrow -\infty$, where $A_{X,T}, B_{X,T}(\nu)$ converge to 0 as $\frac{X}{T \log^2 |X|} \rightarrow -\infty$. Moreover, in the same limit,

$$\begin{aligned} \log J_\sigma(X, T|\emptyset) &= \rho^3 T^2 \left(-\frac{4}{15} (1 - \xi)^{\frac{5}{2}} + \frac{4}{15} - \frac{2}{3} \xi + \frac{1}{2} \xi^2 \right) + O\left(|X|^{\frac{3}{2}} T^{-\frac{1}{2}}\right), \\ V_\sigma(X, T|\emptyset) &= \rho \left(1 - \sqrt{1 - \xi}\right) + O\left(|X|^{-\frac{1}{2}} T^{-\frac{1}{2}}\right), \end{aligned}$$

where $\rho := c_+^2/\pi^2$ and $\xi := X/(\rho T)$.

Observe the specific structure of the ν_j -dependence of the cKdV solution $V_\sigma(X, T|\underline{\nu})$ and the Jánossy density $J_\sigma(X, T|\underline{\nu})$. For $XT^{-\frac{1}{3}} \rightarrow +\infty$, the leading order behaviors of $V_\sigma(X, T|\underline{\nu})$ and $J_\sigma(X, T|\underline{\nu})$ depend on the number of points m but not explicitly on the positions ν_1, \dots, ν_m . On the other hand, for $\frac{X}{T \log^2 |X|} \rightarrow -\infty$, the effect of $\underline{\nu}$ is more prominent. For $J_\sigma(X, T|\underline{\nu})$, it results in a product of factors depending on ν_j , while for $V_\sigma(X, T|\underline{\nu})$, the presence of ν_1, \dots, ν_m results in a superposition of rapidly oscillating terms depending on ν_j .

It is remarkable that both in the left and right tail asymptotics of $V_\sigma(X, T|\underline{\nu})$, we recognize (up to a sub-leading phase shift in the oscillatory terms) a superposition of m 1-soliton solutions whose tail asymptotics are described in Example 1.10, in addition to the leading order and $\underline{\nu}$ -independent contribution coming from $V_\sigma(X, T|\emptyset)$.

For $-KT \log^2 |X| \leq X \leq MT^{\frac{1}{3}}$, the ν_j -dependence is more involved and less explicit, as we will explain in Section 4.4.

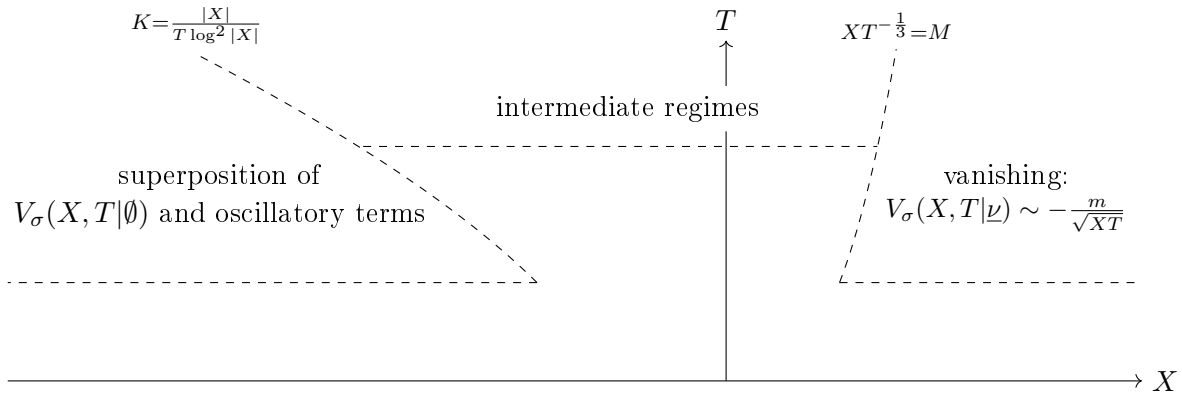


Figure 3: Phase diagram showing the different tail asymptotics for $V_\sigma(X, T|\underline{\nu})$, uniform in the indicated regions for fixed $M, K > 0$.

Methodology and outline

In Section 2, we will gather several properties and identities for the Jánossy densities on which we will rely later, and we will give a probabilistic interpretation to the kernel L_s^σ . Two different factorizations of the Jánossy densities will be of particular importance.

In Section 3, we will characterize the Jánossy densities and other relevant quantities in terms of a 2×2 matrix-valued Riemann–Hilbert (RH) problem, by relying on the Its–Izergin–Korepin–Slavnov method [27]. This RH characterization shows strong similarities with the one from [4] and we give a detailed comparison with the general methods of op. cit. in Section 3.6. Moreover, we establish a connection between the RH problem, the Stark boundary value problem (1.6), and the cKdV equation (1.14). This will enable us to prove Theorem I, Theorem II, and Theorem III.

Section 4 will be devoted to the asymptotic analysis of the RH problem from Section 3. We will distinguish several regions in the (X, T) -plane which will require a different type of asymptotic analysis, and which lead to the results presented in Theorem IV. In this section, we also use previous asymptotic results [14, 15] for the case $m = 0$ in which Jánossy densities reduce to gap probabilities.

2 Preliminaries on Jánossy densities

In this section, we study in more detail the Jánossy densities $j_\sigma(s|\underline{\nu})$ introduced in (1.26). The results could be easily translated into parallel results for $J_\sigma(X, T|\underline{\nu})$ by (1.25) and the observation that $\tilde{\sigma}$ satisfies Assumption A if σ does, but we will omit the details for the sake of brevity.

2.1 Operator preliminaries

For a given $g \in L^\infty(\mathbb{R})$, let \mathcal{M}_g be the multiplication operator on $L^2(\mathbb{R})$ defined by $\mathcal{M}_g f = gf$ for all $f \in L^2(\mathbb{R})$, and let $\mathcal{K}_s^{\text{Ai}}$ be the operator acting on $L^2(\mathbb{R})$ through the shifted Airy kernel,

$$(\mathcal{K}_s^{\text{Ai}} f)(\lambda) = \int_{\mathbb{R}} K_s^{\text{Ai}}(\lambda, \mu) f(\mu) d\mu, \quad K_s^{\text{Ai}}(\lambda, \mu) := K^{\text{Ai}}(\lambda + s, \mu + s), \quad f \in L^2(\mathbb{R}), \quad (2.1)$$

with K^{Ai} defined in (1.3). It is worth recalling that $\mathcal{K}_s^{\text{Ai}}$ is an orthogonal projector which can be represented as $\mathcal{A}_s \mathcal{M}_{1_{(0, +\infty)}} \mathcal{A}_s$ where \mathcal{A}_s is the unitary involution of $L^2(\mathbb{R})$ defined by

$$(\mathcal{A}_s f)(\lambda) = \int_{-\infty}^{+\infty} \text{Ai}(\lambda + \mu + s) f(\mu) d\mu, \quad f \in L^2(\mathbb{R}), \quad (2.2)$$

where the integral in the right-hand side is taken as an L^2 -limit of $\int_{-\Lambda}^{+\infty}$ as $\Lambda \rightarrow +\infty$.

Lemma 2.1. *Let σ satisfy Assumption A. The operator $\mathcal{K}_s^\sigma := \mathcal{M}_{\sqrt{\sigma}} \mathcal{K}_s^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}}$ is trace class on $L^2(\mathbb{R})$ and*

$$j_\sigma(s|\emptyset) = \det_{L^2(\mathbb{R})} (1 - \mathcal{K}_s^\sigma). \quad (2.3)$$

Moreover, $0 < j_\sigma(s|\emptyset) \leq 1$ for all $s \in \mathbb{R}$.

We denote the Fredholm determinant of a trace class perturbation of the identity by $\det_{L^2(\mathbb{R})}$.

Proof. We have $\mathcal{K}_s^\sigma = \mathcal{H}\mathcal{H}^\dagger$ where $\mathcal{H} := \mathcal{M}_{\sqrt{\sigma}} \mathcal{A}_s \mathbf{1}_{(0,+\infty)}$. It follows by the asymptotic properties of the Airy function at $+\infty$ that \mathcal{H} is Hilbert–Schmidt provided σ satisfies Assumption A. Therefore, \mathcal{K}_s^σ is the composition of two Hilbert–Schmidt operators, hence it is trace class on $L^2(\mathbb{R})$. Then, (2.3) follows by the classical formula for Fredholm determinants of operators with an integral kernel. Next, since $\mathcal{K}_s^{\text{Ai}}$ is an orthogonal projector and $0 \leq \sigma \leq 1$, we have $(\mathcal{K}_s^\sigma)^2 \leq \mathcal{K}_s^\sigma$ because

$$(\mathcal{K}_s^\sigma)^2 = \mathcal{M}_{\sqrt{\sigma}} \mathcal{K}_s^{\text{Ai}} \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}} \leq \mathcal{M}_{\sqrt{\sigma}} (\mathcal{K}_s^{\text{Ai}})^2 \mathcal{M}_{\sqrt{\sigma}} = \mathcal{M}_{\sqrt{\sigma}} \mathcal{K}_s^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}} = \mathcal{K}_s^\sigma, \quad (2.4)$$

such that $0 \leq \mathcal{K}_s^\sigma \leq 1$ and so $0 \leq j_\sigma(s|\emptyset) \leq 1$. It remains to show that $j_\sigma(s|\emptyset) \neq 0$, or, equivalently, that 1 is not an eigenvalue of \mathcal{K}_s^σ . For, assume $f \in L^2(\mathbb{R})$ is such that $\mathcal{K}_s^\sigma f = f$. Setting $g := \mathcal{K}_s^{\text{Ai}}(\sqrt{\sigma}f)$, we have $\sqrt{\sigma}g = f$ and so, since $\mathcal{K}_s^{\text{Ai}}$ is an orthogonal projector,

$$\|g\|_2 = \|\mathcal{K}_s^{\text{Ai}}(\sqrt{\sigma}f)\|_2 \leq \|\sqrt{\sigma}f\|_2 = \|\sigma g\|_2. \quad (2.5)$$

Therefore, since $0 \leq \sigma \leq 1$, we have $(\sigma - 1)g = 0$ almost everywhere on \mathbb{R} . Since $\sigma \rightarrow 0$ at $+\infty$ (cf. Assumption A), g has to vanish on some open set of \mathbb{R} . On the other hand, g is the restriction to the real line of an entire function, as it follows from the fact that $\mathcal{A}_s h$ is entire for all h with support bounded below by standard properties of the Airy function. Therefore, g is identically zero, so is f , and 1 is not an eigenvalue of \mathcal{K}_s^σ . \square

Remark 2.2. $\mathcal{M}_{\sqrt{\sigma}} \mathcal{K}_s^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}}$ acts on $L^2(\mathbb{R})$ through the kernel $\sqrt{\sigma(\lambda)\sigma(\mu)} K^{\text{Ai}}(\lambda + s, \mu + s)$, which is a correlation kernel for the σ -thinned shifted Airy point process. It follows from Lemma 2.1 and from the general theory of determinantal point processes [47, Theorem 4] that the σ -thinned shifted and dilated Airy point process has almost surely a finite number of particles. On the other hand, since \mathcal{K}^{Ai} is not trace class, the Airy point process has almost surely an infinite number of particles; it is however trace class once restricted to half-lines $(t, +\infty)$ so that the Airy point process has almost surely a largest particle.

2.2 Conditional ensembles

According to Lemma 2.1, the operator $1 - \mathcal{K}_s^\sigma$ is invertible, and, therefore, so is $1 - \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}}$. Thus, it makes sense to introduce

$$\mathcal{L}_s^\sigma := \mathcal{K}_s^{\text{Ai}} (1 - \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}})^{-1} = \mathcal{K}_s^{\text{Ai}} + \mathcal{K}_s^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}} (1 - \mathcal{K}_s^\sigma)^{-1} \mathcal{M}_{\sqrt{\sigma}} \mathcal{K}_s^{\text{Ai}}. \quad (2.6)$$

As we shall review below following the Its–Izergin–Korepin–Slavnov method [27], \mathcal{L}_s^σ is an integral kernel operator, whose kernel we denote by $L_s^\sigma(\cdot, \cdot)$. It has been proved by the first two authors of this paper [17], building on [10, 11, 12], that this kernel induces a determinantal point process defined as follows. Consider the shifted Airy process and construct a σ -marked point process by assigning to each point λ in a random configuration, independently, a mark 1 with probability $\sigma(\lambda)$ or a mark 0 with probability $1 - \sigma(\lambda)$. Conditioning the marked point process on the event that there are no points with mark 1, it is shown in op. cit. that the resulting conditional ensemble is determinantal, with correlation kernel with respect to the *deformed* reference measure $(1 - \sigma(\lambda)) d\lambda$ given precisely by $L_s^\sigma(\cdot, \cdot)$.

Let us introduce the following notation. Given vectors $\underline{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ and $\underline{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$, introduce the $m \times n$ matrix $K_s^{\text{Ai}}(\underline{u}, \underline{w}) \in \mathbb{R}^{m \times n}$ with entries

$$(K_s^{\text{Ai}}(\underline{u}, \underline{w}))_{i,j} := K_s^{\text{Ai}}(u_i, w_j), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n. \quad (2.7)$$

Lemma 2.3. For any vector $\underline{\nu} = (\nu_1, \dots, \nu_m)$, with $\nu_i \neq \nu_j$ for all $i \neq j$, $K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})$ is positive-definite.

Proof. According to (1.3) and to (2.7), we can rewrite $K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})$ as a Gram matrix

$$(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu}))_{i,j} = \int_0^{+\infty} \text{Ai}(\nu_i + \eta + s) \text{Ai}(\nu_j + \eta + s) d\eta = \langle \text{Ai}(\nu_i + \cdot), \text{Ai}(\nu_j + \cdot) \rangle_{L^2(s, +\infty)}. \quad (2.8)$$

Hence, it suffices to show that the m vectors $\text{Ai}(\nu_i + \cdot) \in L^2(s, +\infty)$, for $1 \leq i \leq m$, are linearly independent when the points ν_1, \dots, ν_m are distinct. In order to obtain a contradiction, let us assume that the linear span of these m vectors is k -dimensional with $k < m$ and, without loss of generality, that $\text{Ai}(\nu_i + \cdot)$ for $i = 1, \dots, k$ form a basis. Then, there exists $c_1, \dots, c_k \in \mathbb{C}$ such that $\text{Ai}(\nu_m + t) = \sum_{i=1}^k c_i \text{Ai}(\nu_i + t)$ identically in t . Subtract $(t + \nu_m)$ times this relation from the second derivative of this relation in t to get, using the Airy equation, $0 = \sum_{i=1}^k (\nu_i - \nu_m) c_i \text{Ai}(\nu_i + t)$. Hence $c_i = 0$ for all $1 \leq i \leq k$ because $\nu_m \neq \nu_i$ for all $i \neq m$, and so $\text{Ai}(\nu_m + t) = 0$ identically in t , a contradiction. \square

According to Lemma 2.3, for distinct points $\nu_1, \dots, \nu_m \in \mathbb{R}$, collected into a vector $\underline{\nu} := (\nu_1, \dots, \nu_m)$, we can introduce the integral kernel operator $\mathcal{H}_s^\underline{\nu}$ acting on $L^2(\mathbb{R})$ through the kernel

$$H_s^\underline{\nu}(\lambda, \mu) := \frac{\det K_s^{\text{Ai}}((\lambda, \underline{\nu}), (\mu, \underline{\nu}))}{\det K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})} = K_s^{\text{Ai}}(\lambda, \mu) - K_s^{\text{Ai}}(\lambda, \underline{\nu}) K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})^{-1} K_s^{\text{Ai}}(\underline{\nu}, \mu), \quad (2.9)$$

where the second equality stems from the well-known formula

$$\det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det(D) \det(A - BD^{-1}C), \quad \text{if } \det D \neq 0, \quad (2.10)$$

for the determinant of a block matrix with lower-right corner invertible. It follows from the results in [46] that $H_s^\underline{\nu}(\lambda, \mu)$ is the kernel of the *reduced Palm measure* of the shifted Airy process at (distinct) points ν_1, \dots, ν_m , which can be interpreted as the shifted Airy point process conditioned on configurations containing points at ν_1, \dots, ν_m and then removing the points ν_1, \dots, ν_m from the configuration.

2.3 Factorizations of Jánossy densities

We can factorize the Jánossy densities $j_\sigma(s|\underline{\nu})$ in two different ways: the first one utilizes the Palm kernels $H_s^\underline{\nu}$, the second one involves the kernels L_s^σ of the conditional ensembles. It is convenient to introduce notations similar to (2.7) for these kernels, namely, given vectors $\underline{u} = (u_1, \dots, u_m) \in \mathbb{R}^m$ and $\underline{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$, we introduce matrices $L_s^\sigma(\underline{u}, \underline{w}) \in \mathbb{R}^{m \times n}$ and $H_s^\underline{\nu}(\underline{u}, \underline{w}) \in \mathbb{R}^{m \times n}$ with entries

$$(L_s^\sigma(\underline{u}, \underline{w}))_{i,j} = L_s^\sigma(u_i, w_j), \quad (H_s^\underline{\nu}(\underline{u}, \underline{w}))_{i,j} = H_s^\underline{\nu}(u_i, w_j), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n. \quad (2.11)$$

Proposition 2.4. For all σ satisfying Assumption A and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for $i \neq j$, we have the identities

$$j_\sigma(s|\underline{\nu}) = \det(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})) \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \mathcal{H}_s^\underline{\nu} \mathcal{M}_{\sqrt{\sigma}} \right), \quad (2.12)$$

$$j_\sigma(s|\underline{\nu}) = \det(L_s^\sigma(\underline{\nu}, \underline{\nu})) \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \mathcal{K}_s^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}} \right) = \det(L_s^\sigma(\underline{\nu}, \underline{\nu})) j_\sigma(s|\emptyset). \quad (2.13)$$

Proof. We start by rewriting (1.24):

$$\begin{aligned} j_\sigma(s|\underline{\nu}) &= \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \det(K_s^{\text{Ai}}((\lambda, \underline{\nu}), (\lambda, \underline{\nu}))) \prod_{i=1}^n \sigma(\lambda_i) d\lambda_i \\ &= \det(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \det(H_s^\underline{\nu}(\lambda, \lambda)) \prod_{i=1}^n \sigma(\lambda_i) d\lambda_i, \end{aligned} \quad (2.14)$$

where we denote $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$, $(\underline{\lambda}, \underline{\nu}) = (\lambda_1, \dots, \lambda_n, \nu_1, \dots, \nu_m)$, and we manipulate the determinant of block matrices using (2.10) and Lemma 2.3 as

$$\begin{aligned} \det(K_s^{\text{Ai}}((\underline{\lambda}, \underline{\nu}), (\underline{\lambda}, \underline{\nu}))) &= \det\left(\begin{array}{c|c} K_s^{\text{Ai}}(\underline{\lambda}, \underline{\lambda}) & K_s^{\text{Ai}}(\underline{\lambda}, \underline{\nu}) \\ \hline K_s^{\text{Ai}}(\underline{\nu}, \underline{\lambda}) & K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu}) \end{array}\right) \\ &= \det(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})) \det(K_s^{\text{Ai}}(\underline{\lambda}, \underline{\lambda}) - K_s^{\text{Ai}}(\underline{\lambda}, \underline{\nu})K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})^{-1}K_s^{\text{Ai}}(\underline{\nu}, \underline{\lambda})) \\ &= \det(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})) \det(H_s^\nu(\underline{\lambda}, \underline{\lambda})). \end{aligned} \quad (2.15)$$

Hence, (2.12) is established. Next, let us introduce the operator $\mathcal{N} := \mathcal{M}_{\sqrt{\sigma}}(\mathcal{K}_s^{\text{Ai}} - \mathcal{H}_s^\nu)\mathcal{M}_{\sqrt{\sigma}}$ such that

$$\det_{L^2(\mathbb{R})}(1 - \mathcal{M}_{\sqrt{\sigma}}\mathcal{H}_s^\nu\mathcal{M}_{\sqrt{\sigma}}) = \det_{L^2(\mathbb{R})}(1 - \mathcal{K}_s^\sigma) \det_{L^2(\mathbb{R})}(1 + (1 - \mathcal{K}_s^\sigma)^{-1}\mathcal{N}). \quad (2.16)$$

From (2.9) we know that the kernel of \mathcal{N} is $N(\lambda, \mu) = \sqrt{\sigma(\lambda)\sigma(\mu)}K_s^{\text{Ai}}(\lambda, \underline{\nu})K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})^{-1}K_s^{\text{Ai}}(\underline{\nu}, \mu)$, such that the kernel of $(I - \mathcal{K}_s^\sigma)^{-1}\mathcal{N}$ is

$$\tilde{L}_s^\sigma(\lambda, \underline{\nu})(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu}))^{-1}K_s^{\text{Ai}}(\underline{\nu}, \mu)\sqrt{\sigma(\mu)} \quad (2.17)$$

where $\tilde{L}_s^\sigma(\cdot, \cdot)$ is the kernel of $(1 - \mathcal{K}_s^\sigma)^{-1}\mathcal{M}_{\sqrt{\sigma}}\mathcal{K}_s^{\text{Ai}}$. By the general formula for the Fredholm determinant of a finite-rank perturbation of the identity, cf. [22, Theorem 3.2], we obtain (I_m denotes the $m \times m$ identity matrix)

$$\begin{aligned} \det_{L^2(\mathbb{R})}(1 + (1 - \mathcal{K}_s^\sigma)^{-1}\mathcal{N}) &= \det\left(I_m + K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu})^{-1} \int_{\mathbb{R}} K_s^{\text{Ai}}(\underline{\nu}, \lambda)\sqrt{\sigma(\lambda)}\tilde{L}_s^\sigma(\lambda, \underline{\nu})d\lambda\right) \\ &= \frac{\det(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu}) + \int_{\mathbb{R}} K_s^{\text{Ai}}(\underline{\nu}, \lambda)\sqrt{\sigma(\lambda)}\tilde{L}_s^\sigma(\lambda, \underline{\nu})d\lambda)}{\det(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu}))} = \frac{\det(L_s^\sigma(\underline{\nu}, \underline{\nu}))}{\det(K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu}))} \end{aligned} \quad (2.18)$$

where we use the second identity in (2.6). Finally, (2.13) follows from (2.12). \square

Remark 2.5. Both factorizations (2.12) and (2.13) have a natural probabilistic interpretation as products of an m -point correlation function with a gap probability. In the first factorization, we have the m -point correlation function in the shifted and rescaled Airy point process, multiplied with the gap probability in the σ -thinning of the Palm measure at points ν_1, \dots, ν_m associated to the shifted and rescaled Airy point process. In the second factorization, we have the m -point correlation function in the conditional ensemble associated to the shifted and rescaled Airy point process introduced above, multiplied with the gap probability in the σ -thinning of the thinned shifted and rescaled Airy point process. In the first factorization, the correlation function is simpler, but the gap probability is on the other hand simpler in the second factorization.

Using the above result, it is now easy to show that Jánossy densities are strictly positive for all distinct ν_1, \dots, ν_m .

Proposition 2.6. For all σ satisfying Assumption A and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for $i \neq j$, we have $\det L_s^\sigma(\underline{\nu}, \underline{\nu}) > 0$ and $j_\sigma(s|\underline{\nu}) > 0$.

Proof. The operator $\mathcal{N} := \mathcal{M}_{\sqrt{\sigma}}(\mathcal{K}_s^{\text{Ai}} - \mathcal{H}_s^\nu)\mathcal{M}_{\sqrt{\sigma}}$ is nonnegative-definite. Indeed, for all $\phi \in L^2(\mathbb{R})$,

$$\langle \mathcal{N}\phi, \phi \rangle = \underline{h}^\dagger (K_s^{\text{Ai}}(\underline{\nu}, \underline{\nu}))^{-1} \underline{h} \geq 0, \quad (2.19)$$

where $\underline{h} = \int_{\mathbb{R}} \sqrt{\sigma(\lambda)}\phi(\lambda)K_s^{\text{Ai}}(\underline{\nu}, \lambda)d\lambda$. Hence, we have proved that

$$1 - \mathcal{M}_{\sqrt{\sigma}}\mathcal{H}_s^\nu\mathcal{M}_{\sqrt{\sigma}} \geq 1 - \mathcal{K}_s^\sigma. \quad (2.20)$$

Since $1 - \mathcal{K}_s^\sigma$ is (strictly) positive-definite by Lemma 2.1, the operator $1 - \mathcal{M}_{\sqrt{\sigma}}\mathcal{H}_s^\nu\mathcal{M}_{\sqrt{\sigma}}$ is also positive-definite, hence invertible. Therefore, $j_\sigma(s|\underline{\nu}) > 0$ by Lemma 2.3 and the first factorization of Jánossy densities (2.12), and therefore $\det(L_s^\sigma(\underline{\nu}, \underline{\nu})) > 0$ by the second one (2.13). \square

3 RH characterization of Jánossy densities

The aim of the section is to give RH characterizations of Jánossy densities, in order to prove Theorems I, II, and III.

3.1 RH problems

The operator $\mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}}$ is *integrable* in the sense of Its–Izergin–Korepin–Slavnov (IIKS) [27], namely it is a kernel operator whose kernel can be expressed as

$$\frac{\mathbf{f}(\lambda; s) \mathbf{h}(\mu; s)}{\lambda - \mu}, \quad \mathbf{f}(\lambda; s) := \sigma(\lambda) \begin{pmatrix} -i \text{Ai}'(\lambda + s) \\ \text{Ai}(\lambda + s) \end{pmatrix}, \quad \mathbf{h}(\mu; s) := \begin{pmatrix} -i \text{Ai}(\mu + s) \\ \text{Ai}'(\mu + s) \end{pmatrix}. \quad (3.1)$$

Therefore, according to op. cit., the resolvent operator $(1 - \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}})^{-1} - 1$ can be characterized in terms of the following RH problem (see proof of Proposition 3.1 below).

RH problem for Y_σ

- (a) $Y_\sigma(\cdot; s) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic for all $s \in \mathbb{R}$.
- (b) The boundary values of $Y_\sigma(\cdot; s)$ are continuous on \mathbb{R} and are related by

$$Y_{\sigma,+}(\lambda; s) = Y_{\sigma,-}(\lambda; s) (I - 2\pi i \mathbf{f}(\lambda; s) \mathbf{h}^\top(\lambda; s)), \quad \lambda \in \mathbb{R}, \quad (3.2)$$

where the subscript $+$ (respectively, $-$) indicates the boundary value from above (respectively, below) the real axis.

- (c) As $\lambda \rightarrow \infty$, we have

$$Y_\sigma(\lambda; s) = I + \frac{1}{\lambda} \begin{pmatrix} \beta_\sigma(s) & i\eta_\sigma(s) \\ i\alpha_\sigma(s) & -\beta_\sigma(s) \end{pmatrix} + O(\lambda^{-2}), \quad (3.3)$$

for some $\alpha_\sigma(s)$, $\beta_\sigma(s)$, and $\eta_\sigma(s)$.

The following result has been proven in [14]. For the reader's convenience, we offer a direct proof based on the IIKS method.

Proposition 3.1. *The RH problem for Y_σ has a unique solution for all $s \in \mathbb{R}$ and we have*

$$\partial_s \log j_\sigma(s|\emptyset) = -\alpha_\sigma(s), \quad (3.4)$$

where α_σ is given in (3.3).

Proof. The upshot of IIKS theory [27] is that the RH problem for Y_σ is uniquely solvable if and only if $1 - \mathcal{M}_s^\sigma \mathcal{K}_s^{\text{Ai}}$ is invertible. The latter condition holds true by Lemma 2.1. Moreover, in this case, the resolvent operator $(1 - \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}})^{-1} - 1$ is also an integral operator with kernel

$$\frac{\mathbf{f}^\top(\lambda; s) Y_\sigma^\top(\lambda; s) Y_\sigma^{-\top}(\mu; s) \mathbf{h}(\mu; s)}{\lambda - \mu}. \quad (3.5)$$

Therefore, using Jacobi variational formula and the identity

$$\partial_s K_s^{\text{Ai}}(\lambda, \mu) = -\text{Ai}(\lambda + s) \text{Ai}(\mu + s), \quad (3.6)$$

which follows directly from (1.3), we compute $\frac{\partial}{\partial s} \log j_\sigma(s|\emptyset)$ as

$$\begin{aligned}
& -\operatorname{tr} \left((1 - \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}})^{-1} \mathcal{M}_\sigma \partial_s \mathcal{K}_s^{\text{Ai}} \right) \\
&= -\operatorname{tr} \left(\left((1 - \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}})^{-1} - 1 \right) \mathcal{M}_\sigma \partial_s \mathcal{K}_s^{\text{Ai}} \right) - \operatorname{tr} \left(\mathcal{M}_\sigma \partial_s \mathcal{K}_s^{\text{Ai}} \right) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mathbf{f}^\top(\lambda; s) Y_\sigma^\top(\lambda; s) Y_\sigma^{-\top}(\mu; s) \mathbf{h}(\mu; s)}{\lambda - \mu} \sigma(\mu) \operatorname{Ai}(\mu + s) \operatorname{Ai}(\lambda + s) d\lambda d\mu + \int_{\mathbb{R}} \sigma(\mu) \operatorname{Ai}(\mu + s)^2 d\mu \\
&= \int_{\mathbb{R}} \left[(i, 0) \int_{\mathbb{R}} \frac{\mathbf{h}(\lambda; s) \mathbf{f}^\top(\lambda; s) Y_\sigma^\top(\lambda; s)}{\lambda - \mu} d\lambda \right] Y_\sigma^{-\top}(\mu; s) \mathbf{h}(\mu; s) \sigma(\mu) \operatorname{Ai}(\mu + s) d\mu + \int_{\mathbb{R}} \sigma(\mu) \operatorname{Ai}(\mu + s)^2 d\mu \\
&\stackrel{(*)}{=} \int_{\mathbb{R}} (i, 0) (I - Y_\sigma^\top(\mu; s)) Y_\sigma^{-\top}(\mu; s) \mathbf{h}(\mu; s) \sigma(\mu) \operatorname{Ai}(\mu + s) d\mu + \int_{\mathbb{R}} \sigma(\mu) \operatorname{Ai}(\mu + s)^2 d\mu \\
&= \int_{\mathbb{R}} (i, 0) Y_\sigma^{-\top}(\mu; s) \mathbf{h}(\mu; s) \mathbf{f}^\top(\mu; s) \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\mu, \tag{3.7}
\end{aligned}$$

where we use the expressions of \mathbf{f} , \mathbf{h} given in (3.1), and in the equality (*) we use the identity

$$Y_\sigma^\top(\mu) = I - \int_{\mathbb{R}} \frac{\mathbf{h}(\lambda; s) \mathbf{f}^\top(\lambda; s) Y_\sigma^\top(\lambda; s)}{\lambda - \mu} d\lambda, \tag{3.8}$$

which follows from the RH problem satisfied by Y_σ and the Sokhotski–Plemelj formula. Finally, (3.7) can be simplified by a residue computation:

$$\begin{aligned}
\int_{\mathbb{R}} (i, 0) Y_\sigma^{-\top}(\mu; s) \mathbf{h}(\mu; s) \mathbf{f}^\top(\mu; s) \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\mu &= i \left(\frac{1}{2\pi i} \int_{\mathbb{R}} (Y_{\sigma,+}^{-\top}(\mu; s) - Y_{\sigma,-}^{-\top}(\mu; s)) d\mu \right)_{1,2} \\
&= i \left(\operatorname{res}_{\mu=\infty} Y_\sigma^{-\top}(\mu; s) \right)_{1,2} = -\alpha_\sigma(s), \tag{3.9}
\end{aligned}$$

using (3.3). \square

Next we consider the following RH problem which depends on $s \in \mathbb{R}$ and on a finite number of distinct points ν_1, \dots, ν_m , $\nu_i \neq \nu_j$ for $i \neq j$, as usual collected into the vector $\underline{\nu} := (\nu_1, \dots, \nu_m)$. When $m = 0$, condition (c) below is empty.

RH problem for Ψ_σ

(a) $\Psi_\sigma(\cdot; s|\underline{\nu}) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic for all $s \in \mathbb{R}$ and all $\underline{\nu}$.

(b) The boundary values of $\Psi_\sigma(\cdot; s|\underline{\nu})$ are continuous on $\mathbb{R} \setminus \{\nu_1, \dots, \nu_m\}$ and are related by

$$\Psi_{\sigma,+}(\lambda; s|\underline{\nu}) = \Psi_{\sigma,-}(\lambda; s|\underline{\nu}) \begin{pmatrix} 1 & 1 - \sigma(\lambda) \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}, \lambda \neq \nu_1, \dots, \nu_m. \tag{3.10}$$

(c) For all $i = 1, \dots, m$, as $\lambda \rightarrow \nu_i$ from either side of the real axis we have

$$\Psi_\sigma(\lambda; s|\underline{\nu})(\lambda - \nu_i)^{-\sigma_3} = O(1). \tag{3.11}$$

(d) As $\lambda \rightarrow \infty$, we have

$$\Psi_\sigma(\lambda; s|\underline{\nu}) = \left(I + \frac{1}{\lambda} \begin{pmatrix} q_\sigma(s|\underline{\nu}) & ir_\sigma(s|\underline{\nu}) \\ ip_\sigma(s|\underline{\nu}) & -q_\sigma(s|\underline{\nu}) \end{pmatrix} + O(\lambda^{-2}) \right) \lambda^{\frac{1}{4}\sigma_3} G e^{\left(-\frac{2}{3}\lambda^{\frac{3}{2}} - s\lambda^{\frac{1}{2}}\right)\sigma_3} C_\delta \tag{3.12}$$

for any $\delta \in (0, \frac{\pi}{2})$. Here we take the principal branches of $\lambda^{\frac{1}{4}\sigma_3}$ and $\lambda^{\frac{1}{2}}$, analytic in $\mathbb{C} \setminus (-\infty, 0]$ and positive for $\lambda > 0$, and

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad G := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad C_\delta := \begin{cases} I, & |\arg \lambda| < \pi - \delta, \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}, & \pi - \delta < \pm \arg \lambda < \pi. \end{cases} \tag{3.13}$$

Remark 3.2. We shall explain in detail in Section 3.5 the relation of this RH problem with the one in [29] related to the inverse scattering for the cKdV equation.

Remark 3.3. The solution to this RH problem is unique by a standard argument in RH problems based on Liouville and Morera theorems. Moreover, as we will show, the solution exists and can be constructed in terms of the solution to the RH problem for Y_σ (by an Airy dressing) and of a suitable matrix-valued rational function (by a Schlesinger transformation [4]).

We first recall the case $m = 0$, which has already been considered in [14]. To this end we introduce the Airy model RH problem in the following form.

RH problem for Φ^{Ai}

- (a) Φ^{Ai} is analytic in $\mathbb{C} \setminus \mathbb{R}$.
- (b) The boundary values of Φ^{Ai} are continuous on \mathbb{R} and are related by

$$\Phi_+^{\text{Ai}}(\lambda) = \Phi_-^{\text{Ai}}(\lambda) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}. \quad (3.14)$$

- (c) As $\lambda \rightarrow \infty$, Φ^{Ai} has the asymptotic behavior

$$\Phi^{\text{Ai}}(\lambda) = \left(I + \lambda^{-1} \begin{pmatrix} 0 & \frac{7i}{48} \\ 0 & 0 \end{pmatrix} + \mathcal{O}(\lambda^{-2}) \right) \lambda^{\frac{1}{4}\sigma_3} G e^{-\frac{2}{3}\lambda^{\frac{3}{2}}\sigma_3} C_\delta, \quad (3.15)$$

for any $0 < \delta < \pi/2$ where G, C_δ are given in (3.13) and the branches of $\lambda^{\frac{1}{4}\sigma_3}$ and $\lambda^{\frac{1}{2}}$ are as in (3.12).

The (unique) solution can be expressed in terms of the Airy function as

$$\Phi^{\text{Ai}}(\lambda) := \begin{cases} -\sqrt{2\pi} \begin{pmatrix} \text{Ai}'(\lambda) & -e^{\frac{2i\pi}{3}} \text{Ai}'(e^{\frac{-2i\pi}{3}}\lambda) \\ i \text{Ai}(\lambda) & -ie^{\frac{-2i\pi}{3}} \text{Ai}(e^{\frac{-2i\pi}{3}}\lambda) \end{pmatrix}, & \text{if } \text{Im } \lambda > 0, \\ -\sqrt{2\pi} \begin{pmatrix} \text{Ai}'(\lambda) & e^{\frac{-2i\pi}{3}} \text{Ai}'(e^{\frac{2i\pi}{3}}\lambda) \\ i \text{Ai}(\lambda) & ie^{\frac{2i\pi}{3}} \text{Ai}(e^{\frac{2i\pi}{3}}\lambda) \end{pmatrix}, & \text{if } \text{Im } \lambda < 0. \end{cases} \quad (3.16)$$

Proposition 3.4. When $m = 0$, the RH problem for Ψ_σ has a unique solution for all $s \in \mathbb{R}$ which can be written as

$$\Psi_\sigma(\lambda; s|\emptyset) = \begin{pmatrix} 1 & \frac{is^2}{4} \\ 0 & 1 \end{pmatrix} Y_\sigma(\lambda; s) \Phi_s^{\text{Ai}}(\lambda) \quad (3.17)$$

where $\Phi_s^{\text{Ai}}(\lambda) := \Phi^{\text{Ai}}(\lambda + s)$. Moreover,

$$p_\sigma(s|\emptyset) = \alpha_\sigma(s) + \frac{s^2}{4}, \quad (3.18)$$

and the kernel $L_s^\sigma(\lambda, \mu)$ of the operator $\mathcal{L}_s^\sigma := \mathcal{K}_s^{\text{Ai}}(1 - \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}})^{-1}$ can be written as

$$L_s^\sigma(\lambda, \mu) = \frac{(\Psi_\sigma(\mu; s|\emptyset)^{-1} \Psi_\sigma(\lambda; s|\emptyset))_{2,1}}{2\pi i(\lambda - \mu)}. \quad (3.19)$$

Proof. As explained in Remark 3.3, uniqueness of the solution follows from standard arguments, so it suffices to verify that (3.17) solves the RH problem. Condition (a) is easily checked, while for condition (b) we use the identity

$$I - 2\pi i \mathbf{f}(\lambda; s) \mathbf{h}^\top(\lambda; s) = \Phi_{s,-}^{\text{Ai}}(\lambda) \begin{pmatrix} 1 & 1 - \sigma(\lambda) \\ 0 & 1 \end{pmatrix} \Phi_{s,+}^{\text{Ai}}(\lambda)^{-1}, \quad (3.20)$$

which follows directly from the identities

$$\mathbf{f}(\lambda; s) = \frac{i\sigma(\lambda)}{\sqrt{2\pi}} \Phi_s^{\text{Ai}}(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{h}(\lambda; s) = -\frac{1}{\sqrt{2\pi}} \Phi_s^{\text{Ai}}(\lambda)^{-\top} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3.21)$$

and from condition (b) in the RH problem for Φ^{Ai} . Finally, combining conditions (c) in the RH problems for Y and Φ^{Ai} , we obtain that as $\lambda \rightarrow \infty$ we have

$$Y_\sigma(\lambda; s) \Phi_s^{\text{Ai}}(\lambda) = \left(I + \lambda^{-1} \begin{pmatrix} \beta(s) & i(\eta(s) + \frac{7}{48}) \\ i\alpha(s) & -\beta(s) \end{pmatrix} + O(\lambda^{-2}) \right) (\lambda + s)^{\frac{1}{4}\sigma_3} G e^{-\frac{2}{3}(\lambda+s)\frac{3}{2}\sigma_3} C_\delta \quad (3.22)$$

and expanding for λ large and s fixed we verify condition (d) in the RH problem for Ψ_σ along with the claimed relation (3.18). Finally, (3.19) follows directly from the expression (3.5) for the kernel of $(1 - \mathcal{M}_\sigma \mathcal{K}_s^{\text{Ai}})^{-1} - 1 = \mathcal{M}_\sigma \mathcal{L}_s^\sigma$, along with the identities (3.17) and (3.21) \square

Proposition 3.5. *The RH problem for Ψ_σ has a unique solution for all $s \in \mathbb{R}$ and all $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for $i \neq j$, which can be expressed as*

$$\Psi_\sigma(\lambda; s|\underline{\nu}) = M(\lambda; s|\underline{\nu}) \Psi_\sigma(\lambda; s|\emptyset), \quad (3.23)$$

where M is a rational function of λ , with poles at $\lambda = \nu_1, \dots, \nu_m$ only, given by

$$M(\lambda; s|\underline{\nu}) = I - \frac{1}{2\pi i} \sum_{i,j=1}^m \frac{(L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i}}{\lambda - \nu_j} \Psi_\sigma(\nu_i; s|\emptyset) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi_\sigma^{-1}(\nu_j; s|\emptyset), \quad (3.24)$$

where we use the notation (2.11).

Proof. By the conditions in the RH problem for Ψ_σ it is straightforward to verify that $M(\lambda; s|\underline{\nu}) := \Psi_\sigma(\lambda; s|\underline{\nu}) \Psi_\sigma(\lambda; s|\emptyset)^{-1}$ is a rational matrix with simple poles at ν_1, \dots, ν_m only and $M(\lambda; s|\underline{\nu}) \rightarrow I$ as $\lambda \rightarrow \infty$. Hence we write

$$M(\lambda; s|\underline{\nu}) = I + \sum_{j=1}^m \frac{M_j(s|\underline{\nu})}{\lambda - \nu_j}. \quad (3.25)$$

Condition (c) in the RH problem for Ψ_σ then translates to the condition

$$M(\lambda; s|\underline{\nu}) \Psi_\sigma(\lambda; s|\emptyset) (\lambda - \nu_j)^{-\sigma_3} = O(1), \quad \text{as } \lambda \rightarrow \nu_j, \quad j = 1, \dots, m, \quad (3.26)$$

and we claim that this condition uniquely determines the coefficients $M_j(s|\underline{\nu})$. Indeed, the expansion at $\lambda \rightarrow \nu_j$ of the left-hand side of (3.26) gives

$$\left(I + \frac{M_j(s|\underline{\nu})}{\lambda - \nu_j} + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} \frac{M_i(s|\underline{\nu})}{\nu_j - \nu_i} + O(\lambda - \nu_j) \right) \left(\Psi_\sigma(\nu_j; s|\emptyset) + \Psi'_\sigma(\nu_j; s|\emptyset)(\lambda - \nu_j) + O((\lambda - \nu_j)^2) \right) (\lambda - \nu_j)^{-\sigma_3} \quad (3.27)$$

where $\Psi'_\sigma(\lambda; s|\underline{\nu}) := \partial_\lambda \Psi_\sigma(\lambda; s|\underline{\nu})$. Vanishing of singular terms in this Laurent series yields

$$M_j(s|\underline{\nu}) \Psi_\sigma(\nu_j; s|\emptyset) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (3.28)$$

$$\left(\left(I + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} \frac{M_i(s|\underline{\nu})}{\nu_j - \nu_i} \right) \Psi_\sigma(\nu_j; s|\emptyset) + M_j(s|\underline{\nu}) \Psi'_\sigma(\nu_j; s|\emptyset) \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.29)$$

Equation (3.28) implies existence of a column vector $\mathbf{a}_j = \mathbf{a}_j(s|\underline{\nu}) \in \mathbb{C}^2$, for every $j = 1, \dots, m$, such that

$$M_j(s|\underline{\nu}) = \mathbf{a}_j(0, 1) \Psi_\sigma^{-1}(\nu_j; s|\emptyset). \quad (3.30)$$

Plugging (3.30) into (3.29) using (3.19) we get

$$\begin{aligned} \Psi_\sigma(\nu_j; s|\emptyset) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{\substack{1 \leq i \leq m \\ i \neq j}} \mathbf{a}_i \underbrace{\frac{(0, 1) \Psi_\sigma^{-1}(\nu_i; s|\emptyset) \Psi_\sigma(\nu_j; s|\emptyset)}{\nu_j - \nu_i}}_{= 2\pi i (L_s^\sigma(\underline{\nu}, \underline{\nu}))_{j,i}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ + \mathbf{a}_j \underbrace{(0, 1) \Psi_\sigma^{-1}(\nu_j; s|\emptyset) \Psi'_\sigma(\nu_j; s|\emptyset)}_{= 2\pi i (L_s^\sigma(\underline{\nu}, \underline{\nu}))_{j,j}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \quad (3.31)$$

and so, cf. Proposition 2.6,

$$\Psi_\sigma(\nu_j; s|\emptyset) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\pi i \sum_{i=1}^m \mathbf{a}_i (L_s^\sigma(\underline{\nu}, \underline{\nu}))_{j,i} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{a}_j = -\frac{1}{2\pi i} \sum_{i=1}^m (L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i} \Psi_\sigma(\nu_i; s|\emptyset) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (3.32)$$

and by (3.30), we finally get (3.24). \square

3.2 Stark equation

It is convenient to introduce the following variant of Ψ_σ , namely

$$\Theta_\sigma(\lambda; s|\underline{\nu}) := \begin{pmatrix} 1 & p_\sigma(s|\underline{\nu}) \\ 0 & 1 \end{pmatrix} e^{\frac{i\pi}{4}\sigma_3} \Psi_\sigma(\lambda; s|\underline{\nu}) e^{-\frac{i\pi}{4}\sigma_3}. \quad (3.33)$$

The RH conditions on Ψ_σ imply that Θ_σ is the unique solution to the following RH problem.

RH problem for Θ_s

- (a) $\Theta_\sigma(\cdot; s|\underline{\nu}) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic for all $s \in \mathbb{R}$ and all finite $\underline{\nu} \subset \mathbb{R}$.
- (b) The boundary values of $\Theta_\sigma(\cdot; s|\underline{\nu})$ are continuous on $\mathbb{R} \setminus \underline{\nu}$ and are related by

$$\Theta_{\sigma,+}(\lambda; s|\underline{\nu}) = \Theta_{\sigma,-}(\lambda; s|\underline{\nu}) \begin{pmatrix} 1 & i(1 - \sigma(\lambda)) \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}, \lambda \neq \nu_1, \dots, \nu_m. \quad (3.34)$$

- (c) For all $i = 1, \dots, m$, as $\lambda \rightarrow \nu_i$ from either side of the real axis we have

$$\Theta_\sigma(\lambda; s|\underline{\nu})(\lambda - \nu_i)^{-\sigma_3} = O(1). \quad (3.35)$$

- (d) As $\lambda \rightarrow \infty$, we have

$$\Theta_\sigma(\lambda; s|\underline{\nu}) = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \left(I + \lambda^{-1} \begin{pmatrix} q & -r \\ p & -q \end{pmatrix} + O(\lambda^{-2}) \right) \lambda^{\frac{1}{4}\sigma_3} \frac{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}{\sqrt{2}} e^{(-\frac{2}{3}\lambda^{\frac{3}{2}} - s\lambda^{\frac{1}{2}})\sigma_3} C_\delta \quad (3.36)$$

for any $\delta \in (0, \frac{\pi}{2})$; here $p = p_\sigma(s|\underline{\nu})$, $q = q_\sigma(s|\underline{\nu})$, and $r = r_\sigma(s|\underline{\nu})$ are the same as in (3.12), C_δ is in (3.13), and the branches of $\lambda^{\frac{1}{4}\sigma_3}$ and $\lambda^{\frac{1}{2}}$ are taken as in (3.12).

The formula (3.19) is equivalent to

$$L_s^\sigma(\lambda, \mu) = \frac{(\Theta_\sigma(\mu; s|\emptyset)^{-1} \Theta_\sigma(\lambda; s|\emptyset))_{2,1}}{2\pi(\lambda - \mu)}. \quad (3.37)$$

Proposition 3.6. For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and for any $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for $i \neq j$, $\Theta_\sigma(\lambda; s|\underline{\nu})$ is differentiable in s , and

$$\partial_s \Theta_\sigma(\lambda; s|\underline{\nu}) = \begin{pmatrix} 0 & \lambda + 2\partial_s p_\sigma(s|\underline{\nu}) \\ 1 & 0 \end{pmatrix} \Theta_\sigma(\lambda; s|\underline{\nu}), \quad (3.38)$$

where $p_\sigma(s|\underline{\nu})$ appears in (3.12).

Proof. The differentiability of Θ_σ in s , and the fact (3.36) continues to hold after differentiating formally in s , can be proved using standard techniques from RH theory, and we refer the reader to [14, Section 3] for details. The matrix function $A(\lambda; s|\underline{\nu}) := \partial_s \Theta_\sigma(\lambda; s|\underline{\nu}) \Theta_\sigma(\lambda; s|\underline{\nu})^{-1}$ is entire in λ ; indeed it has no jump across the real axis and no singularities at $\underline{\nu}$ because of the RH conditions (b) and (c) for Θ_σ . Moreover, condition (d) in the RH problem for Θ_σ implies that

$$A(\lambda; s|\underline{\nu}) = \begin{pmatrix} 0 & \lambda + p^2 + 2q + \partial_s p \\ 1 & 0 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} \star & \star \\ -p^2 - 2q + \partial_s p & \star \end{pmatrix} + O(\lambda^{-2}), \quad \lambda \rightarrow \infty, \quad (3.39)$$

where $p = p_\sigma(s|\underline{\nu})$ and $q = q_\sigma(s|\underline{\nu})$ are as in (3.36) and \star denote expressions which are not relevant to us now. Since $A(\lambda; s|\underline{\nu})$ is entire, Liouville's theorem implies that $A(\lambda; s|\underline{\nu})$ coincides with the linear and constant terms in the Laurent series (3.39) and that higher order terms vanish. This yields

$$p_\sigma(s|\underline{\nu})^2 + 2q_\sigma(s|\underline{\nu}) = \partial_s p_\sigma(s|\underline{\nu}), \quad (3.40)$$

and the proof is complete. \square

From equation (3.38) it follows that

$$\Theta_\sigma(\lambda; s|\underline{\nu}) = -\sqrt{2\pi} \begin{pmatrix} \partial_s \varphi_\sigma(\lambda; s|\underline{\nu}) & \partial_s \chi_\sigma(\lambda; s|\underline{\nu}) \\ \varphi_\sigma(\lambda; s|\underline{\nu}) & \chi_\sigma(\lambda; s|\underline{\nu}) \end{pmatrix}, \quad (3.41)$$

where either $f = \varphi_\sigma(\lambda; s|\underline{\nu})$ or $f = \chi_\sigma(\lambda; s|\underline{\nu})$ solves

$$\left(\partial_s^2 - 2(\partial_s p_\sigma(s|\underline{\nu})) \right) f = \lambda f. \quad (3.42)$$

Proposition 3.7. We have

$$\partial_s L_s^\sigma(\lambda, \mu) = -\varphi_\sigma(\lambda; s|\emptyset) \varphi_\sigma(\mu; s|\emptyset) \quad (3.43)$$

Proof. We use (3.37) to compute

$$\begin{aligned} \partial_s L_s^\sigma(\lambda, \mu) &= \text{tr} \partial_s \left(\frac{\Theta_\sigma(\lambda; s|\emptyset) \mathbf{E}_{12} \Theta_\sigma(\mu; s|\emptyset)^{-1}}{2\pi(\lambda - \mu)} \right) \\ &= \text{tr} \frac{(A(\lambda; s|\emptyset) - A(\mu; s|\emptyset)) \Theta_\sigma(\lambda; s|\emptyset) \mathbf{E}_{12} \Theta_\sigma(\mu; s|\emptyset)^{-1}}{2\pi(\lambda - \mu)} \\ &= \text{tr} \frac{\mathbf{E}_{12} \Theta_\sigma(\lambda; s|\emptyset) \mathbf{E}_{12} \Theta_\sigma(\mu; s|\emptyset)^{-1}}{2\pi}, \end{aligned} \quad (3.44)$$

where we used the cyclic property of the trace and Proposition 3.6 and we denoted

$$\mathbf{E}_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A(\lambda; s|\emptyset) := \begin{pmatrix} 0 & \lambda + 2\partial_s p_\sigma(s|\emptyset) \\ 1 & 0 \end{pmatrix}. \quad (3.45)$$

Finally, it suffices to insert (3.41) into (3.44). \square

We can finally characterize the Jánossy densities in terms of the RH problem for Ψ_σ .

Proposition 3.8. For all $s \in \mathbb{R}$ and all finite sets $\underline{\nu} = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$, we have

$$\partial_s \log j_\sigma(s|\underline{\nu}) = \frac{s^2}{4} - p_\sigma(s|\underline{\nu}) \quad (3.46)$$

where $p_\sigma(s|\underline{\nu})$ appears in (3.12).

Proof. Using Proposition 3.5 we get

$$ip_\sigma(s|\underline{\nu}) - ip_\sigma(s|\emptyset) = (M_\infty^1(s|\underline{\nu}))_{2,1} \quad (3.47)$$

where $M(\lambda; s|\underline{\nu}) = I + \lambda^{-1}M_\infty^1(s|\underline{\nu}) + O(\lambda^{-2})$ as $\lambda \rightarrow \infty$. Using (3.24) we compute

$$M_\infty^1(s|\underline{\nu}) = -\frac{1}{2\pi i} \sum_{i,j=1}^m (L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i} \Psi_\sigma(\nu_i; s|\emptyset) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi_\sigma^{-1}(\nu_j; s|\emptyset) \quad (3.48)$$

and so, using (3.33) and (3.41), we obtain

$$(M_\infty^1(s|\underline{\nu}))_{2,1} = i \sum_{i,j=1}^m (L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i} \varphi_\sigma(\nu_i; s|\emptyset) \varphi_\sigma(\nu_j; s|\emptyset) \quad (3.49)$$

On the other hand, by (3.43) we have

$$\partial_s \log \det L_s^\sigma(\underline{\nu}, \underline{\nu}) = \sum_{i,j=1}^m (L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i} \frac{\partial (L_s^\sigma(\underline{\nu}, \underline{\nu}))_{i,j}}{\partial s} = - \sum_{i,j=1}^m (L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i} \varphi_\sigma(\nu_i; s|\emptyset) \varphi_\sigma(\nu_j; s|\emptyset) \quad (3.50)$$

and the proof now follows from (3.47) because $\log j_\sigma(s|\underline{\nu}) = \log \det L_s^\sigma(\underline{\nu}, \underline{\nu}) + \log j_\sigma(s|\emptyset)$ by (2.13) and because

$$\frac{\partial}{\partial s} \log j_\sigma(s|\emptyset) = \frac{s^2}{4} - p_\sigma(s|\emptyset) \quad (3.51)$$

by (3.4) and (3.18). \square

3.3 Asymptotics as $s \rightarrow +\infty$

The jump matrix of condition (b) in the RH problem for Y can be rewritten, thanks to (3.21), as

$$I - 2\pi i \mathbf{f}(\lambda; s) \mathbf{h}^\top(\lambda; s) = I + \Phi_s^{\text{Ai}}(\lambda) \begin{pmatrix} 0 & \sigma(\lambda) \\ 0 & 0 \end{pmatrix} \Phi_s^{\text{Ai}}(\lambda)^{-1}. \quad (3.52)$$

We now show that this jump matrix is close to the identity in the appropriate norms in order to apply the standard *small-norm* RH theory [26]. To this end, we introduce the following notation for a measurable matrix-valued function $X : \mathbb{R} \rightarrow \mathbb{C}^{m \times n}$:

$$\|X\|_p := \begin{cases} \max_{1 \leq i \leq m, 1 \leq j \leq n} \left\{ \left(\int_{\mathbb{R}} |X_{i,j}(\mu)|^p d\mu \right)^{1/p} \right\}, & p \in [1, +\infty), \\ \max_{1 \leq i \leq m, 1 \leq j \leq n} \left\{ \text{ess sup}_{\mu \in \mathbb{R}} |X_{i,j}(\mu)| \right\}, & p = \infty. \end{cases} \quad (3.53)$$

Lemma 3.9. Let σ satisfy Assumption A. Then, with the same $\kappa > 0$ as in Assumption A, we have

$$\left\| \Phi_s^{\text{Ai}} \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix} (\Phi_s^{\text{Ai}})^{-1} \right\|_p = O(s^{-\kappa}), \quad \text{as } s \rightarrow +\infty, \quad p = 1, 2, \infty. \quad (3.54)$$

Proof. The entries in $\Phi_s^{\text{Ai}}(\lambda) \begin{pmatrix} 0 & \sigma(\lambda) \\ 0 & 0 \end{pmatrix} \Phi_s^{\text{Ai}}(\lambda)^{-1}$ are (possibly, up to a sign) $\sigma(\lambda) \mathcal{B}(\lambda + s)$ where $\mathcal{B}(\lambda)$ is one of the functions $\text{Ai}(\lambda)^2$, $\text{Ai}(\lambda) \text{Ai}'(\lambda)$, or $\text{Ai}'(\lambda)^2$. By Assumption A, there are $\Lambda, C_1 > 0$ such that $\lambda < -\Lambda$ implies $\sigma(\lambda) < C_1 |\lambda|^{-\frac{3}{2}-\kappa}$. Assuming $s \geq 2\Lambda$, we have, using standard asymptotic properties of Ai and Ai':

- for $\lambda \geq -s/2$, $\sigma(\lambda) \leq 1$ and $|\mathcal{B}(\lambda + s)| \leq C_2 \exp(-\lambda - s)$ for some $C_2 > 0$,
- for $\lambda \leq -s/2$, $\sigma(\lambda) \leq C_1 |\lambda|^{-\frac{3}{2}-\kappa}$ and $|\mathcal{B}(\lambda + s)| \leq C_3 |\lambda|^{\frac{1}{2}}$ for some $C_3 > 0$.

Therefore, as $s \rightarrow +\infty$ we get:

$$\|\sigma(\lambda)\mathcal{B}(\lambda + s)\|_1 \leq \int_{-\infty}^{-s/2} C_1 C_3 |\lambda|^{-\kappa-1} d\lambda + \int_{-s/2}^{+\infty} C_2 e^{-\lambda-s} d\lambda = O(s^{-\kappa}), \quad (3.55)$$

$$\|\sigma(\lambda)\mathcal{B}(\lambda + s)\|_2^2 \leq \int_{-\infty}^{-s/2} (C_1 C_3)^2 |\lambda|^{-2\kappa-2} d\lambda + \int_{-s/2}^{+\infty} C_2^2 e^{-\lambda-s} d\lambda = O(s^{-2\kappa-1}), \quad (3.56)$$

$$\|\sigma(\lambda)\mathcal{B}(\lambda + s)\|_\infty \leq \max \left\{ C_1 C_3 \left(\frac{s}{2}\right)^{-\kappa-1}, C_2 e^{-s/2} \right\} = O(s^{-\kappa-1}), \quad (3.57)$$

and the lemma is proved. \square

Proposition 3.10. *If σ satisfies Assumption A, we have*

$$\varphi_\sigma(\lambda; s|\emptyset) = \text{Ai}(\lambda + s)(1 + O(s^{-\frac{1}{2}-\kappa})), \quad s \rightarrow +\infty, \quad (3.58)$$

for all $\lambda \in \mathbb{R}$.

Proof. Rewriting condition (b) in the RH problem for Y as

$$Y_{\sigma,+}(\lambda; s) - Y_{\sigma,-}(\lambda; s) = \Phi_s^{\text{Ai}}(\lambda) \begin{pmatrix} 0 & \sigma(\lambda) \\ 0 & 0 \end{pmatrix} \Phi_s^{\text{Ai}}(\lambda)^{-1} Y_\sigma(\lambda; s), \quad \lambda \in \mathbb{R}, \quad (3.59)$$

standard RH theory implies that we can write

$$\begin{aligned} Y_\sigma(\lambda; s) = I + \frac{1}{2\pi i} \int_{\mathbb{R}} (Y_{\sigma,-}(\mu; s) - I) \Phi_s^{\text{Ai}}(\mu) \begin{pmatrix} 0 & \sigma(\mu) \\ 0 & 0 \end{pmatrix} \Phi_s^{\text{Ai}}(\mu)^{-1} \frac{d\mu}{\mu - \lambda} \\ + \frac{1}{2\pi i} \int_{\mathbb{R}} \Phi_s^{\text{Ai}}(\mu) \begin{pmatrix} 0 & \sigma(\mu) \\ 0 & 0 \end{pmatrix} \Phi_s^{\text{Ai}}(\mu)^{-1} \frac{d\mu}{\mu - \lambda}. \end{aligned} \quad (3.60)$$

Moreover, by Lemma 3.9 and standard small-norm RH theory [?], $Y_{\sigma,-}(\cdot; s) - I$ is in L^2 (entry-wise) for all $s \in \mathbb{R}$ and satisfies

$$\|Y_{\sigma,-}(\cdot; s) - I\|_2 = O(s^{-\kappa}), \quad s \rightarrow +\infty. \quad (3.61)$$

By (3.17), (3.33), and (3.41) we have $\sqrt{2\pi}\varphi_\sigma(\lambda; s|\emptyset) = i(Y_{\sigma,-}(\lambda; s)\Phi_s^{\text{Ai}}(\lambda))_{2,1}$, such that multiplying (3.60) by $\Phi_s^{\text{Ai}}(\lambda)$ and extracting the (2,1)-entry we obtain

$$\begin{aligned} \varphi_\sigma(\lambda; s|\emptyset) = \text{Ai}(\lambda + s) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (0, 1)(Y_{\sigma,-}(\mu) - I)\Phi_s^{\text{Ai}}(\mu) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma(\mu) K_s^{\text{Ai}}(\lambda, \mu) d\mu \\ + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (0, 1)\Phi_s^{\text{Ai}}(\mu) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma(\mu) K_s^{\text{Ai}}(\lambda, \mu) d\mu. \end{aligned} \quad (3.62)$$

Hence, for some $C > 0$,

$$\left| \frac{\varphi_\sigma(\lambda; s|\emptyset)}{\text{Ai}(\lambda + s)} - 1 \right| \leq C \left(\|Y_{\sigma,-} - I\|_2 \frac{\left\| \Phi_s^{\text{Ai}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma K_s^{\text{Ai}}(\cdot, \lambda) \right\|_2}{\text{Ai}(\lambda + s)} + \frac{\left\| \Phi_s^{\text{Ai}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma K_s^{\text{Ai}}(\cdot, \lambda) \right\|_1}{\text{Ai}(\lambda + s)} \right). \quad (3.63)$$

Therefore, denoting \mathcal{A} either Ai or Ai', we need to estimate the $L^p(\mathbb{R}, d\mu)$ -norm (for $p = 1, 2$) of

$$a(\mu) := \mathcal{A}(\mu + s)\sigma(\mu) \frac{K_s^{\text{Ai}}(\lambda, \mu)}{\text{Ai}(\lambda + s)} \quad (3.64)$$

as $s \rightarrow +\infty$, for fixed λ . We can assume s is sufficiently large such that $s > 2|\lambda|$ and $\text{Ai}(\lambda + s) \leq |\text{Ai}'(\lambda + s)|$.

- When $\mu \leq -s/2$, we have $|\mathcal{A}(\mu + s)| = O(|\mu|^{1/4})$, $\sigma(\mu) = O(|\mu|^{-3/2-\kappa})$, and

$$\left| \frac{K_s^{\text{Ai}}(\lambda, \mu)}{\text{Ai}(\lambda + s)} \right| \leq \frac{|\text{Ai}'(\lambda + s)| (|\text{Ai}(\mu + s)| + |\text{Ai}'(\mu + s)|)}{|\lambda - \mu|} = O(s^{-1/2} |\mu|^{1/4}) \quad (3.65)$$

hence

$$a(\mu) = O(|\mu|^{-1-\kappa} s^{-1/2}). \quad (3.66)$$

- When $\mu \geq -s/2$, we have $|\mathcal{A}(\mu + s)| = O(e^{-\mu-s})$, $\sigma(\mu) = O(1)$, and

$$\left| \frac{K_s^{\text{Ai}}(\lambda, \mu)}{\text{Ai}(\lambda + s)} \right| \leq \int_s^{+\infty} \frac{\text{Ai}(\lambda + \eta)}{\text{Ai}(\lambda + s)} \text{Ai}(\mu + \eta) d\eta \leq \int_s^{+\infty} \text{Ai}(\mu + \eta) d\eta = O(e^{-\mu-s}). \quad (3.67)$$

Hence $\|a\|_{L^1(\mathbb{R})} = O(s^{-1/2-\kappa})$ and $\|a\|_{L^2(\mathbb{R})} = O(s^{-1-\kappa})$, so that resuming from (3.63) we get

$$\left| \frac{\varphi_\sigma(\lambda; s|\emptyset)}{\text{Ai}(\lambda + s)} - 1 \right| = O(s^{-1/2-\kappa}), \quad (3.68)$$

and the proof is complete. \square

Corollary 3.11. *We have*

$$L_s^\sigma(\lambda, \mu) = \int_s^{+\infty} \varphi_\sigma(\lambda; r|\emptyset) \varphi_\sigma(\mu; r|\emptyset) dr. \quad (3.69)$$

Proof. Follows directly by integrating (3.43) from s to $+\infty$ thanks to (3.58). \square

3.4 Proofs of Theorems I and II

Proof of Theorem I. The first relation (1.27) is nothing else than (2.13).

The first equality in (1.28) is (3.69) while the second one is a rewriting of (3.37) using (3.41).

That φ_σ solves the Stark boundary value problem (1.6) with potential $v_\sigma(s|\emptyset) := \partial_s^2 \log j_\sigma(s|\emptyset)$ follows from (3.42), (3.51), and (3.58).

Finally, in order to prove (1.29), we first consider the following chain of equalities, where we use (3.51) and the asymptotics as $s \rightarrow +\infty$ of Section 3.3:

$$\log j_\sigma(s|\emptyset) = - \int_s^{+\infty} \partial_r \log j_\sigma(r|\emptyset) dr = \int_s^{+\infty} (r-s) \partial_r^2 \log j_\sigma(r|\emptyset) dr = \int_s^{+\infty} (r-s) v_\sigma(r|\emptyset) dr.$$

In the first step we use $\lim_{r \rightarrow +\infty} j_\sigma(r|\emptyset) = \lim_{r \rightarrow +\infty} \det_{L^2(\mathbb{R})}(1 - \mathcal{K}_s^\sigma) = 1$ because \mathcal{K}_s^σ converges to the zero operator in trace-norm when $s \rightarrow +\infty$. Indeed, \mathcal{K}_s^σ is a non-negative trace-class operator with (jointly) continuous integral kernel so that its trace-norm is

$$\int_{\mathbb{R}} \sigma(\lambda) K_s^{\text{Ai}}(\lambda, \lambda) d\lambda = \int_{-\infty}^{-s/2} \underbrace{\sigma(\lambda) K_s^{\text{Ai}}(\lambda, \lambda)}_{O(|\lambda|^{-1-\kappa})} d\lambda + \int_{-s/2}^{+\infty} \underbrace{\sigma(\lambda) K_s^{\text{Ai}}(\lambda, \lambda)}_{O(\exp(-\lambda-s))} d\lambda = O(s^{-\kappa}) \quad (3.70)$$

as $s \rightarrow +\infty$; here we use that as $s \rightarrow +\infty$ we have $\sigma(\lambda) = O(\lambda^{-3/2-\kappa})$ and $K_s^{\text{Ai}}(\lambda, \lambda) = O(|\lambda|^{1/2})$ for $\lambda < -s/2$, and $\sigma(\lambda) = O(1)$ and $K_s^{\text{Ai}}(\lambda, \lambda) = O(\exp(-\lambda-s))$ for $\lambda > -s/2$ (cf. Assumption A).

The identity $\int \varphi^2(\lambda; s|\emptyset) d\sigma(\lambda) = -v_\sigma(s|\emptyset)$, proved in [14, Proposition 4.1], completes the proof. This identity also follows by setting $\underline{\nu} = \emptyset$ in the more general identity (3.77) below, which will be shown in the proof of Theorem II by an adaptation of the argument in loc. cit. (and not relying on the case $\underline{\nu} = \emptyset$). \square

Proof of Theorem II. Let us introduce

$$\Xi(\lambda; s|\underline{\nu}) := \Theta_\sigma(\lambda; s|\underline{\nu})\xi(\lambda|\underline{\nu})^{-\sigma_3}, \quad \xi(\lambda|\underline{\nu}) := \prod_{i=1}^m (\lambda - \nu_i). \quad (3.71)$$

As it follows from conditions (b) and (c) in the RH problem for Θ_σ , $\Xi(\lambda; s|\underline{\nu})$ is a sectionally analytic matrix-valued function of λ satisfying a jump condition across the real axis of the form

$$\Xi_+(\lambda; s|\underline{\nu}) = \Xi_-(\lambda; s|\underline{\nu}) \begin{pmatrix} 1 & i(1 - \sigma(\lambda))\xi(\lambda|\underline{\nu})^2 \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}. \quad (3.72)$$

It follows that $C(\lambda; s|\underline{\nu}) := (\partial_\lambda \Xi(\lambda; s|\underline{\nu}))\Xi(\lambda; s|\underline{\nu})^{-1}$ is also a sectionally analytic matrix-valued function of λ satisfying a jump condition across the real axis of the form

$$C_+(\lambda; s|\underline{\nu}) - C_-(\lambda; s|\underline{\nu}) = \Xi(\lambda; s|\underline{\nu}) \begin{pmatrix} 0 & i\xi(\lambda|\underline{\nu})^2((1 - \sigma(\lambda))2\partial_\lambda \log \xi(\lambda|\underline{\nu}) - \sigma'(\lambda)) \\ 0 & 0 \end{pmatrix} \Xi(\lambda; s|\underline{\nu})^{-1}, \quad (3.73)$$

for all $\lambda \in \mathbb{R}$. In the right-hand side of this equation we omit the choice of boundary values for Ξ as the expression is independent from this choice, as it can be shown by (3.72). It therefore follows from a contour deformation argument that

$$\int_{\mathbb{R}} \Xi(\lambda; s|\underline{\nu}) \begin{pmatrix} 0 & i\xi(\lambda|\underline{\nu})^2((1 - \sigma(\lambda))2\partial_\lambda \log \xi(\lambda|\underline{\nu}) - \sigma'(\lambda)) \\ 0 & 0 \end{pmatrix} \Xi(\lambda; s|\underline{\nu})^{-1} \frac{d\lambda}{2\pi i} = \lim_{R \rightarrow +\infty} \oint_{c_R} C(\lambda; s|\underline{\nu}) \frac{d\lambda}{2\pi i}, \quad (3.74)$$

where c_R is the *clock-wise* oriented circle $|\lambda| = R$. By the identity

$$C := (\partial_\lambda \Xi)\Xi^{-1} = (\partial_\lambda \Theta_\sigma)\Theta_\sigma^{-1} - (\partial_\lambda \log \xi)\Theta_\sigma \sigma_3 \Theta_\sigma^{-1} \quad (3.75)$$

and the asymptotic relation (3.36) we obtain that, as $\lambda \rightarrow \infty$ uniformly in the complex plane, we have

$$(C(\lambda; s|\underline{\nu}))_{2,1} = 1 + \lambda^{-1} \left(\frac{s}{2} - \partial_s p_\sigma(s|\underline{\nu}) \right) + O(\lambda^{-\frac{3}{2}}). \quad (3.76)$$

Here we also use (3.40). Taking the (2,1)-entry of (3.74), we obtain, also using Proposition 3.8,

$$- \int_{\mathbb{R}} \varphi_\sigma(\lambda; s|\underline{\nu})^2 ((1 - \sigma(\lambda))2\partial_\lambda \log \xi(\lambda|\underline{\nu}) - \sigma'(\lambda)) d\lambda = \partial_s p_\sigma(s|\underline{\nu}) - \frac{s}{2} = -\partial_s^2 \log j_\sigma(s|\underline{\nu}). \quad (3.77)$$

Taking into account that $\xi(\lambda|\underline{\nu}) = \prod_{i=1}^m (\lambda - \nu_i)$, the identity (3.77) we just proved is (1.31).

Next, the Stark equation (1.32) follows from (3.42) and Proposition 3.8.

Finally, extracting the (2,1)-entry in (3.23), using (3.24) and (3.41),

$$\begin{aligned} \varphi_\sigma(\lambda; s|\underline{\nu}) &= \left(1 - \sum_{i,j=1}^m \frac{(L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i}}{\lambda - \nu_j} \varphi_\sigma(\nu_i; s|\emptyset) \partial_s \varphi_\sigma(\nu_j; s|\emptyset) \right) \varphi_\sigma(\lambda; s|\emptyset) \\ &\quad + \left(\sum_{i,j=1}^m \frac{(L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i}}{\lambda - \nu_j} \varphi_\sigma(\nu_i; s|\emptyset) \varphi_\sigma(\nu_j; s|\emptyset) \right) \partial_s \varphi_\sigma(\lambda; s|\emptyset) \\ &= \varphi_\sigma(\lambda; s|\emptyset) - \sum_{i,j=1}^m (L_s^\sigma(\underline{\nu}, \underline{\nu})^{-1})_{j,i} \varphi_\sigma(\nu_i; s|\emptyset) L_s^\sigma(\lambda, \nu_i) \\ &= \frac{1}{\det L_s^\sigma(\underline{\nu}, \underline{\nu})} \det \begin{pmatrix} \varphi_\sigma(\lambda; s|\emptyset) & L_s^\sigma(\lambda, \nu_1) & \cdots & L_s^\sigma(\lambda, \nu_m) \\ \varphi_\sigma(\nu_1, s|\emptyset) & L_s^\sigma(\nu_1, \nu_1) & \cdots & L_s^\sigma(\nu_1, \nu_m) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_\sigma(\nu_m, s|\emptyset) & L_s^\sigma(\nu_m, \nu_1) & \cdots & L_s^\sigma(\nu_m, \nu_m) \end{pmatrix}. \end{aligned} \quad (3.78)$$

where in the second step we use (1.28) and in the third a standard manipulation of the determinant of a block matrix. Therefore (1.34) holds true and the proof is complete. \square

3.5 Comparison with inverse scattering for the Stark operator

We now comment on the connection between our probabilistic construction based on the σ -thinned (shifted) Airy process and the classical inverse scattering problem for the Stark operator, as described in [29], see also [44, 45]. The latter can be formulated through the following RH problem, cf. [29, Definition 2.3].

RH problem for M

- (a) $M(\cdot; \xi) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic for all $\xi \in \mathbb{R}$.
(b) The boundary values of $M(\cdot; \xi)$ are continuous on \mathbb{R} and are related by

$$M_+(\mu; \xi) = \begin{pmatrix} 0 & -s(\mu) \\ s(\mu) & 1 \end{pmatrix} M_-(\mu; \xi), \quad \mu \in \mathbb{R}. \quad (3.79)$$

- (c) As $\mu \rightarrow \infty$, we have

$$M(\mu; \xi) = M_\infty(\mu; \xi)(I + o(1)), \quad M_\infty(\xi; \mu) := \begin{cases} \begin{pmatrix} -w_0(\xi - \mu) & -w'_0(\xi - \mu) \\ w_1(\xi - \mu) & w'_1(\xi - \mu) \end{pmatrix}, & \text{Im } \mu > 0, \\ \begin{pmatrix} w_2(\xi - \mu) & w'_2(\xi - \mu) \\ w_0(\xi - \mu) & w'_0(\xi - \mu) \end{pmatrix}, & \text{Im } \mu < 0, \end{cases} \quad (3.80)$$

where

$$w_0(k) := 2i\sqrt{\pi}\text{Ai}(k), \quad w_1(k) := 2\sqrt{\pi}e^{\frac{\pi i}{6}}\text{Ai}(e^{\frac{2\pi i}{3}}k), \quad w_2(k) := 2\sqrt{\pi}e^{-\frac{\pi i}{6}}\text{Ai}(e^{-\frac{2\pi i}{3}}k). \quad (3.81)$$

In (3.79), $s(\mu) = a(\mu)\overline{a(\mu)}^{-1}$ (for $\mu \in \mathbb{R}$), and $a(\mu)$ is part of the scattering data for the Stark operator. In particular, cf. [29, Theorem 2.2], $a(\mu)$ is analytic and nonzero in the half-plane $\text{Im } \mu < 0$ and $a(\mu) = 1 + o(|\mu|^{-\frac{1}{2}})$ as $\mu \rightarrow \infty$ within $\text{Im } \mu \leq 0$. Then, the matrix

$$\Psi(\lambda; s) := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} \times \begin{cases} M^\top(-\lambda; s) \begin{pmatrix} 0 & a(-\lambda)^{-1} \\ -a(-\lambda) & 0 \end{pmatrix}, & \text{Im } \lambda > 0, \\ M^\top(-\lambda; s) \begin{pmatrix} a(-\bar{\lambda}) & 0 \\ 0 & (\overline{a(-\bar{\lambda})})^{-1} \end{pmatrix}, & \text{Im } \lambda < 0, \end{cases} \quad (3.82)$$

essentially solves the RH problem for Ψ_σ , for $\sigma(\lambda) = 1 - |a(-\lambda)|^{-2}$ and $m = 0$, with the caveat that it only satisfies a slightly weaker normalization at $\lambda = \infty$ in which the sub-leading term is just $o(1)$ rather than $O(\lambda^{-1})$.

Indeed, the expression in the right-hand side of (3.82) is analytic for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ by the above mentioned properties of a , and a direct computation suffices to ascertain that

$$\Phi^{\text{Ai}}(\lambda + s) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix} M_\infty^\top(-\lambda; s) \times \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{Im } \lambda > 0, \\ I, & \text{Im } \lambda < 0, \end{cases} \quad (3.83)$$

such that the normalization at ∞ of the two RH problems match (up to the order of the sub-leading contribution, as we already mentioned). Moreover, a direct computation shows that the jump condition of the RH problem for Ψ_σ is satisfied by the right-hand side of (3.82).

There is however an essential difference between our assumptions on σ , and the assumptions in [29]. Whereas we consider functions σ converging to 0 at $-\infty$, but not necessarily converging to 0 at $+\infty$, cf. Assumption A, it is required in [29, Theorem 2.2(c)] that $\sigma(\lambda) = 1 - |a(-\lambda)|^{-2} \rightarrow 0$ as $\lambda \rightarrow \pm\infty$. Hence, the class of functions σ that we consider, is not included in the class of scattering data considered by classical inverse scattering theory for the Stark operator.

3.6 Connection with the theory of Schlesinger transformations [30, 3, 4].

It is also worth to make a comparison of our setting with the general theory of Schlesinger transformations. For a general RH problem depending on parameters, one can define the *Malgrange–Bertola differential* on the space of parameters [3]. The general definition, applied to the RH problem for $\Gamma(\lambda; s) := \Psi_\sigma(\lambda - s; s|\emptyset)$, specializes to the following one-form in s :

$$\Omega = \omega(s)ds, \quad \omega(s) := \int_{\mathbb{R}} \operatorname{tr} \left[\Gamma_-^{-1}(\lambda; s) \frac{d\Gamma_-(\lambda; s)}{d\lambda} \frac{dJ_\Gamma(\lambda; s)}{ds} J_\Gamma^{-1}(\lambda; s) \right] \frac{d\lambda}{2\pi i}, \quad (3.84)$$

where $J_\Gamma(\lambda; s) = \begin{pmatrix} 1 & 1 - \sigma(\lambda - s) \\ 0 & 1 \end{pmatrix}$. Using the form of J_Γ and the jump condition $\Gamma_+(\lambda; s) = \Gamma_-(\lambda; s)J_\Gamma(\lambda; s)$, the integrand can be rewritten as

$$\operatorname{tr} \left[\Gamma_-^{-1}(\lambda; s) \frac{d\Gamma_-(\lambda; s)}{d\lambda} \begin{pmatrix} 0 & \sigma'(\lambda - s) \\ 0 & 0 \end{pmatrix} \right] = \frac{1}{2} \operatorname{tr} \left(\Gamma_-^{-1}(\lambda; s) \frac{d^2\Gamma_-(\lambda; s)}{d\lambda^2} - \Gamma_+^{-1}(\lambda; s) \frac{d^2\Gamma_+(\lambda; s)}{d\lambda^2} \right) \quad (3.85)$$

and hence a residue computation gives

$$\omega(s) = -\frac{1}{2} \operatorname{res}_{\lambda=\infty} \operatorname{tr} \left(\Gamma^{-1}(\lambda; s) \frac{d^2\Gamma(\lambda; s)}{d\lambda^2} \right) = \frac{s^2}{4} - p_\sigma(s|\emptyset) = \partial_s \log j_\sigma(s|\emptyset). \quad (3.86)$$

The logarithmic potential j_σ of Ω is then termed tau function of the RH problem [3]. Note that a tau function in this sense is defined only up to a multiplicative (integration) constant. Accordingly, we can say that the Fredholm determinant $j_\sigma(s|\emptyset)$ is the tau function associated with the RH problem for $\Gamma(\lambda; s)$.

Pole insertion in a RH problem (*Schlesinger transformation*) and its effect on Ω have been studied in depth in [4] (expanding on [30, 3]) for RH problems with identity normalization at infinity and insertion of poles off the jump contour. The general results of op. cit. formally match with our setting. Namely, in our setting we consider the RH problem for $\Gamma(\lambda; s|\underline{\nu}) := \Psi(\lambda - s; s|\underline{\nu})$ which is obtained from the one for Γ by inserting poles at $\nu_i + s$ such that $\Gamma(\lambda; s|\underline{\nu}) = (\lambda - \nu_i - s)^{-\sigma_3} \mathcal{O}(1)$ for $i = 1, \dots, m$. The *characteristic matrix* of [4, Definition 2.2], such that the logarithmic differential of its determinant expresses the variation between the Malgrange–Bertola differential after pole insertion [4, Theorem 2.2 part (3)], would reduce in the present setting to

$$\operatorname{res}_{\lambda=\nu_i+s} \operatorname{res}_{\mu=\nu_j+s} \frac{(\Gamma^{-1}(\lambda; s)\Gamma(\mu; s))_{2,1}}{(\lambda - \mu)(\lambda - \nu_i - s)(\mu - \nu_j - s)} d\lambda = -2\pi i L_s^\sigma(\nu_i, \nu_j), \quad i, j = 1, \dots, m. \quad (3.87)$$

Hence, the above-mentioned [4, Theorem 2.2 part (3)] predicts that the tau function associated with the RH problem for $\Gamma(\lambda; s|\underline{\nu})$ is (within an absolute multiplicative constant) $j_\sigma(s|\emptyset)$ times the determinant of $L_s^\sigma(\underline{\nu}, \underline{\nu})$, i.e. $j_\sigma(s|\underline{\nu})$ by (2.12), as we showed in Proposition 3.8.

3.7 Isospectral deformation and cKdV: proof of Theorem III

As explained in Section 1, the connection with the cKdV equation is made by studying the shifted and dilated Airy kernel

$$K_{X,T}^{\text{Ai}}(\lambda, \mu) := T^{-\frac{1}{3}} K^{\text{Ai}} \left(T^{-\frac{1}{3}}(\lambda + X), T^{-\frac{1}{3}}(\mu + X) \right), \quad (3.88)$$

where $X \in \mathbb{R}$, $T \geq 0$ are parameters. The corresponding Jánossy density $J_\sigma(X, T|\underline{\nu})$, defined in (1.24), is recovered from $j_\sigma(s|\underline{\nu})$ by (1.25). In view of Proposition 3.8, we have

$$\partial_X \log J_\sigma(X, T|\underline{\nu}) = T^{-\frac{1}{3}} \left. \partial_s \log j_{\tilde{\sigma}}(s|T^{-\frac{1}{3}}\underline{\nu}) \right|_{s=XT^{-\frac{1}{3}}} = \frac{X^2}{4T} - T^{-\frac{1}{3}} p_{\tilde{\sigma}}(XT^{-\frac{1}{3}}; T^{-\frac{1}{3}}\underline{\nu}). \quad (3.89)$$

Throughout this section we use the notation $\tilde{\sigma}(\lambda) = \sigma(T^{\frac{1}{3}}\lambda)$, as in (1.25). It is straightforward to verify by the RH problem for Ψ_σ that the matrix

$$\widehat{\Psi}_\sigma(\lambda; X, T|\underline{\nu}) := T^{\frac{1}{12}\sigma_3} \Psi_{\tilde{\sigma}}(\lambda T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}) \quad (3.90)$$

is the (unique) solution to the following RH problem.

RH problem for $\widehat{\Psi}_\sigma$

(a) $\widehat{\Psi}_\sigma(\cdot; X, T|\underline{\nu}) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}^{2 \times 2}$ is analytic for all $X \in \mathbb{R}$, $T > 0$, and all $\underline{\nu}$.

(b) The boundary values of $\widehat{\Psi}_\sigma(\cdot; X, T|\underline{\nu})$ are continuous on $\mathbb{R} \setminus \{\nu_1, \dots, \nu_m\}$ and are related by

$$\widehat{\Psi}_{\sigma,+}(\lambda; X, T|\underline{\nu}) = \widehat{\Psi}_{\sigma,-}(\lambda; X, T|\underline{\nu}) \begin{pmatrix} 1 & 1 - \sigma(\lambda) \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}, \lambda \neq \nu_i. \quad (3.91)$$

(c) For all $i = 1, \dots, m$, as $\lambda \rightarrow \nu_i$ from either side of the real axis we have

$$\widehat{\Psi}_\sigma(\lambda; X, T|\underline{\nu})(\lambda - \nu_i)^{-\sigma_3} = O(1). \quad (3.92)$$

(d) As $\lambda \rightarrow \infty$, we have

$$\widehat{\Psi}_\sigma(\lambda; X, T|\underline{\nu}) = \left(I + \frac{1}{\lambda} \begin{pmatrix} \widehat{q}_\sigma(X, T|\underline{\nu}) & i\widehat{r}_\sigma(X, T|\underline{\nu}) \\ i\widehat{p}_\sigma(X, T|\underline{\nu}) & -\widehat{q}_\sigma(X, T|\underline{\nu}) \end{pmatrix} + O(\lambda^{-2}) \right) \lambda^{\frac{1}{4}\sigma_3} G e^{-T^{-\frac{1}{2}} \left(\frac{2}{3}\lambda^{\frac{3}{2}} + X\lambda^{\frac{1}{2}} \right) \sigma_3} C_\delta \quad (3.93)$$

for any $\delta \in (0, \frac{\pi}{2})$. Here we take principal branches of the roots of λ as explained after (3.12), and G, C_δ are as in (3.13). Moreover, the coefficients in the sub-leading term are related to the ones in (3.12) by

$$\begin{aligned} \widehat{q}_\sigma(X, T|\underline{\nu}) &= T^{\frac{1}{3}} q_{\overline{\sigma}}(XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}), & \widehat{r}_\sigma(X, T|\underline{\nu}) &= T^{\frac{1}{2}} r_{\overline{\sigma}}(XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}), \\ \widehat{p}_\sigma(X, T|\underline{\nu}) &= T^{\frac{1}{6}} p_{\overline{\sigma}}(XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}). \end{aligned} \quad (3.94)$$

It is convenient to reformulate (3.89) using (3.94), as

$$\partial_X \log J_\sigma(X, T|\underline{\nu}) = \frac{X^2}{4T} - T^{-\frac{1}{2}} \widehat{p}_\sigma(X, T|\underline{\nu}). \quad (3.95)$$

Introduce now, cf. (3.33) and (3.41),

$$\begin{aligned} \widehat{\Theta}_\sigma(\lambda; X, T|\underline{\nu}) &:= T^{\frac{1}{12}\sigma_3} \Theta_{\overline{\sigma}}(\lambda T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}) \\ &= \begin{pmatrix} 1 & \widehat{p}_\sigma(X, T|\underline{\nu}) \\ 0 & 1 \end{pmatrix} e^{\frac{i\pi}{4}\sigma_3} \widehat{\Psi}_\sigma(\lambda; X, T|\underline{\nu}) e^{-\frac{i\pi}{4}\sigma_3} \\ &= -\sqrt{2\pi} \begin{pmatrix} T^{\frac{1}{2}} \partial_X \widehat{\varphi}_\sigma(\lambda; X, T|\underline{\nu}) & T^{\frac{1}{2}} \partial_X \widehat{\chi}_\sigma(\lambda; X, T|\underline{\nu}) \\ \widehat{\varphi}_\sigma(\lambda; X, T|\underline{\nu}) & \widehat{\chi}_\sigma(\lambda; X, T|\underline{\nu}) \end{pmatrix} \end{aligned} \quad (3.96)$$

where we define, cf. (3.41),

$$\widehat{\varphi}_\sigma(\lambda; X, T|\underline{\nu}) = T^{-\frac{1}{12}} \varphi_{\overline{\sigma}}(\lambda T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}), \quad \widehat{\chi}_\sigma(\lambda; X, T|\underline{\nu}) = T^{-\frac{1}{12}} \chi_{\overline{\sigma}}(\lambda T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|T^{-\frac{1}{3}}\underline{\nu}). \quad (3.97)$$

Proposition 3.12. *Let $V_\sigma(X, T|\underline{\nu}) := \partial_X^2 \log J_\sigma(X, T|\underline{\nu})$ and f be in the linear span of $\widehat{\varphi}_\sigma(\lambda; X, T|\underline{\nu})$ and $\widehat{\chi}_\sigma(\lambda; X, T|\underline{\nu})$. We have the ‘‘Lax pair’’*

$$\mathcal{L}f = \lambda f, \quad \mathcal{A}f = \partial_T f, \quad (3.98)$$

where

$$\mathcal{L} := T\partial_X^2 + 2TV_\sigma(X, T|\underline{\nu}) - X, \quad \mathcal{A} := -\frac{1}{3}\partial_X^3 - V_\sigma(X, T|\underline{\nu})\partial_X - \frac{1}{2}\partial_X V_\sigma(X, T|\underline{\nu}) + \frac{1}{3}T^{-1} - \frac{1}{12}T^{-\frac{3}{2}}. \quad (3.99)$$

Proof. Although the first equation $\mathcal{L}f = \lambda f$ follows directly from (3.42) by using (3.97), it is convenient to deduce it again; doing so will provide us with additional information useful in the derivation of the second equation.

We start by noting that the matrix function $A(\lambda; X, T|\underline{\nu}) := (\partial_X \widehat{\Theta}_\sigma(\lambda; X, T|\underline{\nu})) \widehat{\Theta}_\sigma(\lambda; X, T|\underline{\nu})^{-1}$ has no jump across the real axis because the jump condition for $\widehat{\Theta}_\sigma$ across the real axis does not depend on X as it follows from (3.96) along with condition (b) in the RH problem for $\widehat{\Psi}_\sigma$. Once more, we refer the reader to [14, Section 3] for the rigorous justification of the differentiability of the RH solution.

Moreover, it is readily checked, cf. (3.36), that as $\lambda \rightarrow \infty$ we have

$$\begin{aligned} \Theta_\sigma(\lambda; s|\underline{\nu}) &= \begin{pmatrix} 1 & \widehat{p}_\sigma(X, T|\underline{\nu}) \\ 0 & 1 \end{pmatrix} \left(I + \frac{1}{\lambda} \begin{pmatrix} \widehat{q}_\sigma(X, T|\underline{\nu}) & -\widehat{r}_\sigma(X, T|\underline{\nu}) \\ \widehat{p}_\sigma(X, T|\underline{\nu}) & -\widehat{q}_\sigma(X, T|\underline{\nu}) \end{pmatrix} \right) \\ &\quad + \frac{1}{\lambda^2} \begin{pmatrix} \star & \star \\ \widehat{n}_\sigma(X, T|\underline{\nu}) & \star \end{pmatrix} + O(\lambda^{-3}) \Big) \lambda^{\sigma_3/4} \frac{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}{\sqrt{2}} e^{-T^{-\frac{1}{2}} \left(\frac{2}{3} \lambda^{\frac{3}{2}} + X \lambda^{\frac{1}{2}} \right) \sigma_3} C_\delta \end{aligned} \quad (3.100)$$

for any $\delta \in (0, \frac{\pi}{2})$. The notations are as in (3.36) but now we also need to explicitly record the $(2, 1)$ -entry of the second sub-leading term in the asymptotic series, denoted \widehat{n}_σ .

Next, from (3.100), we deduce that A has an expansion for large λ of the form

$$\begin{aligned} A &= T^{-\frac{1}{2}} \begin{pmatrix} 0 & \lambda + \widehat{p}_\sigma^2 + 2\widehat{q}_\sigma + T^{\frac{1}{2}} \partial_X \widehat{p}_\sigma \\ 1 & 0 \end{pmatrix} \\ &\quad + \lambda^{-1} T^{-\frac{1}{2}} \begin{pmatrix} \star & \star \\ T^{\frac{1}{2}} \partial_X \widehat{p}_\sigma - \widehat{p}_\sigma^2 - 2\widehat{q}_\sigma & \widehat{n}_\sigma + \widehat{r}_\sigma + \widehat{p}_\sigma^3 + 3\widehat{p}_\sigma \widehat{q}_\sigma - \frac{1}{2} T^{\frac{1}{2}} \partial_X (\widehat{p}_\sigma^2 + 2\widehat{q}_\sigma) \end{pmatrix} + O(\lambda^{-2}). \end{aligned} \quad (3.101)$$

Liouville theorem guarantees then that A is a polynomial in λ . Consequently, the higher-order Laurent coefficient in this expansion must vanish; we do not need the information coming from the first row (and we have accordingly omitted these terms), while from the second row at order λ^{-1} we obtain

$$\widehat{p}_\sigma^2 + 2\widehat{q}_\sigma = T^{\frac{1}{2}} \partial_X \widehat{p}_\sigma, \quad \widehat{n}_\sigma + \widehat{r}_\sigma + \widehat{p}_\sigma^3 + 3\widehat{p}_\sigma \widehat{q}_\sigma = \frac{1}{2} T \partial_X^2 \widehat{p}_\sigma. \quad (3.102)$$

Summarizing, also thanks to (3.95), we have

$$A = T^{-\frac{1}{2}} \begin{pmatrix} 0 & \lambda + 2T^{\frac{1}{2}} \partial_X \widehat{p}_\sigma \\ 1 & 0 \end{pmatrix} = T^{-\frac{1}{2}} \begin{pmatrix} 0 & \lambda - 2V_\sigma + X \\ 1 & 0 \end{pmatrix}. \quad (3.103)$$

Comparing with (3.96) we obtain $\mathcal{L}f = \lambda f$ whenever f is in the linear span of $\widehat{\varphi}_\sigma, \widehat{\chi}_\sigma$.

Next, the matrix function $B(\lambda; X, T|\underline{\nu}) := (\partial_T \widehat{\Theta}_\sigma(\lambda; X, T|\underline{\nu})) \widehat{\Theta}_\sigma(\lambda; X, T|\underline{\nu})^{-1}$ has no jump across the real axis because the jump condition for $\widehat{\Theta}_\sigma$ across the real axis does not depend on T , and so B is entire in λ . It then follows from an application of Liouville theorem that B is a polynomial in λ . In particular, from (3.100), the $(2, 1)$ -entry of B is expressed as

$$3T^{\frac{3}{2}} B_{2,1} = -\lambda + \left(\widehat{p}_\sigma^2 + 2\widehat{q}_\sigma - \frac{3X}{2} \right) = -\lambda + T^{\frac{1}{2}} \partial_X \widehat{p}_\sigma - \frac{3}{2} X = -\lambda - TV_\sigma - X, \quad (3.104)$$

and, similarly, the $(2, 2)$ -entry of B as

$$3T^{\frac{3}{2}} B_{2,2} = -(\widehat{n}_\sigma + \widehat{r}_\sigma + \widehat{p}_\sigma^3 + 3\widehat{p}_\sigma \widehat{q}_\sigma) = -\frac{1}{2} T \partial_X^2 \widehat{p}_\sigma = \frac{1}{2} T^{\frac{3}{2}} \partial_X V_\sigma - \frac{1}{4}. \quad (3.105)$$

We have used (3.102) and (3.95) to simplify these expressions. Comparing with (3.96), we must have

$$\partial_T f = -\frac{\lambda + TV_\sigma + X}{3T} \partial_X f + \frac{2\partial_X V_\sigma - T^{-\frac{3}{2}}}{12} f, \quad (3.106)$$

for f equal to either $\widehat{\varphi}_\sigma$ or $\widehat{\chi}_\sigma$, and hence for any f in their linear span. By the relation $\mathcal{L}f = \lambda f$ obtained above we can rewrite the last relation by using $\lambda \partial_X f = \partial_X(\mathcal{L}f)$ which finally yields $\partial_T f = \mathcal{A}f$. \square

Corollary 3.13. *The function $V_\sigma(X, T|\underline{\nu}) := \partial_X^2 \log J_\sigma(X, T|\underline{\nu})$ satisfies the cKdV equation (1.14).*

Proof. This is a classical argument [33]. From the compatibility condition of (3.98) we obtain

$$(\partial_T \mathcal{L} + [\mathcal{L}, \mathcal{A}])f = 0. \quad (3.107)$$

A direct computation gives that $\partial_T \mathcal{L} + [\mathcal{L}, \mathcal{A}]$ is the operator of multiplication with the function

$$V_\sigma(X, T|\underline{\nu}) + 2T\partial_T V_\sigma(X, T|\underline{\nu}) + 2TV_\sigma(X, T|\underline{\nu})\partial_X V_\sigma(X, T|\underline{\nu}) + \frac{1}{6}T\partial_X^3 V_\sigma(X, T|\underline{\nu}). \quad (3.108)$$

The equation (3.107) must be true for any f in the linear span of $\widehat{\varphi}_\sigma, \widehat{\chi}_\sigma$. Since $\det \widehat{\Theta}_\sigma = 1$ identically in all variables λ, X, T , the functions $\widehat{\varphi}_\sigma$ and $\widehat{\chi}_\sigma$ never vanish simultaneously, hence (3.108) must vanish identically. \square

This proves Theorem III.

3.8 Generalization to discontinuous σ 's

In this section we briefly explain how to extend the results to a broader class of functions σ , including in particular $\sigma = 1_{(0, +\infty)}$.

Assumption C. *The function $\sigma : \mathbb{R} \rightarrow [0, 1]$ is such that $\sigma = \sigma_0 + \sum_{j=1}^f w_j 1_{(\xi_j, +\infty)}$ for some (finite) integer $f \geq 0$, some $w_1, \dots, w_f > 0$ and some $\xi_1, \dots, \xi_f \in \mathbb{R}$, and a smooth function σ_0 such that $\sigma_0(\lambda) = O(|\lambda|^{-\frac{3}{2}-\kappa})$ as $\lambda \rightarrow -\infty$ for some $\kappa > 0$.*

These are the assumptions made in [14], to which we refer for more details, and they include the setting of [16] which corresponds to the case $\sigma_0 = 0$. Under these more general assumptions, the RH problems for Y_σ and Ψ_σ have to be complemented with the condition that Y_σ and Ψ_σ have, at worst, logarithmic singularities at ξ_j .

Theorem I holds true verbatim except for (1.29), which is to be replaced by

$$j_\sigma(s|\emptyset) = \exp \left(- \int_s^{+\infty} (r-s) \left(\int_{\mathbb{R}} \varphi_\sigma(\lambda; r|\emptyset)^2 \sigma'_0(\lambda) d\lambda + \sum_{j=1}^f \Delta_j \varphi_\sigma(\xi_j; r|\emptyset)^2 \right) \right) \quad (3.109)$$

where $\Delta_j := w_j - w_{j-1}$ for $2 \leq j \leq f$ and $\Delta_1 := w_1$. This follows directly from [14, equations (1.8) and (1.26)].

Moreover, Theorem II holds true verbatim except for (1.31), which is to be replaced by

$$\partial_s^2 \log j_\sigma(s|\underline{\nu}) = \int_{\mathbb{R}} \varphi_\sigma(\lambda; s|\underline{\nu})^2 \left(-\sigma'_0(\lambda) + \sum_{i=1}^m \frac{2(1-\sigma(\lambda))}{\lambda - \nu_i} \right) d\lambda - \sum_{j=1}^f \Delta_j \varphi_\sigma(\xi_j; s|\underline{\nu})^2, \quad (3.110)$$

and Theorem III holds true verbatim. These two generalizations are obtained by studying the local behavior of Ψ_σ near the logarithmic singularities at the points ξ_j , as is done in the end of the proof of [14, Proposition 4.1], cf. equations (4.5) and (4.6) there.

4 Asymptotics

4.1 Outline

The goal of this section is to prove Theorem IV.

The proof of part (i) of Theorem IV will rely on elementary operator estimates, starting from the analogue of the factorization (2.12) in the cKdV variables,

$$J_\sigma(X, T|\underline{\nu}) = \det \left(K_{X,T}^{\text{Ai}}(\underline{\nu}, \underline{\nu}) \right) \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{\mathcal{H}}_{X,T}^\underline{\nu} \mathcal{M}_{\sqrt{\sigma}} \right), \quad (4.1)$$

where $\widehat{\mathcal{H}}_{X,T}^{\text{Ai}}$ is the integral operator with kernel, similarly to (2.9),

$$\widehat{H}_{X,T}^{\nu}(\lambda, \mu) := \frac{\det K_{X,T}^{\text{Ai}}((\lambda, \nu), (\mu, \nu))}{\det K_{X,T}^{\text{Ai}}(\nu, \nu)} = K_{X,T}^{\text{Ai}}(\lambda, \mu) - K_{X,T}^{\text{Ai}}(\lambda, \nu) K_{X,T}^{\text{Ai}}(\nu, \nu)^{-1} K_{X,T}^{\text{Ai}}(\nu, \mu). \quad (4.2)$$

Consequently, by (1.11), we also have

$$V_{\sigma}(X, T|\nu) = V_{\sigma=0}(X, T|\nu) + \partial_X^2 \log \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{\mathcal{H}}_{X,T}^{\nu} \mathcal{M}_{\sqrt{\sigma}} \right). \quad (4.3)$$

We will prove in Section 4.2 that the second factor in (4.1) is close to 1 and that the second term in (4.3) is close to 0, and this will result in part (i) of Theorem IV.

For part (ii) of Theorem IV, we will instead use the analogue of (2.13) in the cKdV variables X, T . Using (1.25), (1.27), and (1.28), we obtain the identity

$$J_{\sigma}(X, T|\nu) = \det(\widehat{L}_{X,T}^{\sigma}(\nu_i, \nu_j))_{i,j=1}^m J_{\sigma}(X, T|\emptyset) \quad (4.4)$$

where

$$\widehat{L}_{X,T}^{\sigma}(\lambda, \mu) := T^{-\frac{1}{3}} L_{XT^{-\frac{1}{3}}}^{\tilde{\sigma}}(T^{-\frac{1}{3}}\lambda, T^{-\frac{1}{3}}\mu), \quad (4.5)$$

with $\tilde{\sigma}(\lambda) = \sigma(T^{\frac{1}{3}}\lambda)$ as in (1.25), which can be rewritten by (1.28) and (3.96)–(3.97) as

$$\begin{aligned} \widehat{L}_{X,T}^{\sigma}(\lambda, \mu) &= T^{\frac{1}{2}} \frac{\widehat{\varphi}_{\sigma}(\lambda; X, T|\emptyset) \partial_X \widehat{\varphi}_{\sigma}(\mu; X, T|\emptyset) - \widehat{\varphi}_{\sigma}(\mu; X, T|\emptyset) \partial_X \widehat{\varphi}_{\sigma}(\lambda; X, T|\emptyset)}{\lambda - \mu} \\ &= \frac{\left(\widehat{\Theta}_{\sigma}(\mu; X, T|\emptyset)^{-1} \widehat{\Theta}_{\sigma}(\lambda; X, T|\emptyset) \right)_{2,1}}{2\pi(\lambda - \mu)}. \end{aligned} \quad (4.6)$$

It also follows that

$$V_{\sigma}(X, T|\nu) = V_{\sigma}(X, T|\emptyset) + \partial_X^2 \log \det(\widehat{L}_{X,T}^{\sigma}(\nu_i, \nu_j))_{i,j=1}^m. \quad (4.7)$$

The asymptotic behavior of the second factor in (4.4) and of the first term in (4.7) has been established in [14, 15], and can be summarized as follows in the cases where $XT^{-\frac{1}{2}} \rightarrow -\infty$.

Theorem 4.1. *Let σ satisfy Assumption B. For any $T_0 > 0$ there exists $K > 0$ such that*

$$\log J_{\sigma}(X, T|\emptyset) = \rho^3 T^2 \left(-\frac{4}{15} (1 - \xi)^{\frac{5}{2}} + \frac{4}{15} - \frac{2}{3} \xi + \frac{1}{2} \xi^2 \right) + O\left(|X|^{\frac{3}{2}} T^{-\frac{1}{2}}\right), \quad (4.8)$$

$$V_{\sigma}(X, T|\emptyset) = \rho \left(1 - \sqrt{1 - \xi} \right) + O\left(|X|^{-\frac{1}{2}} T^{-\frac{1}{2}}\right), \quad (4.9)$$

where $\rho := c_+^2/\pi^2$ and $\xi := X/(\rho T)$, uniformly for $X \leq -KT^{\frac{1}{2}}$ and $T \geq T_0$.

This result is contained in [15, Theorem 1.3].

Therefore, in order to prove part (ii) of Theorem IV we only need to study, in Section 4.3, the additional contributions to (4.4) and (4.7) coming from $\det(\widehat{L}_{X,T}^{\sigma}(\nu_i, \nu_j))_{i,j=1}^m$.

4.2 Right tail: $XT^{-\frac{1}{3}} \rightarrow \infty$

We start with a Fredholm determinant estimate for the operator $\mathcal{M}_{\sqrt{\sigma}} \widehat{\mathcal{H}}_{X,T}^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}}$, not only valid for large positive X , but also for complex large X with $\arg X$ sufficiently small.

Lemma 4.2. *Let σ satisfy Assumption A and let $\nu = (\nu_1, \dots, \nu_m)$ with $\nu_i \neq \nu_j$ for all $i \neq j$. For any $T_0 > 0$ there exist $M, c, \delta > 0$ such that*

$$\det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{\mathcal{H}}_{X,T}^{\nu} \mathcal{M}_{\sqrt{\sigma}} \right) = 1 + O\left(e^{-cXT^{-\frac{1}{3}}}\right), \quad (4.10)$$

uniformly for $|X| \geq MT^{\frac{1}{3}}$, $|\arg X| < \delta$, and $T \geq T_0$.

Proof. Using the integral representation for the Airy kernel in (1.3), (1.10), and the asymptotic behavior for the Airy function, it is straightforward to verify that

$$|K_{X,T}^{\text{Ai}}(\lambda, \mu)| = O\left(|\lambda\mu|^{\frac{1}{4}} e^{-cT^{-\frac{1}{3}}\text{Re}(\lambda+X)_+} e^{-cT^{-\frac{1}{3}}\text{Re}(\mu+X)_+}\right), \quad (4.11)$$

uniformly for $|X| \geq MT^{\frac{1}{3}}$, $|\arg X| < \delta$, and $T \geq T_0$, where $R_+ = \max\{R, 0\}$, for any $c > 0$, and uniformly for $\lambda, \mu \in \mathbb{R}$. Hence, by (4.2),

$$\left|\widehat{H}_{X,T}^{\nu}(\lambda, \mu)\right| = O\left(|\lambda\mu|^{\frac{1}{4}} e^{-cT^{-\frac{1}{3}}\text{Re}(\lambda+X)_+} e^{-cT^{-\frac{1}{3}}\text{Re}(\mu+X)_+}\right), \quad (4.12)$$

uniformly for the same values of X, T, λ, μ . We can now use the triangular inequality in the Fredholm series

$$\det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{H}_{X,T}^{\nu} \mathcal{M}_{\sqrt{\sigma}}\right) - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \det\left(\widehat{H}_{X,T}^{\nu}(\lambda_i, \lambda_j)\right)_{i,j=1}^n \prod_{j=1}^n \sigma(\lambda_j) d\lambda_j, \quad (4.13)$$

in order to obtain

$$\begin{aligned} & \left| \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{H}_{X,T}^{\nu} \mathcal{M}_{\sqrt{\sigma}}\right) - 1 \right| \\ & \leq \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^n} \det(O(1))_{i,j=1}^n \prod_{j=1}^n e^{-2cT^{-\frac{1}{3}}\text{Re}(\lambda_j+X)_+} |\lambda_j|^{\frac{1}{2}} \sigma(\lambda_j) d\lambda_j \\ & = O\left(\sum_{n=1}^{\infty} \frac{n^{\frac{n}{2}} e^{-ncT^{-\frac{1}{3}}\text{Re}X}}{n!} \left(\int_{\mathbb{R}} e^{-cT^{-\frac{1}{3}}\text{Re}(\lambda+X)_+} |\lambda|^{\frac{1}{2}} \sigma(\lambda) d\lambda\right)^n\right) = O\left(\sum_{n=1}^{\infty} \frac{n^{\frac{n}{2}}}{n!} \xi^n\right) \end{aligned}$$

where Hadamard's inequality guarantees that $\det(O(1))_{i,j=1}^n = O(n^{\frac{n}{2}})$, and in the last step we set $\xi := e^{-cT^{-\frac{1}{3}}\text{Re}X} \int_{\mathbb{R}} e^{-cT^{-\frac{1}{3}}\text{Re}(\lambda+X)_+} |\lambda|^{\frac{1}{2}} \sigma(\lambda) d\lambda$. Finally, $\sum_{n=1}^{\infty} \frac{n^{\frac{n}{2}}}{n!} \xi^n$ is a power series in ξ with infinite radius of convergence, hence $\sum_{n=1}^{\infty} \frac{n^{\frac{n}{2}}}{n!} \xi^n = O(\xi)$ when $\xi \rightarrow 0$; since

$$|\xi| \leq e^{-cT^{-\frac{1}{3}}\text{Re}X} \int_{\mathbb{R}} |\lambda|^{\frac{1}{2}} \sigma(\lambda) d\lambda, \quad (4.14)$$

and since the integral on the right-hand side is finite by Assumption A, we have $\xi = O(e^{-cT^{-\frac{1}{3}}\text{Re}X})$ and the proof is complete. \square

Taking logarithms on both sides in (4.1), we obtain

$$\log J_{\sigma}(X, T|\underline{\nu}) = \log \det(K_{X,T}^{\text{Ai}}(\underline{\nu}, \underline{\nu})) + \log \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{H}_{X,T}^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}}\right), \quad (4.15)$$

and it follows from Lemma 4.2 (for real X) that

$$\det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{H}_{X,T}^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}}\right) = 1 + O(e^{-cXT^{-\frac{1}{3}}}) \quad (4.16)$$

as $X, T \rightarrow \infty$ uniformly for $X \geq MT^{\frac{1}{3}}$ and $T \geq T_0$. This implies (1.45).

Taking the second logarithmic X -derivative in (1.45), we obtain

$$V_{\sigma}(X, T|\underline{\nu}) = V_0(X, T|\underline{\nu}) + \partial_X^2 \log \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{H}_{X,T}^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}}\right). \quad (4.17)$$

Since the estimate from Lemma 4.2 holds uniformly for $|X| \geq MT^{\frac{1}{3}}$, $|\arg X| < \delta$, and $T \geq T_0$, we can use Cauchy's integral formula for the second derivative to obtain

$$\partial_X^2 \log \det_{L^2(\mathbb{R})} \left(1 - \mathcal{M}_{\sqrt{\sigma}} \widehat{H}_{X,T}^{\text{Ai}} \mathcal{M}_{\sqrt{\sigma}}\right) = O(e^{-cXT^{-\frac{1}{3}}}), \quad (4.18)$$

and thus we prove (1.46), so the proof of part (i) of Theorem IV is concluded.

4.3 Left tail: $X/T \rightarrow -\infty$

In this section we use the results of [15]; the latter rely on Assumption B, which we assume throughout this section. We recall the transformation $x = -XT^{-\frac{1}{2}}$ and $t = T^{-\frac{1}{2}}$ between the cKdV variables of the present paper and the KdV variables of [15]. For the ease of notations, we will denote $\widehat{\Theta}(\lambda) := \widehat{\Theta}_\sigma(\lambda; X, T|\emptyset)$ throughout this section for the function defined in (3.96). Let us now assume that, for an arbitrary $T_0 > 0$ and for a sufficiently large $K > 0$, we have $X \leq -KT$ and $T \geq T_0$. In this regime (in fact, in the larger regime $X \leq -KT^{\frac{1}{2}}$), the relevant asymptotics have been studied in [15] via a RH analysis involving a series of transformations which we can condense in the relation

$$\begin{aligned} \widehat{\Theta}_\pm(|X|w) &= \begin{pmatrix} 1 & \widehat{p} \\ 0 & 1 \end{pmatrix} e^{\frac{i\pi}{4}\sigma_3} |X|^{\frac{1}{4}\sigma_3} \begin{pmatrix} 1 & -i|X|^{\frac{3}{2}}T^{-\frac{1}{2}}g_1 \\ 0 & 1 \end{pmatrix} R(w) \\ &\quad \times (w-a)^{\frac{1}{4}\sigma_3} G \begin{pmatrix} 1 & 0 \\ \pm e^{-|X|^{\frac{3}{2}}T^{-\frac{1}{2}}\phi_\pm(w)} & 1 \end{pmatrix} e^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)\sigma_3} e^{-\frac{i\pi}{4}\sigma_3}, \end{aligned} \quad (4.19)$$

for w sufficiently close to 0. Here, cf. [15, equations (4.15), (4.17)], with principal branches for the roots,

$$\begin{aligned} g(w) &= \int_a^w g'(s) ds, & g'(w) &= -(w-a)^{\frac{1}{2}} \left(1 + \frac{T^{\frac{1}{2}}}{2\pi|X|^{\frac{1}{2}}} \int_{-\infty}^a \frac{\sigma'(|X|s)}{1-\sigma(|X|s)} \frac{1}{\sqrt{a-s} s-w} ds \right), \\ g_0 &= \frac{T^{\frac{1}{2}} \log(1-\sigma(|X|a))}{2|X|^{\frac{3}{2}}}, & \phi(w) &= 2(g(w)-g_0) + \frac{T^{\frac{1}{2}} \log(1-\sigma(|X|w))}{|X|^{\frac{3}{2}}}. \end{aligned} \quad (4.20)$$

Moreover, G is given in (3.13) and $\widehat{p} = \widehat{p}_\sigma(X, T|\emptyset)$. The value of $a = a(X, T)$ is implicitly defined by the *endpoint condition*, namely $g'_+(w) - g'_-(w) = O((w-a)^{\frac{1}{2}})$ as $w \rightarrow a$ with $w < a$, cf. [15, equation (4.3)]. For $X \leq -KT$ and $T \geq T_0$, a is bounded away from zero and infinity, and by [15, Lemma 4.5],

$$R(w) = I + O(|X|^{-\frac{3}{2}}T^{\frac{1}{2}}), \quad \partial_w R(w) = O(|X|^{-\frac{3}{2}}T^{\frac{1}{2}}), \quad (4.21)$$

uniformly in w , $T \geq T_0$, $X \leq -KT$. Finally, the value of g_1 is given explicitly in [15, equation (4.16)] but it is not needed for our current purposes.

As explained in Section 4.1, in order to describe the behavior of $J_\sigma(X, T|\underline{\nu}), V_\sigma(X, T|\underline{\nu})$ in this regime we will use equations (4.4) and (4.7). Thus what is fundamental to understand is the behavior of $\widehat{L}_{X,T}^\sigma(\nu_i, \nu_j)$. The following two lemmas will show how the kernel behaves on and off the diagonal.

We are interested in values of $\widehat{\Theta}$ at $\nu = |X|w$, hence we assume throughout this section that w is real and small. Therefore, (4.19) implies

$$\widehat{\Theta}_\pm(|X|w) = \begin{pmatrix} 1 & \widehat{p} + \frac{X^2}{T^{\frac{1}{2}}}g_1 \\ 0 & 1 \end{pmatrix} |X|^{\frac{1}{4}\sigma_3} O(1) \begin{pmatrix} e^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)} & 0 \\ \mp i e^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\mp(w)-g_0)} & e^{-|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)} \end{pmatrix} \quad (4.22)$$

where we also use the identity $\phi_\pm = g_\pm - g_\mp$ [15, equation below (4.17)]. From [15, Proposition 4.7 and equation (1.32)] we have

$$\widehat{p} + X^2T^{-\frac{1}{2}}g_1 = O(|X|^{-1}T^{\frac{1}{2}}), \quad (4.23)$$

such that the previous relation implies

$$\widehat{\Theta}_\pm(|X|w) = \begin{pmatrix} 1 & O(|X|^{-1}T^{\frac{1}{2}}) \\ 0 & 1 \end{pmatrix} |X|^{\frac{1}{4}\sigma_3} O(1) \begin{pmatrix} e^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)} & 0 \\ \mp i e^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\mp(w)-g_0)} & e^{-|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(w)-g_0)} \end{pmatrix}. \quad (4.24)$$

Finally, we study the last factor, involving g, g_0 . By (4.20) and the Sokhotski–Plemelj formula, we have

$$g'_\pm(w) = \mp i \sqrt{a-w} \left(1 + \frac{|X|^{-\frac{1}{2}}T^{\frac{1}{2}}}{2\pi} \text{p.v.} \int_{-\infty}^a \frac{\sigma'(|X|s)}{1-\sigma(|X|s)} \frac{1}{\sqrt{a-s} s-w} ds \pm i \frac{T^{\frac{1}{2}}|X|^{-\frac{1}{2}}}{2\sqrt{a-w}} \frac{\sigma'(|X|w)}{1-\sigma(|X|w)} \right), \quad (4.25)$$

where p.v. \int is the principal value integral. It follows that

$$\operatorname{Re} g'_\pm(w) = \frac{T^{\frac{1}{2}}}{2|X|^{\frac{1}{2}}} \frac{\sigma'(|X|w)}{1 - \sigma(|X|w)} \quad (4.26)$$

and so

$$\operatorname{Re} (g_\pm(\zeta) - g_0) = \int_a^w \operatorname{Re} g'_\pm(s) ds - g_0 = -\frac{T^{\frac{1}{2}}}{2|X|^{\frac{3}{2}}} \log(1 - \sigma(|X|w)). \quad (4.27)$$

Finally, let us set $w = |X|^{-1}\nu$, for a fixed ν and sufficiently large $|X|$. It follows from the last estimates and (4.24) that

$$\widehat{\Theta}_\pm(\nu) = \begin{pmatrix} 1 & O(|X|^{-1}T^{\frac{1}{2}}) \\ 0 & 1 \end{pmatrix} |X|^{\frac{1}{4}\sigma_3} O(1) \quad (4.28)$$

because from (4.27) we have

$$\left| e^{|X|^{\frac{3}{2}}T^{-\frac{1}{2}}(g_\pm(\nu|X|^{-1}) - g_0)} \right| = \frac{1}{1 - \sigma(\nu)} \quad (4.29)$$

which is bounded away from 0 and ∞ , uniformly in the regime under consideration. In particular, by (3.96),

$$\widehat{\varphi}_\sigma(\nu; X, T|\emptyset) = O(|X|^{-\frac{1}{4}}), \quad T^{\frac{1}{2}}\partial_X \widehat{\varphi}_\sigma(\nu; X, T|\emptyset) = O(|X|^{\frac{1}{4}}). \quad (4.30)$$

From (4.6) and (4.30), we immediately obtain boundedness of the kernel $\widehat{L}_{X,T}^\sigma(\nu_1, \nu_2)$ for $\nu_1 \neq \nu_2$. Namely, we have proved the following lemma.

Lemma 4.3. *Let $T_0 > 0, \nu_1 \neq \nu_2 \in \mathbb{R}$ be fixed. There exists $K > 0$ such that*

$$\widehat{L}_{X,T}^\sigma(\nu_1, \nu_2) = O(1) \quad (4.31)$$

uniformly for $X \leq -KT$ and $T \geq T_0$.

On the other hand, we now show that on the diagonal, the kernel $\widehat{L}_{X,T}^\sigma(\nu, \nu)$ grows.

Lemma 4.4. *Let $T_0 > 0, \nu \in \mathbb{R}$ be fixed. We have*

$$\widehat{L}_{X,T}^\sigma(\nu, \nu) \sim \frac{|X|^{\frac{1}{2}}}{\pi T^{\frac{1}{2}}} \frac{1}{1 - \sigma(\nu)}, \quad \text{as } \frac{X}{T \log^2 |X|} \rightarrow -\infty, \quad (4.32)$$

uniformly for $T \geq T_0$. In particular, $\widehat{L}_{X,T}^\sigma(\nu, \nu)^{-1} = O(|X|^{-\frac{1}{2}}T^{\frac{1}{2}})$.

Proof. We combine (4.6) (in the confluent limit $\mu, \lambda \rightarrow \nu$) with (4.19) to get

$$\begin{aligned} \widehat{L}_{X,T}^\sigma(\nu, \nu) &= \frac{1}{2\pi} (\widehat{\Theta}_+^{-1}(\nu) \partial_\nu \widehat{\Theta}_+(\nu))_{2,1} \\ &= \frac{1}{2\pi i} \left\{ e^{-\chi(g_+(\frac{\nu}{|X|}) - g_0)\sigma_3} \begin{pmatrix} 1 & 0 \\ -e^{-\chi\phi_+(\frac{\nu}{|X|})} & 1 \end{pmatrix} G^{-1}(-\frac{\nu}{|X|} - a)^{-\frac{1}{4}\sigma_3} R^{-1}(\frac{\nu}{|X|}) \right. \\ &\quad \left. \partial_\nu \left[R(\frac{\nu}{|X|})(-\frac{\nu}{|X|} - a)^{\frac{1}{4}\sigma_3} G \begin{pmatrix} 1 & 0 \\ e^{-\chi\phi_+(\frac{\nu}{|X|})} & 1 \end{pmatrix} e^{\chi(g_+(\frac{\nu}{|X|}) - g_0)\sigma_3} \right] \right\}_{2,1} \\ &= \frac{1}{2\pi i} \left\{ \begin{pmatrix} e^{-\chi(g_+(\frac{\nu}{|X|}) - g_0)} & 0 \\ -e^{\chi(g_-(\frac{\nu}{|X|}) - g_0)} & e^{\chi(g_+(\frac{\nu}{|X|}) - g_0)} \end{pmatrix} G^{-1}(-\frac{\nu}{|X|} - a)^{-\frac{1}{4}\sigma_3} R^{-1}(\frac{\nu}{|X|}) \right. \\ &\quad \left. \partial_\nu \left[R(\frac{\nu}{|X|})(-\frac{\nu}{|X|} - a)^{\frac{1}{4}\sigma_3} G \begin{pmatrix} e^{\chi(g_+(\frac{\nu}{|X|}) - g_0)} & 0 \\ e^{\chi(g_-(\frac{\nu}{|X|}) - g_0)} & e^{-\chi(g_+(\frac{\nu}{|X|}) - g_0)} \end{pmatrix} \right] \right\}_{2,1} \quad (4.33) \end{aligned}$$

where we denote $\chi := |X|^{\frac{3}{2}}T^{-\frac{1}{2}}$ and in the last step we use the relation $\phi_+ = g_+ - g_-$, cf. [15, equation below (4.17)]. As we proved in (4.29), $e^{\chi(g_{\pm}(\frac{\nu}{|X|}) - g_0)}$ is bounded away from 0, ∞ and so the triangular matrices appearing in (4.33) are $O(1)$. Therefore, when the derivative in ν acts in (4.33) it produces terms of order $O(|X|^{-1})$ when it acts on the first two factors, see also (4.21), and another term when it acts on the triangular matrix, which provides the leading asymptotic contribution, yielding

$$\widehat{L}_{X,T}^{\sigma}(\nu, \nu) = \frac{\chi}{2\pi i |X|} e^{\chi(g_+(\frac{\nu}{|X|}) + g_-(\frac{\nu}{|X|}) - 2g_0)} (g'_-(\frac{\nu}{|X|}) - g'_+(\frac{\nu}{|X|})) + O(|X|^{-1}). \quad (4.34)$$

By the construction of g , cf. [15, Section 4.2], we have (recall that $F = 1/(1 - \sigma)$)

$$\chi(g_+(w) + g_-(w) - 2g_0) = (\log F)(|X|w), \quad (4.35)$$

hence we can rewrite the last expression as

$$\widehat{L}_{X,T}^{\sigma}(\nu, \nu) = \frac{|X|^{\frac{1}{2}}}{2\pi i T^{\frac{1}{2}}} F(\nu) (g'_-(\frac{\nu}{|X|}) - g'_+(\frac{\nu}{|X|})) + O(|X|^{-1}). \quad (4.36)$$

Next, we use (4.25), a change of integration variable, and an integration by parts in order to get

$$\begin{aligned} g'_-(w) - g'_+(w) &= 2i\sqrt{a-w} \left(1 + \frac{T^{\frac{1}{2}}}{2\pi|X|^{\frac{1}{2}}} \text{p.v.} \int_{-\infty}^{a|X|} (\log F)'(s) \frac{1}{\sqrt{a-s|X|^{-1}}} \frac{ds}{s - |X|w} \right) \\ &= 2i \left(\sqrt{a-w} + \frac{T^{\frac{1}{2}}}{2\pi|X|^{\frac{1}{2}}} \int_{-\infty}^{a|X|} (\log F)''(s) \log \left| \frac{\sqrt{a-w} + \sqrt{a-s|X|^{-1}}}{\sqrt{a-w} - \sqrt{a-s|X|^{-1}}} \right| ds \right). \end{aligned} \quad (4.37)$$

Now, it is useful to recall the following asymptotic properties for $a = a(X, T)$ from [15, Proposition 4.1]:

$$a = \frac{(\sqrt{1+y} - 1)^2}{y} \Big|_{y=\frac{\pi^2}{4}|X|/T} + O(|X|^{-\frac{3}{2}}T^{\frac{1}{2}}) \quad (4.38)$$

uniformly in $-X/T \leq K$, $T \geq T_0$ (for any $K, T_0 > 0$). Hence,

$$a = 1 - \frac{2c_+T^{\frac{1}{2}}}{\pi|X|^{\frac{1}{2}}} + O(|X|^{-1}T), \quad \sqrt{a - \frac{\nu}{|X|}} = 1 - \frac{c_+T^{\frac{1}{2}}}{\pi|X|^{\frac{1}{2}}} + O(|X|^{-1}T), \quad (4.39)$$

as $X/T \rightarrow -\infty$, $T \geq T_0$. Let us now show that the second term in (4.37) is sub-dominant. To start with, notice that, in the same limit, and uniformly for $s \in (-\infty, a|X|)$,

$$\log \left| \frac{\sqrt{a-\nu|X|^{-1}} + \sqrt{a-s|X|^{-1}}}{\sqrt{a-\nu|X|^{-1}} - \sqrt{a-s|X|^{-1}}} \right| = \log \frac{4a|X|}{|s-\nu|} + O(1) + O(\log(s|X|^{-1})). \quad (4.40)$$

It follows that

$$\widehat{L}_{X,T}^{\sigma}(\nu, \nu) = \frac{|X|^{\frac{1}{2}}}{\pi T^{\frac{1}{2}}} F(\nu) \left[1 - \frac{c_+T^{\frac{1}{2}}}{\pi|X|^{\frac{1}{2}}} + O\left(|X|^{-\frac{1}{2}}T^{\frac{1}{2}} \log |X|\right) \right] \quad \text{as } XT^{-1} \log^{-2} |X| \rightarrow -\infty. \quad (4.41)$$

To achieve this bound, we used

$$\int_{-\infty}^{a|X|} (\log F)''(s) ds = \int_{-\infty}^{+\infty} (\log F)''(s) ds + o(1) = c_+ + o(1) = O(1) \quad (4.42)$$

(note that, by Assumption B, $\int_{-\infty}^{+\infty} (\log F)''(s) ds = \lim_{s \rightarrow +\infty} (\log F)'(s) - \lim_{s \rightarrow -\infty} (\log F)'(s) = c_+$) and

$$\int_{-\infty}^{a|X|} (\log F)''(s) \log \frac{4}{|s-\nu|} ds = \int_{-\infty}^{+\infty} (\log F)''(s) \log \frac{4}{|s-\nu|} ds + o(1) = O(1), \quad (4.43)$$

which, in turn, follow by the Lebesgue dominated convergence theorem, and a similar bound for $\int_{-\infty}^{|X|} (\log F)''(s) \log |s| ds$. Thus we obtain

$$\widehat{L}_{X,T}^\sigma(\underline{\nu}, \underline{\nu}) = \frac{|X|^{\frac{1}{2}}}{\pi T^{\frac{1}{2}}} F(\underline{\nu}) \left[1 + O\left(T^{\frac{1}{2}} |X|^{-\frac{1}{2}} \log |X|\right) \right],$$

and the thesis follows. \square

Remark 4.5. *It is straightforward to adapt the above proof in order to obtain asymptotics in the full region $X/T \rightarrow -\infty$, slightly larger than the region $\frac{X}{T \log^2 |X|} \rightarrow -\infty$. Note however that the error term will then no longer be small, and the asymptotic expression contains several terms, see (4.42) and (4.43). For the sake of simplicity, we present the results only as $\frac{X}{T \log^2 |X|} \rightarrow -\infty$.*

Using the two previous results, we can prove an important decorrelation property: since the matrix $\widehat{L}_{X,T}^\sigma(\underline{\nu}, \underline{\nu})$ is dominated by its diagonal, the m -point correlation function $\det \widehat{L}_{X,T}^\sigma(\underline{\nu}, \underline{\nu})$ decomposes at leading order into a product of one-point correlation functions. Similarly, its second logarithmic derivative decomposes at leading order into a sum of rapidly oscillating terms.

Proposition 4.6. *Let $T_0 > 0$, $\underline{\nu} \in \mathbb{R}^m$. We have*

$$\det \widehat{L}_{X,T}^\sigma(\underline{\nu}, \underline{\nu}) = (1 + O(|X|^{-\frac{1}{2}} T^{\frac{1}{2}})) \prod_{i=1}^m \widehat{L}_{X,T}^\sigma(\nu_i, \nu_i) \quad (4.44)$$

$$\partial_X^2 \log \det \widehat{L}_{X,T}^\sigma(\underline{\nu}, \underline{\nu}) = \frac{1}{\sqrt{|X|T}} \sum_{i=1}^m \cos\left(\frac{4|X|^{\frac{3}{2}}}{3T^{\frac{1}{2}}} (1 + A_{X,T}) - \frac{2|X|^{\frac{1}{2}}}{T^{\frac{1}{2}}} \nu_i (1 + B_{X,T}(\nu_i))\right) + O(|X|^{-1}) \quad (4.45)$$

uniformly for $T \geq T_0$ as $\frac{X}{T \log^2 |X|} \rightarrow -\infty$, where $A_{X,T}, B_{X,T}(\nu)$ converge to 0 as $\frac{X}{T \log^2 |X|} \rightarrow -\infty$.

Proof. For the ease of notation, let us denote $\widehat{L} = (\widehat{L}_{ij})$ for the $m \times m$ matrix with entries $\widehat{L}_{ij} := \widehat{L}_{X,T}^\sigma(\nu_i, \nu_j)$. By Lemma 4.3 and Lemma 4.4 we have

$$\widehat{L}_{ij} = \widehat{L}_{ii}(\delta_{ij} + O(|X|^{-\frac{1}{2}} T^{\frac{1}{2}})), \quad 1 \leq i, j \leq m. \quad (4.46)$$

Taking determinants we get (4.44). Moreover, (4.46) also implies

$$(\widehat{L}^{-1})_{ij} = \frac{1}{\widehat{L}_{ii}} (\delta_{ij} + O(|X|^{-\frac{1}{2}} T^{\frac{1}{2}})) = \frac{\delta_{ij}}{\widehat{L}_{ii}} + O(|X|^{-1} T), \quad 1 \leq i, j \leq m. \quad (4.47)$$

(In the second equality we use again Lemma 4.4.) By a direct computation, we have

$$\partial_X^2 \log \det \widehat{L} = \sum_{i,j=1}^m (\partial_X^2 \widehat{L}_{ij})(\widehat{L}^{-1})_{ji} - \sum_{i,j,k,l=1}^m (\partial_X \widehat{L}_{ij})(\widehat{L}^{-1})_{jk} (\partial_X \widehat{L}_{kl})(\widehat{L}^{-1})_{li}. \quad (4.48)$$

We have the relation $T^{\frac{1}{2}} \partial_X \widehat{L}_{ij} = -\widehat{\varphi}_i \widehat{\varphi}_j$, where we denote $\widehat{\varphi}_i := \widehat{\varphi}_\sigma(\nu_i; X, T | \emptyset)$. This is the analogue, for the full set of cKdV variables X, T , of the relation (3.43). As a consequence, $T^{\frac{1}{2}} \partial_X^2 \widehat{L}_{ij} = -(\partial_X \widehat{\varphi}_i) \widehat{\varphi}_j - \widehat{\varphi}_i (\partial_X \widehat{\varphi}_j)$. Hence, by (4.30), we have

$$\partial_X \widehat{L}_{ij} = O(|X|^{-\frac{1}{2}} T^{-\frac{1}{2}}), \quad \partial_X^2 \widehat{L}_{ij} = O(T^{-1}). \quad (4.49)$$

Therefore, by Lemma 4.4, (4.47), and (4.49), we have the estimates

$$\sum_{i,j=1}^m (\partial_X^2 \widehat{L}_{ij})(\widehat{L}^{-1})_{ji} = \sum_{i=1}^m \frac{\partial_X^2 \widehat{L}_{ii}}{\widehat{L}_{ii}} + O(|X|^{-1}), \quad \sum_{i,j,k,l=1}^m (\partial_X \widehat{L}_{ij})(\widehat{L}^{-1})_{jk} (\partial_X \widehat{L}_{kl})(\widehat{L}^{-1})_{li} = O(|X|^{-2}). \quad (4.50)$$

In the last one we combined (4.47) and Lemma 4.4 to get $(\widehat{L}^{-1})_{ij} = O(|X|^{-\frac{1}{2}}T^{\frac{1}{2}})$. Substituting these estimates into (4.48), we obtain

$$\partial_X^2 \log \det \widehat{L} = -\frac{2}{\sqrt{T}} \sum_{i=1}^m \frac{1}{\widehat{L}_{ii}} \widehat{\varphi}_i \partial_X \widehat{\varphi}_i + O(|X|^{-1}) = -\frac{1}{\pi T} \sum_{i=1}^m \frac{1}{\widehat{L}_{ii}} \left(\widehat{\Theta}(\nu_i) \mathbf{E}_{12} \widehat{\Theta}(\nu_i)^{-1} \right)_{2,2} + O(|X|^{-1}), \quad (4.51)$$

where the elementary unit matrix \mathbf{E}_{12} is defined in (3.45), and where we used (3.96).

Using (4.19), we can write after straightforward computations

$$\left(\widehat{\Theta}(\nu) \mathbf{E}_{12} \widehat{\Theta}(\nu)^{-1} \right)_{2,2} = \mathbf{v} B \mathbf{w}, \quad (4.52)$$

where, writing $\chi := |X|^{\frac{3}{2}} T^{-\frac{1}{2}}$ as before,

$$\mathbf{v} = (0, 1) R\left(\frac{\nu}{|X|}\right), \quad \mathbf{w} = R\left(\frac{\nu}{|X|}\right)^{-1} \begin{pmatrix} i|X|^{-\frac{1}{2}} \widehat{p} + i\chi g_1 \\ 1 \end{pmatrix}, \quad (4.53)$$

and,

$$B = \frac{e^{2\chi(g_+(\frac{\nu}{|X|}) - g_0)}}{i} \left(\frac{\nu}{|X|} - a\right)^{\frac{\sigma_3}{4}} G \begin{pmatrix} 1 & 0 \\ e^{-\chi\phi_+(\frac{\nu}{|X|})} & 1 \end{pmatrix} \mathbf{E}_{12} \begin{pmatrix} 1 & 0 \\ -e^{-\chi\phi_+(\frac{\nu}{|X|})} & 1 \end{pmatrix} G^{-1} \left(\frac{\nu}{|X|} - a\right)^{-\frac{\sigma_3}{4}}. \quad (4.54)$$

Using (4.21) and the asymptotic estimate (4.23), we obtain

$$\mathbf{v} = (O(\chi^{-1}), 1 + O(\chi^{-1})), \quad \mathbf{w} = \begin{pmatrix} O(\chi^{-1}) \\ 1 + O(\chi^{-1}) \end{pmatrix}, \quad \text{as } X/T \rightarrow -\infty, \quad (4.55)$$

and in particular as $\frac{X}{T \log^2 |X|} \rightarrow -\infty$. By (4.20), we can simplify the expression for B and obtain

$$\begin{aligned} B &= -iF(\nu) \left(\frac{\nu}{|X|} - a\right)^{\frac{\sigma_3}{4}} G \begin{pmatrix} -1 & e^{\chi\phi_+(\frac{\nu}{|X|})} \\ -e^{-\chi\phi_+(\frac{\nu}{|X|})} & 1 \end{pmatrix} G^{-1} \left(\frac{\nu}{|X|} - a\right)^{-\frac{\sigma_3}{4}} \\ &= \frac{F(\nu)}{2} \left(\frac{\nu}{|X|} - a\right)^{\frac{\sigma_3}{4}} \begin{pmatrix} e^{\chi\phi_+(\frac{\nu}{|X|})} + e^{-\chi\phi_+(\frac{\nu}{|X|})} & -2 - ie^{\chi\phi_+(\frac{\nu}{|X|})} + ie^{-\chi\phi_+(\frac{\nu}{|X|})} \\ 2 - ie^{\chi\phi_+(\frac{\nu}{|X|})} + ie^{-\chi\phi_+(\frac{\nu}{|X|})} & -e^{\chi\phi_+(\frac{\nu}{|X|})} - e^{-\chi\phi_+(\frac{\nu}{|X|})} \end{pmatrix} \left(\frac{\nu}{|X|} - a\right)^{-\frac{\sigma_3}{4}}. \end{aligned} \quad (4.56)$$

Since $\phi_+(\frac{\nu}{|X|})$ is purely imaginary and B is bounded and bounded away from 0, we have

$$B = \frac{1}{2} F(\nu) \begin{pmatrix} O(1) & O(1) \\ O(1) & -e^{\chi\phi_+(\frac{\nu}{|X|})} - e^{-\chi\phi_+(\frac{\nu}{|X|})} \end{pmatrix}. \quad (4.57)$$

Hence

$$\left(\widehat{\Theta}(\nu) \mathbf{E}_{12} \widehat{\Theta}(\nu)^{-1} \right)_{2,2} = \mathbf{v} B \mathbf{w} = -F(\nu) \cos\left(\chi|\phi_+(\frac{\nu}{|X|})|\right) + O(\chi^{-1}). \quad (4.58)$$

We finally obtain

$$\begin{aligned} \partial_X^2 \log \det \widehat{L} &= \frac{1}{\pi T} \sum_{i=1}^m \frac{F(\nu_i)}{\widehat{L}_{ii}} \cos\left(\chi|\phi_+(\frac{\nu_i}{|X|})|\right) + O(|X|^{-1}) \\ &= \frac{1}{\sqrt{|X|T}} \sum_{i=1}^m \cos\left(|X|^{\frac{3}{2}} T^{-\frac{1}{2}} |\phi_+(\frac{\nu_i}{|X|})|\right) + O(|X|^{-1}), \end{aligned} \quad (4.59)$$

where we used Lemma 4.4. It remains to compute

$$|\phi_+(\frac{\nu_i}{|X|})| = \int_{\nu_i/|X|}^a |(g_+ - g_-)'(s)| ds. \quad (4.60)$$

For this, we recall (4.37) and the estimates below that equation; in the same way as in the proof of Lemma 4.4, we then obtain

$$|\phi_+(0)| \rightarrow \frac{4}{3}, \quad |\phi_+(\frac{\nu_i}{|X|})| - |\phi_+(0)| \sim -2\frac{\nu_i}{|X|}, \quad (4.61)$$

as $\frac{X}{T \log^2 |X|} \rightarrow -\infty$. The argument of the cosine is thus equal to

$$\frac{4|X|^{\frac{3}{2}}}{3\sqrt{T}}(1 + A_{X,T}) - \frac{2\sqrt{|X|}}{\sqrt{T}}\nu_i(1 + B_{X,T}(\nu_i)), \quad (4.62)$$

with $A_{X,T} \rightarrow 0$, $B_{X,T}(\nu_i) \rightarrow 0$ as $\frac{X}{T \log^2 |X|} \rightarrow -\infty$, and the result follows. \square

Combining the above result with (4.4), and then substituting the asymptotics from Lemma 4.4, we complete the proof of part (ii) of Theorem IV.

With some more effort, we could obtain asymptotics in the slightly bigger asymptotic region where $X/T \rightarrow -\infty$, as already mentioned in Remark 4.5.

4.4 Intermediate regimes: $-KT \leq X \leq MT^{\frac{1}{3}}$

We will now discuss the asymptotic behavior as $T \rightarrow \infty$ of various relevant quantities in the intermediate regimes where $-KT \leq X \leq MT^{\frac{1}{3}}$ for sufficiently large constants $K, M > 0$. The asymptotics for the Jánossy density $J_\sigma(X, T|\underline{\nu})$ and the cKdV solution $V_\sigma(X, T|\underline{\nu})$ become, unfortunately, rather involved and implicit. In order to understand the mechanisms behind these asymptotics, an interesting and relevant object to consider, is the kernel $\widehat{L}_{X,T}^\sigma(\nu_1, \nu_2)$. Indeed, in view of the factorization (4.4), determinants of this kernel describe the effect of the points ν_1, \dots, ν_m on the Jánossy densities $J_\sigma(X, T|\underline{\nu})$, and the second logarithmic X -derivative of such determinants describe the effect of the points ν_1, \dots, ν_m on the cKdV solutions $V_\sigma(X, T|\underline{\nu})$. Recall that the kernel $\widehat{L}_{X,T}^\sigma(\nu_1, \nu_2)$ is expressed in terms of the RH solution $\widehat{\Theta}$ through (4.6). We distinguish three further asymptotic regimes.

Left-intermediate regime: $-KT \leq X \leq -K'T^{\frac{1}{2}}$ for any $K, K' > 0$. The asymptotic analysis of the RH problem for $\widehat{\Theta}$ has been carried through in [15] and is very similar to the one utilized for the left tail. However, there is an important difference in that the decorrelation property from Proposition 4.6 no longer holds. For that reason, even if we could obtain asymptotics for $\widehat{L}_{X,T}^\sigma(\nu_1, \nu_2)$, the explicit asymptotic behavior of the determinants $\det \widehat{L}_{X,T}^\sigma(\underline{\nu}, \underline{\nu})$ and their logarithmic derivatives becomes cumbersome for $m > 1$.

Right-intermediate regime: $-MT^{\frac{1}{3}} \leq X \leq MT^{\frac{1}{3}}$ for any $M > 0$. In this case, it was proved in [14, Theorem 1.15] that there exists a (sufficiently large) $T_0 > 0$ such that for all $M > 0$ we have

$$\log J_\sigma(X, T|\emptyset) = \log F_{\text{TW}}(XT^{-\frac{1}{3}}) + O(T^{-\frac{1}{6}}), \quad (4.63)$$

$$V_\sigma(X, T|\emptyset) = -T^{-\frac{2}{3}}y_{\text{HM}}(XT^{-\frac{1}{3}})^2 + O(T^{-1}), \quad (4.64)$$

uniformly for $|X| \leq MT^{\frac{1}{3}}$ and $T \geq T_0$, where y_{HM} is the Hastings–McLeod solution of the Painlevé II equation, and F_{TW} is the Tracy–Widom distribution (see also Example 1.3).

The asymptotic analysis of $\widehat{\Theta}$ has also been obtained in [14], and it implies that the leading order asymptotics of $\widehat{L}_{X,T}^\sigma(\lambda, \mu)$ are determined by the soft-to-hard edge transition kernel $L_s^{1(0,+\infty)}$ from Example 1.7, as we prove next.

Proposition 4.7. *Let $M > 0$. As $T \rightarrow \infty$, we have uniformly for $-MT^{\frac{1}{3}} \leq X \leq MT^{\frac{1}{3}}$, and uniformly for ν_1, ν_2 in compact subsets of the real line that*

$$\widehat{L}_{X,T}^\sigma(\nu_1, \nu_2) = T^{-\frac{1}{3}}L_{s=XT^{-\frac{1}{3}}}^{1(0,+\infty)}(T^{-\frac{1}{3}}\nu_1, T^{-\frac{1}{3}}\nu_2) + O(T^{-\frac{2}{3}}), \quad (4.65)$$

and this error estimate continues to hold upon differentiating an arbitrary number of times with respect to ν_1 and ν_2 .

Proof. The proof relies on the RH analysis performed in [14, Section 6]: the result is that, for every fixed $\nu \in \mathbb{C}$, we have the factorization

$$\widehat{\Theta}_\sigma(\nu; X, T|\emptyset) = \begin{pmatrix} 1 & \widehat{p}_\sigma \\ 0 & 1 \end{pmatrix} e^{\frac{i\pi}{4}\sigma_3} T^{\frac{1}{12}\sigma_3} S(\nu T^{-\frac{1}{3}}) \Psi_{1(0,+\infty)}(\nu T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|\emptyset) \begin{pmatrix} 1 & a(\nu T^{-\frac{1}{3}}) \\ 0 & 1 \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3}, \quad (4.66)$$

where $\widehat{p}_\sigma = \widehat{p}_\sigma(X, T|\emptyset)$, a has an explicit expression which is not needed for our purposes (cf. [14, equation (6.10)]). The matrix S should be interpreted as an error term: it satisfies a *small-norm* RH problem, which means that, provided $|w| < 1$, $S(w) = I + O(T^{-\frac{1}{3}})$ and $\partial_w S(w) = O(T^{-\frac{1}{3}})$, uniformly for $-MT^{\frac{1}{3}} \leq X \leq MT^{\frac{1}{3}}$ (for any $M > 0$). In particular, for every fixed $\nu_1, \nu_2 \in \mathbb{R}$ and T sufficiently large we also have

$$S(\nu_2 T^{-\frac{1}{3}})^{-1} S(\nu_1 T^{-\frac{1}{3}}) = I + O(T^{-\frac{2}{3}}(\nu_1 - \nu_2)). \quad (4.67)$$

Combining this with (4.6) and (4.66), we get

$$\begin{aligned} \widehat{L}_{X,T}^\sigma(\nu_1, \nu_2) &= \frac{(\Psi_{1(0,+\infty)}(\nu_2 T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|\emptyset)^{-1} S(\nu_2 T^{-\frac{1}{3}})^{-1} S(\nu_1 T^{-\frac{1}{3}}) \Psi_{1(0,+\infty)}(\nu_1 T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|\emptyset))_{2,1}}{2\pi i(\nu_1 - \nu_2)} \\ &= T^{-\frac{1}{3}} L_{s=XT^{-\frac{1}{3}}}^{1(0,+\infty)}(T^{-\frac{1}{3}}\nu_1, T^{-\frac{1}{3}}\nu_2) \\ &\quad + \frac{(\Psi_{1(0,+\infty)}(\nu_2 T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|\emptyset)^{-1} O(T^{-\frac{2}{3}}) \Psi_{1(0,+\infty)}(\nu_1 T^{-\frac{1}{3}}; XT^{-\frac{1}{3}}|\emptyset))_{2,1}}{2\pi i}, \end{aligned} \quad (4.68)$$

where we also use the identity

$$L_s^{1(0,+\infty)}\left(T^{-\frac{1}{3}}\nu_1, T^{-\frac{1}{3}}\nu_2\right) = \frac{\left(\Psi_{1(0,+\infty)}(\nu_2 T^{-\frac{1}{3}}; s|\emptyset)^{-1} \Psi_{1(0,+\infty)}(\nu_1 T^{-\frac{1}{3}}; s|\emptyset)\right)_{2,1}}{2\pi i T^{-\frac{1}{3}}(\nu_1 - \nu_2)}, \quad (4.69)$$

which is a special case of (3.19). The last term is $O(T^{-\frac{2}{3}})$ since it is a combination of the entries of the first column of $\Psi_{1(0,+\infty)}(\nu_i T^{-\frac{1}{3}})$, $i = 1, 2$, which are both entire. The above identities extend to ν_1, ν_2 in compact subsets of the complex plane, hence we can apply Cauchy's formula to differentiate, without affecting the error term. \square

A first, crucial, obstruction for obtaining explicit asymptotics for the Jánossy densities $J_\sigma(X, T|\underline{\nu})$ lies in the fact that the kernel $L_{s=XT^{-\frac{1}{3}}}^{1(0,+\infty)}$ is itself a transcendental object, which we cannot evaluate explicitly. However, we can proceed in the hope of describing $J_\sigma(X, T|\underline{\nu})$ asymptotically in terms of the σ -independent quantity $L_{s=XT^{-\frac{1}{3}}}^{1(0,+\infty)}$. For $m = 1$, we immediately find by (4.4) that

$$J_\sigma(X, T|\nu) \sim T^{-\frac{1}{3}} L_{s=XT^{-\frac{1}{3}}}^{1(0,+\infty)}(0, 0) J_\sigma(X, T|\emptyset), \quad (4.70)$$

where the asymptotics for $J_\sigma(X, T|\emptyset)$ are given by (4.63). For $m > 1$, we can estimate $J_\sigma(X, T|\underline{\nu})$ as follows.

Proposition 4.8. *Let $M > 0$, $\underline{\nu} = (\nu_1, \dots, \nu_m) \in \mathbb{R}^m$, and $\nu_i \neq \nu_j$ for $i \neq j$. As $T \rightarrow \infty$, we have uniformly for $-MT^{\frac{1}{3}} \leq X \leq MT^{\frac{1}{3}}$ that*

$$J_\sigma(X, T|\underline{\nu}) \sim C_m T^{-\frac{m^2}{3}} \prod_{1 \leq j < k \leq m} (\nu_k - \nu_j)^2 J_{1(0,+\infty)}(X, T|\emptyset), \quad (4.71)$$

for some constant $C_m > 0$ possibly depending on m but not on $\sigma, \underline{\nu}, X, T$.

Proof. Let us abbreviate $L = \widehat{L}_{X,T}^\sigma$ and $\widetilde{L} = L_{s=XT^{-\frac{1}{3}}}^{1(0,+\infty)}$. By (4.4), we have

$$J_\sigma(X, T|\underline{\nu}) = \det(L(\nu_i, \nu_j))_{i,j=1}^m J_\sigma(X, T|\emptyset), \quad (4.72)$$

hence by (4.63), it remains to prove that

$$\det(L(\nu_i, \nu_j))_{i,j=1}^m \sim C_m T^{-\frac{m^2}{3}} \prod_{1 \leq j < k \leq m} (\nu_k - \nu_j)^2, \quad (4.73)$$

in the relevant limit. Since $L(\cdot, \cdot)$ is entire in its variables, we have

$$L(\nu_i, \nu_j) = \sum_{a,b \geq 0} L^{(a,b)}(0,0) \frac{\nu_i^a \nu_j^b}{a!b!} \quad (4.74)$$

hence

$$\begin{aligned} & \begin{pmatrix} L(\nu_1, \nu_1) & L(\nu_1, \nu_2) & \cdots & L(\nu_1, \nu_m) \\ L(\nu_2, \nu_1) & L(\nu_2, \nu_2) & \cdots & L(\nu_2, \nu_m) \\ \vdots & \vdots & \ddots & \vdots \\ L(\nu_m, \nu_1) & L(\nu_m, \nu_2) & \cdots & L(\nu_m, \nu_m) \end{pmatrix} \\ &= \begin{pmatrix} 1 & \nu_1 & \frac{\nu_1^2}{2!} & \cdots \\ 1 & \nu_2 & \frac{\nu_2^2}{2!} & \cdots \\ \vdots & \vdots & \cdots & \cdots \\ 1 & \nu_m & \frac{\nu_m^2}{2!} & \cdots \end{pmatrix} \begin{pmatrix} L^{(0,0)}(0,0) & L^{(1,0)}(0,0) & L^{(2,0)}(0,0) & \cdots \\ L^{(0,1)}(0,0) & L^{(1,1)}(0,0) & L^{(2,1)}(0,0) & \cdots \\ L^{(0,2)}(0,0) & L^{(1,2)}(0,0) & L^{(2,2)}(0,0) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \nu_1 & \nu_2 & \cdots & \nu_m \\ \frac{\nu_1^2}{2!} & \frac{\nu_2^2}{2!} & \cdots & \frac{\nu_m^2}{2!} \\ \vdots & \vdots & \cdots & \cdots \end{pmatrix}. \end{aligned} \quad (4.75)$$

Then we use Proposition 4.7 to obtain

$$L^{(i-1, j-1)}(0,0) \sim T^{-\frac{i+j-1}{3}} \widetilde{L}^{(i-1, j-1)}(0,0), \quad \text{as } T \rightarrow \infty, |XT^{-\frac{1}{3}}| \leq M. \quad (4.76)$$

Expanding (4.75) by the Binet–Cauchy identity, we immediately see that the leading order as $T \rightarrow \infty$ is given by

$$\begin{aligned} & \det_{i,j=1}^m \left(\frac{\nu_i^{j-1}}{(j-1)!} \right) \det_{i,j=1}^m \left(T^{-\frac{i+j-1}{3}} L^{(i-1, j-1)}(0,0) \right) \det_{i,j=1}^m \left(\frac{\nu_i^{j-1}}{(j-1)!} \right) \\ &= T^{-\frac{m^2}{3}} \prod_{k=1}^{m-1} \frac{1}{k!^2} \det_{i,j=1}^m \left(\widetilde{L}^{(i-1, j-1)}(0,0) \right) \det_{i,j=1}^m \left(\nu_i^{j-1} \right)^2. \end{aligned} \quad (4.77)$$

In the latter, we recognize the Vandermonde determinant, and the result follows. \square

Middle-intermediate regime: $-K'T^{\frac{1}{2}} \leq X \leq -MT^{\frac{1}{3}}$ for some $K', M > 0$. Here, the asymptotic analysis of the RH problem for $\widehat{\Theta}$ has also been completed in [14], but the asymptotics are implicit and described in terms of the solution of an integro-differential generalization of the fifth Painlevé equation.

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