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# Local discontinuous Galerkin schemes for an ultrasonic propagation equation with fractional attenuation <sup>\*</sup>

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## Abstract

The goal of this article is to develop local discontinuous Galerkin (LDG) schemes for solving a time fractional equation describing the ultrasonic wave in a homogeneous isotropic porous material. Two novel semi-discrete LDG schemes are designed for the considered model. The semi-discrete LDG schemes are constructed by splitting the original model into a coupled system. The first semi-discrete scheme follows the traditional LDG method by splitting second-order space derivative. The second one splits the original model for both time and space derivatives. The discontinuous Galerkin is used for the spatial discretization. Two kinds of fully discrete LDG schemes are presented by using the Grünwald-Letnikov and L1 approximation formulas for the time fractional derivatives. The  $L^2$  norm stability and convergence analysis are carried out for both semi-discrete and fully discrete LDG schemes. The stability analysis reveals that the numerical schemes are unconditionally stable in  $L^2$  norm and convergence with optimal convergence rate. Finally, numerical examples are presented to test the effectiveness of the proposed schemes and the correctness of the theoretical analysis.

**Keywords:** local discontinuous Galerkin methods, ultrasonic propagation equation, stability, convergence.

## 1 Introduction

It is known that the wave equations play an important role in describing wave propagation phenomena in various kinds of materials. Different kinds of wave equations are frequently constructed in different physical situation, especially, in the sound propagation in the porous materials and soft matter. To well understand the wave propagation in this kind of material, many quantities are taken into account [1, 4, 17, 19]. Two important factors refer to the attenuation and dispersion of the medium. Many experiments show that different factors can effect the attenuation of acoustic wave propagation in porous materials, including diffusion, scattering and absorption (e.g., viscous and relaxation). In solid-fluid media, to describe the interactions between the fluid and the structure, there are usually need consider two frequency dependent response factors which appear in the attenuation functions [19, 15]. These functions are derived from the behavior of the fluid in the free space as the frequency increases. For the components of the high frequencies, these factors appear in the form of temporal convolution operators and their asymptotic expressions are given by a fractional derivative in the time domain. The corresponding models involve the fractional derivative which can rationally explain the relaxation and frequency dependence of waves. It is widely used in describing wave propagation of porous materials, including air-saturated porous media and fluid-saturated porous media. The pressure of ultrasonic in a dispersive medium with a rigid frame can be depicted by the following ultrasonic propagation equation [12, 13, 14]

$$Au_{tt}(x, t) + B_0^C D_t^{3/2} u(x, t) + Cu_t(x, t) = u_{xx}(x, t), \quad (1)$$

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where the coefficients  $A = \frac{1}{c^2}$ ,  $B = \frac{\alpha_\infty \sqrt{\rho_f \eta}}{K_a} \left( \frac{1}{\Lambda} + \frac{\gamma-1}{\sqrt{Pr\Lambda'}} \right)$ ,  $C = \frac{4\alpha_\infty(\gamma-1)\eta}{K_a \Lambda \Lambda' \sqrt{Pr}}$ , and  ${}_0^C D_t^{3/2}$  denotes the fractional operator

$${}_0^C D_t^{3/2} u(x, t) = \frac{1}{\Gamma(1/2)} \int_0^t \frac{u_{ss}(x, s)}{\sqrt{t-s}} ds.$$

In time-domain ultrasonic propagation model (1), the parameter  $\rho_f$  denotes the frame of density,  $\eta$  denotes the frame of density,  $\gamma$  denotes the adiabatic constant,  $Pr$  denotes the Prandtl number,  $\alpha_\infty$  denotes the tortuosity,  $\Lambda$  and  $\Lambda'$  are the characteristic lengths, and  $K_a$  expresses the bulk modulus of air. The time domain model (1) is derived by Fellah et al. in [12, 13, 14] based on the two frequency response factors suggested by reference [19]. For its derivation and the analytical solution of model (1) with special initial values, see [12, 13, 14].

Various of time fractional wave models have been derived and they have been widely used in many fields, including geoscience, biology, rock and soil mechanics, see [12, 33, 23, 6, 41, 15, 35, 17] and references therein. Numerical methods such as finite difference methods [34, 42, 28, 18, 27], finite element methods [25, 20, 36] and spectral methods [30] have been developed for the numerical solutions of time fractional diffusion-wave models. So far, only a few efficient numerical investigation have been devoted to simulate the fractional wave propagation models. For example, by using the Grünwald-Letnikov formula for time fractional derivative and central-difference approximations for space variable, Carcione et al. [6] investigate the numerical results of Kjartansson's model. However, the detailed error estimates of the proposed algorithms are not given. Wilson et al. [46] developed a time-domain formulation for sound propagation in rigid-frame porous media, including waveform attenuation and dispersion, and discussed the dispersive and attenuative nature of sound propagation by finite-difference time-domain technique.

As one of high order numerical methods for fractional differential equations, the discontinuous Galerkin (DG) method has been applied to solve the many kinds of fractional differential equations [32, 44, 29, 21, 8, 10, 3, 26, 39, 48, 49]. In the meantime, different kinds of DG methods are developed, see [5, 7, 16, 45, 40]. The use of these new DG methods for solving ultrasonic propagation equation (1) has not been addressed. In this paper, we try to solve ultrasonic propagation equation (1) in time domain by using local DG methods developed by Cockburn and Shu [7]. Different from the existing works, the local DG schemes are presented by applying the characteristic of fractional derivative. The integral and differential characteristics of Caputo fractional derivative ensure we can split the original model into a coupled system of equations. The coupled system contains two coupled low order equations which are similar to the Euler and constitutive equations [1]. The semi-discrete and fully discrete local DG methods are designed for the coupled system. The time fractional derivative in the new local DG formulation is approximated by both the L1 approximation and shift weighted Grünwald-Letnikov formulas in the present fully discrete schemes.

The outline of this article is as follows. In section 2, the stability of an initial boundary value problem of fractional ultrasonic propagation equation is discussed. In Section 3, two kinds of semi-discrete local DG methods are constructed for the considered model. And the stability and error estimate of the semi-discrete numerical schemes are established. In Section 4, the two fully discrete local DG schemes for equation (1) are established. The detailed theoretical analysis are provided for the present full-discrete local DG schemes. Finally, in Section 5, some numerical examples that confirm the theoretical analysis are presented.

## 2 Stability of initial-boundary value problem

We begin by discussing the stability of for the considered model (1) subjects to the corresponding initial and boundary conditions. Without loss of generality, we consider the initial-boundary value problem

$$\begin{cases} u_{tt} + {}_0^C D_t^{3/2} u + u_t = u_{xx} + f, x \in \Omega, t \in (0, T], \\ u(x, 0) = \phi(x), u_t(x, 0) = \varphi(x), x \in \Omega, \\ u(x, t) = u(x + L, t), x \in \Omega, \end{cases} \quad (2)$$

where  $\Omega = (0, L)$  is a bounded open subinterval of  $\mathbb{R}$ , and  $f = f(x, t)$  is a known function. Let us now define some suitable functional spaces which will be used in our theory analysis. Throughout this paper, we denote  $L^p(\Omega)$  the Lebesgue space of  $p$ -integrable functions on  $\Omega$ , equipped with the

norm  $\|u\|_{L^p(\Omega)} = (\int_{\Omega} |u|^p dx)^{1/p}$ . Particularly, for  $p = 2$ , we denote the inner product and the associated norm of  $L^2(\Omega)$  by

$$(u(x), v(x)) = \int_{\Omega} u(x)v(x)dx, \quad \|u\| = (u(x), u(x))^{1/2}.$$

Denote  $H^m(\Omega)$  be the classic Sobolev spaces equipped with related semi-norms and norms [11]. Let  $X$  be a real Banach function space, the space  $L^p(0, T; X)$  and  $H^1(0, T; X)$  denote the standard Sobolev spaces, see [11].

**Definition 1.** Let  $v(t)$  be an absolutely integrable function defined on  $[0, t]$  and  $\sigma \geq 0$ . The left Riemann-Liouville fractional integral of function  $v(t)$  is defined by

$${}_0I_t^\sigma v(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} v(s) ds,$$

where  $\Gamma(\cdot)$  denotes the Gamma function.

**Definition 2.** [38] For a positive integer  $n$ , let  $v(t) \in C^n[0, t]$  and  $\alpha \in (n-1, n)$ . The left Caputo fractional derivative is defined by

$${}_0^C D_t^\alpha v(t) = {}_0I_t^{(n-\alpha)}(v^{(n)}(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} v^{(n)}(s) ds.$$

**Lemma 1.** [2] For function  $v(t)$  absolutely continuous on  $[0, t]$ , holds the following inequality

$$\frac{1}{2} {}_0^C D_t^\alpha (v^2(t)) \leq v(t) {}_0^C D_t^\alpha (v(t)), \quad 0 < \alpha < 1. \quad (3)$$

For the initial-boundary value problem (2), we have the following stability

**Theorem 1.** Let  $u(x, t) \in L^2(0, T; H^1(\Omega))$  be the solution of problem (2), for  $t \in [0, T]$ , there holds the prior estimate

$$\|u_t\|^2 + \|u_x\|^2 \leq e^t (\|\varphi\|^2 + \|\phi_x\|^2 + \frac{t^{\frac{1}{2}} \|\varphi\|^2}{\Gamma(3/2)}) + \int_0^t \|f(s)\| ds. \quad (4)$$

*Proof.* By taking the inner product of (2) with  $2u_t$ , we have

$$2(u_{tt} + \delta_0^C D_t^{3/2} u + u_t - u_{xx}, u_t) = 2(f, u_t).$$

Recalling

$$(u_{tt}, u_t) = \frac{1}{2} \frac{d}{dt} \|u_t\|^2, \quad (u_x, u_{xt}) = \frac{1}{2} \frac{d}{dt} \|u_x\|^2.$$

we deduce that

$$\frac{d}{dt} (\|u_t\|^2 + \|u_x\|^2) + 2({}_0^C D_t^{3/2} u, u_t) + 2\|u_t\|^2 = 2(f, u_t).$$

By using Lemma 1, we can check that

$$({}_0^C D_t^{3/2} u, u_t) = ({}_0^C D_t^{1/2} u_t, u_t) \geq \frac{1}{2} {}_0^C D_t^{1/2} \|u_t\|^2. \quad (5)$$

Applying (5) and the Cauchy-Schwarz inequality, we get

$$\frac{dy(t)}{dt} + {}_0^C D_t^{1/2} (\|u_t\|^2) \leq 2\|f(t)\| \|u_t\| \leq \|f(t)\|^2 + \|u_t\|^2,$$

where  $y(t) = \|u_t\|^2 + \|u_x\|^2$ . In view of  $\|u_t\|^2 \leq \|u_t\|^2 + \|u_x\|^2$ , we obtain

$$\frac{dy(t)}{dt} + {}_0^C D_t^{1/2} (\|u_t\|^2) \leq y(t) + \|f(t)\|^2,$$

for  $t \in [0, T]$ . Furthermore, integrating above inequality from 0 to  $t$ ,

$$y(t) + \int_0^t {}_0^C D_s^{1/2} \|u_s\|^2 ds \leq y(0) + \int_0^t y(s) ds + \int_0^t \|f(s)\|^2 ds,$$

Using the composite property of the Caputo fractional derivative and integral of  $v(t)$  [38, 50]

$$\int_0^t {}_0^C D_s^\alpha v(s) ds = -\frac{t^{1-\alpha} v(0)}{\Gamma(2-\alpha)} + {}_0 I_t^{1-\alpha} v(t), \quad 0 < \alpha < 1. \quad (6)$$

we get

$$y(t) + {}_0 I_t^{\frac{1}{2}} (\|u_t\|^2) \leq y(0) + \frac{t^{\frac{1}{2}} \|\varphi\|^2}{\Gamma(3/2)} + \int_0^t y(s) ds + \int_0^t \|f(s)\|^2 ds.$$

Moreover, using the positivity of  ${}_0 I_t^{\frac{1}{2}} (\|u_t\|^2)$  and the Gronwall's inequality [37, 11], we have (4).  $\square$

### 3 The semi-discrete local DG schemes

Now, we present and analyze a local discontinuous Galerkin method for problem (2). We use the notations used in reference [7]. We first divide the domain  $\Omega = [0, L]$  into  $N$  cells  $0 = x_{1/2} < x_{3/2} < \dots < x_{N-1/2} < x_{N+1/2} = L$ . We define the mesh  $\mathcal{T}_h = \{I_j = (x_{j-1/2}, x_{j+1/2}), j = 1, 2, \dots, N\}$  with interval cell  $I_j = (x_{j-1/2}, x_{j+1/2})$ ,  $x_j = (x_{j-1/2} + x_{j+1/2})/2$ , and step  $h_j = x_{j+1/2} - x_{j-1/2}$ ,  $h = \max_{1 \leq j \leq N} h_j$ . Furthermore, the finite element space is defined by

$$V_h^k = \{v_h(x) : v_h(x)|_{I_j} \in P^k(I_j), j = 1, 2, \dots, N\},$$

where  $P^k(I_j)$  denotes the set of all polynomials of degree at most  $k$  on cell  $I_j$ . Let  $v|_{j+1/2}$  denotes the values of the function  $v = v(x)$  at  $x = x_{j+1/2}$ , and  $v_{j+1/2}^-, v_{j+1/2}^+$  denote the left limit and the right limit of the function  $v$  at the discontinuity point  $x_{j+1/2}$ , i.e.,  $v_{j+1/2}^- = v^-(x_{j+1/2}) = \lim_{s \rightarrow 0^-} v(x_{j+1/2} + s)$ , and  $v_{j+1/2}^+ = v^+(x_{j+1/2}) = \lim_{s \rightarrow 0^+} v(x_{j+1/2} + s)$ .

To define the local DG method, we rewrite the fractional wave equation given in (2) as the following coupled system

$$\begin{aligned} u_{tt} + {}_0^C D_t^{3/2} u + u_t &= p_x + f, \\ p - u_x &= 0. \end{aligned} \quad (7)$$

The local DG method to the coupled system (7) gives: find  $(u_h, p_h) \in H^1(0, T; V_h^k) \times L^2(0, T; V_h^k)$  such that

$$\begin{aligned} &((u_h)_{tt}, v_h)_{I_j} + ({}_0^C D_t^{3/2} u_h, v_h)_{I_j} + ((u_h)_t, v_h)_{I_j} + (p_h, (v_h)_x)_{I_j} \\ &- (\widehat{p}_h v_h^-)_{j+\frac{1}{2}} + (\widehat{p}_h v_h^+)_{j-\frac{1}{2}} = (f_h, v_h)_{I_j}, \\ &(p_h, w_h)_{I_j} + (u_h, (w_h)_x)_{I_j} - (\widehat{u}_h w_h^-)_{j+\frac{1}{2}} + (\widehat{u}_h w_h^+)_{j-\frac{1}{2}} = 0, \\ &(u(x, 0), v_h)_{I_j} = (\phi_h, v_h)_{I_j}, (u_t(x, 0), v_h)_{I_j} = (\varphi_h, v_h)_{I_j}, \end{aligned} \quad (8)$$

for all  $(v_h, w_h) \in H^1(0, T; V_h^k) \times L^2(0, T; V_h^k)$ . The initial values  $u_h(x, 0) = \phi_h(x) \in V_h^k$  and  $(u_h)_t(x, 0) = \varphi_h(x) \in V_h^k$  are approached by special projections of the exact initial conditions  $u(x, 0)$  and  $u_t(x, 0)$ . This particular projection will be introduced later. The 'hat' terms in (8) are the numerical probability density fluxes. We chose the alternating numerical fluxes [22]

$$\widehat{u}_h = u_h^-, \widehat{p}_h = p_h^+. \quad (9)$$

Next, we present the stability and convergence analysis for the semi-discrete scheme (8) in  $L^2$ -norm sense. To do so, we follow the technique used by Cockburn and Shu [7]. In what follows, as the distinction between inner product  $(\cdot, \cdot)$ , we denote the inner product defined on cell  $I_j$  as  $(\cdot, \cdot)_{I_j}$ .

**Theorem 2.** *The semi-discrete local DG scheme (8) with the numerical fluxes (9) is  $L^2$ -stable, i.e.*

$$\|(u_h)_t\|^2 + \|p_h\|^2 \leq e^t (\|\varphi_h\|^2 + \|(\phi_h)_x\|^2) + \frac{t^{\frac{1}{2}} \|\varphi_h\|^2}{\Gamma(3/2)} + \int_0^t \|f_h(s)\|^2 ds. \quad (10)$$

*Proof.* Taking the time derivative of the first equation in scheme (8), we denote

$$\begin{aligned} B_j(u_h, p_h; v_h, w_h) := &((u_h)_{tt}, v_h)_{I_j} + ({}_0^C D_t^{3/2} u_h, v_h)_{I_j} + ((u_h)_t, v_h)_{I_j} \\ &+ (p_h, (v_h)_x)_{I_j} - [(\widehat{p}_h v_h^-)_{j+\frac{1}{2}} - (\widehat{p}_h v_h^+)_{j-\frac{1}{2}}] \\ &+ ((u_h)_t, (w_h)_x)_{I_j} + ((p_h)_t, w_h)_{I_j} \\ &- [((\widehat{u}_h)_t w_h^-)_{j+\frac{1}{2}} - ((\widehat{p}_h)_t w_h^+)_{j-\frac{1}{2}}]. \end{aligned}$$

Then, scheme (8) can be rewritten as

$$B_j(u_h, p_h; v_h, w_h) = (f_h, v_h)_{I_j}. \quad (11)$$

Taking  $v_h = (u_h)_t, w_h = p_h$  in (11), we have

$$\begin{aligned} B_j(u_h, p_h; (u_h)_t, p_h) &= ((u_h)_{tt}, (u_h)_t)_{I_j} + ({}_0^C D_t^{3/2} u_h, (u_h)_t)_{I_j} + ((u_h)_t, (u_h)_t)_{I_j} \\ &\quad + (p_h, (u_h)_{tx})_{I_j} - [(\widehat{p}_h(u_h)_t^-)_{j+\frac{1}{2}} - (\widehat{p}_h(u_h)_t^+)_{j-\frac{1}{2}}] \\ &\quad + ((u_h)_t, (p_h)_x)_{I_j} + ((p_h)_t, p_h)_{I_j} \\ &\quad - [(\widehat{u}_h t p_h^-)_{j+\frac{1}{2}} - ((\widehat{u}_h)_t p_h^+)_{j-\frac{1}{2}}] \\ &= (f_h, (u_h)_t)_{I_j}. \end{aligned}$$

Summing index  $j$  over to  $N$ , and combining the numerical flux defined by (9), we have

$$\begin{aligned} B(u_h, p_h; (u_h)_t, p_h) &= \sum_{j=1}^N B_j(u_h, p_h; (u_h)_t, p_h) \\ &= ((u_h)_{tt}, (u_h)_t) + ({}_0^C D_t^{3/2} u_h, (u_h)_t) + ((u_h)_t, (u_h)_t) + ((p_h)_t, p_h) \\ &\quad - [(p_h^+(u_h)_t^-)_{N+\frac{1}{2}} - (p_h^+(u_h)_t^-)_{\frac{1}{2}}] \\ &= (f_h, (u_h)_t). \end{aligned}$$

Using the periodic boundary condition given in (2) we have that the semi-discrete LDG scheme (8) with the flux  $\widehat{u}_h, \widehat{p}_h$  defined in (9), satisfies

$$((u_h)_{tt}, (u_h)_t) + ({}_0^C D_t^{3/2} u_h, (u_h)_t) + ((u_h)_t, (u_h)_t) + ((p_h)_t, p_h) = (f_h, (u_h)_t). \quad (12)$$

or

$$\frac{1}{2} \frac{d}{dt} (\|(u_h)_t\|^2 + \|p_h\|^2) + ({}_0^C D_t^{1/2} ((u_h)_t), (u_h)_t) + \|(u_h)_t\|^2 = (f_h, (u_h)_t). \quad (13)$$

Finally, using the Cauchy-Schwarz inequality, Lemma 1, and relation (6), with the similar argument given in Theorem , we arrive at (10).  $\square$

Remembering the definition of Caputo fractional derivative given in Definition 2, we observe that

$${}_0^C D_t^\alpha u(t) = {}_0 I_t^{1-\alpha}(u_t), \quad 0 < \alpha < 1,$$

and

$${}_0^C D_t^\alpha u(t) = {}_0^C D_t^{1-\alpha}(u_t) = {}_0 I_t^{2-\alpha}(u_{tt}), \quad 1 < \alpha < 2. \quad (14)$$

Thus, we can also rewrite model (2) as the following coupled system

$$\begin{aligned} p_t + {}_0^C D_t^{1/2} p + p &= u_x + g, \\ u_t - p_x &= 0, \end{aligned} \quad (15)$$

where  $g(x, t) = \int_0^x f(x, t) dx$ . In the formwork of (15), we get a new local discontinuous Galerkin method of the system (15), gives: find  $(u_h, p_h) \in H^1(0, T; V_h^k) \times L^2(0, T; V_h^k)$ , for all  $(v_h, w_h) \in H^1(0, T; V_h^k) \times L^2(0, T; V_h^k)$ , such that

$$\begin{aligned} &((p_h)_t, w_h)_{I_j} + ({}_0^C D_t^{1/2} p_h, w_h)_{I_j} + (p_h, w_h)_{I_j} + (u_h, (w_h)_x)_{I_j} \\ &- (\widehat{u}_h w_h^-)_{j+\frac{1}{2}} + (\widehat{u}_h w_h^+)_{j-\frac{1}{2}} = (g, w_h)_{I_j}, \\ &((u_h)_t, v_h)_{I_j} + (p_h, (v_h)_x)_{I_j} - (\widehat{p}_h v_h^-)_{j+\frac{1}{2}} + (\widehat{p}_h v_h^+)_{j-\frac{1}{2}} = 0, \\ &(u(x, 0), v_h)_{I_j} = (\phi_h, v_h)_{I_j}, (u_t(x, 0), v_h)_{I_j} = (\varphi_h, v_h)_{I_j}. \end{aligned} \quad (16)$$

For the initial conditions  $u_h(x, 0) = \phi_h(x) \in V_h^k$  and  $(u_h)_t(x, 0) = \varphi_h(x) \in V_h^k$ , we use a special projection of the exact initial conditions  $u(x, 0)$  and  $u_t(x, 0)$  which is the same as DG scheme (8). In addition, we also chose (9) as the numerical fluxes.

**Remark 1.** Compared with the local DG procedure (8), we need the regularity assumption  $u_{tx} = u_{xt}$  in semi-discrete local DG scheme (16). The local DG procedure (16) works if provides the initial value of  $p_h(x, 0)$ , in actual calculations, we can use the second formula given in (15) to get  $p_h(x, 0)$ , i.e.,  $p_h(x, 0) = \int_0^x (u_h)_t(y, 0) dy$ .

Similar to the semi-discrete LDG scheme (8), we can prove the stability of numerical scheme (16).

**Theorem 3.** ( $L^2$ -stability) The semi-discrete LDG scheme (16) with the flux choice (9) is  $L^2$ -stable, i.e.

$$\|u_h\|^2 + \|p_h\|^2 \leq \|\phi_h\|^2 + \|p_h(0)\|^2 + \int_0^t \|g_h\|^2 ds. \quad (17)$$

*Proof.* We first prove that the semi-discrete LDG scheme (16) with the flux  $\hat{u}_h, \hat{p}_h$  defined in (9), holds

$$((u_h)_t, u_h) + ((p_h)_t, p_h) + ({}_0^C D_t^{1/2}(p_h), p_h) + (p_h, p_h) = (g_h, p_h). \quad (18)$$

For simplicity, we denote

$$\begin{aligned} \mathcal{B}_j(u_h, p_h; v_h, w_h) := & ((u_h)_t, v_h)_{I_j} + (p_h, (v_h)_x)_{I_j} \\ & - [(\hat{p}_h)_{j+\frac{1}{2}}(v_h)_{j+\frac{1}{2}}^- - (\hat{p}_h v_h^+)_{j-\frac{1}{2}}] \\ & + ((p_h)_t, w_h)_{I_j} + ({}_0^C D_t^{1/2} p_h, w_h)_{I_j} + (p_h, w_h)_{I_j} \\ & + (u_h, (w_h)_x)_{I_j} - [(\hat{u}_h w_h^-)_{j+\frac{1}{2}} - (\hat{u}_h w_h^+)_{j-\frac{1}{2}}]. \end{aligned} \quad (19)$$

Then, scheme (16) can be rewritten as

$$\mathcal{B}_j(u_h, p_h; v_h, w_h) = (g_h, w_h)_{I_j}. \quad (20)$$

Taking  $v_h = u_h, w_h = p_h$  in (20) and using the numerical flux defined by (9), we have

$$\begin{aligned} \mathcal{B}_j(u_h, p_h; u_h, p_h) = & ((u_h)_t, u_h)_{I_j} + (p_h, (u_h)_x)_{I_j} \\ & - [(p_h^+ u_h^-)_{j+\frac{1}{2}} - (p_h^+ u_h^+)_{j-\frac{1}{2}}] \\ & + ((p_h)_t, p_h)_{I_j} + ({}_0^C D_t^{1/2} p_h, p_h)_{I_j} + (p_h, p_h)_{I_j} \\ & + (u_h, (p_h)_x)_{I_j} - [(u_h^- p_h^-)_{j+\frac{1}{2}} - (u_h^- p_h^+)_{j-\frac{1}{2}}] \\ = & (g_h, p_h)_{I_j}. \end{aligned}$$

Summing over all  $j$ , we obtain

$$\begin{aligned} \mathcal{B}(u_h, p_h; u_h, p_h) &= \sum_{j=1}^N \mathcal{B}_j(u_h, p_h; u_h, p_h) \\ &= ((u_h)_t, u_h) + ((p_h)_t, p_h) + ({}_0^C D_t^{1/2}(p_h), p_h) + (p_h, p_h) \\ &\quad - [(p_h^+(u_h)_t^-)_{N+\frac{1}{2}} - (p_h^+(u_h)_t^-)_{\frac{1}{2}}] \\ &= (g_h, p_h)_{I_j}. \end{aligned}$$

Using the periodic boundary condition given in (2), we arrive at (18) which equals to

$$\frac{1}{2} \frac{d}{dt} (\|u_h\|^2 + \|p_h\|^2) + ({}_0^C D_t^{1/2}(p_h), p_h) + \|p_h\|^2 = (g_h, p_h).$$

Moreover, using the Cauchy-Schwarz inequality and Lemma 1, we get

$$\frac{1}{2} \frac{d}{dt} (\|u_h\|^2 + \|p_h\|^2) + {}_0^C D_t^{1/2} (\|p_h\|^2) + \frac{1}{2} \|p_h\|^2 \leq \frac{1}{2} \|g_h\|^2. \quad (21)$$

Using relation (6), integrating inequality (21) from 0 to  $t$ , we have

$$\|u_h\|^2 + \|p_h\|^2 + 2{}_0 I_t^{\frac{1}{2}} (\|p_h\|^2) + \int_0^t \|p_h\|^2 ds \leq \|\phi_h\|^2 + \|p_h(0)\|^2 + \int_0^t \|g_h\|^2 ds. \quad (22)$$

Dropping the positive terms in (22), we arrive at (17).  $\square$

**Remark 2.** Particularly, if  $A = 1, B = 0, C = 0$  in model (1), we can get the prior estimate

$$\|u_h\|^2 + \|p_h\|^2 = \|u_h(0)\|^2 + \|p_h(0)\|^2,$$

which means the semi-discrete method (16) is the energy conservation which is consistent with reference [47].

Now, we given the convergence of above semi-discrete schemes in  $L^2$  norm. In order to give the error estimates of present numerical schemes, we introduce the standard  $L^2$  projection operator

$$(\mathbb{P}_h w(x) - w(x), v(x))_{I_j} = 0, \forall v \in P^k(I_j), \quad (23)$$

and the following two special projections operators [7]

$$(\mathbb{P}_h^- w(x) - w(x), v(x))_{I_j} = 0, \forall v(x) \in P^{k-1}(I_j), \mathbb{P}_h^- w(x_{j+1/2}^-) = w(x_{j+1/2}), \quad (24)$$

$$(\mathbb{P}_h^+ w(x) - w(x), v(x))_{I_j} = 0, \forall v(x) \in P^{k-1}(I_j), \mathbb{P}_h^+ w(x_{j-1/2}^+) = w(x_{j-1/2}). \quad (25)$$

**Lemma 2.** [7, 16, 45] For projection operators  $\mathbb{P}_h^\pm$ , the following estimate holds

$$\|w - \pi_h w\| + h\|w - \pi_h w\|_\infty + h^{\frac{1}{2}}\|w - \pi_h w\|_{\Gamma_h} \leq Ch^{k+1}\|v\|_{H^{k+1}}, \quad (26)$$

where  $\pi_h = \mathbb{P}_h^-$  or  $\mathbb{P}_h^+$ ,  $C$  is a positive constant depending  $u$  and its derivatives but independent of  $h$ , and  $\Gamma_h$  denotes the set of boundary points of all cells.

To analysis the error estimates of the local DG schemes, we divide the error terms  $e$  and  $\bar{e}$  into two parts

$$\begin{aligned} e &= u - u_h = (\mathbb{P}_h^- u - u_h) - (\mathbb{P}_h^- u - u) = e_h - \varepsilon_h, \\ \bar{e} &= p - p_h = (\mathbb{P}_h^+ p - p_h) - (\mathbb{P}_h^+ p - p) = \bar{e}_h - \bar{\varepsilon}_h. \end{aligned} \quad (27)$$

where  $e = u - u_h, \bar{e} = p - p_h$ . For the special projection of the initial conditions, using the similar techniques given by reference [47], we have

**Lemma 3.** Assume the initial conditions of the LDG scheme (8) are given

$$u_h(x, 0) = \mathbb{P}_h^- u(x, 0), \quad u_{ht}(x, 0) = \mathbb{P}_h u_t(x, 0). \quad (28)$$

there holds the following error estimate

$$\|e_h(0)\| = 0, \|\bar{e}_h(0)\| \leq Ch^{k+1}, \|\varepsilon_{ht}(0)\| \leq Ch^{k+1}, \quad (29)$$

and

$$(e_t, v)_{I_j} = 0, \forall v \in V_h^k. \quad (30)$$

*Proof.* From Eq.(8), the error  $\bar{e}$  satisfies

$$(\bar{e}, w_h)_{I_j} + (e, (w_h)_x)_{I_j} - (e^- w_h^-)_{j+\frac{1}{2}} + (e^- w_h^+)_{j-\frac{1}{2}} = 0, \forall w_h \in V_h^k.$$

By the properties of the projection  $\mathbb{P}_h^-$ , we get

$$(\bar{e}, w_h)_{I_j} + (\varepsilon_h, (w_h)_x)_{I_j} - (\varepsilon_h^- w_h^-)_{j+\frac{1}{2}} + (\varepsilon_h^- w_h^+)_{j-\frac{1}{2}} = 0. \quad (31)$$

In view of  $u_h(x, 0) = \mathbb{P}_h^- u(x, 0)$ , we have  $(\bar{e}, w_h)_{I_j} = 0$ . Taking  $w_h = \bar{e}(0)$  in Eq.(31), we obtain

$$\|\bar{e}(0)\|^2 = (\bar{e}(0), \bar{e}(0)) = (\bar{e}(0), \varepsilon_h(0)) \leq \frac{1}{2}\|\bar{e}(0)\|^2 + \frac{1}{2}\|\varepsilon_h(0)\|^2,$$

which yields

$$\|\bar{e}(0)\| \leq \|\varepsilon_h(0)\| \leq Ch^{k+1}.$$

We can prove the other error estimate with the similar method, we omit it here.  $\square$



**Theorem 4.** Let  $(u_h, p_h)$  to be the solutions of semi-discrete LDG scheme (8), and  $(u, p)$  be the exact solutions of system (7), the following error estimate holds

$$\|u_t(\cdot, t) - (u_h)_t(\cdot, t)\| \leq Ch^{k+1}, \|p(\cdot, t) - p_h(\cdot, t)\| \leq Ch^{k+1}, \quad (32)$$

where constant  $C$  is independent of mesh step  $h$ .

*Proof.* With the notations used in Lemma 2, we directly get

$$B(u_h, p_h; v_h, w_h) = (f, v_h), \quad \forall v_h, w_h \in V_h^k, \quad (33)$$

and

$$B(u, p; v_h, w_h) = (f, v_h), \quad \forall v_h, w_h \in V_h^k, \quad (34)$$

Subtracting (34) from (33), we obtain the error equation

$$B(e, \bar{e}; v_h, w_h) = 0, \quad \forall v_h, w_h \in V_h^k. \quad (35)$$

Taking  $v_h = e_{ht}, w_h = \bar{e}_h$  in (35), we get

$$B(e_h, \bar{e}_h; (e_h)_t, \bar{e}_h) = B(\varepsilon_h, \bar{\varepsilon}_h; (e_h)_t, \bar{e}_h). \quad (36)$$

For the left side of Eq.(36), using the Eq.(12) in Lemma 2, we have

$$B(e_h, \bar{e}_h; (e_h)_t, \bar{e}_h) = ((e_h)_{tt}, (e_h)_t) + ({}_0^C D_t^{3/2} e_h, (e_h)_t) + ((e_h)_t, (e_h)_t) + ((\bar{e}_h)_t, \bar{e}_h).$$

Obviously, the right of (36) can be written as

$$\begin{aligned} B(\varepsilon_h, \bar{\varepsilon}_h; (e_h)_t, \bar{e}_h) &= ((\varepsilon_h)_{tt}, (e_h)_t) + ({}_0^C D_t^{3/2} \varepsilon_h, (e_h)_t) + ((\varepsilon_h)_t, (e_h)_t) \\ &\quad + (\bar{\varepsilon}_h, (e_h)_{tx}) - \sum_{j=1}^N [(\bar{\varepsilon}_h^+(e_h)_t^-)_{j+\frac{1}{2}} - (\bar{\varepsilon}_h^+(e_h)_t^+)_{j-\frac{1}{2}}] \\ &\quad + ((\varepsilon_h)_t, (\bar{e}_h)_x) + ((\bar{\varepsilon}_h)_t, \bar{e}_h) \\ &\quad - \sum_{j=1}^N [((\varepsilon_h)_t^- \bar{e}_h^-)_{j+\frac{1}{2}} - ((\bar{\varepsilon}_h)_t^- \bar{e}_h^+)_{j-\frac{1}{2}}]. \end{aligned}$$

Since  $(e_{ht})_x$  and  $(\bar{e}_h)_x$  are polynomials of degree at most  $k-1$ , applying the properties (24) and (25) of the projections  $\mathbb{P}_h^\pm$ , we obtain

$$(\bar{\varepsilon}_h, (e_h)_{tx})_{I_j} = 0 \quad \text{and} \quad ((\varepsilon_h)_t, (\bar{e}_h)_x)_{I_j} = 0.$$

The projections  $\mathbb{P}_h^\pm$  implies

$$(\varepsilon_h)_{j+\frac{1}{2}}^- = (\mathbb{P}u)_{j+\frac{1}{2}}^- - u_{j+\frac{1}{2}} = 0 \quad \text{and} \quad (\bar{\varepsilon}_h)_{j-\frac{1}{2}}^+ = (\mathbb{P}p)_{j-\frac{1}{2}}^+ - p_{j-\frac{1}{2}} = 0.$$

Furthermore, applying the Cauchy's inequality, Eq.(36) produces

$$\begin{aligned} &((e_h)_{tt}, (e_h)_t) + ({}_0^C D_t^{3/2} (e_h), (e_h)_t) + ((e_h)_t, (e_h)_t) + ((\bar{e}_h)_t, \bar{e}_h) \\ &\leq \frac{1}{2} \|(\varepsilon_h)_{tt}\|^2 + \frac{1}{2} \|{}_0^C D_t^{3/2} \varepsilon_h\|^2 + \frac{1}{2} \|(\varepsilon_h)_t\|^2 + \frac{3}{2} \|(e_h)_t\|^2 + \frac{1}{2} \|\bar{\varepsilon}_h\|^2 + \frac{1}{2} \|\bar{e}_h\|^2. \end{aligned}$$

From the Lemma 2, we conclude that

$$((e_h)_{tt}, (e_h)_t) + ({}_0^C D_t^{3/2} (e_h), (e_h)_t) + ((\bar{e}_h)_t, \bar{e}_h) \leq \frac{1}{2} \|(e_h)_t\|^2 + Ch^{2k+2}.$$

Using the inequality (3), we have

$$\frac{1}{2} \frac{d}{dt} (\|(e_h)_t\|^2 + \|\bar{e}_h\|^2) + \frac{1}{2} {}_0^C D_t^{1/2} (\|(e_h)_t\|^2) \leq \frac{1}{2} \|(e_h)_t\|^2 + Ch^{2k+2}.$$

or

$$\frac{d}{dt} (\|(e_h)_t\|^2 + \|\bar{e}_h\|^2) + {}_0^C D_t^{1/2} (\|(e_h)_t\|^2) \leq \|(e_h)_t\|^2 + Ch^{2k+2}.$$

Combining the composite property (6) and by integrating the above inequality with respect to time between 0 and  $t$ , we arrive at

$$\begin{aligned} \|(e_h)_t\|^2 + \|\bar{e}_h\|^2 + oI_t^{\frac{1}{2}}(\|(e_h)_t\|^2) &\leq \left(1 + \frac{2t^{1/2}}{\Gamma(3/2)}\right) \|(e_h)_t(0)\|^2 + \|\bar{e}_h(0)\|^2 \\ &\quad + \int_0^t \|(e_h)_s\|^2 ds + Cth^{2k+2}. \end{aligned}$$

Using Lemma 3, and the positivity of integral term in Eq.(37), we have

$$\|(e_h)_t\|^2 + \|\bar{e}_h\|^2 \leq \int_0^t \|(e_h)_s\|^2 ds + CTh^{2k+2}.$$

Furthermore, using the Gronwall's inequality [37, 11], we have

$$\|(e_h)_t\|^2 + \|\bar{e}_h\|^2 \leq Ce^T h^{2k+2}.$$

Applying triangle inequality  $\|e\| \leq \|e_h\| + \|\varepsilon_h\|$ , we get the desired results.  $\square$

**Theorem 5.** *Let  $(u_h, p_h)$  to be the solution of semi-discrete LDG scheme (16), and  $(u, p)$  be the exact solutions of system (15), with the periodic boundary, the following error estimate holds*

$$\|u(\cdot, t) - u_h(\cdot, t)\| \leq Ch^{k+1}, \|p(\cdot, t) - p_h(\cdot, t)\| \leq Ch^{k+1}, \quad (37)$$

where constant  $C$  is independent of mesh step  $h$ .

*Proof.* Using (19), we have

$$\mathcal{B}(u_h, p_h; v_h, w_h) = (g_h, w_h), \forall v_h, w_h \in V_h, \quad (38)$$

and

$$\mathcal{B}(u, p; v_h, w_h) = (g_h, w_h), \forall v_h, w_h \in V_h. \quad (39)$$

Subtracting (39) from (38), then we obtain the error equation

$$\mathcal{B}(e, \bar{e}; v_h, w_h) = 0, \forall v_h, w_h \in V_h, \quad (40)$$

where we denote  $e = u - u_h, \bar{e} = p - p_h$ . Taking  $v_h = e_h, w_h = \bar{e}_h$  in (40), we get

$$\mathcal{B}(e_h, \bar{e}_h; e_h, \bar{e}_h) = \mathcal{B}(\varepsilon_h, \bar{\varepsilon}_h; e_h, \bar{e}_h). \quad (41)$$

For the left side of (41), using the Eq. (12) in Lemma 2, we have

$$\mathcal{B}(e_h, \bar{e}_h; e_h, \bar{e}_h) = ((e_h)_t, e_h) + ((\bar{e}_h)_t, \bar{e}_h) + ({}_0^C D_t^{1/2}(\bar{e}_h), \bar{e}_h) + (\bar{e}_h, \bar{e}_h).$$

Obviously, the right of (41) can be written as

$$\begin{aligned} \mathcal{B}(\varepsilon_h, \bar{\varepsilon}_h; e_h, \bar{e}_h) &= ((\varepsilon_h)_t, e_h) + (\bar{\varepsilon}_h, (e_h)_x) - \sum_{j=1}^N [(\bar{\varepsilon}_h^+ e_h^-)_{j+\frac{1}{2}} - (\bar{\varepsilon}_h^+ e_h^+)_{j-\frac{1}{2}}] \\ &\quad + ((\bar{\varepsilon}_h)_t, \bar{e}_h) + ({}_0^C D_t^{1/2}(\bar{\varepsilon}_h), \bar{e}_h) + (\bar{\varepsilon}_h, \bar{e}_h) \\ &\quad + (\varepsilon_h, (\bar{e}_h)_x) - \sum_{j=1}^N [(\varepsilon_h^- \bar{e}_h^-)_{j+\frac{1}{2}} - (\varepsilon_h^- w_h^+)_{j-\frac{1}{2}}]. \end{aligned}$$

Applying the properties (24) and (25) of the projections  $\mathbb{P}_h^\pm$ , we obtain

$$\begin{aligned} &((e_h)_t, e_h) + ((\bar{e}_h)_t, \bar{e}_h) + ({}_0^C D_t^{1/2}(\bar{e}_h), \bar{e}_h) + (\bar{e}_h, \bar{e}_h) \\ &\leq \frac{1}{2} \|\bar{\varepsilon}_h\|^2 + \frac{1}{2} \|e_h\|^2 + \frac{3}{2} \|\bar{\varepsilon}_h\|^2 + \frac{1}{2} \|{}_0^C D_t^{1/2}(\bar{\varepsilon}_h)\|^2 + \frac{1}{2} \|\bar{e}_h\|^2. \end{aligned}$$

From the Lemma 2, we conclude that

$$((e_h)_t, e_h) + ((\bar{e}_h)_t, \bar{e}_h) + ({}_0^C D_t^{1/2} \bar{e}_h, \bar{e}_h) \leq \frac{1}{2} \|e_h\|^2 + Ch^{2k+2}.$$

Using the inequality (3), we have

$$\frac{1}{2} \frac{d}{dt} (\|e_h\|^2 + \|\bar{e}_h\|^2) + \frac{1}{2} {}_0^C D_t^{1/2} (\|\bar{e}_h\|^2) \leq \frac{1}{2} \|e_h\|^2 + Ch^{2k+2}.$$

Hence

$$\frac{d}{dt} (\|e_h\|^2 + \|\bar{e}_h\|^2) \leq \|e_h\|^2 + Ch^{2k+2}.$$

Combining the composite properties (6) and the definition of Riemann-Liouville fractional integral, we arrive at

$$\|e_h\|^2 + \|\bar{e}_h\|^2 \leq \int_0^t \|e_h\|^2 ds + \int_0^t Ch^{2k+2} ds.$$

Furthermore, using the classic Gronwall's inequality [37, 11], we have

$$\|e_h\|^2 + \|\bar{e}_h\|^2 \leq Ce^T h^{2k+2}. \quad (42)$$

Finally, combining (27), (42) and triangle inequality, we get error estimate (37).  $\square$

## 4 Fully discrete LDG schemes

Next we discrete the time variable in the semi-discrete scheme by virtue of efficient difference approximations. Let  $0 = t_0 < t_1 < \dots < t_n < t_{n+1} < \dots < t_M = T$  be the subdivision of the time interval  $[0, T]$ , with the time step  $\tau = t_{n+1} - t_n$  and we denote  $v^n = v(t_n)$ . Adopt the following notation at discrete time levels

$$v^{n+\frac{1}{2}} = \frac{1}{2}(v^{n+1} + v^n), \quad \partial_t v^{n+1} = \frac{v^{n+1} - v^n}{\tau}, \quad (43)$$

$$\partial_t^2 v^{n+1} = \frac{v^{n+1} - 2v^n + v^{n-1}}{\tau^2} = \frac{\partial_t v^{n+\frac{1}{2}} - \partial_t v^{n-\frac{1}{2}}}{\tau}. \quad (44)$$

For  $1 < \alpha < 2$ , the linear interpolation formula of Caputo fractional derivative gives [30, 42]

$${}_0^C D_t^\alpha v(t)|_{t=t_n} = \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[ a_0 Q^n - \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j}) Q^j - a_{n-1} Q^0 \right] + T_j^n,$$

where  $a_j = (j+1)^{2-\alpha} - j^{2-\alpha}$ ,  $Q^n = v(t_n)$ . It is proved that the truncation error  $T_j^n$  satisfy [30, 42]

$$|T_j^n| \leq c_\alpha \tau^{3-\alpha} \max_{t \in [0, t_n]} |Q''(t)|.$$

The constant  $c_\alpha$  is only dependent on parameter  $\alpha$ . Discrete  $Q^n = v(t_n)$  as  $Q^n = \partial_t v^n + \mathcal{O}(\tau)$ , omit the truncation error, we get

$${}_0^C D_t^\alpha v(t)|_{t=t_n} = \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[ a_0 \partial_t v^n - \sum_{j=1}^{n-1} (a_{n-j-1} - a_{n-j}) \partial_t v^j - a_{n-1} \partial_t v^0 \right]. \quad (45)$$

Let  $u_h^n$  and  $p_h^n \in V_h^k$  be the approximate solution of  $u(x, t_n)$  and  $p(x, t_n)$ , respectively. Using the discrete formula (45) to approximate the Caputo fractional derivative  ${}_0^C D_t^{3/2}$  and central difference formula (44) to the second temporal derivative, we get the fully discrete DG scheme of the first order system (8), i.e.,

$$\begin{aligned} & (\partial_t^2 u_h^{n+1}, v_h)_{I_j} + ({}_0^C D_t^{3/2} u_h^{n+1}, v_h)_{I_j} + (\partial_t u_h^{n+1}, v_h)_{I_j} + (p_h^{n+1}, (v_h)_x)_{I_j} \\ & - (\widehat{p}_h^{n+1} v_h^-)_{j+\frac{1}{2}} + (\widehat{p}_h^{n+1} v_h^+)_{j-\frac{1}{2}} = (f_h^{n+1}, v_h), \\ & (p_h^{n+1}, w_h)_{I_j} + (u_h^{n+1}, (w_h)_x)_{I_j} - (\widehat{u}_h^{n+1} w_h^-)_{j+\frac{1}{2}} + (\widehat{u}_h^{n+1} w_h^+)_{j-\frac{1}{2}} = 0, \end{aligned} \quad (46)$$

for all  $v_h, w_h \in V_h^k$ .

On the other hand, the relation of the Riemann-Liouville (RL) and Caputo fractional derivative [38]

$${}_0^C D_t^\alpha v(t) = {}_0 D_t^\alpha (v(t) - v(0)) = {}_0 D_t^\alpha v(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} v(0), \quad 0 < \alpha < 1, \quad (47)$$

suggests that we can discrete RL fractional derivative instead of Caputo ones. For a function with homogeneous initial value, relation (47) gives the equivalence of RL and Caputo fractional derivative. For the numerical approximation of RL fractional derivative, there are various kinds of effective methods includes convolution quadrature, Grünwald-Letnikov (GL) approximation and some variants [38, 24]. In order to develop second order approximation of the RL fractional derivative, we employ the shifted GL approximation to the RL fractional derivative given by [43, 39]

$${}^C D_t^\alpha v(t)|_{t=t_{n+1}} = \tau^{-\alpha} \sum_{k=0}^{n+1} w_k^{(\alpha)} (v^{n+1-k} - v^0) + \tilde{T}_j^n, 0 < \alpha < 1,$$

where

$$w_0^{(\alpha)} = \frac{2+\alpha}{2} g_0^{(\alpha)}, \quad w_k^{(\alpha)} = \frac{2+\alpha}{2} g_k^{(\alpha)} - \frac{\alpha}{2} g_{k-1}^{(\alpha)}, \quad k \geq 1. \quad (48)$$

with  $g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}$  for  $k \geq 0$ . And the truncation error satisfies [43]

$$|\tilde{T}_j^n| \leq C_\alpha \tau^2 \max_{t \in [0, t_n]} |{}_0 D_t^{\alpha+2} v(t)|. \quad (49)$$

For simplify, the shifted GL approximation is abbreviated as

$${}_0 \mathcal{D}_t^\alpha v^{n+1} = \tau^{-\alpha} \sum_{k=0}^{n+1} w_k^{(\alpha)} (v^{n+1-k} - v^0). \quad (50)$$

**Lemma 4.** [43] Let  $\{w_n^{(\alpha)}\}_{n=0}^\infty$  be defined as in (48), then for any positive integer  $k$  and real vector  $(v_1, v_2, \dots, v_k)^T \in \mathbb{R}^k$ , it holds that

$$\sum_{n=0}^{k-1} \left( \sum_{m=0}^n w_m^{(\alpha)} v_{n+1-m} \right) v_{n+1} \geq 0.$$

Applying GL formula (50) to approach the fractional derivative  ${}_0^C D_t^{1/2}$ , we propose the fully discrete local DG schemes of (16) as follows: find  $u_h^n, p_h^n \in V_h^k$ ,

$$\begin{aligned} & (\partial_t p_h^{n+1}, w_h)_{I_j} + ({}_0 \mathcal{D}_t^{1/2} p_h^{n+\frac{1}{2}}, w_h)_{I_j} + (p_h^{n+\frac{1}{2}}, w_h)_{I_j} + (u_h^{n+\frac{1}{2}}, (w_h)_x)_{I_j} \\ & - (\widehat{u}_h^{n+\frac{1}{2}} w_h^-)_{j+\frac{1}{2}} + (\widehat{u}_h^{n+\frac{1}{2}} w_h^+)_{j-\frac{1}{2}} = (g^{n+\frac{1}{2}}, w_h), \\ & (\partial_t u_h^{n+\frac{1}{2}}, v_h)_{I_j} + (p_h^{n+\frac{1}{2}}, (v_h)_x)_{I_j} - (\widehat{p}_h^{n+\frac{1}{2}} v_h^-)_{j+\frac{1}{2}} + (\widehat{p}_h^{n+\frac{1}{2}} v_h^+)_{j-\frac{1}{2}} = 0. \end{aligned} \quad (51)$$

for all  $v_h, w_h \in V_h^k$ ,  $j = 1, 2, \dots, N$ . We take numerical flux to be  $\widehat{u}_h^n = (u_h^n)^-$ ,  $\widehat{q}_h^n = (q_h^n)^+$  with the same choice of (9).

**Theorem 6.** For small time step  $\tau$ , the fully discrete local DG scheme (51) of system (15) is unconditional stability and holds

$$\|p_h^M\|^2 + \|u_h^M\|^2 \leq C_{w,\alpha} \|p_h^0\|^2 + \|u_h^0\|^2 + \tau \sum_{n=0}^{M-1} \|g^{n+\frac{1}{2}}\|^2, \quad (52)$$

where  $C_{w,\alpha} = 1 + \tau^{\frac{1}{2}} \sum_{k=0}^n w_k^{(\frac{1}{2})}$ .

*Proof.* Setting  $v_h = \beta u_h^{n+\frac{1}{2}}$ ,  $w_h = p_h^{n+\frac{1}{2}}$  in (51) and summing over all elements, we obtain

$$\begin{aligned} & \frac{1}{2} \|p_h^{n+1}\|^2 + \frac{1}{2} \|u_h^{n+1}\|^2 + \tau^{\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} p_h^{n+\frac{1}{2}-k}, p_h^{n+\frac{1}{2}} \right) + \tau (p_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) \\ & + \tau \sum_{j=1}^N [\mathcal{F}_{j+\frac{1}{2}}(u_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) - \mathcal{F}_{j-\frac{1}{2}}(u_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) + \Theta_{j-\frac{1}{2}}(u_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})] \\ & = \frac{1}{2} \|p_h^n\|^2 + \frac{1}{2} \|u_h^n\|^2 + \tau^{\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} p_h^0, p_h^{n+\frac{1}{2}} \right) + \tau (g^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}), \end{aligned}$$

where  $\mathcal{F}(u_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) = (p_h^{n+\frac{1}{2}})^-(u_h^{n+\frac{1}{2}})^- - (\widehat{u}_h^{n+\frac{1}{2}})(p_h^{n+\frac{1}{2}})^- - (u_h^{n+\frac{1}{2}})^-(\widehat{p}_h^{n+\frac{1}{2}})$ , and

$$\begin{aligned}\Theta(u_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) &= (p_h^{n+\frac{1}{2}})^-(u_h^{n+\frac{1}{2}})^- + (\widehat{u}_h^n)(p_h^n)^+ + (u_h^{n+\frac{1}{2}})^+(\widehat{p}_h^{n+\frac{1}{2}}) \\ &\quad - (u_h^{n+\frac{1}{2}})^+(p_h^{n+\frac{1}{2}})^+ - (\widehat{u}_h^{n+\frac{1}{2}})(p_h^{n+\frac{1}{2}})^- - (u_h^n)^-(\widehat{p}_h^{n+\frac{1}{2}}).\end{aligned}$$

Recalling the numerical fluxes in (9), we have  $\Theta(u_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) = 0$ . On the other hand, in view of the periodic boundary condition given in (2), we get

$$\sum_{j=1}^N [\mathcal{F}_{j+1/2}(u_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}}) - \mathcal{F}_{j-1/2}(u_h^{n+\frac{1}{2}}, p_h^{n+\frac{1}{2}})] = 0.$$

In view of the positivity of  $\sum_{k=0}^n w_k^{\frac{1}{2}}$  (see [31]), application of Cauchy inequality gives

$$\begin{aligned}\|p_h^{n+1}\|^2 + \|u_h^{n+1}\|^2 + 2\tau^{\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} p_h^{n+\frac{1}{2}-k}, p_h^{n+\frac{1}{2}} \right) + \tau \|p_h^{n+\frac{1}{2}}\|^2 \\ \leq \|p_h^n\|^2 + \|u_h^n\|^2 + \tau^{\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \right) \|p_h^0\|^2 + \tau^{\frac{1}{2}} \|p_h^{n+\frac{1}{2}}\|^2 + \tau \|g^{n+\frac{1}{2}}\|^2.\end{aligned}$$

Summing up for  $0 \leq n \leq M-1$  and using Lemma 4, we have (52).  $\square$

**Theorem 7.** *Let  $(u(x, t_n), p(x, t_n))$  be the exact solution of the system (15),  $(u_h^n, p_h^n)$  be the numerical solution of the fully discrete LDG scheme (51), then there holds the following error estimates*

$$\|u(x, t_n) - u_h^n\| \leq C(\tau^2 + h^{k+1}), \|p(x, t_n) - p_h^n\| \leq C(\tau^2 + h^{k+1}), n \geq 1, \quad (53)$$

where  $C$  is a constant depending on the regularity of  $u$  but independent of  $\tau$  and  $h$ .

*Proof.* To simplify the notation, we decompose the errors as follows:

$$\begin{aligned}e_u^n &= (\mathbb{P}_h^- u(x, t_n)) - u_h^n - (\mathbb{P}_h^- u(x, t_n) - u(x, t_n)) = \eta_u^n - \theta_u^n, \\ e_p^n &= (\mathbb{P}_h^+ p(x, t_n) - p_h^n) - (\mathbb{P}_h^+ p(x, t_n) - p(x, t_n)) = \eta_p^n - \theta_p^n.\end{aligned} \quad (54)$$

The weak form of the coupled system (15) at  $(x, t_{n+\frac{1}{2}})$  can be rewritten as

$$\begin{aligned}(\partial_t p(x, t_{n+\frac{1}{2}}), w_h)_{I_j} + \tau^{-\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} p(x, t_{n+\frac{1}{2}-k}), w_h \right)_{I_j} + (p(x, t_{n+\frac{1}{2}}), w_h)_{I_j} \\ + (u(x, t_{n+\frac{1}{2}}), (w_h)_x)_{I_j} - (\widehat{u}(x, t_{n+\frac{1}{2}}) w_h^-)_{j+\frac{1}{2}} + (\widehat{u}(x, t_{n+\frac{1}{2}}) w_h^+)_{j-\frac{1}{2}} = 0, \\ (\partial_t u(x, t_{n+\frac{1}{2}}), v_h)_{I_j} + (p(x, t_{n+\frac{1}{2}}), (v_h)_x)_{I_j} - (\widehat{p}(x, t_{n+\frac{1}{2}}) v_h^-)_{j+\frac{1}{2}} + (\widehat{p}(x, t_{n+\frac{1}{2}}) v_h^+)_{j-\frac{1}{2}} \\ = 0.\end{aligned}$$

System (54) minus system (51), we can get the error equation

$$\begin{aligned}(\partial_t e_p^{n+\frac{1}{2}}, w_h) + \tau^{-\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} e_p^{n+\frac{1}{2}-k}, w_h \right) + (e_p^{n+\frac{1}{2}}, w_h) \\ + (e_u^{n+\frac{1}{2}}, (w_h)_x) - \sum_{j=1}^N [(\widehat{e}_u^{n+\frac{1}{2}} w_h^-)_{j+\frac{1}{2}} - (\widehat{e}_u^{n+\frac{1}{2}} w_h^+)_{j-\frac{1}{2}}] + (\partial_t e_u^{n+\frac{1}{2}}, v_h) \\ + (e_p^{n+\frac{1}{2}}, (v_h)_x) - \sum_{j=1}^N [(\widehat{e}_p^{n+\frac{1}{2}} v_h^-)_{j+\frac{1}{2}} - (\widehat{e}_p^{n+\frac{1}{2}} v_h^+)_{j-\frac{1}{2}}] = (\widetilde{T}^{n+\frac{1}{2}}, w_h).\end{aligned}$$

Assume that no error is introduced in the initial value (i.e.,  $e_u^0 = 0$ ), combining the numerical flux

given in (9), we have

$$\begin{aligned}
& (\partial_t(\eta_p^{n+\frac{1}{2}}), w_h) + \tau^{-\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \eta_p^{n+\frac{1}{2}-k}, w_h \right) + (\eta_p^{n+\frac{1}{2}}, w_h) \\
& + (\partial_t(\eta_u^{n+\frac{1}{2}}), v_h) + (\eta_u^{n+\frac{1}{2}}, (w_h)_x) + (\eta_p^{n+\frac{1}{2}}, (v_h)_x) \\
& - \sum_{j=1}^N [(\eta_u^{n+\frac{1}{2}})^-_{j+\frac{1}{2}} (w_h)^-_{j+\frac{1}{2}} - (\eta_u^{n+\frac{1}{2}})^-_{j-\frac{1}{2}} (w_h)^+_{j-\frac{1}{2}}] \\
& - \sum_{j=1}^N [(\eta_p^{n+\frac{1}{2}})^+_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} - (\eta_p^{n+\frac{1}{2}})^+_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}}] \\
& = (\tilde{T}^{n+\frac{1}{2}}, w_h) + (\partial_t \theta_p^{n+\frac{1}{2}}, w_h) + \tau^{-\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \theta_p^{n+\frac{1}{2}-k}, w_h \right)_\Omega \\
& + (\theta_p^{n+\frac{1}{2}}, w_h) + (\theta_u^{n+\frac{1}{2}}, (w_h)_x) + (\partial_t(\theta_u^{n+\frac{1}{2}}), v_h) + (\theta_p^{n+\frac{1}{2}}, (v_h)_x)_\Omega \\
& - \sum_{j=1}^N [(\theta_u^{n+\frac{1}{2}})^-_{j+\frac{1}{2}} (w_h)^-_{j+\frac{1}{2}} - (\theta_u^{n+\frac{1}{2}})^-_{j-\frac{1}{2}} (w_h)^+_{j-\frac{1}{2}}] \\
& - \sum_{j=1}^N [(\theta_p^{n+\frac{1}{2}})^+_{j+\frac{1}{2}} (v_h)^-_{j+\frac{1}{2}} - (\theta_p^{n+\frac{1}{2}})^+_{j-\frac{1}{2}} (v_h)^+_{j-\frac{1}{2}}]
\end{aligned} \tag{55}$$

Taking  $v_h = \eta_u^{n+\frac{1}{2}}$ ,  $w_h = \eta_p^{n+\frac{1}{2}}$  in (55), we get

$$\begin{aligned}
& \frac{1}{2} \|\eta_u^{n+1}\|^2 + \frac{1}{2} \|\eta_p^{n+1}\|^2 + \tau^{\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \eta_p^{n+\frac{1}{2}-k}, \eta_p^{n+\frac{1}{2}} \right) + \tau \|\eta_p^{n+\frac{1}{2}}\|^2 \\
& = \frac{1}{2} \|\eta_u^n\|^2 + \frac{1}{2} \|\eta_p^n\|^2 + \tau (\tilde{T}^{n+\frac{1}{2}}, \eta_p^{n+\frac{1}{2}}) \\
& + \tau (\partial_t(\theta_p^{n+\frac{1}{2}}), \eta_p^{n+\frac{1}{2}}) + \tau \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \theta_p^{n+\frac{1}{2}-k}, \theta_p^{n+\frac{1}{2}} \right) \\
& + \tau (\theta_p^{n+\frac{1}{2}}, \eta_p^{n+\frac{1}{2}}) + \tau (\theta_u^{n+\frac{1}{2}}, (\eta_p^{n+\frac{1}{2}})_x) \\
& + \tau \sum_{j=1}^N [(\theta_u^{n+\frac{1}{2}})^-_{j+\frac{1}{2}} (\eta_p^{n+\frac{1}{2}})^-_{j+\frac{1}{2}} - (\theta_u^{n+\frac{1}{2}})^-_{j-\frac{1}{2}} (\eta_p^{n+\frac{1}{2}})^+_{j-\frac{1}{2}}] \\
& + \tau (\partial_t(\theta_u^{n+\frac{1}{2}}), \eta_u^{n+\frac{1}{2}}) + \tau (\theta_p^{n+\frac{1}{2}}, (\eta_u^{n+\frac{1}{2}})_x) \\
& - \tau \sum_{j=1}^N [(\theta_p^{n+\frac{1}{2}})^+_{j+\frac{1}{2}} (\eta_u^{n+\frac{1}{2}})^-_{j+\frac{1}{2}} - (\theta_p^{n+\frac{1}{2}})^+_{j-\frac{1}{2}} (\eta_u^{n+\frac{1}{2}})^+_{j-\frac{1}{2}}].
\end{aligned}$$

Using the properties of projections  $\mathbb{P}_h^\pm$  in Eq. (24) with (25), we can further get

$$\begin{aligned}
& \frac{1}{2} \|\eta_u^{n+1}\|^2 + \frac{1}{2} \|\eta_p^{n+1}\|^2 + \tau^{\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \eta_p^{n+\frac{1}{2}-k}, \eta_p^{n+\frac{1}{2}} \right) + \tau \|\eta_p^{n+\frac{1}{2}}\|^2 \\
& = \frac{1}{2} \|\eta_u^n\|^2 + \frac{1}{2} \|\eta_p^n\|^2 + \tau (\tilde{T}^{n+\frac{1}{2}}, \eta_p^{n+\frac{1}{2}}) + \tau (\partial_t(\theta_p^{n+\frac{1}{2}}), \eta_p^{n+\frac{1}{2}}) \\
& + \tau \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \theta_p^{n+\frac{1}{2}-k}, \eta_p^{n+\frac{1}{2}} \right) + \tau (\theta_p^{n+\frac{1}{2}}, \eta_p^{n+\frac{1}{2}}) + \tau (\partial_t(\theta_u^{n+\frac{1}{2}}), \eta_u^{n+\frac{1}{2}}).
\end{aligned}$$

Combining the Cauchy inequality and Young's inequality, we have

$$\begin{aligned}
& \frac{1}{2} \|\eta_u^{n+1}\|^2 + \frac{1}{2} \|\eta_p^{n+1}\|^2 + \tau^{\frac{1}{2}} \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \eta_p^{n+\frac{1}{2}-k}, \eta_p^{n+\frac{1}{2}} \right) + \tau \|\eta_p^{n+\frac{1}{2}}\|^2 \\
& \leq \frac{1}{2} \|\eta_u^n\|^2 + \frac{\tau}{4} \|\tilde{T}^{n+\frac{1}{2}}\|^2 + \|\partial_t(\theta_p^{n+\frac{1}{2}})\|^2 + \tau \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \theta_p^{n+\frac{1}{2}-k}, \eta_p^{n+\frac{1}{2}} \right) \\
& + \tau \|\theta_p^{n+\frac{1}{2}}\|^2 + \tau \|\eta_p^{n+\frac{1}{2}}\|^2 + \frac{\tau}{2} \|\eta_p^{n+\frac{1}{2}}\|^2 + \tau (\partial_t(\theta_u^{n+\frac{1}{2}}), \eta_u^{n+\frac{1}{2}}).
\end{aligned}$$

Using the equation (49) and Lemma 2, we obtain

$$\begin{aligned} & \frac{1}{2} \|\eta_u^{n+1}\|^2 + \frac{1}{2} \|\eta_p^{n+1}\|^2 + \tau \left( \sum_{k=0}^n w_k^{(\frac{1}{2})} \eta_p^{n+\frac{1}{2}-k}, \eta_p^{n+\frac{1}{2}} \right) \\ & \leq \frac{1}{2} \|\eta_p^n\|^2 + \frac{1}{2} \|\eta_p^n\|^2 + \frac{\tau}{2} \|\eta_u^{n+\frac{1}{2}}\| + C(\tau^2 + h^{k+1})^2. \end{aligned}$$

Summing up for  $n$ , and using Lemma 4, we have

$$\begin{aligned} & \frac{1}{2} \sum_{n=0}^{M-1} \|\eta_u^{n+1}\|^2 + \frac{1}{2} \sum_{n=0}^{M-1} \|\eta_p^{n+1}\|^2 \\ & \leq \frac{1}{2} \sum_{n=0}^{M-1} \|\eta_u^n\|^2 + \frac{1}{2} \sum_{n=0}^{M-1} \|\eta_p^n\|^2 + \frac{\tau}{4} \sum_{n=0}^{M-1} (\|\eta_u^{n+1}\|^2 + \|\eta_u^n\|^2) + C(\tau^2 + h^{k+1})^2. \end{aligned}$$

Furthermore, above inequality can be written as

$$\begin{aligned} \frac{1}{2} \|\eta_u^M\|^2 + \frac{1}{2} \|\eta_p^M\|^2 & \leq \left( \frac{1}{2} + \frac{\tau}{4} \right) \|\eta_u^0\|^2 + \frac{1}{2} \|\eta_p^0\|^2 + \frac{\tau}{2} \sum_{n=1}^{N-1} \|\eta_u^n\|^2 + \frac{\tau}{4} \|\eta_u^M\|^2 \\ & + C(\tau^2 + h^{k+1})^2. \end{aligned}$$

Applying Lemma 3, we arrive at

$$\|\eta_u^M\|^2 + \|\eta_p^M\|^2 \leq 2\tau \sum_{n=1}^{M-1} \|\eta_u^n\|^2 + C(\tau^2 + h^{k+1})^2.$$

By virtue of classic discrete Gronwall's inequality [37], we have  $\|\eta_u^M\| \leq C(\tau^2 + h^{k+1})$ . Then the desired result can be obtained by triangle inequality.  $\square$

## 5 Numerical experiments

In this section, we present an examples to validate the effectiveness of the two kinds of full discrete local discontinuous methods investigated in previous sections. The comparison between full discrete local discontinuous schemes (46) and (51) are measured in the  $L^2$  norm. And the final time  $T$  is taken as  $T = 1$  in the numerical example. We consider problem (2) with exact solution  $u(x, t) = t^\gamma \sin(2\pi x)$  on finite domain  $\Omega = (0, 1)$ . The force term  $f(x, t)$  is calculated by the exact solution with the corresponding initial-boundary conditions.

Firstly, we use the schemes (46) and (51) to solve problem (2) with smooth solution, i.e.,  $\gamma = 3$ . Tables 1-3 display the computational errors and convergence orders in  $L^2$ -norm for the discrete local DG schemes (46) and (51). In Table 1, the numerical results is obtained by choosing the different time step  $\tau$  with the  $P^1$  element, and we take the space size  $h$  sufficiently small as  $h = 1/2000$ . We observe the expected convergence rate of scheme (51). The third column of Table 1 shows that the convergence rate of scheme (46) is  $\mathcal{O}(\tau)$ . In Tables 2-3, we list the numerical results that calculated the convergence rate in space with  $P^k$  ( $k = 0, 1, 2$ ) elements. Here, we take the space-time steps as  $\tau = h^{k+1}$  and  $\tau = h^{(k+1)/2}$  for schemes (46) and (51), respectively. The numerical results show that the order of accuracy in time and space coincide with our theoretical analysis very well.

Table 1: The  $L^2$  errors and convergence orders of scheme (46) and scheme (51) for  $P^1$  element with  $h = 1/2000, \gamma = 3$ .

$\tau$	Scheme (46)		Scheme (51)	
	$\ \cdot\ $ -error	order	$\ \cdot\ $ -error	order
1/5	5.0164e-02	*	3.4866e-04	*
1/10	2.4845e-02	1.0137	1.2763e-04	1.4499
1/20	1.1776e-02	1.0771	3.2934e-05	1.9542
1/40	5.6086e-03	1.0701	8.3057e-06	1.9874
1/80	2.7134e-03	1.0475	2.1335e-06	1.9609

Table 2: The  $L^2$  errors and convergence orders of scheme (46) for different  $P^k$  elements with  $\tau = h^{k+1}, \gamma = 3$ .

$h$	$P^0$		$P^1$		$P^2$	
	$\ \cdot\ $ -error	order	$\ \cdot\ $ -error	order	$\ \cdot\ $ -error	order
1/5	2.7810e-01	*	6.7699e-02	*	6.8549e-03	*
1/10	1.3414e-01	1.0518	1.7074e-02	1.9873	8.7451e-04	2.9706
1/20	6.5965e-02	1.0240	4.2764e-03	1.9973	1.0989e-04	2.9925
1/40	3.2726e-02	1.0113	1.0696e-03	1.9994	1.3751e-05	2.9984

Table 3: The  $L^2$  errors and convergence orders of scheme (51) for different  $P^k$  elements with  $\tau = h^{(k+1)/2}, \gamma = 3$ .

$h$	$P^0$		$P^1$		$P^2$	
	$\ \cdot\ $ -error	order	$\ \cdot\ $ -error	order	$\ \cdot\ $ -error	order
1/5	1.1735e+00	*	6.4446e-02	*	8.2638e-03	*
1/10	5.1205e-01	1.1965	1.6766e-02	1.9426	8.8156e-04	3.2287
1/20	1.7857e-01	1.5198	4.2333e-03	1.9857	1.0884e-04	3.0178
1/40	8.6859e-02	1.0397	1.0609e-03	1.9964	1.3375e-05	3.0249

Table 4: The  $L^2$  errors and convergence orders of schemes (46) and (51) for  $P^1$  element with  $h = 1/2000, \gamma = 2$ .

$\tau$	Scheme (46)		Scheme (51)	
	$\ \cdot\ $ -error	order	$\ \cdot\ $ -error	order
1/5	1.0574e-02	*	3.5121e-04	*
1/10	5.8509e-03	0.8537	3.4066e-04	0.0440
1/20	2.7058e-03	1.1126	1.3991e-04	1.2838
1/40	1.2358e-03	1.1306	5.1111e-05	1.4528
1/80	5.7308e-04	1.1087	1.8255e-05	1.4853

Secondly, we take  $\gamma = 2$  in exact solution to test requirements of the regularity of problem (2) for the present numerical schemes. Table 4 display the computational error and convergence orders of time direction. The numerical results show that local DG scheme (46) can get the theoretical convergence rate. However, local DG scheme (51) can't get our expected convergence rate  $\mathcal{O}(\tau^2)$  due to the low regularity of solution.

## 6 Conclusions

In this paper, we have presented two novel local DG methods to solve an ultrasonic propagation equation in a porous material. The key idea to our methods is to write the ultrasonic propagation equation as a coupled system. The first semi-discrete scheme is constructed by splitting second-order space derivative. The second one is constructed by splitting for both time and space derivatives. The main idea of second scheme is modified by the fractional feature. In the second scheme, it is interesting see that the two coupled equations are similar to the Euler equation and the law of the mass conservation. The coefficients of fractional approximation formulas are crucial to ensure both the stability and convergence of the fully discrete schemes. For this purpose, we adopt the L1 approximation and shift weighted Grünwald-Letnikov formulas to discrete the fractional derivative. If the stability can be ensured, the semi-discrete schemes coupling with the high order time discretizations of fractional derivative are also possible used to solve the considered model. Due to above reason, the convergence rate of the first fully discrete scheme is only tested by the numerical results, the details of theoretical analysis will be discussed in our further work. And the numerical schemes discussed under the assumption that the model has smooth solution. We will furthermore investigate the regularity of solution of the considered model, and investigate the efficient time-stepping methods for ultrasonic propagation equation with non-smooth solution.



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