ϵ-hypercyclic operators that are not δ-hypercyclic for $\delta < \epsilon$

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ε-HYPERCYCLIC OPERATORS THAT ARE NOT δ-HYPERCYCLIC FOR $\delta < \varepsilon$

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Abstract. For every fixed $\varepsilon \in (0,1)$, we construct an operator on the separable Hilbert space which is δ-hypercyclic for all $\delta \in (\varepsilon, 1)$ and which is not δ-hypercyclic for all $\delta \in (0, \varepsilon)$.

1. Introduction

Let X be a separable infinite dimensional Banach space. During the last decades the properties of the orbits of operators acting on X have been widely studied. In particular, the notion of hypercyclic operators, namely operators with a dense orbit, has drawn the attention of many mathematicians (see for instance [3]). It seems natural in this context to investigate operators having orbits with a property slightly weaker than denseness. Does this imply that the operator admits a dense orbit? For instance, N. Feldman has shown in [4] that if there is an orbit of $T \in \mathcal{L}(X)$ which meets every ball of radius $d > 0$, then T is hypercyclic.
The following definition concerning operators admitting an orbit which intersects every cone of aperture ε has been introduced in [1].

Definition 1.1. Let $\varepsilon \in (0,1)$. A vector $x \in X$ is called an *ε-hypercyclic vector* for $T \in \mathcal{L}(X)$ provided for every non-zero vector $y \in X$, there exists an integer $n \in \mathbb{N}$ such that $\|T^n x - y\| \leq \varepsilon \|y\|$. The operator T is called *ε-hypercyclic* if it admits an ε-hypercyclic vector.

In [1], the authors have shown that for every $\varepsilon \in (0,1)$, there exists an ε-hypercyclic operator on $\ell^1(\mathbb{N})$ which is not hypercyclic. This was refined in [2] and [5] where similar examples are given on $\ell^2(\mathbb{N})$ and on more general spaces. Moreover it is pointed out in [5, Remark 4.7] that the ε-hypercyclic operator which is considered in that paper is not even δ-hypercyclic for some $\delta \in (0, \varepsilon)$.
This leaves open the following natural question: let X be a Banach space, let $0 < \delta < \varepsilon < 1$. Can we distinguish the class of δ-hypercyclic operators and that of ε-hypercyclic operators acting on X? We give a positive answer for a large class of Banach spaces. To state our result we recall some terminology. Let $(e_n)_{n\geq 0}$ be a basis of X (namely every $x \in X$ writes uniquely $\sum_{n\geq 0} x_n e_n$) and let $C \geq 1$. We say that $(e_n)_{n\geq 0}$ is C-unconditional if for any $N \geq 0$, for any finite sequences of scalars $(a_n)_{n=0,\dots,N}$ and $(b_n)_{n=0,\dots,N}$ such that

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$|b_n|\leq |a_n|$ for all $n=0,\dots,N$, then

$$\left\|\sum_{n=0}^N b_ne_n\right\|\leq C\left\|\sum_{n=0}^N a_ne_n\right\|.$$

Let us fix now X and Y two Banach spaces and suppose that (f_n) is a 1-unconditional basis of Y. We denote by $\bigoplus_Y X$ the vector space

$$\bigoplus_Y X:=\left\{(x_n)\in X^{\mathbb N}:\ \sum_{n=0}^{+\infty}\|x_n\|_X f_n\in Y\right\}$$

and we endow it by

$$\|(x_n)\|=\left\|\sum_{n=0}^{+\infty}\|x_n\|_X f_n\right\|_Y.$$

It is standard that $\bigoplus_Y X$ is a Banach space.
Our main theorem now reads.

Theorem 1.2. *Let X be an infinite dimensional separable Banach space with a 1-unconditional basis, let Y be an infinite dimensional separable Banach space with a normalized 1-unconditional basis such that the associated backward shift operator is continuous. For all $\varepsilon\in(0,1)$, there exists an operator on $Z=\bigoplus_Y X$ which is not δ-hypercyclic for all $\delta\in(0,\varepsilon)$ and which is δ-hypercyclic for all $\delta\in(\varepsilon,1)$.*

Observe that if X is either $c_0(\mathbb N)$ or $\ell^p(\mathbb N)$, $p\in[1,+\infty)$, then X is isometric to $\bigoplus_X X$ by using the canonical basis of X. Therefore, it satisfies the assumptions of the previous theorem. Recall also that if T is ε-hypercyclic for all $\varepsilon\in(0,1)$, then it is hypercyclic (see [1, Theorem 1.3]).
We will need a way to prove that an operator is ε-hypercyclic. We state here a variant of the ε-hypercyclicity criterion given in [5, Theorem 1.2]. Its proof is completely similar.

Theorem 1.3. *Let X be an infinite dimensional separable Banach space, let $T\in\mathcal L(X)$ and let $\varepsilon\in(0,1)$. Assume that there exist a dense subset $\mathcal D$ of X, a sequence $(u(k))$ dense in X such that that, for all $k\geq 0$, $u(k)=u(l)$ for infinitely many integers l, a sequence $(v(k))$ of vectors in X and an increasing sequence of positive integers (n_k) such that*

- $\lim_{k\to+\infty}\|T^{n_k}x\|=0$ *for all $x\in\mathcal D$;*
- $\lim_{k\to+\infty}\|v(k)\|=0$;
- *for all $k\geq 0$, $\|T^{n_k}v(k)-u(k)\|\leq\varepsilon\|u(k)\|$.*

Then T is δ-hypercyclic for all $\delta>\varepsilon$.

The remaining part of the paper is devoted to the proof of Theorem 1.2.

2. Proofs

2.1. **A geometric lemma in dimension 2.** The construction ultimately relies on the following fact regarding normed spaces of dimension 2. It deals with the distance of some fix vector to lines depending on a parameter.

Lemma 2.1. *Let F be a normed space of dimension 2, let (u,v) be a normalized basis of F, let (u^*,v^*) be the dual basis and assume that $\|u^*\|=\|v^*\|=1$. For all $\varepsilon\in(0,1)$, there exists $\omega\in[\varepsilon,\varepsilon(1-\varepsilon)^{-1}]$ such that*

$$\min_{y\in\mathbb{C}}\|(y-1)u+y\omega v\|=\varepsilon.$$

Proof. When $\omega=\varepsilon$,

$$\min_{y\in\mathbb{C}}\|(y-1)u+y\omega v\|\leq\|\omega v\|\leq\varepsilon.$$

When $\omega=\varepsilon(1-\varepsilon)^{-1}$, for all $y\in\mathbb{C}$,

$$\|(y-1)u+y\omega v\|\geq\max(|y-1|,|y|\omega).$$

Now, if $|y|\geq 1/(1+\omega)$, $|y|\omega\geq\omega/(1+\omega)\geq\varepsilon$ and if $|y|\leq 1/(1+\omega)$,

$$|y-1|\geq 1-\frac{1}{1+\omega}=\varepsilon.$$

Therefore, $\min_{y\in\mathbb{C}}\|(y-1)u+y\omega v\|\geq\varepsilon$. The result follows by continuity of $\omega\mapsto\min_{y\in\mathbb{C}}\|(y-1)u+y\omega v\|$. □

Remark 2.2. If $\|au+bv\|=(|a|^p+|b|^p)^{1/p}$ for some $p\in(1,+\infty)$, then it is easy to prove that the value of ω is given by

$$\frac{\omega}{(1+\omega^{\frac{p}{p-1}})^{\frac{p-1}{p}}}=\varepsilon$$

and that the minimum is attained at

$$y=\frac{1}{1+\omega^{\frac{p}{p-1}}}.$$

When $p=1$, $\omega=\varepsilon$ and $y=1$. When $p=\infty$, $\omega=\frac{\varepsilon}{1-\varepsilon}$ and $y=\frac{1}{1+\omega}=1-\varepsilon$. This corresponds to the extremal cases of Lemma 2.1.

2.2. **The construction of a sequence of operators on X.** As the previous constructions of ε-hypercyclic operators which are not δ-hypercyclic, our operator will be an operator weighted shift. The next part of the proof consists in defining his weights. We denote by (e_n) (resp. (f_n)) the 1-unconditional basis of X (resp. Y) which appears in the statement of Theorem 1.2. We may assume that (e_n) is normalized which implies (by 1-unconditionality) that (e_n^*) is normalized too.

The strategy is the following. At each step k we will define weights $A_{m_k+1},\cdots,A_{m_{k+1}}$ such that the products $A_{m_k+1}\cdots A_j$, $j=m_k+1,\dots,m_{k+1}$ leave e_0 invariant, send e_k onto the line defined by Lemma 2.1 and e_l onto a multiple of e_l for $l\neq k$. Therefore, provided $e_0^*(u)$ is small, $A_{m_k+1}\cdots A_j(u)$ can be close to e_0, but not too close.

We proceed with the details. We set

$$\lambda=\frac{3}{\varepsilon(1-\varepsilon)}\text{ and }\kappa=(1+\lambda)+\max(1+\varepsilon(1-\varepsilon)^{-1},2/\varepsilon).$$

We exhibit two sequences of integers $(m_k)_{k\geq1}$ and $(r_k)_{k\geq1}$ and a sequence of operators $(A_j)_{j\geq1}$ on X such that, for all $k\geq1$,

(i) $A_ne_0=e_0$ for all $n=m_k+1,\dots,m_{k+1}$;
(ii) A_n is invertible, $\|A_n\|\leq\kappa$ for all $n=m_k+1,\dots,m_{k+1}$;

(iii) $A_{m_k+1}\cdots A_{m_{k+1}}=\mathrm{Id}$.

We initialize the construction by setting $m_1=0$. We assume that the construction has been done until step $k-1$ to do it at step $k\geq 1$. We thus have to define m_{k+1}, r_k and $(A_j)_{j=m_k+1,\dots,m_{k+1}}$. We set $F_k=\mathrm{span}(e_0,e_k)$ and $G_k=\overline{\mathrm{span}}(e_l:\ l\neq 0,k)$ so that $X=F_k\oplus G_k$. Let $\omega_k\in[\varepsilon,\varepsilon(1-\varepsilon)^{-1}]$ be given by Lemma 2.1 for $F=F_k$ and let $y_k\in\mathbb{C}$ minimizing $y\mapsto\|(y-1)e_0+y\omega_k e_k\|$. Since (e_n) is a 1-unconditional basis of X, we deduce from the definition of ω_k and y_k that

$$\min_{y\in\mathbb{C},\ w\in G_k}\|(y-1)e_0+y\omega_k e_k+w\|=\|(y_k-1)e_0+y_k\omega_k e_k\|=\varepsilon. \tag{1}$$

Let $r_k>0$ be a very large integer (more precise conditions on r_k will be given later) and let us set $m_{k+1}=m_k+r_k+k+1$. For $j=1,\dots,r_k+k+1$, we define A_{m_k+j} by

- $A_{m_k+j}(e_0)=e_0$.
- $$\left\{\begin{array}{rcl} A_{m_k+1}(e_k) &=& e_0+\omega_k e_k\\ A_{m_k+2}(e_k) &=& \cdots\ =\ A_{m_k+r_k}(e_k)=2e_k\\ A_{m_k+r_k+1}(e_k) &=& \cdots\ =A_{m_k+r_k+k-1}(e_k)=e_k\\ A_{m_k+r_k+k}(e_k) &=& \frac{1}{2^{r_k-1}}e_k\\ A_{m_k+r_k+k+1}(e_k) &=& -\frac{1}{\omega_k}e_0+\frac{1}{\omega_k}e_k. \end{array}\right.$$
- for $l\neq 0,k$,
$$\left\{\begin{array}{rcl} A_{m_k+j}(e_l) &=& \lambda e_l,\ j=1,\dots,r_k+k,\\ A_{m_k+r_k+k+1}(e_l) &=& \frac{1}{\lambda^{r_k+k}}e_l. \end{array}\right.$$

The invertibility of each A_n comes from the invertibility of its restriction to F_k and to G_k. Furthermore we prove $\|A_n\|\leq\kappa$. For $n=m_k+1,\dots,m_{k+1}$, for $a,b\in\mathbb{C}$ and $w\in G_k$,

$$\begin{aligned}\big\|A_n(ae_0+be_k+w)\big\| &\leq |a|+|b|\max(1+\omega_k,2,2/\omega_k)+\lambda\|w\|\\ &\leq\kappa\|ae_0+be_k+w\|\end{aligned}$$

where we have taken into account that $\omega_k\in[\varepsilon,\varepsilon(1-\varepsilon)^{-1}]$.

To go further with the properties of (A_j) we need to compute $A_{m_k+1}\cdots A_{m_k+j}e_k$ for $j=1,\dots,r_k+k+1$. We find

$$A_{m_k+1}\cdots A_{m_k+j}e_k=\left\{\begin{array}{ll} e_0+\omega_k e_k & j=1\\ 2^{j-1}e_0+2^{j-1}\omega_k e_k & j=2,\dots,r_k\\ 2^{r_k-1}e_0+2^{r_k-1}\omega_k e_k & j=r_k+1,\dots,r_k+k-1\\ e_0+\omega_k e_k & j=r_k+k\\ e_k & j=r_k+k+1. \end{array}\right.$$

We then deduce the following formula, which will be equally important:

$$A_{m_k+j}^{-1}\cdots A_{m_k+1}^{-1}e_k=\left\{\begin{array}{ll} -\frac{1}{\omega_k}e_0+\frac{1}{\omega_k}e_k & j=1\\ -\frac{1}{\omega_k}e_0+\frac{1}{2^{j-1}\omega_k}e_k & j=2,\dots,r_k\\ -\frac{1}{\omega_k}e_0+\frac{1}{2^{r_k-1}\omega_k}e_k & j=r_k+1,\dots,r_k+k-1\\ -\frac{1}{\omega_k}e_0+\frac{1}{\omega_k}e_k & j=r_k+k\\ e_k & j=r_k+k+1. \end{array}\right.$$

2.3. **The operator.** We now glue together these maps. We formally define T on $Z = \bigoplus_Y X$ by

$$T(u_0, u_1, \dots) = (A_1 u_1, A_2 u_2, \dots).$$

Let K_1 be the norm of the backward shift operator associated to (f_n). Then for $u = (u_0, u_1, \dots)$,

$$\begin{aligned}
\|Tu\| &= \left\| \sum_{n=1}^{+\infty} \|A_n u_n\|_X f_{n-1} \right\|_Y \\
&\leq K_1 \left\| \sum_{n=1}^{+\infty} \|A_n\| \cdot \|u_n\|_X f_n \right\|_Y \\
&\leq K_1 \kappa \|u\|
\end{aligned}$$

which implies that T is well defined and maps boundedly Z into itself.

2.4. T **is not** δ**-hypercyclic for any** $\delta \in (0, \varepsilon)$**.** By contradiction, assume that T is δ-hypercyclic for some $\delta \in (0, \varepsilon)$ and let $u = (u_0, u_1, \dots)$ be a δ-hypercyclic vector for T. Observe that $\|u_n\| \to 0$ so that $u_{n,0} := e_0^*(u_n) \to 0$. Therefore it is possible to fix $K > 0$ such that

$$|K - u_{n,0}|\varepsilon > \delta K \text{ for any } n \geq 0.$$

We set $v = (Ke_0, 0, \dots)$. Let $n \geq 1$ be such that $\|v - T^n u\| \leq \delta \|v\|$ and let $k \geq 1$ be such that $n \in [m_k + 1, m_{k+1}]$. Let us write $u_n = u_{n,0} e_0 + w_n$ with $e_0^*(w_n) = 0$. Then by using (i) and (iii),

$$\begin{aligned}
\|v - T^n u\| &\geq \|K e_0 - A_1 \cdots A_n(u_n)\| \\
&\geq \|K e_0 - A_{m_k+1} \cdots A_n(u_{n,0} e_0 + w_n)\| \\
&\geq \|(K - u_{n,0}) e_0 - A_{m_k+1} \cdots A_n(w_n)\|.
\end{aligned}$$

If $n = m_{k+1}$, then $A_{m_k+1} \cdots A_n(w_n) = w_n$ and

$$\|v - T^n u\| \geq |K - u_{n,0}| \geq \varepsilon |K - u_{n,0}| > \delta \|v\|.$$

If $n \neq m_{k+1}$, then $A_{m_k+1} \cdots A_n(w_n) = x_n e_0 + x_n \omega_k e_k + w_n'$ for some $w_n' \in G_k$ and some $x_n \in \mathbb{C}$. Therefore

$$\begin{aligned}
\|v - T^n u\| &\geq \left\| \big((K - u_{n,0}) - x_n\big) e_0 + x_n \omega_k e_k \right\| \\
&\geq \varepsilon |K - u_{n,0}| > \delta \|v\|
\end{aligned}$$

where we have used (1). In both cases, we find a contradiction.

2.5. T **is** δ**-hypercyclic for all** $\delta \in (\varepsilon, 1)$**.** Let $\delta \in (\varepsilon, 1)$ and let us prove that T is δ-hypercyclic by applying Theorem 1.3. Let $(u(k))$ be a dense sequence in Z such that each $u(k)$ may be written $u(k) = (u_0(k), \dots, u_{k-1}(k), 0, \dots)$ with $u_j(k) \in \operatorname{span}(e_0, \dots, e_{k-1})$ and $\|u_j(k)\| \leq k$. Moreover for any $k \geq 1$, we assume that there exist infinitely many integers ℓ with $u(k) = u(\ell)$.

We want to find a sequence of vectors $(v(k))$ in Z and a sequence of integers (n_k) such that $\|v(k)\| \to 0$ and $\|T^{n_k} v(k) - u(k)\| \leq \varepsilon \|u(k)\|$ for all $k \geq 1$. We will define $v(k) =$

$(0,\dots,0,v_0(k),\dots,v_{k-1}(k),0,\dots)$ where $v_0(k)$ is at the (m_k+r_k)-th position. Let $k\geq 1$, let $j\in\{0,\dots,k-1\}$ and let us write

$$u_j(k)=\sum_{s=0}^{k-1}u_{j,s}(k)e_s.$$

Let l be the unique integer such that $m_l\leq j<m_{l+1}$. We will search $v_j(k)$ under the form

$$v_j(k)=\sum_{\substack{s=1\\ s\neq l}}^{k-1}\frac{u_{j,s}(k)}{\lambda^{r_k+j}}\lambda^{j-m_l}e_s+xe_l+ye_k$$

where x and y will be chosen so that

$$\|A_{j+1}\cdots A_{m_k+r_k+j}(v_j(k))-u_j(k)\|\leq\varepsilon\|u_j(k)\|$$

and $\|v_j(k)\|\leq k^{-2}$. Upon this has been done, we can easily apply Theorem 1.3 to deduce that T is δ-hypercyclic for $\delta>\varepsilon$. Indeed, T has a dense generalized kernel and, for all $k\geq 1$,

$$\begin{aligned}\|v(k)\|&=\left\|\sum_{j=0}^{k-1}\|v_j(k)\|f_{m_k+r_k+j}\right\|\\&\leq\sum_{j=0}^{k-1}\|v_j(k)\|\leq\frac{1}{k}.\end{aligned}$$

Moreover

$$T^{m_k+r_k}(v(k))=(A_1\cdots A_{m_k+r_k}(v_0(k)),\dots,A_k\cdots A_{m_k+r_k+k-1}(v_{k-1}(k)),0,\dots).$$

Therefore,

$$\begin{aligned}\|u(k)-T^{m_k+r_k}(v(k))\|&=\left\|\sum_{j=0}^{k-1}\|A_{j+1}\cdots A_{m_k+r_k+j}(v_j(k))-u_j(k)\|\,f_j\right\|\\&\leq\varepsilon\left\|\sum_{j=0}^{k-1}\|u_j(k)\|f_j\right\|\\&\leq\varepsilon\|u(k)\|.\end{aligned}$$

So let us compute $A_{j+1}\cdots A_{m_k+r_k+j}(v_j(k))=:z_j(k)$.

$$\begin{aligned}z_j(k)&=A_j^{-1}\cdots A_1^{-1}A_1\cdots A_{m_k+r_k+j}(v_j(k))\\&=A_j^{-1}\cdots A_{m_l+1}^{-1}A_{m_k+1}\cdots A_{m_k+r_k+j}(v_j(k))\\&=A_j^{-1}\cdots A_{m_l+1}^{-1}\left(\sum_{\substack{s=1\\ s\neq l}}^{k-1}\lambda^{j-m_l}u_{j,s}(k)e_s+\lambda^{r_k+j}xe_l+2^{r_k-1}ye_0+2^{r_k-1}y\omega_ke_k\right).\end{aligned}$$

The easiest case is when $j=m_l$. In that case,

$$z_j(k)=2^{r_k-1}ye_0+\sum_{\substack{s=1\\ s\neq l}}^{k-1}u_{j,s}(k)e_s+\lambda^{r_k+j}xe_l+2^{r_k-1}y\omega_k e_k.$$

We simply choose $x=\frac{1}{\lambda^{r_k+j}}u_{j,l}(k)$ and $y=\frac{y_k}{2^{r_k-1}}u_{j,0}(k)$ so that by (1)

$$\begin{aligned}\|z_j(k)-u_j(k)\|&=|u_{j,0}(k)|\cdot\|(y_k-1)e_0+y_k\omega_k e_k\|\\ &\leq\varepsilon|u_{j,0}(k)|\leq\varepsilon\|u_j(k)\|\end{aligned}$$

whereas

$$\|v_j(k)\|\leq\sum_{s=1}^{k-1}\frac{\|u_j(k)\|}{\lambda^{r_k}}+\frac{|y_k|\cdot\|u_j(k)\|}{2^{r_k-1}}\leq k^{-2}$$

provided r_k is sufficiently large.
Let us now turn to $j>m_l$. In that case, there exists $0\leq t_j\leq j-m_l$ such that

$$z_j(k)=\sum_{\substack{s=1\\ s\neq l}}^{k-1}u_{j,s}(k)e_s+\frac{\lambda^{r_k+j}}{\omega_l 2^{t_j}}xe_l+\left(2^{r_k-1}y-\frac{\lambda^{r_k+j}x}{\omega_l}\right)e_0+\frac{2^{r_k-1}\omega_k y}{\lambda^{j-m_l}}e_k.$$

If we set to simplify the notations

$$\begin{array}{rclcrcl}\lambda_j&=&\lambda^{j-m_l}&\qquad&\mu_j&=&2^{t_j}\\ x'&=&\dfrac{\lambda^{r_k+j}}{\omega_l 2^{t_j}}x&&y'&=&\dfrac{2^{r_k-1}}{\lambda^{j-m_l}}y\\ a&=&u_{j,0}(k)&&b&=&u_{j,l}(k)\end{array}$$

then

$$z_j(k)-u_j(k)=\big(\lambda_j y'-\mu_j x'-a\big)e_0+(x'-b)e_l+y'\omega_k e_k.$$

We are now ready to choose x' and y', namely x and y. We indeed set

$$x'=b\text{ and }y'=\frac{a+\mu_j b}{\lambda_j}$$

so that

$$z_j(k)-u_j(k)=\left(\frac{a+\mu_j b}{\lambda_j}\right)\omega_k e_k.$$

Therefore, by 1-unconditionality of (e_k), since $t_j\leq j-m_l$, $\lambda\geq 2$ and $\omega_k\leq(1-\varepsilon)^{-1}$,

$$\begin{aligned}\|z_j(k)-u_j(k)\|&\leq\frac{1+\mu_j}{\lambda_j}\omega_k\|u_j(k)\|\\ &\leq\left(\frac{1}{\lambda}+\frac{2}{\lambda}\right)\omega_k\|u_j(k)\|\\ &\leq\frac{3}{\lambda(1-\varepsilon)}\|u_j(k)\|\leq\varepsilon\|u_j(k)\|\end{aligned}$$

by our choice of λ. Finally observe that

$$\begin{aligned}\|v_j(k)\| &\leq \sum_{\substack{s=1\\ s\neq l}}^{k-1} \frac{\|u_j(k)\|}{\lambda^{r_k}} + \frac{\omega_l 2^{t_j}}{\lambda^{r_k}}\|u_j(k)\| + \frac{\lambda^{j-m_l}}{2^{r_k-1}} \times \frac{3}{\lambda}\|u_j(k)\| \\ &\leq k\left(\sum_{s=1}^{k-1}\frac{1}{\lambda^{r_k}} + \frac{\omega_l 2^{m_k}}{\lambda^{r_k}} + \frac{3\lambda^{m_k-1}}{2^{r_k-1}}\right) \leq k^{-2}\end{aligned}$$

provided r_k has been chosen large enough.

2.6. **Concluding remarks and questions.** Observe that in Theorem 1.2, we assume a priori that (f_n) is normalized which was not the case for (e_n). Indeed, if we normalize (f_n), this could destroy the continuity of the associated backward shift operator.
Following [5], we can slightly enlarge the scale of spaces where it is possible to produce such an example. Indeed, observe that, by adjusting (m_k) and (r_k) during the construction (we may ask that r_k is bigger than any prescribed value), it is possible to ensure that T satisfies the ε-hypercyclicity criterion with the sequence (n_k) chosen as a subsequence of a prescribed sequence (p_k). Arguing like in [5, Theorem 4.10], we get therefore the following corollary.

Corollary 2.3. *Let Z be a separable Banach space. Assume that Z admits an infinite dimensional complemented subspace $\bigoplus_Y X$, where X is an infinite dimensional separable Banach space with a 1-unconditional basis and Y is an infinite dimensional separable Banach space with a normalized 1-unconditional basis such that the associated backward shift operator is continuous. Then for all $\varepsilon \in (0,1)$, there exists an operator on Z which is not δ-hypercyclic for all $\delta \in (0,\varepsilon)$ and which is δ-hypercyclic for all $\delta \in (\varepsilon, 1)$.*

Writing $V = \bigoplus_Y X$ and $Z = V \oplus W$, the main step is to define $T = T_1 \oplus T_2$ where T_2 is a hypercyclic operator satisfying the hypercyclicity criterion and T_1 is the operator defined above. In particular, if Z contains a complemented copy of $c_0(\mathbb{N})$ or of $\ell^p(\mathbb{N})$, $p \in [1, +\infty)$, then it satisfies the assumptions of Corollary 2.3.
To conclude, we observe that we can give an additional property of our operator when it is defined on ℓ^1.

Theorem 2.4. *Assume that $X = Y = \ell^1$. Then T is not ε-hypercyclic.*

Proof. By contradiction assume that u is an ε-hypercyclic vector for T. Let us introduce $M = \max |u_{n,0}|$, $v = (e_0, 0, \dots)$, $I = \left[\frac{2M}{1-\varepsilon}, +\infty\right)$ and for $n \in \mathbb{N}$,

$$I_n = I \cap \{a \in \mathbb{R} : \ \|T^n u - av\| \leq \varepsilon a\}$$

so that $I = \bigcup_n I_n$. We first observe that if $n = m_k$ for some k, then I_n is empty. Indeed, for these values of n, since $A_1 \cdots A_n u_n = u_n$, we get

$$\begin{aligned} \|T^n u - av\| &\geq \|u_n - ae_0\| \\ &\geq |u_{n,0} - a| \\ &\geq a - \frac{1-\varepsilon}{2} a \\ &\geq \left(\frac{1}{2} + \frac{\varepsilon}{2}\right) a > \varepsilon a = \varepsilon \|v\|. \end{aligned}$$

Let $\mathcal{N} = \{n \in \mathbb{N} : \ I_n \neq \varnothing\}$. For $n \in \mathcal{N}$, let $a \in I_n$. Since $n \neq m_k$, we know that $A_1 \cdots A_n(u_n) = (u_{n,0} + x_n)e_0 + x_n \varepsilon e_k + w_n$ for some $x_n \in \mathbb{C}$ and some $w_n \in \ell^1$ with $e_0^*(w_n) = e_k^*(w_n) = 0$. Therefore

$$\begin{aligned} \varepsilon a &\geq \|T^n u - av\| \\ &\geq |u_{n,0} + x_n - a| + |x_n|\varepsilon. \end{aligned}$$

Arguing as above, we find $\Re e(x_n) \geq 0$ so that

$$\varepsilon a \geq |\Re e(u_{n,0}) + \Re e(x_n) - a| + \Re e(x_n)\varepsilon.$$

We thus find

$$\begin{cases} \varepsilon a \geq \Re e(u_{n,0}) + \Re e(x_n) - a + \Re e(x_n)\varepsilon \\ \varepsilon a \geq a - \Re e(x_n) - \Re e(u_{n,0}) + \Re e(x_n)\varepsilon \end{cases}$$

which in turn yields

$$\Re e(x_n) + \frac{\Re e(u_{n,0})}{1+\varepsilon} \leq a \leq \Re e(x_n) + \frac{\Re e(u_{n,0})}{1-\varepsilon}.$$

In particular, $\Re e(u_{n,0})$ must be positive and I_n is contained in an interval of length $c\Re e(u_{n,0})$ for some $c > 0$. But since we are working on ℓ^1, $\sum_n |\Re e(u_{n,0})| < +\infty$, which contradicts that $I = \bigcup_n I_n$ has infinite length. $\square$

Corollary 2.5. *Let $\varepsilon \in (0,1)$. There exists an operator T on ℓ^1 which is δ-hypercyclic operator for all $\delta \in (\varepsilon, 1)$ and which is not ε-hypercyclic.*

This leads to the following natural question:

Question 2.6. *Let $\varepsilon \in (0,1)$. Does there exist an operator which is δ-hypercyclic if and only if $\delta \in [\varepsilon, 1)$?*

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