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# Spectrum of the Laplacian with mixed boundary conditions in a chamfered quarter of layer

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**Abstract.** We investigate the spectrum of a Laplace operator with mixed boundary conditions in an unbounded chamfered quarter of layer. The geometry depends on two parameters gathered in some vector  $\kappa = (\kappa_1, \kappa_2)$  which characterizes the domain at the edges. We identify the essential spectrum and establish different results concerning the discrete spectrum with respect to  $\kappa$ . In particular, we show that for a given  $\kappa_1 > 0$ , there is some  $h(\kappa_1) > 0$  such that discrete spectrum exists for  $\kappa_2 \in (-\kappa_1, 0) \cup (h(\kappa_1), \kappa_1)$  whereas it is empty for  $\kappa_2 \in [0; h(\kappa_1)]$ . The proofs rely on classical arguments of spectral theory such as the max-min principle. The main originality lies rather in the delicate use of the features of the geometry.

**Key words.** Laplacian with mixed boundary conditions, chamfered quarter of layer, Fichera layer, discrete spectrum, trapped modes.

**Mathematics Subject Classification.** 35J05, 35P15, 35Q40, 81Q10, 65N25

## 1 Formulation of the problem

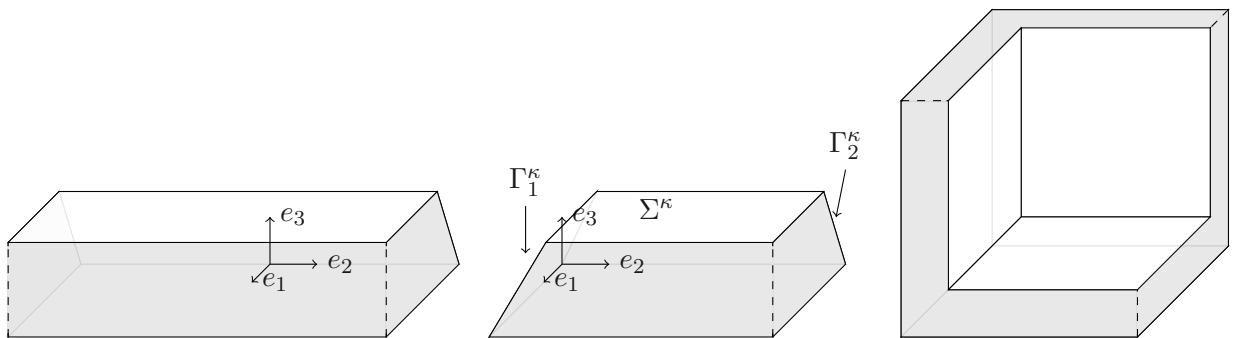


Figure 1: Domains  $\mathcal{B}^{\kappa_1}$  (left) and  $\Omega^\kappa$  (centre). Fichera layer  $\mathcal{F}$  (right).

In this work, we study the spectrum of a Laplace operator with mixed boundary conditions in a chamfered quarter of layer characterized by two parameters  $\kappa_1, \kappa_2 \in \mathbb{R}$  gathered in some vector  $\kappa = (\kappa_1, \kappa_2)$ . To describe the geometry, referring to carpentry and locksmith tools, we first define the “blade”

$$\mathcal{B}^{\kappa_1} := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 > \kappa_1 x_3, x_2 \in \mathbb{R}, x_3 \in (0, 1)\} \quad (1)$$

(see Figure 1 left). Then we introduce the “incisor”

$$\Omega^\kappa := \{x \in \mathcal{B}^{\kappa_1} \mid x_2 > \kappa_2 x_3\} \quad (2)$$

(see Figure 1 centre). Let us give names to the different components of the boundary  $\partial\Omega^\kappa$  of  $\Omega^\kappa$ . First, denote by  $\Sigma^\kappa$  the union of the two “horizontal” quadrants:

$$\Sigma^\kappa := \{x \in \partial\Omega^\kappa \mid x_3 = 0 \text{ or } x_3 = 1\}.$$

Then consider the laterals sides of the incisor. Set

$$\Gamma_1^\kappa := \partial\Omega^\kappa \cap \mathcal{B}^{\kappa_1}, \quad \Gamma_2^\kappa := \partial\Omega^\kappa \setminus \{\overline{\Gamma_1^\kappa} \cup \overline{\Sigma^\kappa}\}. \quad (3)$$

We study the spectral problem with mixed boundary conditions

$$\begin{cases} -\Delta_x u = \lambda u & \text{in } \Omega^\kappa \\ u = 0 & \text{on } \Sigma^\kappa \\ \partial_\nu u = 0 & \text{on } \Gamma_1^\kappa \cup \Gamma_2^\kappa, \end{cases} \quad (4)$$

where  $\partial_\nu$  is the outward normal derivative on  $\partial\Omega^\kappa$ . Denote by  $H_0^1(\Omega^\kappa; \Sigma^\kappa)$  the Sobolev space of functions of  $H^1(\Omega^\kappa)$  vanishing on  $\Sigma^\kappa$ . Classically (see e.g. [12, 13]), the variational formulation of Problem (4) writes

$$\begin{cases} \text{Find } (\lambda, u) \in \mathbb{R} \times H_0^1(\Omega^\kappa; \Sigma^\kappa) \setminus \{0\} \text{ such that} \\ (\nabla_x u, \nabla_x \psi)_{\Omega^\kappa} = \lambda (u, \psi)_{\Omega^\kappa} \quad \forall \psi \in H_0^1(\Omega^\kappa; \Sigma^\kappa), \end{cases} \quad (5)$$

where for a domain  $\Xi$ ,  $(\cdot, \cdot)_\Xi$  stands for the inner product of the Lebesgue spaces  $L^2(\Xi)$  or  $(L^2(\Xi))^3$  according to the case. Using the homogeneous Dirichlet condition on  $\Sigma^\kappa$  for the functions in  $H_0^1(\Omega^\kappa; \Sigma^\kappa)$ , one can prove that there holds the Friedrichs inequality

$$\|u; L^2(\Omega^\kappa)\|^2 \leq c_\kappa \|\nabla_x u; L^2(\Omega^\kappa)\|^2 \quad \forall u \in H_0^1(\Omega^\kappa; \Sigma^\kappa).$$

Classically (see e.g [3, §10.1], [18, Ch. VIII.6]), the variational problem (5) gives rise to the unbounded operator  $A^\kappa$  of  $L^2(\Omega^\kappa)$  such that

$$\begin{aligned} A^\kappa : \mathcal{D}(A^\kappa) &\rightarrow L^2(\Omega^\kappa) \\ u &\mapsto A^\kappa u = -\Delta u, \end{aligned}$$

with  $\mathcal{D}(A^\kappa) := \{u \in H_0^1(\Omega^\kappa; \Sigma^\kappa) \mid \Delta u \in L^2(\Omega^\kappa) \text{ and } \partial_\nu u = 0 \text{ on } \Gamma_1^\kappa \cup \Gamma_2^\kappa\}$ . The operator  $A^\kappa$  is positive definite and selfadjoint. Since  $\Omega^\kappa$  is unbounded, the embedding  $H_0^1(\Omega^\kappa; \Sigma^\kappa) \subset L^2(\Omega^\kappa)$  is not compact and  $A^\kappa$  has a non-empty essential component  $\sigma_{\text{ess}}(A^\kappa)$  ([3, Thm. 10.1.5]). Let us mention that the case

$$\kappa_1 = \kappa_2 = 1$$

plays a particular role. Indeed in this situation, if  $u$  is an eigenfunction associated with an eigenvalue of  $A^\kappa$ , by extending  $u$  via even reflections with respect to the faces  $\Gamma_1^\kappa, \Gamma_2^\kappa$ , one gets an eigenvalue of the Dirichlet Laplacian in the Fichera layer

$$\mathcal{F} = \bigcup_{j=1,2,3} \{x \mid x_j \in (0, 1), x_k > 0, k \neq j\}$$

represented in Figure 1 right (see [8] for the original article that gave rise to the name). This latter problem has been studied in [5, 1]. More precisely, in [5] the authors give a characterization of the essential spectrum of the Dirichlet Laplacian and show that the discrete spectrum has at most a finite number of eigenvalues. The existence of discrete spectrum is proved in [1, Thm. 2].

The goal of this paper is to get similar information for the operator  $A^\kappa$  with respect to the parameter  $\kappa$ . Observe that by exchanging the axes or modifying their orientations, there is no loss of generality to restrict our study to the cases

$$\kappa_1 \geq 0, \quad |\kappa_2| \leq \kappa_1. \quad (6)$$

Note that we consider mixed boundary conditions in (3) because we have in mind to study the spectrum of the Dirichlet Laplacian in geometries similar to the Fichera layer of Figure 1. Indeed, playing with symmetries, we can reduce the analysis of certain of these problems to the one of (3) (see Figure 13 for an example). On the other hand, let us mention that the spectrum of the Dirichlet Laplacian in  $\Omega^\kappa$  has a rather simple structure with only essential spectrum and no discrete spectrum. This is a consequence of Proposition 8.1 in Appendix.

This note is organized as follows. In Section 2, we describe the essential spectrum of  $A^\kappa$  (Theorem 2.1). Then in Section 3, we state the results for the discrete spectrum of  $A^\kappa$  (the main outcome of the present work is Theorem 3.2). The next four sections contain the proof of the different items of Theorem 3.2. In Section 8, we illustrate the theory with some numerical results. Finally we establish the above mentioned result related to the Dirichlet Laplacian in  $\Omega^\kappa$  in the Appendix.

## 2 Essential spectrum

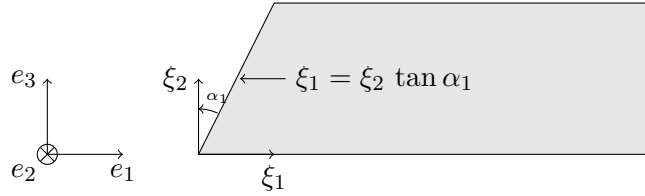


Figure 2: Domain  $\Pi^{\alpha_1}$  corresponding with a cut of the blade  $\mathcal{B}^{\kappa_1}$  in the plane  $x_2 = 0$ .

Introducing the angle  $\alpha_1 \in [0, \pi/2)$  such that  $\kappa_1 = \tan \alpha_1$ , the blade (1) can also be defined as

$$\mathcal{B}^{\tan \alpha_1} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_3) \in \Pi^{\alpha_1}\}$$

where  $\Pi^{\alpha_1}$  stands for the 2D pointed strip

$$\Pi^{\alpha_1} := \{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1 > \xi_2 \tan \alpha_1, \xi_2 \in (0, 1)\} \quad (7)$$

(see Figure 2). To describe  $\sigma_{\text{ess}}(A^\kappa)$ , we need information on the spectrum of the auxiliary planar problem

$$\begin{cases} -\Delta_\xi v = \mu v & \text{in } \Pi^{\alpha_1} \\ v = 0 & \text{on } \zeta^{\alpha_1} \\ \partial_\nu v = 0 & \text{on } \gamma^{\alpha_1} \end{cases} \quad (8)$$

where  $\zeta^{\alpha_1} := \{\xi \in \partial \Pi^{\alpha_1} \mid \xi_2 = 0 \text{ or } \xi_2 = 1\}$  denotes the horizontal part of  $\partial \Pi^{\alpha_1}$  and  $\gamma^{\alpha_1} := \partial \Pi^{\alpha_1} \setminus \zeta^{\alpha_1}$  stands for the oblique part of  $\partial \Pi^{\alpha_1}$ .

The continuous spectrum of Problem (8) coincides with the ray  $[\pi^2, +\infty)$ . When  $\alpha_1 = 0$  (straight end), working with the decomposition in Fourier series in the vertical direction, one can prove that the discrete spectrum is empty. On the other hand, for all  $\alpha_1 \in (0, \pi/2)$ , it has been shown in [10] that there is at least one eigenvalue below the continuous spectrum (see also [15] for more general shapes). Notice that by extending  $\Pi^{\alpha_1}$  by reflection with respect to  $\gamma^{\alpha_1}$ , we obtain a broken strip that we can also call a V-shaped domain. This allows us to exploit all the results from [7, 17] (see also [6] as well as the amendments in [16]) to get information on  $\mu_1^{\alpha_1}$ , the smallest eigenvalue of (8). In particular, the function  $\alpha_1 \mapsto \mu_1^{\alpha_1}$  is smooth and strictly decreasing on  $(0, \pi/2)$ . Additionally, we have

$$\lim_{\alpha_1 \rightarrow 0^+} \mu_1^{\alpha_1} = \pi^2, \quad \lim_{\alpha_1 \rightarrow (\pi/2)^-} \mu_1^{\alpha_1} = \frac{\pi^2}{4} \quad (9)$$

(see Figure 8 for a numerical approximation of  $\alpha_1 \mapsto \mu_1^{\alpha_1}$ ). By adapting the approach proposed in [5, §3.1], one establishes the next assertion. The only point to be commented here is that there holds

$$\lambda_{\dagger}^{\kappa} := \mu_1^{\arctan \kappa_1} \leq \mu_1^{\arctan |\kappa_2|} \quad (10)$$

because  $|\kappa_2| \leq \kappa_1$  implies  $\arctan |\kappa_2| \leq \arctan \kappa_1$  and because  $\alpha_1 \mapsto \mu_1^{\alpha_1}$  is decreasing.

**Theorem 2.1.** *The essential spectrum  $\sigma_{\text{ess}}(A^{\kappa})$  of the operator  $A^{\kappa}$  coincides with the ray  $[\lambda_{\dagger}^{\kappa}, +\infty)$  where  $\lambda_{\dagger}^{\kappa}$  is defined in (10).*

**Remark 2.2.** *Thus the lower bound of  $\sigma_{\text{ess}}(A^{\kappa})$  is characterized by the sharpest edge of  $\Omega^{\kappa}$ .*

### 3 Discrete spectrum

For the discrete spectrum  $\sigma_{\text{d}}(A^{\kappa})$ , our main results are as follows:

**Theorem 3.1.** *For  $\kappa_1 = \kappa_2 = 0$  (straight edges),  $\sigma_{\text{d}}(A^{\kappa})$  is empty.*

**Theorem 3.2.** *Assume that  $\kappa_1 > 0$ .*

1)  $\sigma_{\text{d}}(A^{\kappa})$  is non-empty for  $\kappa_2 \in [-\kappa_1, 0)$ .

2) There exists  $h(\kappa_1) > 0$  such that:

- i)  $\sigma_{\text{d}}(A^{\kappa})$  is empty for  $\kappa_2 \in [0, h(\kappa_1)]$ ;
- ii)  $\sigma_{\text{d}}(A^{\kappa})$  is non-empty for  $\kappa_2 \in (h(\kappa_1), \kappa_1]$ .

3) For  $\kappa_2 \in [-\kappa_1, 0) \cup (h(\kappa_1), \kappa_1]$ , denote by  $\lambda_1^{\kappa}$  the first (smallest) eigenvalue of  $\sigma_{\text{d}}(A^{\kappa})$ . The function  $\kappa_2 \mapsto \lambda_1^{\kappa}$  is strictly increasing on  $[-\kappa_1, 0)$  and strictly decreasing on  $(h(\kappa_1), \kappa_1]$ .

4) For  $\kappa_2 \in (-\kappa_1; \kappa_1)$ ,  $\sigma_{\text{d}}(A^{\kappa})$  contains at most a finite number of eigenvalues.

The items 1)-3) of Theorem 3.2 are illustrated by Figure 3. Note in particular that we have the following mechanism for positive  $\kappa_2$ : diminishing  $\kappa_2$  from the value  $\kappa_1$  makes the eigenvalue  $\lambda_1^{\kappa}$  to reach the threshold  $\lambda_{\dagger}^{\kappa} = \mu_1^{\arctan \kappa_1}$  at a certain  $\kappa_2 = h(\kappa_1) \in (0, \kappa_1)$ . The rest of the present note is dedicated to the proof of the above statements.

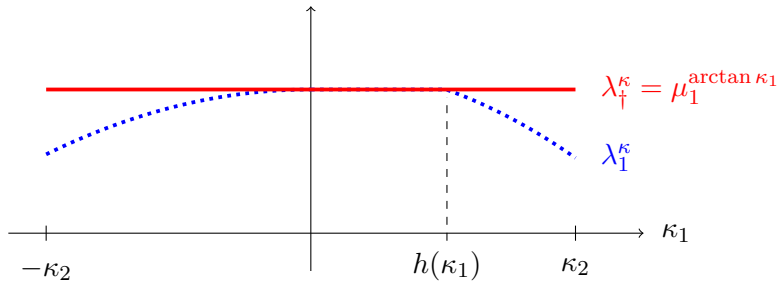


Figure 3: Picture of the behaviour of  $\lambda_1^{\kappa}$ , the smallest eigenvalue of  $\sigma_{\text{d}}(A^{\kappa})$ , for a given  $\kappa_1 > 0$  and  $\kappa_2 \in [-\kappa_1, 0) \cup (h(\kappa_1), \kappa_1]$ .

### 4 Discrete spectrum for negative $\kappa_2$

In this section, we prove the item 1) of Theorem 3.2 and so we consider the case  $\kappa_2 < 0$ . A direct application of the minimum principle, see e.g. [3, Thm. 10.2.1], [18, Thm. XIII.3], shows that the

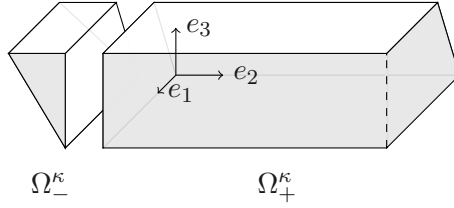


Figure 4: Domains  $\Omega_-^\kappa$  and  $\Omega_+^\kappa$ .

discrete spectrum of  $A^\kappa$  contains an eigenvalue  $\lambda_1^\kappa$  if one can find a trial function  $\psi \in H_0^1(\Omega^\kappa; \Sigma^\kappa)$  such that

$$\|\nabla_x \psi; L^2(\Omega^\kappa)\|^2 < \lambda_1^\kappa \|\psi; L^2(\Omega^\kappa)\|^2. \quad (11)$$

Let us construct a function satisfying (11). To proceed, first divide the incisor  $\Omega^\kappa$  into the two domains

$$\Omega_-^\kappa := \{x \in \Omega^\kappa \mid x_2 < 0\}, \quad \Omega_+^\kappa := \{x \in \Omega^\kappa \mid x_2 > 0\} = \Omega^{(0, \kappa_2)} \quad (12)$$

(see Figure 4). Then for  $\varepsilon > 0$  small, define  $\psi^\varepsilon$  such that

$$\psi^\varepsilon(x) = \begin{cases} v(x_1, x_3) & \text{in } \Omega_-^\kappa \\ e^{-\varepsilon x_2} v(x_1, x_3) & \text{in } \Omega_+^\kappa \end{cases} \quad (13)$$

where  $v$  is an eigenfunction of the 2D problem (8) associated with  $\mu_1^{\alpha_1}$ , the smallest eigenvalue, and  $\alpha_1 = \arctan \kappa_1$ . To set ideas, we choose  $v$  such that  $\|v; L^2(\Pi^{\alpha_1})\| = 1$ . Note that  $\psi^\varepsilon$  satisfies the homogeneous Dirichlet condition on  $\Sigma^\kappa$  and decays exponentially at infinity. Using (10), we obtain

$$\begin{aligned} & \|\nabla_x \psi^\varepsilon; L^2(\Omega_+^\kappa)\|^2 - \lambda_1^\kappa \|\psi^\varepsilon; L^2(\Omega_+^\kappa)\|^2 \\ &= (\|\nabla_\xi v; L^2(\Pi^{\alpha_1})\|^2 + (\varepsilon^2 - \mu_1^{\alpha_1}) \|v; L^2(\Pi^{\alpha_1})\|^2) \int_0^\infty e^{-2\varepsilon x_2} dx_2 = \frac{\varepsilon}{2}. \end{aligned} \quad (14)$$

As for the integral over the prism  $\Omega_-^\kappa$  with triangular cross-sections and the bevelled end, we integrate by parts and take into account the boundary conditions of (4), which yields

$$\begin{aligned} \|\nabla_x \psi^\varepsilon; L^2(\Omega_-^\kappa)\|^2 - \lambda_1^\kappa \|\psi^\varepsilon; L^2(\Omega_-^\kappa)\|^2 &= - \int_{\Omega_-^\kappa} v(x_1, x_3) (\Delta_x v(x_1, x_3) + \mu_1^{\alpha_1} v(x_1, x_3)) dx \\ &+ \int_{\Gamma_1^\kappa} v(x_1, x_3) \partial_\nu v(x_1, x_3) ds =: I_{\Omega_-^\kappa} + I_{\Gamma_1^\kappa}. \end{aligned} \quad (15)$$

Owing to (8), there holds  $I_{\Omega_-^\kappa} = 0$ . Now we focus our attention on the term  $I_{\Gamma_1^\kappa}$ . Let  $(e_1, e_2, e_3)$  denote the canonical basis of  $\mathbb{R}^3$ . Set  $\alpha_2 := \arctan \kappa_2 \in (-\pi/2, 0)$  and define the new orthonormal basis  $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$  with

$$\tilde{e}_1 = e_1; \quad \tilde{e}_2 = \cos(\alpha_2)e_2 - \sin(\alpha_2)e_3; \quad \tilde{e}_3 = \sin(\alpha_2)e_2 + \cos(\alpha_2)e_3. \quad (16)$$

Observe that the component  $\Gamma_1^\kappa$  of the boundary of the incisor  $\Omega^\kappa$  is included in the plane  $(O, \tilde{e}_1, \tilde{e}_3)$ .

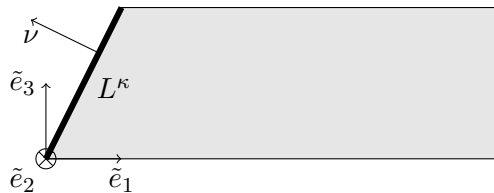


Figure 5: Domain  $\Gamma_1^\kappa$  in the plane  $(O, \tilde{e}_1, \tilde{e}_3)$ .

Let  $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$  denote the coordinates in the basis (16). We have

$$I_{\Gamma_1^\kappa} = -\frac{1}{2} \int_{\Gamma_1^\kappa} \frac{\partial(v^2)}{\partial \tilde{x}_2} d\tilde{x}_1 d\tilde{x}_3. \quad (17)$$

Using that  $v$  is independent of  $x_2$ , we obtain

$$0 = \frac{\partial(v^2)}{\partial x_2} = \cos \alpha_2 \frac{\partial(v^2)}{\partial \tilde{x}_2} + \sin \alpha_2 \frac{\partial(v^2)}{\partial \tilde{x}_3}. \quad (18)$$

Combining (18) and (17), we find

$$I_{\Gamma_1^\kappa} = \frac{\tan \alpha_2}{2} \int_{\Gamma_1^\kappa} \frac{\partial(v^2)}{\partial \tilde{x}_3} d\tilde{x}_1 d\tilde{x}_3 = \frac{\tan \alpha_2}{2} \int_{\partial \Gamma_1^\kappa} v^2 \nu \cdot \tilde{e}_3 dl = \frac{\tan \alpha_2}{2} \int_{L^\kappa} v^2 \nu \cdot \tilde{e}_3 dl, \quad (19)$$

where  $L^\kappa := \{x \in \mathbb{R}^3 \mid x_j = \kappa_j x_3, j = 1, 2, x_3 \in (0, 1)\}$  and where  $\nu$  stands for the outward unit normal vector to  $\partial \Gamma_1^\kappa$  (in the plane  $(O, \tilde{e}_1, \tilde{e}_3)$ ). Using that  $\alpha_2 \in (-\pi/2, 0)$ ,  $\nu \cdot \tilde{e}_3 > 0$  on  $L^\kappa$  (see Figure 5) and  $v \not\equiv 0$  on  $L^\kappa$ , we deduce that  $I_{\Gamma_1^\kappa} < 0$ . Note also that the quantity  $I_{\Gamma_1^\kappa}$  is independent of  $\varepsilon$ . Gathering (14) and (15), we infer that the inequality (11) holds for  $\varepsilon > 0$  small enough. This is enough to guarantee that  $\sigma_d(A^\kappa)$  is non-empty for negative  $\kappa_2$ .

## 5 Absence of eigenvalues for small positive $\kappa_2$

The goal of this section is to prove an intermediate result to establish the item 2) of Theorem 3.2. Therefore we assume that  $\kappa_2 \geq 0$ . In that situation, the integral  $I_{\Gamma_1^\kappa}$  in (19) is positive because  $\alpha_2 \in [0, \pi/2)$  and our argument of the previous section does not work for showing the existence of discrete spectrum. Of course, this does not yet guarantee that  $\sigma_d(A^\kappa)$  is empty. Actually we will see in Section 6 that  $\sigma_d(A^\kappa)$  is non-empty for certain  $\kappa$  with  $\kappa_2 > 0$ . For the moment, combining the calculations of Section 4 with the approach of [14], we show the following result.

**Proposition 5.1.** *For all  $\kappa_1 > 0$ , there exists  $\delta(\kappa_1) > 0$  such that  $\sigma_d(A^\kappa)$  is empty for*

$$\kappa_2 \in [0, \delta(\kappa_1)). \quad (20)$$

*Proof.*

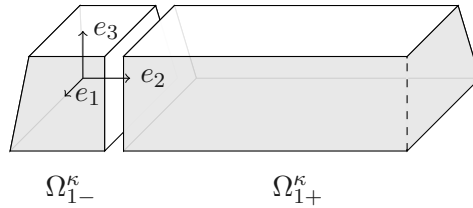


Figure 6: Domains  $\Omega_{1-}^\kappa$  and  $\Omega_{1+}^\kappa$ .

Fix  $\kappa_2 \in [0, \min(1, \kappa_1))$  and divide  $\Omega^\kappa$  into the two domains

$$\Omega_{1-}^\kappa := \{x \in \Omega^\kappa \mid x_2 < 1\} \quad \text{and} \quad \Omega_{1+}^\kappa := \{x \in \Omega^\kappa \mid x_2 > 1\} \quad (21)$$

(see Figure 6). Since  $\Omega_{1+}^\kappa = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_3) \in \Pi^{\arctan \kappa_1}, x_2 > 1\}$ , there holds

$$\|\nabla_x \psi; L^2(\Omega_{1+}^\kappa)\|^2 \geq \lambda_+^\kappa \|\psi; L^2(\Omega_{1+}^\kappa)\|^2 \quad (22)$$

for all  $\psi \in H_0^1(\Omega^\kappa; \Sigma^\kappa)$ . Below we show that there is some  $\varrho > 0$  such that for  $\kappa_2$  small enough, there holds, for all  $\psi \in H_0^1(\Omega^\kappa; \Sigma^\kappa)$ ,

$$\|\nabla_x \psi; L^2(\Omega_{1-}^\kappa)\|^2 \geq (\lambda_+^\kappa + \kappa_1 \varrho) \|\psi; L^2(\Omega_{1-}^\kappa)\|^2. \quad (23)$$

Combining (22) and (23) with the minimum principle yields the result of the proposition.

**Remark 5.2.** Note that estimate (22) implies that  $\sigma_d(A^\kappa)$  is empty for all  $\kappa = (\kappa_1, 0)$  with  $\kappa_1 \geq 0$ . This proves in particular the result of Theorem 3.1.

The remaining part of the proof is dedicated to establishing (23). Consider the mixed boundary-value problem

$$\begin{cases} -\Delta_x w = \tau w & \text{in } \Omega_{1-}^\kappa \\ w = 0 & \text{on } \{x \in \partial\Omega_{1-}^\kappa \mid x_3 = 0 \text{ or } x_3 = 1\} \\ \partial_\nu w = 0 & \text{on } \{x \in \partial\Omega_{1-}^\kappa \mid x_3 \in (0, 1)\}. \end{cases} \quad (24)$$

As pictured in Figure 6 left, the domain  $\Omega_{1-}^\kappa$  in (21) is a semi-infinite prism with a trapezoidal cross-section and a skewed end. When  $\kappa_2 = 0$ , the trapezoid is simply the unit square and the continuous spectrum of the problem (24) coincides with the ray  $[\pi^2, +\infty)$ . In that situation, the problem (24) admits an eigenvalue at  $\mu_1^{\alpha_1} \in (0, \pi^2)$  with  $\alpha_1 = \arctan \kappa_1$  (see the text above (9) for the definition of that quantity), a corresponding eigenfunction being  $w$  such that

$$w(x) = v(x_1, x_3),$$

where  $v$  is an eigenfunction of (8) associated with  $\mu_1^{\alpha_1}$ . Now let us consider the situation  $\kappa_2 > 0$  small. Then the map

$$\Omega_{1-}^\kappa \ni x \mapsto \left( x_1, \frac{x_2 - \kappa_2 x_3}{1 - \kappa_2 x_3}, x_3 \right) \in \Omega_{1-}^{(\kappa_1, 0)} \quad (25)$$

is a diffeomorphism whose Jacobian matrix is close to the identity and whose Hessian matrix is small. Using these properties, we deduce that the discrete spectrum of the problem (24) is still non-empty for  $\kappa_2$  small enough. This comes from the fact that the cut-off point of the essential spectrum satisfies the estimate

$$|\tau_1^{\kappa_2} - \pi^2| \leq C_+ \kappa_2$$

and the first (smallest) eigenvalue of the discrete spectrum, which is simple, admits the expansion

$$\tau_1^{\kappa_2} = \mu_1^{\alpha_1} + \kappa_2 \tau_1' + \tilde{\tau}_1^{\kappa_2} \quad (26)$$

with  $|\tilde{\tau}_1^{\kappa_2}| \leq C \kappa_2^2$ . Here  $C > 0$  is a constant independent of  $\kappa_2$ . These properties can be justified using classical results of the perturbation theory for linear operators, see e.g. [11, Ch. 7], [3, Ch. 10], [18, Ch. XII]. From the minimum principle, to establish (23), we see that it suffices to show that

$$\tau_1' = \left. \frac{d\tau_1^{\kappa_2}}{d\kappa_2} \right|_{\kappa_2=0} > 0. \quad (27)$$

Let  $w_1^{\kappa_2}$  be an eigenfunction of Problem (24) associated with  $\tau_1^{\kappa_2}$ . Together with (29), consider the asymptotic ansatz

$$w_1^{\kappa_2}(x) = v(x_1, x_3) + \kappa_2 w_1'(x) + \tilde{w}_1^{\kappa_2}(x) \quad (28)$$

where  $\tilde{w}_1^{\kappa_2}$  is a small remainder. Insert (29), (28) into (24) and collect the terms of order  $\kappa_2$ . We obtain

$$\begin{cases} -\Delta_x w_1' - \mu_1^{\alpha_1} w_1' = \tau_1' v & \text{in } \Omega_{1-}^{(\kappa_1, 0)} \\ w_1' = 0 & \text{on } \{x \in \partial\Omega_{1-}^\kappa \mid x_3 = 0 \text{ or } x_3 = 1\}. \end{cases} \quad (29)$$

As for the Neumann boundary condition of (24), using in particular that on  $\Gamma_1^\kappa$ ,

$$\partial_\nu \cdot = (1 + \kappa_2^2)^{-1/2} \left( -\frac{\partial \cdot}{\partial x_2} + \kappa_2 \frac{\partial \cdot}{\partial x_3} \right),$$

at order  $\kappa_2$ , we find

$$-\frac{\partial w_1'}{\partial x_2}(x_1, 0, x_3) = -\frac{\partial v}{\partial x_3}(x_1, x_3), \quad \frac{\partial w_1'}{\partial x_2}(x_1, 1, x_3) = 0, \quad (x_1, x_3) \in \Pi^{\alpha_1}. \quad (30)$$



Since the smallest eigenvalue  $\mu_1^{\alpha_1}$  is simple, there exists only one compatibility condition to satisfy to ensure that the problem (29)–(30) has a non trivial solution. It can be written as

$$\begin{aligned}\tau_1' &= \tau_1' \|v; L^2(\Pi^{\alpha_1})\|^2 \\ &= - \int_{\Omega_{1-}^{(\kappa_1, 0)}} v(\Delta_x w_1' + \mu_1^{\alpha_1} w_1') dx = \int_{\Gamma_1^{(\kappa_1, 0)}} v(x_1, x_3) \frac{\partial w_1'}{\partial x_2}(x_1, 0, x_3) ds \\ &= \int_{\Pi^{\alpha_1}} v(\xi_1, \xi_2) \frac{\partial v}{\partial \xi_2}(\xi_1, \xi_2) d\xi_1 d\xi_2 = \frac{1}{2} \cos \alpha_1 \int_{L^{(\kappa_1, 0)}} v^2 d\ell > 0\end{aligned}$$

where  $L^{(\kappa_1, 0)} := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1 = \xi_2 \tan \alpha_1, \xi_2 \in (0, 1)\}$ . This shows (27) which guarantees that estimate (23) is valid according to the minimum principle. Therefore the proof of Proposition 5.1 is complete.  $\square$

## 6 Existence of eigenvalues for $\kappa_2$ close to $\kappa_1 > 0$

We start this section by proving that the discrete spectrum  $\sigma_d(A^\kappa)$  of the operator  $A^\kappa$  can also be non-empty for certain positive  $\kappa_2$ . This happens for example in the case  $\kappa_1 = \kappa_2$ , which we now assume. We adapt the proof of [1, Thm. 2] and exhibit a function  $\varphi \in H_0^1(\Omega^\kappa; \Sigma^\kappa)$  satisfying (11). First, note that for  $\kappa_1 = \kappa_2$ , the domain  $\Omega^\kappa$  is symmetric with respect to the “bisector” cross-section

$$\Upsilon^\kappa := \{x \in \Omega^\kappa \mid x_1 = x_2\}.$$

Let us divide  $\Omega^\kappa$  into the two congruent domains

$$\Omega_\lambda^\kappa := \{x \in \Omega^\kappa \mid x_1 > x_2\} \quad \text{and} \quad \Omega_\ltimes^\kappa := \{x \in \Omega^\kappa \mid x_2 > x_1\}. \quad (31)$$

Accordingly, we set

$$\psi^\varepsilon(x) = \begin{cases} e^{-\varepsilon x_1} v(x_2, x_3) & \text{in } \Omega_\lambda^\kappa \\ e^{-\varepsilon x_2} v(x_1, x_3) & \text{in } \Omega_\ltimes^\kappa \end{cases} \quad (32)$$

where  $v$  is as in (13). Since  $\psi^\varepsilon$  is continuous on  $\Upsilon^\kappa$  and decays exponentially at infinity, it belongs to  $H_0^1(\Omega^\kappa; \Sigma^\kappa)$ . Moreover, we have

$$\begin{aligned}& \|\nabla_x \psi; L^2(\Omega_\lambda^\kappa)\|^2 - \mu_1^{\alpha_1} \|\psi; L^2(\Omega_\lambda^\kappa)\|^2 \\ &= - \int_{\Omega_\lambda^\kappa} e^{-2\varepsilon x_1} v(x_2, x_3) (\Delta_x v(x_2, x_3) + (\mu_1^{\alpha_1} + \varepsilon^2) v(x_2, x_3)) dx \\ & \quad + \int_{\Upsilon^\kappa} e^{-\varepsilon x_1} v(x_2, x_3) \partial_\nu (e^{-\varepsilon x_1} v(x_2, x_3)) ds =: I_{\Omega^\kappa}^\varepsilon + I_{\Upsilon^\kappa}^\varepsilon.\end{aligned} \quad (33)$$

Using that  $v$  solves (8), we get  $I_{\Omega^\kappa}^\varepsilon = O(\varepsilon)$ . Now consider the integral  $I_{\Upsilon^\kappa}^\varepsilon$ . Define the new orthonormal basis  $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$  with

$$\hat{e}_1 = \frac{\sqrt{2}}{2} e_1 + \frac{\sqrt{2}}{2} e_2; \quad \hat{e}_2 = -\frac{\sqrt{2}}{2} e_1 + \frac{\sqrt{2}}{2} e_2; \quad \hat{e}_3 = e_3. \quad (34)$$

Remark that  $\Upsilon^\kappa$  is included in the plane  $(O, \hat{e}_1, \hat{e}_3)$ . Let  $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$  denote the coordinates in the basis (34). We have

$$\begin{aligned}I_{\Upsilon^\kappa}^\varepsilon &= \int_{\Upsilon^\kappa} e^{-\varepsilon \sqrt{2}(\hat{x}_1 - \hat{x}_2)/2} v \frac{\partial}{\partial \hat{x}_2} (e^{-\varepsilon \sqrt{2}(\hat{x}_1 - \hat{x}_2)/2} v) d\hat{x}_1 d\hat{x}_3 \\ &= \int_{\Upsilon^\kappa} \frac{\varepsilon \sqrt{2}}{2} e^{-\varepsilon \sqrt{2}(\hat{x}_1 - \hat{x}_2)} v^2 d\hat{x}_1 d\hat{x}_3 + \int_{\Upsilon^\kappa} \frac{1}{2} e^{-\varepsilon \sqrt{2}(\hat{x}_1 - \hat{x}_2)} \frac{\partial (v^2)}{\partial \hat{x}_2} d\hat{x}_1 d\hat{x}_3.\end{aligned} \quad (35)$$

Exploiting the exponential decay of  $v(\xi)$  as  $\xi_1 \rightarrow +\infty$ , (see Section 2), one finds that the first integral of the right hand side above is  $O(\varepsilon)$ . For the second one, using that  $v$  is independent of  $x_1$ , we can write

$$0 = \frac{\partial(v^2)}{\partial x_1} = \frac{\sqrt{2}}{2} \frac{\partial(v^2)}{\partial \hat{x}_1} - \frac{\sqrt{2}}{2} \frac{\partial(v^2)}{\partial \hat{x}_2}.$$

Remarking also that  $\hat{x}_1 = 0$  on  $\Upsilon^\kappa$ , this gives

$$\begin{aligned} & \int_{\Upsilon^\kappa} \frac{1}{2} e^{-\varepsilon\sqrt{2}(\hat{x}_1 - \hat{x}_2)} \frac{\partial(v^2)}{\partial \hat{x}_2} d\hat{x}_1 d\hat{x}_3 = \int_{\Upsilon^\kappa} \frac{1}{2} e^{-\varepsilon\sqrt{2}\hat{x}_1} \frac{\partial(v^2)}{\partial \hat{x}_1} d\hat{x}_1 d\hat{x}_3 \\ & = \int_{\Upsilon^\kappa} \frac{\varepsilon\sqrt{2}}{2} e^{-\varepsilon\sqrt{2}\hat{x}_1} v^2 d\hat{x}_1 d\hat{x}_3 + \int_{L^\kappa} \frac{1}{2} e^{-\varepsilon\sqrt{2}\hat{x}_1} v^2 \nu \cdot \hat{e}_1 d\ell \end{aligned} \quad (36)$$

where  $L^\kappa := \{x \in \mathbb{R}^3 \mid x_j = \kappa_j x_3, j = 1, 2, x_3 \in (0, 1)\}$  and where  $\nu$  stands for the outward unit normal vector to  $\Upsilon^\kappa$  (in the plane  $(O, \hat{e}_1, \hat{e}_3)$ ). Using that  $\kappa_1 = \kappa_2 > 0$ , we find  $\nu \cdot \hat{e}_1 < 0$  on  $L^\kappa$ . Since there holds  $v \not\equiv 0$  on  $L^\kappa$ , gathering (35) and (36), we deduce that we have  $I_{\Upsilon^\kappa}^\varepsilon < 0$  for  $\varepsilon$  small enough. From (33), we deduce

$$\|\nabla_x \psi; L^2(\Omega_\lambda^\kappa)\|^2 - \mu_1^{\alpha_1} \|\psi; L^2(\Omega_\lambda^\kappa)\|^2 < 0$$

for  $\varepsilon$  small enough. Then by symmetry, we obtain

$$\|\nabla_x \psi; L^2(\Omega^\kappa)\|^2 - \mu_1^{\alpha_1} \|\psi; L^2(\Omega^\kappa)\|^2 = 2\|\nabla_x \psi; L^2(\Omega_\lambda^\kappa)\|^2 - 2\mu_1^{\alpha_1} \|\psi; L^2(\Omega_\lambda^\kappa)\|^2 < 0.$$

We conclude that the inequality (11) is satisfied by the function (32) which proves the following statement.

**Theorem 6.1.** *For  $\kappa_1 = \kappa_2 > 0$ , the discrete spectrum  $\sigma_d(A^\kappa)$  of the operator  $A^\kappa$  is not empty.*

Since the eigenvalues of the discrete spectrum are stable with respect to small perturbations of the operator, Theorem 6.1 and diffeomorphisms similar to (25) imply that  $\sigma_d(A^\kappa)$  is not empty for  $\kappa_2$  in a neighbourhood of  $\kappa_1$ . With Proposition 5.1, this allows us to introduce  $h(\kappa_1) \in (0, \kappa_1)$  as the infimum of the numbers  $\delta$  such that  $\sigma_d(A^\kappa)$  is non-empty for all  $\kappa_2 \in (\delta, \kappa_1]$ .

On the other hand, we have the following monotonicity result:

**Proposition 6.2.** *Consider some  $\kappa = (\kappa_1, \kappa_2)$  with  $\kappa_1 > 0$  and  $\kappa_2 \in (0, \kappa_1]$  such that  $A^\kappa$  has a non-empty discrete spectrum. Let  $\lambda_1^\kappa$  denote the first (smallest) eigenvalue of  $\sigma_d(A^\kappa)$ . For  $\varepsilon > 0$  small, set  $\kappa^\varepsilon := (\kappa_1, \kappa_2 + \varepsilon)$  and denote by  $\lambda_1^{\kappa^\varepsilon}$  the first eigenvalue of  $\sigma_d(A^{\kappa^\varepsilon})$ . Then, we have*

$$\lambda_1^{\kappa^\varepsilon} < \lambda_1^\kappa. \quad (37)$$

*Proof.* Using again the minimum principle, we can write

$$\lambda_1^{\kappa^\varepsilon} = \min_{\psi^\varepsilon \in H_0^1(\Omega^{\kappa^\varepsilon}; \Sigma^{\kappa^\varepsilon}) \setminus \{0\}} \frac{\|\nabla_x \psi^\varepsilon; L^2(\Omega^{\kappa^\varepsilon})\|^2}{\|\psi^\varepsilon; L^2(\Omega^{\kappa^\varepsilon})\|^2}. \quad (38)$$

Now define the function  $\psi^\varepsilon$  such that

$$\psi^\varepsilon(x) = u^\kappa\left(x_1, \frac{\kappa_2 x_2}{\kappa_2 + \varepsilon}, x_3\right),$$

where  $u^\kappa \in H_0^1(\Omega^\kappa; \Sigma^\kappa)$  is an eigenfunction associated with the first eigenvalue of  $\sigma_d(A^\kappa)$ . Clearly  $\psi^\varepsilon$  is a non zero element of  $H_0^1(\Omega^{\kappa^\varepsilon})$ . Besides, we find

$$\|\psi^\varepsilon; L^2(\Omega^{\kappa^\varepsilon})\|^2 = \frac{\kappa_2 + \varepsilon}{\kappa_2} \|u^\kappa; L^2(\Omega^\kappa)\|^2$$

$$\text{and} \quad \|\nabla_x \psi^\varepsilon; L^2(\Omega^{\kappa^\varepsilon})\|^2 = \frac{\kappa_2}{\kappa_2 + \varepsilon} \left\| \frac{\partial u^\kappa}{\partial x_2}; L^2(\Omega^\kappa) \right\|^2 + \frac{\kappa_2 + \varepsilon}{\kappa_2} \sum_{j=1,3} \left\| \frac{\partial u^\kappa}{\partial x_j}; L^2(\Omega^\kappa) \right\|^2.$$

According to (38), these identities imply

$$\begin{aligned} \lambda_1^{\kappa^\varepsilon} &\leq \|u^\kappa; L^2(\Omega^\kappa)\|^{-2} \left( \frac{\kappa_2^2}{(\kappa_2 + \varepsilon)^2} \left\| \frac{\partial u^\kappa}{\partial x_2}; L^2(\Omega^\kappa) \right\|^2 + \sum_{j=1,3} \left\| \frac{\partial u^\kappa}{\partial x_j}; L^2(\Omega^\kappa) \right\|^2 \right) \\ &\leq \frac{\|\nabla_x u^\kappa; L^2(\Omega^\kappa)\|^2}{\|u^\kappa; L^2(\Omega^\kappa)\|^2} = \lambda_1^\kappa. \end{aligned}$$

The strict inequality in (37) follows from the fact that the derivative  $\partial u^\kappa / \partial x_2$  cannot be null in the whole domain  $\Omega^\kappa$ . This completes the proof of the proposition.  $\square$

According to relation (37), the function  $\kappa_2 \mapsto \lambda_1^\kappa$  is strictly decreasing on  $(h(\kappa_1), \kappa_1)$ . Besides, Proposition 6.2 ensures that  $\sigma_d(A^\kappa)$  cannot be non-empty for some  $\tilde{h}(\kappa_1) \in (0, h(\kappa_1))$  otherwise  $\sigma_d(A^\kappa)$  would be non-empty for all  $\kappa_2 \in (\tilde{h}(\kappa_1), \kappa_1]$  which contradicts the definition of  $h(\kappa_1)$ . This completes the proof of the item 2) of Theorem 3.2.

**Remark 6.3.** For  $\kappa_2 \in [-\kappa_1, 0)$ , we have seen in Section 4 that  $\sigma_d(A^\kappa)$  is non-empty. Let  $\lambda_1^\kappa$  denote the smallest eigenvalue of  $\sigma_d(A^\kappa)$ . By adapting the proof of Proposition 6.2, one establishes that the map  $\kappa_2 \mapsto \lambda_1^\kappa$  is strictly increasing on  $[-\kappa_1, 0)$ . Together with Proposition 6.2, this shows the item 3) of Theorem 3.2.

## 7 Finiteness of the discrete spectrum

Finally, we establish the item 4) of Theorem 3.2 and so assume that  $\kappa_2 \in (-\kappa_1, \kappa_1)$ . Set again  $\alpha_1 = \arctan \kappa_1$ ,  $\alpha_2 = \arctan \kappa_2$ . Since  $|\alpha_2| < \alpha_1$ , similarly to (10), we have

$$\lambda_{\dagger}^\kappa = \mu_1^{\alpha_1} < \mu_1^{|\alpha_2|} = \mu_1^{\alpha_2}. \quad (39)$$

We remind the reader that  $\mu_1^{\alpha_j}$  stands for the smallest eigenvalue of the 2D problem (8) set in the pointed strip  $\Pi^{\alpha_j}$  appearing in (7). Observe that  $\Pi^{\alpha_2}$  can be obtained from  $\Pi^{-\alpha_2}$  by a symmetry with respect to the line  $\xi_2 = 1/2$  and a translation, which ensures that  $\mu_1^{-\alpha_2} = \mu_1^{\alpha_2}$  and so  $\mu_1^{|\alpha_2|} = \mu_1^{\alpha_2}$ .

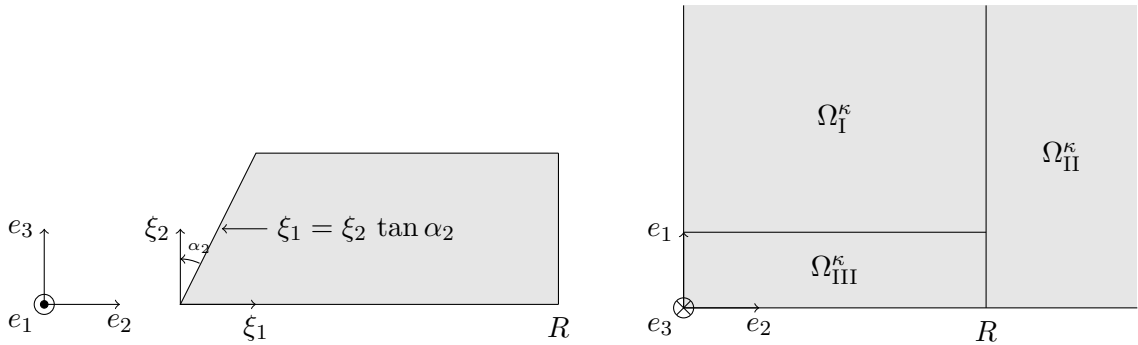


Figure 7: Left: truncated pointed strip  $\Pi^{\alpha_2}(R)$ . Right: bottom view of the decomposition of  $\Omega^\kappa$ .

For  $R > 0$ , define the truncated pointed strip

$$\Pi^{\alpha_2}(R) := \{(\xi_1, \xi_2) \in \Pi^{\alpha_2} \mid \xi_1 < R\}$$

(see Figure 7 left) and consider the problem

$$\begin{cases} -\Delta_\xi v = \mu v & \text{in } \Pi^{\alpha_2}(R) \\ v = 0 & \text{on } \{\xi \in \partial \Pi^{\alpha_2}(R) \mid \xi_2 = 0 \text{ or } \xi_2 = 1\} \\ \partial_\nu v = 0 & \text{on } \{\xi \in \partial \Pi^{\alpha_2}(R) \mid \xi_2 \neq 0 \text{ and } \xi_2 \neq 1\}. \end{cases} \quad (40)$$

Denote by  $\mu_1^{\alpha_2}(R)$  the smallest eigenvalue of (40). Since  $\mu_1^{\alpha_2}(R)$  converges to  $\mu_1^{\alpha_2}$  as  $R \rightarrow +\infty$ , according to (39), we can fix  $R > |\kappa_2|$  such that

$$\mu_1^{\alpha_2}(R) > \lambda_{\dagger}^{\kappa}. \quad (41)$$

Then let us divide  $\Omega^{\kappa}$  into the three domains

$$\begin{aligned} \Omega_{\text{I}}^{\kappa} &:= \{x \in \Omega^{\kappa} \mid x_1 > \kappa_1 \text{ and } x_2 < R\}, & \Omega_{\text{II}}^{\kappa} &:= \{x \in \Omega^{\kappa} \mid x_2 > R\}, \\ \Omega_{\text{III}}^{\kappa} &:= \{x \in \Omega^{\kappa} \mid x_1 < \kappa_1 \text{ and } x_2 < R\} \end{aligned}$$

(see the representation of Figure 7 right). Using (41), we obtain

$$\|\nabla_x \psi; L^2(\Omega_{\text{I}}^{\kappa})\|^2 \geq \lambda_{\dagger}^{\kappa} \|\psi; L^2(\Omega_{\text{I}}^{\kappa})\|^2 \quad \forall \psi \in H_0^1(\Omega^{\kappa}; \Sigma^{\kappa}). \quad (42)$$

On the other hand, from (22), we get

$$\|\nabla_x \psi; L^2(\Omega_{\text{II}}^{\kappa})\|^2 \geq \lambda_{\dagger}^{\kappa} \|\psi; L^2(\Omega_{\text{II}}^{\kappa})\|^2 \quad \forall \psi \in H_0^1(\Omega^{\kappa}; \Sigma^{\kappa}). \quad (43)$$

Besides, since  $\Omega_{\text{III}}^{\kappa}$  is bounded, the max-min principle ([3, Thm. 10.2.2]) guarantees that there is  $n \in \mathbb{N} := \{0, 1, 2, \dots\}$  such that

$$\lambda_{\dagger}^{\kappa} \leq \max_{E \subset \mathcal{E}_n} \inf_{\psi \in E \setminus \{0\}} \frac{\int_{\Omega_{\text{III}}^{\kappa}} |\nabla \psi|^2 dx}{\int_{\Omega_{\text{III}}^{\kappa}} \psi^2 dx}, \quad (44)$$

where  $\mathcal{E}_n$  denotes the set of subspaces of  $H_0^1(\Omega_{\text{III}}^{\kappa}; \Sigma_0 \cap \partial\Omega^{\kappa}) := \{\varphi \in H^1(\Omega_{\text{III}}^{\kappa}) \mid \varphi = 0 \text{ on } \Sigma_0 \cap \partial\Omega^{\kappa}\}$  of codimension  $n$ . Gathering (42)–(44), we deduce that there holds

$$\lambda_{\dagger}^{\kappa} \leq \max_{E \subset \tilde{\mathcal{E}}_n} \inf_{\psi \in E \setminus \{0\}} \frac{\int_{\Omega^{\kappa}} |\nabla \psi|^2 dx}{\int_{\Omega^{\kappa}} \psi^2 dx},$$

where this times  $\tilde{\mathcal{E}}_n$  stands for the set of subspaces of  $H_0^1(\Omega^{\kappa}; \Sigma_0)$  of codimension  $n$ . From the max-min principle, this proves that  $\sigma_{\text{d}}(A^{\kappa})$  contains at most  $n$  (depending on  $\kappa$ ) eigenvalues.

**Remark 7.1.** *Our simple proof above does not work to show that  $\sigma_{\text{d}}(A^{\kappa})$  is discrete when  $\kappa_2 = \pm\kappa_1$ . However we do not expect particular phenomenon and think the result also holds in this case. It is proved in [5, Thm. 1.2] when  $\kappa_2 = \kappa_1 = 1$ .*

## 8 Numerics and discussion

In this section, we illustrate some of the results above. In Figure 8, we represent an approximation of the first eigenvalue of the 2D problem (8) set in the pointed strip  $\Pi^{\alpha_1}$  with respect to  $\alpha_1 \in (0; 9\pi/20)$ . We use a rather crude method which consists in truncating the domain at  $\xi_2 = 12$  (see the picture of Figure 7 left) and imposing homogeneous Dirichlet boundary condition on the artificial boundary. Then we compute the spectrum in this bounded geometry by using a classical P2 finite element method. To proceed, we use the library `FreeFem++` [9] and display the results with `Matlab`<sup>1</sup> and `Paraview`<sup>2</sup>. The values we get are coherent with the ones recalled in (9).

In Figure 9–11, we fix  $\kappa_1 = 1$  (equivalently  $\alpha_1 = \pi/4$ ) and compute the first eigenvalue of  $\sigma_{\text{d}}(A^{\kappa})$  for different  $\kappa_2$ . For  $\kappa_1 = 1$ , the bound of the essential spectrum of  $A^{\kappa}$  is  $\lambda_{\dagger}^{\kappa} \approx 0.929\pi^2$  (see

<sup>1</sup>Matlab, <http://www.mathworks.com/>.

<sup>2</sup>Paraview, <http://www.paraview.org/>.

Figure 8 as well as [5]). For each of the three  $\kappa_2$ , in agreement with Theorem 3.2, we find an eigenvalue below the essential spectrum. Actually, in each situation our numerical experiments seem to indicate that there is only one eigenvalue in the discrete spectrum, which is a result that we have not proved. Interestingly, for  $\kappa_2 = -0.1$  (Figure 10), the eigenfunction is not particularly localized at the intersection of the oblique sides. We emphasize that here we simply compute the spectrum of the Laplace operator with mixed boundary conditions in the bounded domain  $\{x \in \Omega^\kappa \mid x_1 < 6 \text{ and } x_2 < 6\}$ . At  $x_1 = 6$  and  $x_2 = 6$ , we impose homogeneous Neumann boundary condition. Admittedly, this is a very naive approximation, especially in the case of Figure 10 where the eigenfunction has a poor decay at infinity. In the latter situation, the corresponding eigenvalue is very close to  $\lambda_\dagger^\kappa$  and we do not give its value because our approximation probably lacks precision.

In Figure 12, we represent eigenfunctions associated with two different eigenvalues of  $\sigma_d(A^\kappa)$  for  $\kappa = (3, -3)$ . For the 2D problem (8) in the pointed strip, the cardinal of the discrete spectrum can be made as large as desired by considering sufficiently sharp angles. We imagine that a similar phenomenon occurs in our geometry  $\Omega^\kappa$ . However to prove such a result is an open problem. At least the numerics of Figure 12 suggest that we can have more than one eigenvalue in  $\sigma_d(A^\kappa)$ .

As mentioned in the introduction, the fact that the discrete spectrum of  $A^\kappa$  for  $\kappa = (1, 1)$  is not empty ensures that the Dirichlet Laplacian in the so-called Fichera layer  $\mathcal{F}$  of Figure 1 right admits an eigenvalue below the essential spectrum. This can be proved by playing with symmetries and reconstructing  $\mathcal{F}$  from three versions of  $\Omega^{(1,1)}$ . Now, gluing six domains  $\Omega^{(1,-1)}$ , we can create the cubical structure pictured in Figure 13. Then, from Theorem 3.2 which guarantees that  $\sigma_d(A^\kappa)$  contains at least one eigenvalue, we deduce that the Dirichlet Laplacian in this geometry has at least one eigenvalue.

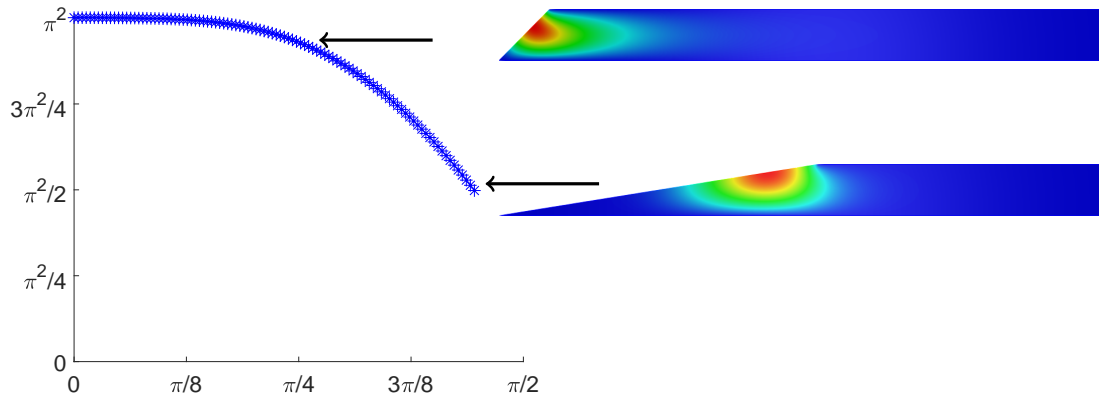


Figure 8: Curve  $\alpha_1 \mapsto \mu_1^{\alpha_1}$  for  $\alpha_1 \in (0; 9\pi/20)$ . According to Theorem 2.1, this gives the bound  $\lambda_\dagger^\kappa$  of the essential spectrum of  $A^\kappa$ . The two pictures correspond to eigenfunctions associated with  $\mu_1^{\alpha_1}$  for  $\alpha_1 = \pi/4$  and  $\alpha_1 = 9\pi/20$ .

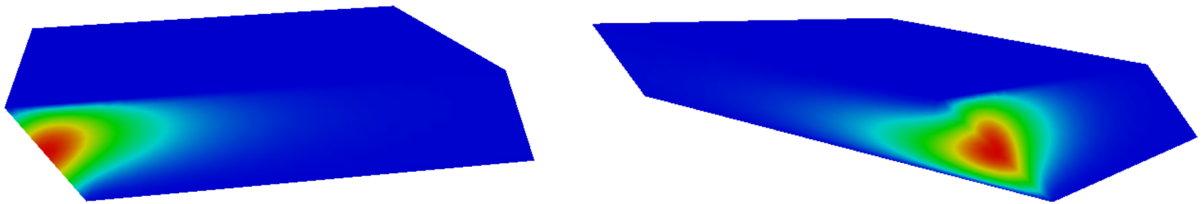


Figure 9: Two views of an eigenfunction associated with the first eigenvalue of  $\sigma_d(A^\kappa)$  for  $\kappa = (1, -1)$ . We find  $\lambda_1^\kappa \approx 0.81\pi^2$ .

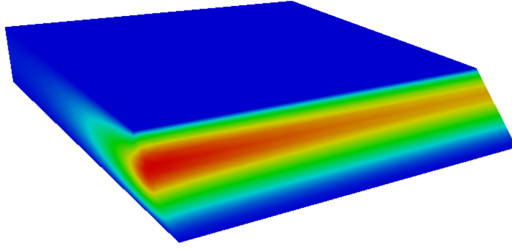


Figure 10: Eigenfunction associated with the first eigenvalue of  $\sigma_d(A^\kappa)$  for  $\kappa = (1, -0.1)$ .

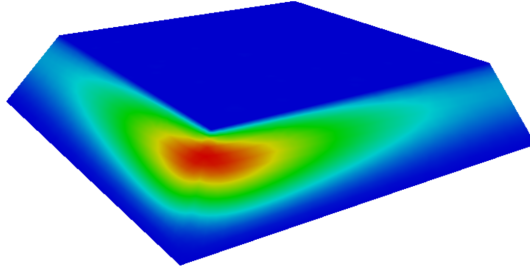


Figure 11: Eigenfunction associated with the first eigenvalue of  $\sigma_d(A^\kappa)$  for  $\kappa = (1, 1)$ . We find  $\lambda_1^\kappa \approx 0.90\pi^2$ .

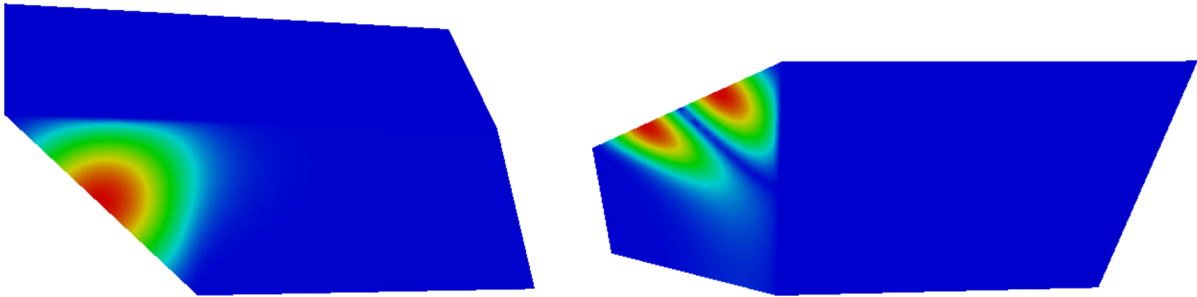


Figure 12: Eigenfunctions associated with two different eigenvalues of  $\sigma_d(A^\kappa)$  for  $\kappa = (3, -3)$ .

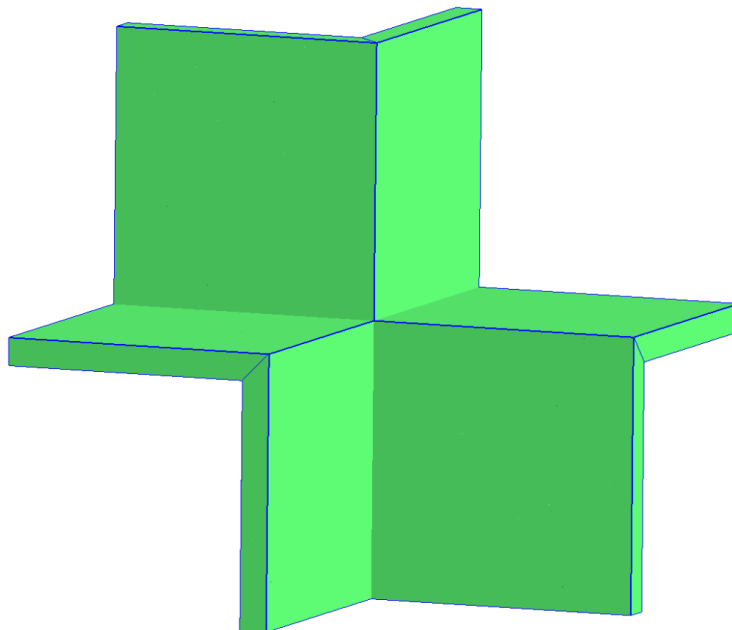


Figure 13: Cubical structure obtained by gluing six domains  $\Omega^\kappa$  with  $\kappa = (1, -1)$ .

## Appendix

Here we show that the Dirichlet Laplacian in  $\Omega^\kappa$  has no isolated eigenvalue nor eigenvalues embedded in the essential spectrum.

**Proposition 8.1.** *Fix  $\kappa = (\kappa_1, \kappa_2) \in \mathbb{R}^2$ . Assume that  $u \in H^1(\Omega^\kappa)$  satisfies*

$$\begin{cases} -\Delta_x u = \lambda u & \text{in } \Omega^\kappa \\ u = 0 & \text{on } \partial\Omega^\kappa \end{cases} \quad (45)$$

for some  $\lambda \in \mathbb{C}$ . Then there holds  $u \equiv 0$  in  $\Omega^\kappa$ .

*Proof.* The result is clear when  $\lambda \in \mathbb{C} \setminus (0, +\infty)$ . Let us apply the Rellich trick [19] to deal with the case  $\lambda > 0$ . If  $u$  solves (45), the function  $\partial_{x_1} u$  satisfies

$$\begin{cases} -\Delta_x(\partial_{x_1} u) = \lambda \partial_{x_1} u & \text{in } \Omega^\kappa \\ \partial_{x_1} u = 0 & \text{on } \Sigma^\kappa. \end{cases} \quad (46)$$

Note that since  $\Omega^\kappa$  is convex, classical regularity results (see e.g., the second basic inequality in [12, §II.6]) ensure that  $u$  belongs to  $H^2(\Omega^\kappa)$  so that  $\partial_{x_1} u$  falls in  $H^1(\Omega^\kappa)$ . Multiplying (45) by  $\partial_{x_1} u$ , (46) by  $u$ , integrating by parts and taking the difference, we obtain

$$0 = \int_{\Gamma_2^\kappa} \frac{\partial u}{\partial \nu} \frac{\partial u}{\partial x_1} ds \quad (47)$$

(note that  $\partial_{x_1} u = 0$  on  $\Gamma_1^\kappa$  because  $u = 0$  on  $\Gamma_1^\kappa$ ). On  $\Gamma_2^\kappa$ , the fact that  $u = 0$  implies

$$-\kappa_1 \frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x_3}.$$

Therefore we get

$$\frac{\partial u}{\partial \nu} = (1 + \kappa_1^2)^{-1/2} \left( -\frac{\partial u}{\partial x_1} + \kappa_1 \frac{\partial u}{\partial x_3} \right) = -\sqrt{1 + \kappa_1^2} \frac{\partial u}{\partial x_1}.$$

Using this in (47) gives  $\partial u / \partial \nu = 0$  on  $\Gamma_2^\kappa$ . Since we also have  $u = 0$  on  $\Gamma_2^\kappa$  and  $\Delta_x u + \lambda u = 0$  in  $\Omega^\kappa$ , the theorem of unique continuation (see e.g. [2], [4, Thm. 8.6]) guarantees that  $u \equiv 0$  in  $\Omega^\kappa$ .  $\square$

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